Free probability aspect of irreducible meandric systems, and some related observations about meanders

Alexandru Nica, [1]
University of Waterloo

Abstract

We consider the concept of irreducible meandric system introduced by Lando and Zvonkin. We place this concept in the lattice framework of $NC(n)$. As a consequence, we show that the even generating function for irreducible meandric systems is the R-transform of $\xi\eta$, where ξ and η are classically (commuting) independent random variables, and each of ξ, η has centred semicircular distribution of variance 1. Following this point of view, we make some observations about the symmetric linear functional on $\mathbb{C}[X]$ which has R-transform given by the even generating function for meanders.

1. Introduction

A *closed meandric system on* $2n$ *bridges* is a picture obtained by independently drawing two non-crossing pairings (a.k.a. "arch-diagrams") of $\{1, \ldots, 2n\}$, one of them above and the other one below a horizontal line, as exemplified in Figure 1. The combined arches of the two non-crossing pairings create a family of disjoint closed curves which wind up and down the horizontal line. If this family consists of precisely one curve going through all the points $\{1, \ldots, 2n\}$, then the meandric system in question is called a *closed meander*.

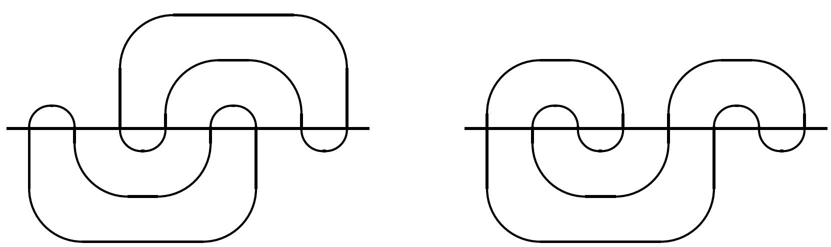

Figure 1. *Two closed meandric systems on* 8 *bridges, where one of them (on the right) is a closed meander.*

Let $\underline{m}_n^{(1)}$ denote the number of closed meanders on $2n$ bridges. Determining the asymptotic behaviour of the sequence $(\underline{m}_n^{(1)})_{n=1}^{\infty}$ is known to be a difficult problem – see e.g. [3], or Section 3.4 of the monograph [9]. In particular, the constant

$$c^{(1)} := \limsup_{n\to\infty} \Big(\underline{m}_n^{(1)} \Big)^{1/n} \tag{1.1}$$

[1] Research supported by a Discovery Grant from NSERC, Canada.

(reciprocal of radius of convergence for the generating function of the $\underline{m}_n^{(1)}$) is not known precisely. Numerical experimentation gives $c^{(1)} \approx 12.26$.

In the paper [8], Lando and Zvonkin considered the concept [2] of *irreducible meandric system* on $2n$ bridges. Every meander is in particular an irreducible meandric system; hence the number $\underline{m}_n^{(irr)}$ of irreducible meandric systems on $2n$ bridges is an upper bound for $\underline{m}_n^{(1)}$, and the constant

$$c^{(irr)} := \limsup_{n\to\infty} \Big(\underline{m}_n^{(irr)} \Big)^{1/n} \tag{1.2}$$

is an upper bound for $c^{(1)}$ of (1.1). Interestingly enough, Lando and Zvonkin could determine $c^{(irr)}$ precisely, namely

$$c^{(irr)} = \big(\pi/(4-\pi)\big)^2 \approx 13.39 \tag{1.3}$$

The equality (1.3) was obtained by finding a functional equation satisfied by the power series

$$1 + \sum_{n=1}^{\infty} \underline{m}_n^{(irr)} \, z^n, \tag{1.4}$$

which could then be used to determine the radius of convergence of the series.

In the present paper we place the concept of irreducible meandric system in the framework of lattice operations on $NC(n)$, the lattice of non-crossing partitions of $\{1, \ldots, n\}$. This is done via a natural bijective correspondence ("the doubling construction") between $NC(n)$ and the set of non-crossing pairings of $\{1, \ldots, 2n\}$, and leads to the following:

Theorem 1.1. *For every $n \in \mathbb{N}$, the number $\underline{m}_n^{(irr)}$ of irreducible meandric systems on $2n$ bridges can be described as*

$$\underline{m}_n^{(irr)} = \Big| \, \{(\pi,\rho) \in NC(n)^2 \mid \pi \vee \rho = 1_n \ \textit{and} \ \pi \wedge \rho = 0_n\} \, \Big|, \tag{1.5}$$

where "$\vee$" and "$\wedge$" are the join and respectively meet operations on $NC(n)$, while $0_n, 1_n$ are the minimal and respectively maximal element of $NC(n)$.

In connection to the above, it turns out that a close relative of the power series from (1.4) has a neat free probabilistic interpretation, as an R-transform (the counterpart in free probability for the concept of characteristic function of a random variable). More precisely, denoting

$$f_{\mathrm{irr}}(z) := \sum_{n=1}^{\infty} \underline{m}_n^{(irr)} \, z^{2n}, \tag{1.6}$$

one has the following:

Theorem 1.2. *The series f_{irr} from (1.6) is the R-transform of the product $\xi\eta$, where ξ and η are classically (commuting) independent random variables, and each of ξ and η has centred semicircular distribution of variance* 1.

[2] Henceforth we will implicitly assume the adjective "closed", and we will just write "meandric system" and "meander" to mean "closed meandric system" and "closed meander", respectively.

Theorem 1.2 can be obtained as a rather straightforward application of a result of Biane and Dehornoy [1].

We note that, in view of Theorem 1.2, the functional equation found by Lando and Zvonkin (when written for the series f_{irr}) becomes precisely the functional equation which is known to always be satisfied by the R-transform of a real random variable – see e.g. the discussion on pages 269-270 of the monograph [11]. Moreover, the calculation of radius of convergence made in [8] suggests a method for determining, more generally, the radius of convergence for R-transforms of certain random variables with "nice" moment-generating functions.

Returning to the analogy between the sequences $(\underline{m}_n^{(1)})_{n=1}^{\infty}$ and $(\underline{m}_n^{(irr)})_{n=1}^{\infty}$, it is then natural to consider the power series f_1 which is analogous to f_{irr} from Equation (1.6), but has the meander number $\underline{m}_n^{(1)}$ (instead of $\underline{m}_n^{(irr)}$) as coefficient of z^{2n}. Theorem 1.2 suggests that we write f_1 as an R-transform. We can in any case do that on an algebraic level – that is, we can get f_1 as R-transform of a linear functional $\nu : \mathbb{C}[X] \to \mathbb{C}$ which is *defined* via the requirement that $R_\nu = f_1$. The final section of the paper is devoted to making some observations about this functional ν: on the one hand we identify some sets of "strictly non-crossing" meandric systems which are counted by the even moments of ν, and on the other hand we observe that

$$\nu = \lim_{t \to 0} \nu_t^{\boxplus 1/t} \quad \text{(limit in moments)}$$

where $\boxplus$ refers to the operation of free additive convolution and $(\nu_t)_{t \in (0,\infty)}$ (defined precisely in Notation 5.5 of the paper) is a family of linear functionals of independent interest.

Besides the present introduction, the paper has four other sections. After a brief review of $NC(n)$ in Section 2, the proof of Theorem 1.1 is given in Section 3, then the proof and some comments around Theorem 1.2 are given in Section 4. The final Section 5 presents the related observations about meanders that were mentioned in the preceding paragraph.

2. Background on non-crossing partitions

In this section we do a brief review, mostly intended for setting the notations, of a few basic facts about the lattices of non-crossing partitions $NC(n)$. For a more detailed discussion of this topic, we refer the reader to Lectures 9 and 10 of the monograph [11].

Notation 2.1. Let n be a positive integer.

1^o We will work with partitions of the set $\{1, \ldots, n\}$. Our typical notation for such a partition is $\pi = \{V_1, \ldots, V_k\}$, where $V_1, \ldots, V_k$ (the *blocks* of π) are non-empty, pairwise disjoint sets with $\cup_{i=1}^k V_i = \{1, \ldots, n\}$. Occasionally, we will use the notation "$V \in \pi$" to mean that V is one of the blocks of the partition π. The number of blocks of π is denoted as $|\pi|$.

2^o We say that a partition π of $\{1, \ldots, n\}$ is *non-crossing* when it is not possible to find two distinct blocks $V, W \in \pi$ and numbers $a < b < c < d$ in $\{1, \ldots, n\}$ such that $a, c \in V$

and $b, d \in W$. This condition amounts precisely to the fact that one can draw the blocks of π without crossings in a picture of the kind exemplified in Figure 2 below.

3^o The set of all non-crossing partitions of $\{1, \ldots, n\}$ is denoted as $NC(n)$. This is one of the many combinatorial structures counted by Catalan numbers – indeed, it is not hard to verify that

$$|NC(n)| = C_n := \frac{(2n)!}{n!(n+1)!} \quad (n\text{-th Catalan number}).$$

4^o On $NC(n)$ we will use the partial order given by *reverse refinement*: for π, ρ we put

$$\Big(\pi \leq \rho \Big) \Leftrightarrow \left(\begin{array}{c} \text{for every } V \in \pi \text{ there} \\ \text{exists } W \in \rho \text{ such that } V \subseteq W \end{array} \right). \tag{2.1}$$

We denote by 0_n the partition of $\{1, \ldots, n\}$ into n blocks of 1 element, and we denote by 1_n the partition of $\{1, \ldots, n\}$ into 1 block of n elements. These are the minimum and respectively the maximum element in $\big(NC(n), \leq \big)$ (one has $0_n \leq \pi \leq 1_n$ for every $\pi \in NC(n)$).

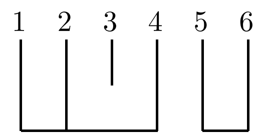

Figure 2. *Picture of the partition*
$\pi = \{ \{1, 2, 4\}, \{3\}, \{5, 6\} \} \in NC(6)$.

Notation and Remark 2.2. *(Lattice properties of* $(NC(n), \leq)$*).*
Let n be a positive integer, and consider the partially ordered set $(NC(n), \leq)$ from Notation 2.1.

1^o The *meet* of $\pi, \rho \in NC(n)$ is the partition $\pi \wedge \rho$ of $\{1, \ldots, n\}$ defined as

$$\pi \wedge \rho := \{V \cap W \mid V \in \pi, W \in \rho, V \cap W \neq \emptyset\}. \tag{2.2}$$

It is easily verified that $\pi \wedge \rho$ belongs to $NC(n)$, and is uniquely determined by its properties that:

$$\begin{cases} \bullet \quad \pi \wedge \rho \leq \pi \text{ and } \pi \wedge \rho \leq \rho; \\ \\ \bullet \quad \text{If } \lambda \in NC(n) \text{ is such that } \lambda \leq \pi \text{ and } \lambda \leq \rho, \\ \quad\;\; \text{then it follows that } \lambda \leq \pi \wedge \rho. \end{cases}$$

2^o For every $\pi, \rho \in NC(n)$ there exists a partition $\pi \vee \rho \in NC(n)$, called the *join* of π and ρ, which is uniquely determined by its properties that:

$$\begin{cases} \bullet \quad \pi \vee \rho \geq \pi \text{ and } \pi \vee \rho \geq \rho; \\ \\ \bullet \quad \text{If } \lambda \in NC(n) \text{ is such that } \lambda \geq \pi \text{ and } \lambda \geq \rho, \\ \quad\;\; \text{then it follows that } \lambda \geq \pi \vee \rho. \end{cases}$$

Unlike for $\pi \wedge \rho$, there is no simple explicit formula describing the blocks of $\pi \vee \rho$. (It is instructive to check, for instance, that the join of $\{\,\{1,3\},\ \{2\},\ \{4\}\,\}$ and $\{\,\{1\},\ \{3\},\ \{2,4\}\,\}$ in $NC(4)$ is the partition with one block 1_4.)

Notation 2.3. *(Permutation associated to $\pi \in NC(n)$.)*
Let n be a positive integer and let $\mathcal{S}_n$ denote the group of permutations of $\{1, \ldots, n\}$.

1^o For $\tau \in \mathcal{S}_n$, we will use the notation $\mathrm{Orb}(\tau)$ for the partition of $\{1, \ldots, n\}$ into orbits of τ (thus i and j are in the same block of $\mathrm{Orb}(\tau)$ if and only if there exists $p \in \mathbb{N}$ such that $\tau^p(i) = j$). We denote

$$\#(\tau) := |\mathrm{Orb}(\tau)| \quad \text{(number of orbits of the permutation } \tau).$$

2^o For $\pi \in NC(n)$ we will denote by P_π the permutation in $\mathcal{S}_n$ which has $\mathrm{Orb}(P_\pi) = \pi$, and performs an increasing cycle on every block of π: if $V = \{i_1, i_2, \ldots, i_k\} \in \pi$ with $i_1 < i_2 < \cdots < i_k$, then we have $P_\pi(i_1) = i_2, \ldots, P_\pi(i_{k-1}) = i_k$, $P_\pi(i_k) = i_1$.

Notation and Remark 2.4. *(Non-crossing pairings and the doubling construction.)*
Let n be a positive integer. We denote

$$NCP(2n) := \{\sigma \in NC(2n) \mid \text{every block } W \text{ of } \sigma \text{ has } |W| = 2\}.$$

The partitions in $NCP(2n)$ are called *non-crossing pairings*, or *arch-diagrams* on $2n$ points.

It is not hard to verify that $|NCP(2n)| = C_n$, the n-th Catalan number. Hence $NCP(2n)$ has precisely the same cardinality as $NC(n)$. One has in fact a natural bijection

$$NC(n) \ni \pi \mapsto A(\pi) \in NCP(2n), \tag{2.3}$$

which goes essentially by "doubling the points" in the picture of π, and will therefore be called *the doubling construction* (sometimes also referred to as "the fattening construction").

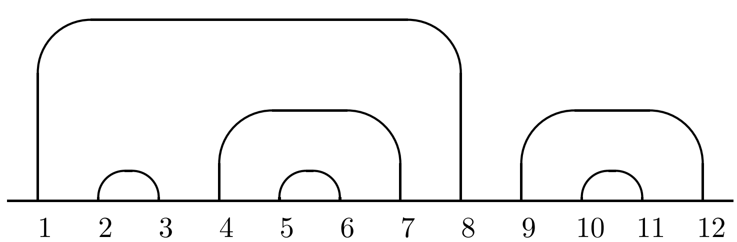

Figure 3. *The arch-diagram $A(\pi) \in NCP(12)$ obtained by performing the doubling construction on the partition π from Figure 2. (For $1 \le i \le 6$, the point i in the picture of π becomes the interval $[2i-1, 2i]$ in the picture of $A(\pi)$.)*

Formally, the arch-diagram $A(\pi)$ can be introduced by indicating how the permutation $P_{A(\pi)} \in \mathcal{S}_{2n}$ is described in terms of the permutation $P_\pi \in \mathcal{S}_n$. The formula doing this is:

$$\left\{ \begin{array}{lcll} P_{A(\pi)}(2i) & = & 2P_\pi(i) - 1, & \\ P_{A(\pi)}(2i-1) & = & 2P_\pi^{-1}(i), & 1 \leq i \leq n. \end{array} \right. \tag{2.4}$$

Indeed, it is easy to check that the assignment

$$2i \mapsto 2P_\pi(i) - 1, \ \ 2i-1 \mapsto 2P_\pi^{-1}(i), \text{ for } 1 \leq i \leq n,$$

defines a permutation $\tau \in \mathcal{S}_{2n}$ such that the orbit partition $\mathrm{Orb}(\tau)$ is in $NCP(2n)$; thus it makes sense to define $A(\pi)$ as the unique arch-diagram having $P_{A(\pi)} = \tau$.

From (2.4) it is clear that P_π can be retrieved from $P_{A(\pi)}$. This shows that the map $\pi \mapsto A(\pi)$ from (2.3) is one-to-one (hence bijective, since $|NC(n)| = |NCP(2n)|$).

3. Meanders and irreducible meandric systems

Definition 3.1. Let n be a positive integer, and let π, ρ be in $NC(n)$.

1^o The *meandric system* associated to π and ρ is the permutation $M_{\pi,\rho} \in \mathcal{S}_{2n}$ defined as follows:

$$\left\{ \begin{array}{ccccll} M_{\pi,\rho}(2i-1) & = & P_{A(\pi)}(2i-1) & = & 2P_\pi^{-1}(i), & \\ M_{\pi,\rho}(2i) & = & P_{A(\rho)}(2i) & = & 2P_\rho(i) - 1, & 1 \leq i \leq n. \end{array} \right. \tag{3.1}$$

The number of orbits $\#(M_{\pi,\rho})$ is called *number of components* of the meandric system.

2^o We will say that $M_{\pi,\rho}$ *is a meander* to mean that $\#(M_{\pi,\rho}) = 1$.

3^o We will say that $M_{\pi,\rho}$ *is reducible* to mean that there exists a proper subinterval $J = \{a, \ldots, b\} \subset \{1, \ldots, 2n\}$ (with $a \leq b$ in $\{1, \ldots, 2n\}$ having $b - a < 2n - 1$) such that J is invariant under the action of $M_{\pi,\rho}$. We will say that $M_{\pi,\rho}$ *is irreducible* to mean that it is not reducible.

Remark 3.2. 1^o Let π, ρ be as in the preceding definition. We record here, for further use, the following immediate consequence of the definition of $M_{\pi,\rho}$: for a set $S \subseteq \{1, \ldots, 2n\}$ one has that

$$\left(\begin{array}{c} S \text{ is invariant} \\ \text{for } M_{\pi,\rho} \end{array} \right) \Leftrightarrow \left(\begin{array}{c} S \text{ is at the same time} \\ \text{a union of blocks (pairs) of } A(\pi) \\ \text{and a union of blocks of } A(\rho) \end{array} \right). \tag{3.2}$$

Note that (3.2) implies, in particular, that every set $S \subseteq \{1, \ldots, 2n\}$ which is invariant for $M_{\pi,\rho}$ must have even cardinality.

2^o Recall from the introduction that for every $n \in \mathbb{N}$ we have denoted:

$$\underline{m}_n^{(1)} := \Big| \ \{(\pi, \rho) \in NC(n)^2 \mid M_{\pi,\rho} \text{ is a meander } \} \Big|, \tag{3.3}$$

and

$$\underline{m}_n^{(irr)} := \Big| \ \{(\pi,\rho) \in NC(n)^2 \mid M_{\pi,\rho} \text{ is irreducible } \} \Big| . \tag{3.4}$$

It is clear that every meander is in particular an irreducible meandric system, but the converse is not true (for instance, the meandric system depicted on the left side of Figure 1 is irreducible). Hence $\underline{m}_n^{(irr)} \geq \underline{m}_n^{(1)}$, where the inequality is generally strict. The smallest n for which $\underline{m}_n^{(irr)} > \underline{m}_n^{(1)}$ is $n = 4$ – the reader may find it amusing to verify that there exist precisely 4 irreducible meandric systems on 8 bridges which are not meanders, and this leads to $\underline{m}_4^{(irr)} = 46 = \underline{m}_4^{(1)} + 4$.

Lemma 3.3. *Let n be a positive integer, let π be a partition in $NC(n)$, and consider the corresponding arch-diagram $A(\pi) \in NCP(2n)$.*

1^o *For $1 \leq p \leq q \leq n$ one has that*

$$\left(\begin{array}{c} [2p-1, 2q] \cap \mathbb{Z} \textit{ is a} \\ \textit{union of blocks of } A(\pi) \end{array} \right) \Leftrightarrow \left(\begin{array}{c} [p,q] \cap \mathbb{Z} \textit{ is a} \\ \textit{union of blocks of } \pi \end{array} \right).$$

2^o *For $1 \leq p < q \leq n$ one has that*

$$\left(\begin{array}{c} [2p, 2q-1] \cap \mathbb{Z} \textit{ is a} \\ \textit{union of blocks of } A(\pi) \end{array} \right) \Leftrightarrow \left(\begin{array}{c} p \textit{ and } q \textit{ belong to} \\ \textit{the same block of } \pi \end{array} \right).$$

Proof. 1^o "$\Rightarrow$" We must prove that that if $i \in [p,q] \cap \mathbb{Z}$, then $P_\pi(i)$ still belongs to $[p,q]$. And indeed, for such i we have $2i \in [2p-1, 2q] \cap \mathbb{Z}$, hence our current hypothesis implies $P_{A(\pi)}(i) \in [2p-1, 2q]$. But then $P_\pi(i) = (P_{A(\pi)}(i) + 1)/2 \in [p, q + \frac{1}{2}]$, so (since $P_\pi(i)$ is an integer), we conclude that $P_\pi(i) \in [p,q] \cap \mathbb{Z}$, as required.

1^o "$\Leftarrow$" Here we must prove that if $m \in [2p-1, 2q] \cap \mathbb{Z}$, then $P_{A(\pi)}(m)$ still belongs to $[2p-1, 2q]$. We distinguish two cases.

Case 1: m is even. In this case we have $m = 2i$ with $i \in [p,q] \cap \mathbb{Z}$. The current hypothesis entails that $P_\pi(i) \in [p,q]$, so we find that

$$P_{A(\pi)}(m) = P_{A(\pi)}(2i) = 2P_\pi(i) - 1 \in [2p-1, 2q-1] \subseteq [2p-1, 2q], \quad \text{as required.}$$

Case 2: m is odd. In this case we have $m = 2i - 1$ with $i \in [p,q] \cap \mathbb{Z}$. The current hypothesis entails that $P_\pi^{-1}(i) \in [p,q]$, so we find that

$$P_{A(\pi)}(m) = P_{A(\pi)}(2i-1) = 2P_\pi^{-1}(i) \in [2p, 2q] \subseteq [2p-1, 2q], \quad \text{as required.}$$

2^o "$\Rightarrow$" We claim there exist $k \geq 1$ and $p = p_0 < p_1 < \cdots < p_k = q$ such that

$$P_{A(\pi)}(2p_{i-1}) = 2p_i - 1, \quad \forall\, 1 \leq i \leq k. \tag{3.5}$$

The points p_i are found recursively, in the way described as follows. We start with $p_0 = p$ and we look at $P_{A(\pi)}(2p) =: 2p_1 - 1$. The current hypothesis gives us that $2p_1 - 1 \in [2p, 2q-1]$, hence that $p < p_1 \leq q$. If $p_1 = q$ then we take $k = 1$ in (3.5) and we are done; so let us assume that $p_1 < q$. In this case we remark that $[2p, 2p_1 - 1] \cap \mathbb{Z}$ is a union of blocks of $A(\pi)$ (because $A(\pi)$ is non-crossing), hence the set-difference

$$[2p_1, 2q-1] \cap \mathbb{Z} = \Big([2p, 2q-1] \cap \mathbb{Z} \Big) \setminus \Big([2p, 2p_1 - 1] \cap \mathbb{Z} \Big)$$

must be a union of blocks of $A(\pi)$ as well. We can thus repeat the same procedure as above: we look at $P_{A(\pi)}(2p_1) =: 2p_2 - 1$, and from the invariance of $[2p_1, 2q-1] \cap \mathbb{Z}$ under $A(\pi)$ we infer that $p_1 < p_2 \leq q$. If $p_2 = q$ then we take $k = 2$ in (3.5) and we are done; while if $p_2 < q$, then we look at the invariant set $[2p_2, 2q-1] \cap \mathbb{Z}$ and consider $P_{A(\pi)}(2p_2) =: 2p_3 - 1$, and so on (where, of course, the process of finding new points p_i must stop after finitely many steps).

We next compare (3.5) against the formula $P_{A(\pi)}(2p_{i-1}) = 2P_\pi(p_{i-1}) - 1$ from the definition of $P_{A(\pi)}$, and we see that the points $p_0, p_1, \ldots, p_k$ must satisfy $P_\pi(p_{i-1}) = p_i$, for all $1 \leq i \leq k$. This implies that all of $p_0, p_1, \ldots, p_k$ belong to the same block of π, and (since $p_0 = p$ and $p_k = q$) the required conclusion follows.

2^o "$\Leftarrow$" From the definition of the permutation P_π it follows that there exist $k \geq 1$ and $p = p_0 < p_1 < \cdots < p_k = q$ such that $P_\pi(p_{i-1}) = p_i$, $1 \leq i \leq k$. We then have

$$[2p, 2q-1] \cap \mathbb{Z} = \cup_{i=1}^k \Big([2p_{i-1}, 2p_i - 1] \cap \mathbb{Z} \Big) = \cup_{i=1}^k \Big([2p_{i-1}, P_{A(\pi)}(2p_{i-1})] \cap \mathbb{Z} \Big).$$

This in turn implies (by taking into account that $A(\pi)$ is non-crossing) that $[2p, 2q-1] \cap \mathbb{Z}$ is a union of blocks of $A(\pi)$, as required. ■

Proposition 3.4. *Let n be a positive integer, let π, ρ be in $NC(n)$, and let us consider the arch-diagrams $A(\pi), A(\rho) \in NCP(2n)$ and the meandric system $M_{\pi,\rho} \in \mathcal{S}_{2n}$. The following three statements are equivalent:*

(1) $M_{\pi,\rho}$ is irreducible.

(2) $A(\pi) \vee A(\rho) = 1_{2n}$ (join considered in $NC(2n)$).

(3) $\pi \vee \rho = 1_n$ and $\pi \wedge \rho = 0_n$ (join and meet considered in $NC(n)$).

Proof. We will verify the equivalence of the complementary statements that:

($\overline{1}$) $M_{\pi,\rho}$ is reducible; ($\overline{2}$) $A(\pi) \vee A(\rho) \neq 1_{2n}$; ($\overline{3}$) $\pi \vee \rho \neq 1_n$ or $\pi \wedge \rho \neq 0_n$.

"($\overline{1}$) $\Rightarrow$ ($\overline{2}$)". Let J be a proper subinterval of $\{1, \ldots, 2n\}$ which is invariant under the action of $M_{\pi,\rho}$. Thus J is, at the same time, a union of blocks of $A(\pi)$ and a union of blocks of $A(\rho)$. Obviously, the same is true for $\overline{J} = \{1, \ldots, 2n\} \setminus J$, which implies that the partition $\sigma := \{J, \overline{J}\} \in NC(2n)$ is such that $\sigma \geq A(\pi)$ and $\sigma \geq A(\rho)$. It follows that $A(\pi) \vee A(\rho) \leq \sigma$ and hence that $A(\pi) \vee A(\rho) \neq 1_{2n}$, as required.

"($\overline{2}$) $\Rightarrow$ ($\overline{3}$)". Let us denote $A(\pi) \vee A(\rho) =: \sigma \in NC(2n)$. Every non-crossing partition has interval blocks, hence we can find $1 \leq a \leq b \leq 2n$ such that $J := [a, b] \cap \mathbb{Z}$ is a block of σ. Observe that $J \neq \{1, \ldots, 2n\}$ (since $\sigma \neq 1_{2n}$). Thus J is a proper subinterval of $\{1, \ldots, 2n\}$ which is, at the same time, a union of blocks of $A(\pi)$ and a union of blocks of $A(\rho)$. We distinguish two possible cases.

Case 1: $\min(J)$ *is an odd number.* In this case, J must be of the form $J := [2p-1, 2q] \cap \mathbb{Z}$ for some $1 \leq p \leq q \leq n$. Lemma 3.3.1 gives us that $V := [p, q] \cap \mathbb{Z}$ is at the same time a union of blocks of π and a union of blocks of ρ. Note that $V \neq \{1, \ldots, n\}$, since $J \neq \{1, \ldots, 2n\}$. Then $\lambda := \{V, \{1, \ldots, n\} \setminus V\}$ is in $NC(n)$, has $|\lambda| = 2$, and is such that $\pi \leq \lambda$ and $\rho \leq \lambda$; hence $\pi \vee \rho \leq \lambda$, implying $\pi \vee \rho \neq 1_n$, and $(\overline{3})$ holds.

Case 2: $\min(J)$ *is an even number.* In this case, J must be of the form $J := [2p, 2q-1] \cap \mathbb{Z}$ for some $1 \leq p < q \leq n$. Lemma 3.3.2 gives us that p and q belong to the same block of π, and also that they belong to the same block of ρ. This implies $\pi \wedge \rho \neq 0_n$ (as p, q are in the same block of $\pi \wedge \rho$), and $(\overline{3})$ holds in this case as well.

"$(\overline{3}) \Rightarrow (\overline{1})$". Here we must verify that either of the hypotheses $\pi \vee \rho \neq 1_n$ or $\pi \wedge \rho \neq 0_n$ imply the reducibility of $M_{\pi,\rho}$.

Claim 1. If $\pi \vee \rho \neq 1_n$, then $M_{\pi,\rho}$ is reducible.

Verification of Claim 1. Let us denote $\pi \vee \rho =: \lambda$. Every non-crossing partition has interval blocks, hence we can find $1 \leq p \leq q \leq n$ such that $[p, q] \cap \mathbb{Z}$ is a block of λ. Since $\pi \leq \lambda$, it follows that $[p, q] \cap \mathbb{Z}$ is a union of blocks of π, and Lemma 3.3.1 then gives us that $J := [2p-1, 2q] \cap \mathbb{Z}$ is a union of blocks of $A(\pi)$. In the same way we obtain that J is a union of blocks of $A(\rho)$. Note that $J \neq \{1, \ldots, 2n\}$ (from $J = \{1, \ldots, 2n\}$ we would infer $p = 1, q = n$, hence that $\lambda = 1_n$). Thus J is a proper subinterval of $\{1, \ldots, 2n\}$ which is invariant under $M_{\pi,\rho}$, and Claim 1 follows.

Claim 2. If $\pi \wedge \rho \neq 0_n$, then $M_{\pi,\rho}$ is reducible.

Verification of Claim 2. $\pi \wedge \rho$ has blocks that are not singletons, hence we can find $1 \leq p < q \leq n$ such that p and q are in the same block of $\pi \wedge \rho$. These p and q belong to the same block of π, hence Lemma 3.3.2 gives us that $J := [2p, 2q-1] \cap \mathbb{Z}$ is a union of blocks of $A(\pi)$. In the same way we find that J is a union of blocks of $A(\rho)$. Thus J is a proper subinterval of $\{1, \ldots, 2n\}$ which is invariant under $M_{\pi,\rho}$, and Claim 2 follows. ■

Remark 3.5. 1^o Theorem 1.1 follows from Proposition 3.4, by equating the cardinalities of the sets of (π, ρ)'s that are considered in the statements (1) and (3) of that proposition.

2^o Condition (2) of Proposition 3.4 has a nice interpretation supporting the idea that irreducible meandric systems truly are some kind of counterparts of meanders. To be specific, let $\mathcal{P}(n)$ denote the set of *all* partitions of $\{1, \ldots, n\}$, crossing or non-crossing, and on $\mathcal{P}(n)$ let us consider the partial order by reverse refinement (defined exactly as in formula (2.1) of Notation 2.1.4). Then $(\mathcal{P}(n), \leq)$ turns out to be a lattice, with meet operation "$\wedge$" defined exactly as for $NC(n)$, by block intersections (same formula as (2.2) of Notation 2.2). However, the join operation of $\mathcal{P}(n)$ no longer coincides with the "$\vee$" of $NC(n)$, and we will denote it (slightly differently) as "$\widetilde{\vee}$". For instance, the reader may find it instructive to note that $\{\{1,3\}, \{2\}, \{4\}\} \,\widetilde{\vee}\, \{\{1\}, \{3\}, \{2,4\}\} = \{\{1,3\}, \{2,4\}\} \in \mathcal{P}(4)$, in contrast to the comment about "$\vee$" which appeared in the last sentence of Remark 2.2.

Once the join $\widetilde{\vee}$ on $\mathcal{P}(n)$ is put into evidence, it is easy to verify that for every $n \in \mathbb{N}$ and $\pi, \rho \in NC(n)$ one has:

$$\left(\begin{array}{c} M_{\pi,\rho} \text{ is a meander} \\ \text{(in the sense of Def. 3.1.2)} \end{array} \right) \Leftrightarrow \left(A(\pi) \,\widetilde{\vee}\, A(\rho) = 1_{2n} \right), \tag{3.6}$$

with $A(\pi), A(\rho) \in NCP(2n)$ denoting the arch-diagrams associated to π and ρ, respectively. By comparing (3.6) to condition (2) of Proposition 3.4 we see that the concept of irreducible meandric system is indeed analogous to the one of meander – we only change the lattice where the join operation is being considered.

4. Counting irreducible meandric systems with free cumulants

The goal of this section is to explain how the power series $f_{\text{irr}}(z) = \sum_{n=1}^{\infty} \underline{m}_n^{(irr)} z^{2n}$ appears as R-transform of a nice probability distribution (viewed here as a linear functional on $\mathbb{C}[X]$). In order to make the presentation self-contained, we first review the relevant facts needed about free cumulants and R-transforms.

Definition and Remark 4.1. Let $\mu : \mathbb{C}[X] \to \mathbb{C}$ be linear with $\mu(1) = 1$.

1^o We will use the notation $(\kappa_n(\mu))_{n=1}^{\infty}$ for the sequence of *free cumulants* of μ. This is the sequence of complex numbers which is uniquely determined by the requirement that

$$\mu(X^n) = \sum_{\pi \in NC(n)} \Big(\prod_{V \in \pi} \kappa_{|V|}(\mu) \Big), \quad \forall\, n \in \mathbb{N}. \tag{4.1}$$

Equation (4.1) goes under the name of "moment-(free) cumulant" formula. For instance for $n \leq 3$ it says that

$$\mu(X) = \kappa_1(\mu),\ \mu(X^2) = \kappa_2(\mu) + \kappa_1(\mu)^2,\ \mu(X^3) = \kappa_3(\mu) + 3\kappa_1(\mu)\kappa_2(\mu) + \kappa_1(\mu)^3,$$

which then yields explicit expressions for the first free cumulants:

$$\kappa_1(\mu) = \mu(X),\ \kappa_2(\mu) = \mu(X^2) - \mu(X)^2,\ \kappa_3(\mu) = \mu(X^3) - 3\mu(X)\mu(X^2) + 2\mu(X)^3. \tag{4.2}$$

One can write a formula like in (4.2) for $\kappa_n(\mu)$ with general $n \in \mathbb{N}$, where the occurring coefficients are understood in terms of the Möbius function of $NC(n)$; but we will not need this here (the interested reader may check pp. 175-176 in Lecture 11 of the monograph [11]).

2^o The power series $R_\mu(z) := \sum_{n=1}^{\infty} \kappa_n(\mu) z^n$ is called the *R-transform* of μ.

3^o The *functional equation of the R-transform* says that

$$R_\mu\big(z(1 + M_\mu(z)) \big) = M_\mu(z), \tag{4.3}$$

with R_μ as above and $M_\mu(z) := \sum_{n=1}^{\infty} \mu(X^n) z^n$ (moment-generating series for μ). For the derivation of (4.3) out of the moment-cumulant formula (4.1), see e.g Theorem 10.23 in [11].

4^o The functional μ is said to be *symmetric* when it has $\mu(X^{2n-1}) = 0$, $\forall\, n \in \mathbb{N}$. An immediate consequence of the moment-cumulant formula (4.1) is that μ is symmetric if and only if $\kappa_{2n-1}(\mu) = 0$, $\forall\, n \in \mathbb{N}$.

Definition and Remark 4.2. In this definition, $(\Omega, \mathcal{F}, P)$ is a probability space and $\xi, \eta : \Omega \to \mathbb{R}$ are random variables with finite moments of all orders.

1^o Let $\mu : \mathbb{C}[X] \to \mathbb{C}$ be the linear functional determined by the requirement that

$$\mu(X^n) = \int \xi^n \, dP, \quad \forall \, n \in \mathbb{N} \cup \{0\}.$$

We will refer to μ as the *distribution* of ξ. The free cumulants of μ (as introduced in the preceding definition) are also called *free cumulants of* ξ, and we will use the notation

$$\kappa_n(\xi) := \kappa_n(\mu), \quad n \in \mathbb{N}.$$

2^o We will say that the random variable ξ is *centred semicircular of variance* 1 to mean that its distribution is $(2\pi)^{-1}\sqrt{4-t^2}\, dt$ on $[-2, 2]$, i.e that for every $n \in \mathbb{N}$ one has

$$\int \xi^n \, dP = \frac{1}{2\pi} \int_{-2}^{2} t^n \sqrt{4-t^2} \, dt = \begin{cases} 0, & \text{if } n \text{ is odd,} \\ C_{n/2} \text{ (Catalan number),} & \text{if } n \text{ is even.} \end{cases} \tag{4.4}$$

It is easy to verify (by using the equalities $|NCP(2n)| = C_n$, $n \in \mathbb{N}$) that the free cumulants of a ξ as in (4.4) are

$$\kappa_n(\xi) = \begin{cases} 1, & \text{if } n = 2, \\ 0, & \text{otherwise.} \end{cases} \tag{4.5}$$

3^o Suppose that the random variables ξ and η are independent, hence that the product $\xi\eta$ has moments

$$\int (\xi\eta)^n \, dP = \int \xi^n \, dP \cdot \int \eta^n \, dP, \quad n \in \mathbb{N}.$$

Theorem 1.2 of [1] gives the following formula for calculating the free cumulants of $\xi\eta$ in terms of those of ξ and of η: for every $n \in \mathbb{N}$ one has

$$\kappa_n(\xi\eta) = \sum_{\substack{\pi, \rho \in NC(n) \ such \\ that \ \pi \vee \rho = 1_n}} \Big(\prod_{V \in \pi} \kappa_{|V|}(\xi) \Big) \Big(\prod_{W \in \rho} \kappa_{|W|}(\eta) \Big). \tag{4.6}$$

The next proposition is a rephrasing of Theorem 1.2 from the introduction.

Proposition 4.3. *Let $\xi, \eta : \Omega \to \mathbb{R}$ be independent random variables (as in Remark 4.2.3), where each of ξ, η is centred semicircular of variance 1 (as in Remark 4.2.2). Then the free cumulants of the product $\xi\eta$ are*

$$\kappa_{2n-1}(\xi\eta) = 0 \ \textit{and} \ \kappa_{2n}(\xi\eta) = \underline{m}_n^{(irr)}, \quad n \in \mathbb{N}. \tag{4.7}$$

Proof. It is clear that $\xi\eta$ has vanishing odd moments, which implies that $\kappa_{2n-1}(\xi\eta) = 0$ for all $n \in \mathbb{N}$. For an even free cumulant $\kappa_{2n}(\xi\eta)$ we calculate as follows:

$$\begin{aligned}
\kappa_{2n}(\xi\eta) = & \sum_{\substack{\sigma,\theta \in NC(2n) \text{ such} \\ \text{that } \sigma\vee\theta = 1_{2n}}} \Big(\prod_{V\in\sigma} \kappa_{|V|}(\xi) \Big) \Big(\prod_{W\in\theta} \kappa_{|W|}(\theta) \Big) \\
& \text{(by Theorem 1.2 of [1] – Equation (4.6))} \\
= & \sum_{\substack{\sigma,\theta \in NCP(2n) \text{ such} \\ \text{that } \sigma\vee\theta = 1_{2n}}} 1 \quad \text{(by Equation (4.5))} \\
= & \,|\, \{(\sigma,\theta) \in NCP(2n)^2 \mid \sigma \vee \theta = 1_{2n}\} \,| \\
= & \,|\, \{(\pi,\rho) \in NC(n)^2 \mid A(\pi) \vee A(\rho) = 1_{2n}\} \,| \\
& \text{(by writing } \sigma = A(\pi), \theta = A(\rho)) \\
= & \, \underline{m}_n^{(irr)} \quad \text{(due to “(1) } \Leftrightarrow \text{ (2)” in Proposition 3.4).}
\end{aligned}$$

■

Remark 4.4. The paper [1] pays special attention to a sequence of numbers denoted as $(b^*_{n,2})_{n=1}^{\infty}$, where one puts

$$b^*_{n,2} := \Big| \{(\pi,\rho) \in NC(n)^2 \mid \pi \wedge \rho = 0_n\} \Big|, \quad n \in \mathbb{N}. \tag{4.8}$$

One of the main points made in [1] about the above sequence is that it can be neatly identified as a sequence of free cumulants:

$$b^*_{n,2} = \kappa_n(\, (\xi\eta)^2 \,), \quad n \in \mathbb{N}, \tag{4.9}$$

for the same ξ, η as considered in Proposition 4.3.

Now, one has a non-trivial result about how the free cumulants of the square of a symmetric random variable (here $(\xi\eta)^2$) are expressed in terms of the even free cumulants of the random variable itself. This is done via an equation which resembles the moment-cumulant formula, and says in the case at hand that

$$\kappa_n(\, (\xi\eta)^2 \,) = \sum_{\pi \in NC(n)} \Big(\prod_{V\in\pi} \kappa_{2|V|}(\xi\eta) \Big), \quad \forall\, n \in \mathbb{N}. \tag{4.10}$$

For the proof of (4.10), see Proposition 11.25 in [11].

In view of the interpretations we have for the free cumulants on the two sides of Equation (4.10), we thus arrive to a formula which relates the numbers $b^*_{n,2}$ to irreducible meandric systems, namely

$$b^*_{n,2} = \sum_{\pi \in NC(n)} \Big(\prod_{V\in\pi} \underline{m}_{|V|}^{(irr)} \Big), \quad \forall\, n \in \mathbb{N}. \tag{4.11}$$

The numbers $b^*_{n,2}$ are part of a larger collection of numbers $b^*_{n,d}$ with $n, d \in \mathbb{N}$ – see Equation (2.6) of [1] for a description of $b^*_{n,d}$ given in terms of non-crossing partitions. We note that for $d = 3$ one has

$$b^*_{n,3} = \Big| \{(\pi_1, \pi_2, \pi_3) \in NC(n)^3 \mid \pi_1 \wedge \pi_2 = 0_n \text{ and } \pi_2 \vee \pi_3 = 1_n\} \Big|, \quad n \in \mathbb{N}. \tag{4.12}$$

There is a slight resemblance of Equation (4.12) with (1.5) of Theorem 1.1, which prompts the question if one could also find a formula relating the numbers $b^*_{n,3}$ to meandric systems.

Remark 4.5. 1^o The linear functional involved in Proposition 4.3 (that is, the distribution of $\xi\eta$) has vanishing moments of odd order, while its even moment of order $2n$ is

$$\int (\xi\eta)^{2n}\, dP = \int \xi^{2n}\, dP \cdot \int \eta^{2n}\, dP = C_n^2, \quad n \in \mathbb{N}.$$

Upon writing the functional equation of the R-transform (Equation (4.3)) for this particular functional, one thus gets that

$$f_{\text{irr}}\Big(z(1 + \sum_{n=1}^{\infty} C_n^2 z^{2n}) \Big) = \sum_{n=1}^{\infty} C_n^2 z^{2n}.$$

Modulo some trivial transformations, this is the same functional equation as found by Lando and Zvonkin in [8].

2^o The method used in [8] for obtaining the radius of convergence of f_{irr} points to a class of functionals $\mu : \mathbb{C}[X] \to \mathbb{C}$ with tractable radius of convergence for R_μ, as follows. Suppose that:

(i) All the free cumulants $(\kappa_n(\mu))_{n=1}^{\infty}$ are real non-negative numbers.
(ii) The moment series $M_\mu(z)$ has a finite positive radius of convergence r_o.
(iii) There exist $c > 0$ and $\beta > 1$ such that (with r_o from (ii)) one has $\mu(X^n) \le c r_o^{-n} n^{-\beta}$, for all $n \in \mathbb{N}$.

Then it makes sense to consider the finite value $M_\mu(r_o) := \sum_{n=1}^{\infty} \mu(X^n) r_o^n \in (0, \infty)$, and the radius of convergence of the R-transform R_μ is equal to r_1, where

$$r_1 = r_o(1 + M_\mu(r_o)). \tag{4.13}$$

The reason for occurrence of this specific value r_1 is that upon writing the functional equation of the R-transfom as

$$M_\mu(z) = R_\mu(w) \quad \text{for } w = z(1 + M_\mu(z)),$$

and upon letting z and w grow along the positive semiaxes of the z-plane and w-plane, they will hit at the same time the singularities that are closest to origin for M_μ and R_μ, respectively.

In the specific case of Proposition 4.3 (when μ is the distribution of $\xi\eta$), one has $M_\mu(z) = \sum_{n=1}^{\infty} C_n^2 z^{2n}$ with radius of convergence $r_o = 1/4$. From the asymptotics $C_n \sim c4^n n^{-3/2}$

(which folows e.g. from Stirling's formula) it follows that in (iii) above we may take $\beta = 3$. The radius of convergence for $R_\mu = f_{\mathrm{irr}}$ thus comes out as

$$r_1 = \frac{1}{4} \cdot \Big(1 + \sum_{n=1}^{\infty} C_n^2 (1/4)^{2n} \Big). \tag{4.14}$$

As shown in [8], one can determine precisely that $1 + \sum_{n=1}^{\infty} C_n^2 (1/4)^{2n} = 4(4-\pi)/\pi$, which leads to $r_1 = (4-\pi)/\pi$, and to the value of $c^{(irr)}$ indicated in Equation (1.3).

5. Counting meanders with free cumulants?

In this section we look at the framework analogous to the one of Theorem 1.2, but where instead of the power series f_{irr} from Theorem 1.2 we consider the series

$$f_1(z) := \sum_{n=1}^{\infty} \underline{m}_n^{(1)} z^{2n}, \tag{5.1}$$

with $\underline{m}_n^{(1)}$ counting the meanders on $2n$ bridges, $n \in \mathbb{N}$. More precisely, Theorem 1.2 says that "$f_{\mathrm{irr}} = R_\mu$", where $\mu : \mathbb{C}[X] \to \mathbb{C}$ is the symmetric linear functional with $\mu(X^{2n}) = C_n^2$, $n \in \mathbb{N}$; so we consider the analogous equation "$f_1 = R_\nu$", which is now used as a definition, for a functional ν. The fact that a linear functional on $\mathbb{C}[X]$ can be defined by prescribing its R-transform follows immediately from the moment-cumulant formula (see e.g. Exercise 16.21 in [11]).

Notation 5.1. We denote as $\nu : \mathbb{C}[X] \to \mathbb{C}$ the linear functional with $\nu(1) = 1$ and such that $R_\nu = f_1$, the series from Equation (5.1). That is, ν is uniquely determined by the requirement that its free cumulants are

$$\kappa_{2n-1}(\nu) = 0 \ \text{ and } \ \kappa_{2n}(\nu) = \underline{m}_n^{(1)}, \ \ \forall\, n \in \mathbb{N}. \tag{5.2}$$

In order to give an alternative description of ν in terms of its moments, we introduce the following concept.

Definition 5.2. Let n be a positive integer, let π, ρ be in $NC(n)$, and consider the meandric system $M_{\pi,\rho} \in \mathcal{S}_{2n}$. Let $\sigma := \mathrm{Orb}\big(M_{\pi,\rho} \big)$, the partition of $\{1, \ldots, 2n\}$ into orbits of $M_{\pi,\rho}$. If σ is non-crossing, then we will say that the meandric system $M_{\pi,\rho}$ is *strictly non-crossing.*

[For a concrete example, the meandric system depicted on the left side of Figure 1 is not strictly non-crossing, since it has $\mathrm{Orb}(M_{\pi,\rho}) = \{ \{1,2,5,6\}, \ \{3,4,7,8\} \} \notin NC(8)$.]

Proposition 5.3. *For every $n \in \mathbb{N}$, the functional ν introduced in Notation 5.1 has $\nu(X^{2n-1}) = 0$ and*

$$\nu(X^{2n}) = \Big| \, \{(\pi, \rho) \in NC(n)^2 \mid M_{\pi,\rho} \text{ is strictly non-crossing}\} \, \Big|. \tag{5.3}$$

Proof. The vanishing of odd moments of ν follows from the vanishing of its odd free cumulants, as mentioned in Remark 4.1.4. Here we fix $n \in \mathbb{N}$ and we address the calculation of $\nu(X^{2n})$. We start from the right-hand side of Equation (5.3), which we write as

$$\sum_{\sigma \in NC(2n)} \mid \{(\pi, \rho) \in NC(n)^2 \mid \mathrm{Orb}(M_{\pi,\rho}) = \sigma\} \mid .$$

Since all orbits of $M_{\pi,\rho}$ have even cardinality, the above summation reduces to

$$\sum_{\sigma \in NCE(2n)} \mid \{(\pi, \rho) \in NC(n)^2 \mid \mathrm{Orb}(M_{\pi,\rho}) = \sigma\} \mid, \tag{5.4}$$

where we denoted

$$NCE(2n) := \{\sigma \in NC(2n) \mid \ |W| \text{ is even, for all } W \in \sigma\}. \tag{5.5}$$

Let us momentarily fix a partition $\sigma = \{W_1, \ldots, W_k\} \in NCE(2n)$. To every $(\pi, \rho) \in NC(n)^2$ such that $\mathrm{Orb}(M_{\pi,\rho}) = \sigma$ we can associate a k-tuple of meanders on $|W_1|$, respectively $|W_2|, \ldots$, respectively $|W_k|$ bridges, in the way described as follows. For every $1 \leq i \leq k$, the set W_i is at the same time a union of blocks of $A(\pi)$ and a union of blocks of $A(\rho)$. We can thus consider the restrictions $A(\pi) \mid W_i$ and $A(\rho) \mid W_i$, which become non-crossing pairings $\sigma_i, \theta_i \in NCP(|W_i|)$ upon the re-numbering of the elements of W_i as $1, \ldots, |W_i|$. We then write $\sigma_i = A(\pi_i)$, $\theta_i = A(\rho_i)$ with $\pi_i, \rho_i \in NC(|W_i|/2)$, and we note that M_{π_i,ρ_i} is a meander (due to the fact that W_i is an orbit of $M_{\pi,\rho}$).

The preceding paragraph has put into evidence a natural map

$$(\pi, \rho) \mapsto \Big((\pi_1, \rho_1), \ldots, (\pi_k, \rho_k) \Big),$$

going from $\{(\pi, \rho) \in NC(n)^2 \mid \mathrm{Orb}(M_{\pi,\rho}) = \sigma\}$ to

$$\prod_{i=1}^{k} \{(\pi_i, \rho_i) \in NC(|W_i|/2)^2 \mid M_{\pi_i,\rho_i} \text{ is a meander}\}. \tag{5.6}$$

This map is in fact a bijection. Indeed, if we start with a k-tuple $\big((\pi_1, \rho_1), \ldots, (\pi_k, \rho_k) \big)$ from the set in (5.6), then every $(A(\pi_i), A(\rho_i))$ can be re-numbered into a meander on W_i, and the k meanders thus created will combine together into a meandric system with orbit-partition equal to σ. (A detail to be emphasized at this point is that, when putting together the k meanders, we don't get any crossings. This holds because σ was picked to be in $NC(2n)$. Indeed, from the fact that the blocks of σ don't cross it follows that there can't be crossings among the re-numbered $A(\pi_i)$'s, and likewise for the re-numbered $A(\rho_i)$'s.)

The conclusion of the preceding two paragraphs is that, for a fixed $\sigma \in NCE(2n)$, we have a bijection between $\{(\pi, \rho) \in NC(n)^2 \mid \mathrm{Orb}(M_{\pi,\rho}) = \sigma\}$ and the set from (5.6). Upon equating cardinalities, we infer that

$$\mid \{(\pi, \rho) \in NC(n)^2 \mid \mathrm{Orb}(M_{\pi,\rho}) = \sigma\} \mid = \prod_{W \in \sigma} \underline{m}^{(1)}_{|W|/2}. \tag{5.7}$$

We now unfix σ, and plug the equality (5.7) into (5.4), to find that the right-hand side of (5.3) can be written as

$$\sum_{\sigma \in NCE(2n)} \Big(\prod_{W \in \sigma} \underline{m}^{(1)}_{|W|/2} \Big).$$

By taking into account what are the free cumulants of ν, we see that the latter expression equals

$$\sum_{\sigma \in NC(2n)} \Big(\prod_{W \in \sigma} \kappa_{|W|}(\nu) \Big),$$

which gives $\nu(X^{2n})$, as required. ∎

Remark 5.4. The online encyclopedia of integer sequences gives, following the paper [7], the meander numbers $\underline{m}_n^{(1)}$ for $1 \leq n \leq 24$ (see www.oeis.org, sequence A005315). Starting from these values, one can use the moment-cumulant formula (4.1) in order to calculate [3] the even moments of ν up to order 48, as listed in Table 1 on the next page. An interesting problem concerning these moments is to find non-trivial lower bounds for the radius of convergence of the series $M_\nu(z) = \sum_{n=1}^{\infty} \nu(X^{2n}) z^{2n}$. This, in turn, could give non-trivial upper bounds for the constant $c^{(1)}$ in Equation (1.1), via an argument like the one mentioned in Remark 4.5.2.

The moments listed in Table 1 show that (unfortunately) ν is not positive definite – for instance the determinant of the matrix $[\, \nu(X^{2i+2j})\,]_{0 \leq i,j \leq 9}$ is negative. It would nevertheless be of interest to pursue the study of ν as an analytic object (as a signed measure, perhaps).

Another observation about ν is that it relates to a family of functionals which are interesting in their own right, and are defined as follows.

Notation 5.5. 1^o For every $n \in \mathbb{N}$ and $k \in \{1, \ldots, n\}$ we denote

$$\underline{m}_n^{(k)} := \Big| \{ (\pi, \rho) \in NC(n)^2 \mid M_{\pi,\rho} \text{ has exactly } k \text{ orbits} \} \Big| .$$

(For $k = 1$, this agrees with the notation $\underline{m}_n^{(1)}$ used since the introduction.)

2^o Let t be a parameter in $(0, \infty)$. We will denote as $\nu_t : \mathbb{C}[X] \to \mathbb{C}$ the linear functional with $\nu_t(1) = 1$ and which has moments given by

$$\nu_t(X^{2n-1}) = 0 \ \text{ and } \ \nu_t(X^{2n}) = \sum_{k=1}^{n} \underline{m}_n^{(k)} t^k, \quad n \in \mathbb{N}. \tag{5.8}$$

Remark 5.6. In order to state, in the next proposition, the connection between ν and the ν_t's, let us review some more (rather standard) bits of terminology.

(a) $\boxplus$-powers. Let $\mu : \mathbb{C}[X] \to \mathbb{C}$ be linear with $\mu(1) = 1$, and let t be in $(0, \infty)$. We denote as $\mu^{\boxplus t}$ the linear functional $\widetilde{\mu} : \mathbb{C}[X] \to \mathbb{C}$ which has $\widetilde{\mu}(1) = 1$ and is uniquely determined by the requirement that $R_{\widetilde{\mu}}(z) = t R_\mu(z)$. The exponential notation $\mu^{\boxplus t}$ is meaningful in connection to the operation $\boxplus$ of free additive convolution, see for instance pp. 231-233 in Lecture 14 of [11].

[3] I am grateful to Mathieu Guay-Paquet and Franz Lehner for their help with computer-aided calculations.

(b) Convergence in moments. Let $(\mu_t)_{t\in(0,\infty)}$ and μ be linear maps from $\mathbb{C}[X]$ to $\mathbb{C}$, which send 1 to 1. We will write

$$\text{“}\lim_{t\to 0}\mu_t = \mu, \text{ in moments”} \tag{5.9}$$

to mean that $\lim_{t\to 0}\mu_t(X^n) = \mu(X^n)$ for all $n \in \mathbb{N}$. Upon invoking the moment-cumulant formula (4.1) it is immediate that, equivalently, one can define (5.9) via the requirement that $\lim_{t\to 0}\kappa_n(\mu_t) = \kappa_n(\mu)$ for all $n \in \mathbb{N}$.

n	Free cumulant $\kappa_{2n}(\nu) = \underline{m}_n^{(1)}$	Moment $\nu(X^{2n})$	Ratio $\nu(X^{2n})/C_n^2$
1	1	1	1.00000
2	2	4	1.00000
3	8	25	1.00000
4	42	192	0.97959
5	262	1664	0.94331
6	1828	15626	0.89681
7	13820	155439	0.84459
8	110954	1615208	0.78987
9	933458	17371372	0.73486
10	8152860	192116692	0.68101
11	73424650	2174556080	0.62925
12	678390116	25101780538	0.58013
13	6405031050	294692569630	0.53396
14	61606881612	3510877767198	0.49085
15	602188541928	42371895120585	0.45081
16	5969806669034	517281396522616	0.41377
17	59923200729046	6380271752428956	0.37960
18	608188709574124	79428025047086276	0.34816
19	6234277838531806	997137221492794404	0.31926
20	64477712119584604	12614196796924143524	0.29276
21	672265814872772972	160696941192856063186	0.26845
22	7060941974458061392	2060412248079723985072	0.24619
23	74661728661167809752	26575640310738797507800	0.22581
24	794337831754564188184	344671815256362419882958	0.20715

Table 1. *Even free cumulants and even moments of the linear functional ν, up to order 48. The rightmost column of the table shows the probability that a random meandric system on $2n$ bridges is strictly non-crossing, for $1 \le n \le 24$.*

Proposition 5.7. *One has* $\lim_{t\to 0} \nu_t^{\boxplus 1/t} = \nu$, *in moments, where* ν_t *and* ν *are as in Notations 5.5 and 5.1, respectively.*

Proof. We will prove the convergence of free cumulants,

$$\lim_{t\to 0} \kappa_n(\nu_t^{\boxplus 1/t}) = \kappa_n(\nu), \quad \forall\, n \in \mathbb{N}. \tag{5.10}$$

For n oddd, (5.10) holds trivially, because ν and the ν_t's are symmetric functionals. For n even, $n = 2p$, the limit in (5.10) amounts to

$$\lim_{t\to 0} \frac{1}{t}\kappa_{2p}(\nu_t) = \underline{m}_p^{(1)}. \tag{5.11}$$

We will obtain this as a consequence of the following stronger claim.

Claim. For every $p \in \mathbb{N}$, there exists a polynomial $Q_p \in \mathbb{Z}[t]$, with $Q_p(0) = 0$ and $Q_p'(0) = \underline{m}_p^{(1)}$, such that $\kappa_{2p}(\nu_t) = Q_p(t)$ for all $t \in (0, \infty)$.

Verification of Claim. By induction on p. For $p = 1$ we have $\kappa_2(\nu_t) = \nu_t(X^2) - \nu_t(X)^2 = t$, hence we can take $Q_1(t) = t = \underline{m}_1^{(1)} t$.

Induction step: we fix $p \geq 2$ and we verify the claim for this p, by assumming it was already verified for $1, \ldots, p-1$. For every $t \in (0, \infty)$, the moment-cumulant formula says that

$$\nu_t(X^{2p}) = \sum_{\sigma \in NC(2p)} \Big(\prod_{W \in \sigma} \kappa_{|W|}(\nu_t) \Big).$$

Since ν_t is symmetric, the latter sum has in fact only contributions from partitions in $NCE(2p)$ (same notation as in Equation (5.5) from the proof of Proposition 5.3). By separating the term which corresponds to $\sigma = 1_{2p}$, we find that

$$\kappa_{2p}(\nu_t) = \nu_t(X^{2p}) - \sum_{\substack{\sigma \in NCE(2p) \\ \sigma \neq 1_{2p}}} \Big(\prod_{W \in \sigma} \kappa_{|W|}(\nu_t) \Big). \tag{5.12}$$

The induction hypothesis allows us to replace the sum which is subtracted in (5.12) with

$$\sum_{\substack{\sigma \in NCE(2p) \\ \sigma \neq 1_{2p}}} \Big(\prod_{W \in \sigma} Q_{|W|/2}(t) \Big) =: U(t),$$

where $U \in \mathbb{Z}[t]$ has $U(0) = U'(0) = 0$. If on the right-hand side of (5.12) we also substitute $\nu_t(X^{2p}) = \underline{m}_p^{(1)} t + \sum_{k=2}^{p} \underline{m}_p^{(k)} t^k$, it clearly follows that $\kappa_{2p}(\nu_t)$ has indeed the form required by the claim. ■

Remark 5.8. It is natural to ask: for what values of t is ν_t positive definite? Proposition 5.7 shows this cannot hold for $t \to 0$ (if there would exist a sequence $t_n \to 0$ with ν_{t_n} positive definite, then it would follow that ν is positive definite as well). On the other hand, there are values of $t \geq 1$ for which ν_t is sure to be positive definite because it admits an *operator* *model* (that is, it arises as scalar spectral measure for a selfadjoint operator on Hilbert space). The largest known set of such t's appears to be $\{2\cos\frac{\pi}{n} \mid n \geq 3\} \cup [2, \infty)$; for t in this set, an operator model for ν_t is described in [2] (see discussion preceding Proposition 3.1 of that paper). Some other operator models (or random matrix models) for ν_t are known in the special case when $t \in \mathbb{N}$: see Section 4 of [5], or Section 4 of the physics paper [10]; the latter model is also described in Section 6.2 of [4].

The next proposition presents a version of the model from [10], [4], which is placed in the framework of a $*$-probability space (that is, $\mathcal{A}$ is a unital $*$-algebra over $\mathbb{C}$, and $\varphi : \mathcal{A} \to \mathbb{C}$ is linear with $\varphi(1_{\mathcal{A}}) = 1$ and $\varphi(a^*a) \geq 0$ for all $a \in \mathcal{A}$). The interesting point of the proposition is that it involves tensors products of elements from a free family – a mixture of classical and free probability which may provide a good setting for further study of meandric systems.

Proposition 5.9. *Let d be a positive integer. Suppose that $a_1, \ldots, a_d$ is a free family of selfadjoint elements in a $*$-probability space $(\mathcal{A}, \varphi)$, such that every a_i ($1 \leq i \leq d$) has centred semicircular distribution of variance* 1. *Consider the $*$-probability space $(\mathcal{A}\otimes\mathcal{A}, \varphi\otimes\varphi)$, and the selfadjoint element*

$$x = a_1 \otimes a_1 + a_2 \otimes a_2 + \cdots + a_d \otimes a_d \in \mathcal{A} \otimes \mathcal{A}. \tag{5.13}$$

Then x has distribution ν_d with respect to $\varphi \otimes \varphi$.

Proof. The conclusion of the proposition amounts to the fact that for every $n \in \mathbb{N}$ one has

$$(\varphi \otimes \varphi)(x^{2n-1}) = 0 \ \text{ and } \ (\varphi \otimes \varphi)(x^{2n}) = \sum_{k=1}^{n} \underline{m}_n^{(k)} d^k. \tag{5.14}$$

Throughout the proof we fix an $n \in \mathbb{N}$ for which we will verify the second formula (5.14) (the easy verification that $(\varphi \otimes \varphi)(x^{2n-1}) = 0$ is left to the reader).

We start by expanding $(a_1 \otimes a_1 + a_2 \otimes a_2 + \cdots + a_d \otimes a_d)^{2n}$, and by applying $\varphi \otimes \varphi$ to the result, to get

$$(\varphi \otimes \varphi)(x^{2n}) = \sum_{i(1),\ldots,i(2n)=1}^{d} \Big(\varphi(a_{i(1)} \cdots a_{i(2n)}) \Big)^2. \tag{5.15}$$

Let us momentarily fix a $(2n)$-tuple $(i(1), \ldots, i(2n)) \in \{1, \ldots, d\}^{2n}$. The moment-cumulant formula for several variables (for which we refer to Lecture 11 of [11]) expresses the moment $\varphi(a_{i(1)} \cdots a_{i(2n)})$ as a certain summation over $NC(2n)$,

$$\varphi(a_{i(1)} \cdots a_{i(2n)}) = \sum_{\sigma \in NC(2n)} \mathrm{term}_\sigma. \tag{5.16}$$

Due to the free independence of $a_1, \ldots, a_d$ and to the special form of the free cumulants of the a_i's (namely $\kappa_2(a_i) = 1$ and $\kappa_p(a_i) = 0$ for $p \neq 2$), it turns out that in (5.16) we always have $\mathrm{term}_\sigma \in \{0, 1\}$, with

$$(\mathrm{term}_\sigma = 1) \ \Leftrightarrow \ \left(\begin{array}{c} \sigma \in NCP(2n), \text{ and for every} \\ W = \{p, q\} \in \sigma \text{ one has } i(p) = i(q) \end{array} \right).$$

It comes in handy to introduce here a notation, say "$\sigma \leq \ker i$" to mean [4] that σ is in $NCP(2n)$ and fulfills the compatibility condition $(W = \{p,q\} \in \sigma) \Rightarrow i(p) = i(q)$. With this notation, (5.16) becomes

$$\varphi(a_{i(1)} \cdots a_{i(2n)}) = |\ \{\sigma \in NCP(2n) \mid \sigma \leq \ker i\}\ | \,. \tag{5.17}$$

We now unfix $(i(1), \ldots, i(2n))$ and return to Equation (5.15). We find that

$$(\varphi \otimes \varphi)(x^{2n}) = \sum_{i(1),\ldots,i(2n)=1}^{d} \left|\ \Big\{(\sigma,\theta) \in NCP(2n)^2 \mid \sigma \leq \ker i \text{ and } \theta \leq \ker i\Big\}\ \right|$$

$$= \sum_{\sigma,\theta \in NCP(2n)} \left|\ \Big\{(i(1),\ldots,i(2n)) \in \{1,\ldots,d\}^{2n} \mid \sigma \leq \ker i \text{ and } \theta \leq \ker i\Big\}\ \right|, \tag{5.18}$$

where (5.18) is obtained via change of order of summation in the suitable sum of 0's and 1's indexed by the aggregated σ, θ and $(i(1), \ldots, i(2n))$.

Let us now momentarily fix $\sigma, \theta \in NCP(2n)$, which we write as $A(\pi)$ and respectively $A(\rho)$, with $\pi, \rho \in NC(n)$. It is immediate that for a tuple $(i(1), \ldots, i(2n)) \in \{1, \ldots, d\}^{2n}$, the condition "$\sigma \leq \ker i$ and $\theta \leq \ker i$" is equivalent to asking that $i : \{1, \ldots, 2n\} \to \{1, \ldots, d\}$ is constant along the orbits of the permutation $M_{\pi,\rho}$. This clearly implies

$$\Big|\{(i(1),\ldots,i(2n)) \mid \sigma \leq \ker i \text{ and } \theta \leq \ker i\}\Big| = d^{\#(M_{\pi,\rho})}. \tag{5.19}$$

We finally let σ, θ run in $NCP(2n)$ (equivalently, we let π, ρ run in $NC(n)$) and we replace (5.19) into (5.18), to obtain

$$\begin{aligned}(\varphi \otimes \varphi)(x^{2n}) &= \sum_{\pi,\rho \in NC(n)} d^{\#(M_{\pi,\rho})} \\ &= \sum_{k=1}^{n} |\ \{(\pi,\rho) \in NC(n)^2 \mid \#(M_{\pi,\rho}) = k\}\ | \cdot d^k \\ &= \sum_{k=1}^{n} \underline{m}_n^{(k)} d^k,\end{aligned}$$

as had to be proved. ■

Remark 5.10. I am grateful to Roland Speicher for bringing to my attention the following fact: one can easily adjust the proof of Proposition 5.9 in order to find combinatorial interpretations for the moments (with respect to $\varphi \otimes \varphi$) of more general elements of the form $a_1 \otimes b_1 + \cdots + a_d \otimes b_d \in \mathcal{A} \otimes \mathcal{A}$, where each of $a_1, \ldots, a_d$ and $b_1, \ldots, b_d$ is a free family of elements of $(\mathcal{A}, \varphi)$. Here are two nice examples obtained on these lines.

[4] It is customary to denote by $\ker i$ the partition of $\{1, \ldots, 2n\}$ defined via the requirement that for $1 \leq p, q \leq 2n$ one has: "$(p, q$ belong to the same block of $\ker i) \Leftrightarrow i(p) = i(q)$". The notation "$\sigma \leq \ker i$" can thus be construed as an inequality with respect to the reverse refinement order (cf. Definition 2.1.3, Remark 3.5) on the set $\mathcal{P}(2n)$ of all partitions of $\{1, \ldots, 2n\}$.

(a) Let $(\mathcal{A}, \varphi)$ and $a_1, \ldots, a_d \in \mathcal{A}$ be exactly as in Proposition 5.9, and let us put

$$y := a_1^2 \otimes a_1^2 + \cdots + a_d^2 \otimes a_d^2 \in \mathcal{A} \otimes \mathcal{A}.$$

It is known that a_i^2 has free cumulants $\kappa_p(a_i^2) = 1$ for all $p \in \mathbb{N}$. By using this fact and by repeating the method of calculation from the proof of Proposition 5.9, one finds that

$$(\varphi \otimes \varphi)(y^n) = \sum_{\pi, \rho \in NC(n)} d^{|\pi \widetilde{\vee} \rho|}, \quad n \in \mathbb{N}, \tag{5.20}$$

where "$\widetilde{\vee}$" is the join operation for the lattice $\mathcal{P}(n)$ (cf. Remark 3.5). The occurrence of the operation $\widetilde{\vee}$ in connection to partitions from $NC(n)$ may seem a bit strange, but matrices of the form $\left[q^{|\pi \widetilde{\vee} \rho|} \right]_{\pi, \rho \in NC(n)}$ do appear in the research literature – see e.g. [6].

(b) With $a_1, \ldots, a_d$ still being exactly as in Proposition 5.9, let us put

$$z := a_1 \otimes a_1^2 + \cdots + a_d \otimes a_d^2 \in \mathcal{A} \otimes \mathcal{A}.$$

It is immediate that one has $(\varphi \otimes \varphi)(z^{2n-1}) = 0$ for all $n \in \mathbb{N}$. For the even moments of z, the method of calculation from the proof of Proposition 5.9 (and the combined knowledge of the free cumulants of a_i and a_i^2) leads to the formula

$$(\varphi \otimes \varphi)(z^{2n}) = \sum_{\substack{\sigma \in NC(2n) \\ \theta \in NCP(2n)}} d^{|\sigma \widetilde{\vee} \theta|}, \quad n \in \mathbb{N}, \tag{5.21}$$

a version of (5.20) which now mixes together non-crossing pairings with general non-crossing partitions from $NC(2n)$.

Acknowledgement

This paper was written while I was on a long-term visit to MAP5, the Applied Mathematics Department of Université Paris 5. I acknowledge the hospitality of MAP5, and many interesting discussions with Florent Benaych-Georges, Thierry Cabanal-Duvillard and Camille Male around the topic of the paper.

References

[1] P. Biane, P. Dehornoy. Dual Garside structure of braids and free cumulants of products, *Séminaire Lotharingien de Combinatoire* B72b (2014), 15 pp. Also available as arXiv:1407.1604.

[2] S. Curran, V.F.R. Jones, D. Shlyakhtenko. On the symmetric enveloping algebra of planar algebra subfactors, *Transactions of the American Mathematical Society* 366 (2014), 113-133. Also available as arXiv:1105.1721.

[3] P. Di Francesco, O. Golinelli, E. Guitter. Meanders, folding and arch statistics, in *Special Issue: Combinatorics and Physics, Math. and Comp. Modelling* 26 (1997), 97-147. Also available as arXiv:hep-th/9506030.

[4] P. Di Francesco. Matrix model combinatorics: applications to folding and coloring, in *Random matrix models and their applications*, volume 40 of *Math. Sci. Res. Inst. Publ.*, pages 111-170, Cambridge University Press, 2001. Also available as arXiv:math-ph/9911002.

[5] M. Fukuda, P. Sniady. Partial transpose of random quantum states: exact formulas and meanders, *Journal of Mathematical Physics* 54 (2013), 042202. Also available as arXiv:1211.1525.

[6] D.M. Jackson. The lattice of non-crossing partitions and the Birkhoff-Lewis equations, *European Journal of Combinatorics* 15 (1994), 245-250.

[7] I. Jensen. Enumeration of plane meanders, arXiv:cond-mat/9910313. See also the expanded version appearing in *Journal of Physics A* 33 (2000), 5953-5963.

[8] S.K. Lando, A.K. Zvonkin. Meanders, *Selecta Mathematica Sovietica* 11 (1992), 117-144.

[9] S.K. Lando, A.K. Zvonkin. *Graphs on surfaces and their applications*, Encyclopaedia of Mathematical Sciences, volume 141, Springer, 2004.

[10] Y. Makeenko. Strings, matrix models, and meanders, arXiv:hep-th/9512211.

[11] A. Nica, R. Speicher. *Lectures on the combinatorics of free probability*, London Mathematical Society Lecture Note Series 335, Cambridge University Press, 2006.

Alexandru Nica
Department of Pure Mathematics, University of Waterloo,
Waterloo, Ontario N2L 3G1, Canada.
Email: anica@uwaterloo.ca