Upper Bounds for Prime Gaps Related to Firoozbakht's Conjecture

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Abstract

We study two kinds of conjectural bounds for the prime gap after the kth prime p_k: (A) $p_{k+1} < (p_k)^{1+1/k}$ and (B) $p_{k+1} - p_k < \log^2 p_k - \log p_k - b$ for $k > 9$. The upper bound (A) is equivalent to Firoozbakht's conjecture. We prove that (A) implies (B) with $b = 1$; on the other hand, (B) with $b = 1.17$ implies (A). We also give other sufficient conditions for (A) that have the form (B) with $b \to 1$ as $k \to \infty$.

1 Introduction

In 1982 Firoozbakht proposed the following conjecture [6, p. 185]:

Firoozbakht's Conjecture. If p_k is the kth prime, the sequence $(p_k^{1/k})_{k\in\mathbb{N}}$ is decreasing. Equivalently, for all $k \ge 1$, the prime p_{k+1} is bounded by the inequality

$$p_{k+1} < (p_k)^{1+1/k}. \tag{1}$$

Several authors [7, 8, 10, 11] have observed that

- Firoozbakht's conjecture (1) implies *Cramér's conjecture* $p_{k+1} - p_k = O(\log^2 p_k)$ [2].
- If conjecture (1) is true and k is large, then

$$p_{k+1} - p_k < \log^2 p_k - \log p_k. \tag{2}$$

(Sun [10, 11] gives a variant of (2) with a larger right-hand side, $\log^2 p_k - \log p_k + 1$.)

In Section 2 we prove that (1) implies a sharper bound than (2):

$$p_{k+1} - p_k \;<\; \log^2 p_k - \log p_k - b \quad \text{for all } k > 9, \tag{3}$$

with $b = 1$. If the exact value of $k = \pi(p_k)$ is not available, then a violation of (2) or (3) might be used to *disprove* Firoozbakht's conjecture (1). However, given a pair of primes p_k, p_{k+1}, the validity of (2) alone is not enough for the verification of (1). We discuss this in more detail in Section 3; see also [4]. In Section 4 we prove that (3) with $b = 1.17$ implies (1); we also give other sufficient conditions for (1). Probabilistic considerations [2, 3, 4, OEIS A235402] suggest that bounds (1), (2), (3) hold for *almost all maximal gaps* between primes.

2 A corollary of Firoozbakht's conjecture

Theorem 1. *If conjecture* (1) *is true, then*

$$p_{k+1} - p_k \;<\; \log^2 p_k - \log p_k - 1 \quad \textit{for all } k > 9.$$

Proof. It is easy to check that

$$\frac{x + \log^2 x}{\log x - 1 - \frac{1}{\log x}} \;<\; \frac{x}{\log x - 1 - \frac{1}{\log x} - \frac{1}{\log^2 x}} \quad \text{for } x \geq 285967. \tag{4}$$

Denote by $\pi(x)$ the prime-counting function. From Axler [1, Corollary 3.6] we have

$$\frac{x}{\log x - 1 - \frac{1}{\log x} - \frac{1}{\log^2 x}} \;<\; \pi(x) \quad \text{for } x \geq 1772201. \tag{5}$$

Taking the log of both sides of (1) we find that Firoozbakht's conjecture (1) is equivalent to

$$k \;<\; \frac{\log p_k}{\log p_{k+1} - \log p_k}. \tag{6}$$

Let $k \geq 133115$. Then $p_k \geq 1772201$. By setting $x = p_k$ in (4) and (5), we see that inequalities (4), (5), (6) form a chain. Therefore, if Firoozbakht's conjecture is true, then

$$\frac{p_k + \log^2 p_k}{\log p_k - 1 - \frac{1}{\log p_k}} \;<\; \frac{\log p_k}{\log p_{k+1} - \log p_k} \quad \text{for } p_k \geq 1772201. \tag{7}$$

Cross-multiplying, we get

$$(\log p_{k+1} - \log p_k)(p_k + \log^2 p_k) \;<\; \log^2 p_k - \log p_k - 1. \tag{8}$$

We have

$$\frac{y}{x+y} \;<\; \log(x+y) - \log x \qquad \text{for every } x, y > 0. \tag{9}$$

Setting $x = p_k$ and $y = p_{k+1} - p_k$, we can replace the left-hand side of (8) by a smaller quantity $(p_{k+1} - p_k)(p_k + \log^2 p_k)/p_{k+1}$ to obtain the inequality

$$\frac{(p_{k+1} - p_k)(p_k + \log^2 p_k)}{p_{k+1}} \ < \ \log^2 p_k - \log p_k - 1,$$

which is equivalent to

$$(p_{k+1} - p_k)(p_k + \log^2 p_k) \ < \ (p_k + (p_{k+1} - p_k))(\log^2 p_k - \log p_k - 1),$$

$$p_{k+1} - p_k \ < \ \frac{p_k}{p_k + \log p_k + 1}(\log^2 p_k - \log p_k - 1).$$

This proves the theorem for every $k \geq 133115$ because $p_k/(p_k + \log p_k + 1) < 1$. Separately, for $9 < k < 133115$ we verify the desired inequality by direct computation. □

3 Does a given prime gap confirm or disprove Firoozbakht's conjecture?

Given p_k and p_{k+1}, where the prime gap $p_{k+1} - p_k$ is "large" and $k = \pi(p_k)$ is *not* known, can we decide whether this gap confirms or disproves Firoozbakht's conjecture? The answer is, in most cases, *yes*. We showed this in [4, Sect. 3] and established the following theorem:

Theorem 2. ([4, Sect. 4]). *Firoozbakht's conjecture* (1) *is true for all primes* $p_k < 4 \times 10^{18}$.

In the verification of (1) for $p_k < 4 \times 10^{18}$ we have *not* used bound (2) or (3); see [4]. Indeed, (2) is a corollary of (1); as such, (2) might be true even when (1) is false. Here is a more detailed discussion. Define (see Table 1):

$$\begin{aligned} f_k &= p_k^{1+1/k} - p_k && \text{(the upper bound for } p_{k+1} - p_k \text{ predicted by (1));} \\ \ell_k &= \log^2 p_k - \log p_k && \text{(the upper bound for } p_{k+1} - p_k \text{ predicted by (2)).} \end{aligned}$$

One can prove that $f_k < \ell_k$ when $k \to \infty$; moreover, $f_k = \ell_k - 1 + o(1)$ (see *Appendix*). Computation shows that $f_k < \ell_k$ for $p_k \geq 11783$ $(k \geq 1412)$. Suppose there is a prime $q \in [p_k + f_k,\ p_k + \ell_k]$; for example, there is such a prime, $q = 2010929$, when $p_k = 2010733$ (see line 7 in Table 1). Now what if there were no other primes between p_k and q? Then we would have $p_{k+1} = q$, Firoozbakht's conjecture (1) would be *false*, while (2) would still be *true*. So (2) is not particularly useful for *verifying* (1). On the other hand, any violation of (2) would immediately disprove Firoozbakht's conjecture (1). Clearly, similar reasoning is valid for (3) with $b \leq 1$. However, in the next section we prove that (3) with $b = 1.17$ is a *sufficient condition* for Firoozbakht's conjecture (1). We will also give a few other sufficient conditions that have the form (3) with $b \to 1$ as $k \to \infty$.

k	p_k	$p_{k+1} - p_k$	$f_k = p_k^{1+1/k} - p_k$	$\ell_k = \log^2 p_k - \log p_k$
6	13	4	6.934	4.014
9	23	6	9.586	6.696
30	113	14	19.286	17.621
217	1327	34	44.709	44.515
3385	31397	72	96.188	96.861
31545	370261	112	150.529	151.581
149689	2010733	148	194.972	196.142
1319945	20831323	210	265.959	267.137
1094330259	25056082087	456	548.237	549.389
94906079600	2614941710599	652	787.801	788.925
662221289043	19581334192423	766	904.982	906.097
6822667965940	218209405436543	906	1055.966	1057.071
49749629143526	1693182318746371	1132	1193.418	1194.516

Table 1: Upper bounds for prime gaps $p_{k+1} - p_k$ predicted by (1) and (2); $p_k \in$ A111943 [9]

4 Sufficient conditions for Firoozbakht's conjecture

Theorem 3. *If*

$$p_{k+1} - p_k \;<\; \log^2 p_k - \log p_k - 1.17 \quad \textit{for all } k > 9 \;\; (p_k \ge 29), \tag{10}$$

then Firoozbakht's conjecture (1) *is true.*

Proof. From Axler [1, Corollary 3.5] we have

$$\log x - 1 - \frac{1.17}{\log x} \;<\; \frac{x}{\pi(x)} \quad \text{for every } x \ge 5.43. \tag{11}$$

Let $k > 9$. Multiplying both sides of (11) by $\log x$, taking $x = p_k$, and using (10), we get

$$p_{k+1} - p_k \;<\; \log^2 p_k - \log p_k - 1.17 \;<\; \frac{p_k \log p_k}{k} \quad \text{for } p_k \ge 29; \tag{12}$$

therefore,

$$\frac{p_{k+1} - p_k}{p_k} \;<\; \frac{\log p_k}{k} \qquad \text{for } p_k \ge 29. \tag{13}$$

We have

$$\log(x+y) - \log x \;<\; \frac{y}{x} \qquad \text{for every } x, y > 0.$$

Setting $x = p_k$ and $y = p_{k+1} - p_k$, we can replace the left-hand side of (13) by a smaller quantity $\log p_{k+1} - \log p_k$ to obtain the inequality

$$\log p_{k+1} - \log p_k \;<\; \frac{\log p_k}{k}, \tag{14}$$

which is equivalent to

$$\log_{p_k} \frac{p_{k+1}}{p_k} < \frac{1}{k}.$$

Now, exponentiation with base p_k yields Firoozbakht's conjecture (1) for $p_k \geq 29$. This completes the proof since for small p_k conjecture (1) holds unconditionally [4]. □

Other sufficient conditions for (1). Based on the $\pi(x)$ formula of Panaitopol [5], Axler gives a family of upper bounds for $\pi(x)$ [1, Corollary 3.5]:

$$\begin{aligned}
\pi(x) &< \frac{x}{\log x - 1 - \frac{1.17}{\log x}} && \text{for } x \geq 5.43,\\
\pi(x) &< \frac{x}{\log x - 1 - \frac{1}{\log x} - \frac{3.83}{\log^2 x}} && \text{for } x \geq 9.25,\\
\pi(x) &< \frac{x}{\log x - 1 - \frac{1}{\log x} - \frac{3.35}{\log^2 x} - \frac{15.43}{\log^3 x}} && \text{for } x \geq 14.36,\\
\pi(x) &< \frac{x}{\log x - 1 - \frac{1}{\log x} - \frac{3.35}{\log^2 x} - \frac{12.65}{\log^3 x} - \frac{89.6}{\log^4 x}} && \text{for } x \geq 21.95.
\end{aligned}$$

Just as in Theorem 3, we can transform the above upper bounds into sufficient conditions for Firoozbakht's conjecture (1) and obtain our next theorem.

Theorem 4. *If one or more of the following conditions hold for all* $p_k > 4 \times 10^{18}$:

$$\begin{aligned}
p_{k+1} - p_k &< \log^2 p_k - \log p_k - 1.17,\\
p_{k+1} - p_k &< \log^2 p_k - \log p_k - 1 - \frac{3.83}{\log p_k},\\
p_{k+1} - p_k &< \log^2 p_k - \log p_k - 1 - \frac{3.35}{\log p_k} - \frac{15.43}{\log^2 p_k},\\
p_{k+1} - p_k &< \log^2 p_k - \log p_k - 1 - \frac{3.35}{\log p_k} - \frac{12.65}{\log^2 p_k} - \frac{89.6}{\log^3 p_k},
\end{aligned}$$

then Firoozbakht's conjecture (1) *is true.*

In the statement of Theorem 4, we have taken into account that for $p_k < 4 \times 10^{18}$ conjecture (1) holds unconditionally [4]. We do not give a proof of Theorem 4; it is fully similar to the proof of Theorem 3.

Remarks.

(i) In inequality (10) the right-hand side is an increasing function of p_k. Therefore, if (10) holds for a *maximal prime gap* with $p_k =$ A002386(n), then (10) must also be true for *every* p_k between A002386(n) and A002386($n+1$). So an easy way to prove Theorem 2 is to check (1) directly for all primes $p_k \leq 89$, then verify (10) just for maximal prime gaps with $p_k =$ A002386(n) ≥ 89.

(ii) In Theorem 4, the coefficients of $(\log p_k)^{-n}$ approximate the terms of OEIS sequence A233824: a recurrent sequence in Panaitopol's formula for $\pi(x)$ [5].

5 Appendix: An asymptotic formula for $p_k^{1+1/k} - p_k$

Theorem 5. *Let p_k be the k-th prime, and let $f_k = p_k^{1+1/k} - p_k$, then*

$$f_k = \log^2 p_k - \log p_k - 1 + o(1) \quad \text{as } k \to \infty \qquad \text{(cf. OEIS A246778).}$$

Proof. From Axler [1, Corollaries 3.5, 3.6] we have

$$\frac{x}{\log x - 1 - \frac{1}{\log x} - \frac{1}{\log^2 x}} < \pi(x) < \frac{x}{\log x - 1 - \frac{1}{\log x} - \frac{3.83}{\log^2 x}} \quad \text{for } x \geq 1772201. \tag{15}$$

By definition of f_k, we have $\log_{p_k}(p_k + f_k) = 1 + 1/k$, so $k = \pi(p_k) = \dfrac{\log p_k}{\log(p_k + f_k) - \log p_k}$. Therefore, for $x = p_k \geq 1772201$, we can rewrite (15) as

$$\frac{p_k}{\log p_k - 1 - \frac{1}{\log p_k} - \frac{1}{\log^2 p_k}} < \frac{\log p_k}{\log(p_k + f_k) - \log p_k} < \frac{p_k}{\log p_k - 1 - \frac{1}{\log p_k} - \frac{3.83}{\log^2 p_k}}. \tag{16}$$

An upper bound for f_k. We combine (4) with the left inequality of (16) to get

$$\frac{p_k + \log^2 p_k}{\log p_k - 1 - \frac{1}{\log p_k}} < \frac{\log p_k}{\log(p_k + f_k) - \log p_k} \quad \text{for } p_k \geq 1772201. \tag{17}$$

Cross-multiplying and using (9), similar to Theorem 1, we obtain

$$\frac{f_k(p_k + \log^2 p_k)}{p_k + f_k} < (\log(p_k + f_k) - \log p_k)(p_k + \log^2 p_k) < \log^2 p_k - \log p_k - 1,$$

$$f_k(p_k + \log^2 p_k) < (p_k + f_k)(\log^2 p_k - \log p_k - 1),$$

$$f_k < \frac{p_k}{p_k + \log p_k + 1}(\log^2 p_k - \log p_k - 1) < \log^2 p_k - \log p_k - 1.$$

A lower bound for f_k. From the right inequality of (16) we get

$$\frac{\log^2 p_k - \log p_k - 1 - \frac{3.83}{\log p_k}}{p_k} < \log(p_k + f_k) - \log p_k < \frac{f_k}{p_k},$$

$$\log^2 p_k - \log p_k - 1 - \frac{3.83}{\log p_k} < f_k.$$

Together, the upper and lower bounds yield the desired asymptotic formula for $k \to \infty$. ☐

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