

# CHABAUTY WITHOUT THE MORDELL-WEIL GROUP

MICHAEL STOLL

ABSTRACT. Based on ideas from recent joint work with Bjorn Poonen, we describe an algorithm that can in certain cases determine the set of rational points on a curve  $C$ , given only the  $p$ -Selmer group  $S$  of its Jacobian (or some other abelian variety  $C$  maps to) and the image of the  $p$ -Selmer set of  $C$  in  $S$ . The method is more likely to succeed when the genus is large, which is when it is usually rather difficult to obtain generators of a finite-index subgroup of the Mordell-Weil group, which one would need to apply Chabauty's method in the usual way. We give some applications, for example to generalized Fermat equations of the form  $x^5 + y^5 = z^p$ .

## 1. INTRODUCTION

When one is faced with the task of determining the set of rational points on a (say) hyperelliptic curve  $C: y^2 = f(x)$ , then the usual way to proceed is in the following steps. We denote the Jacobian variety of  $C$  by  $J$ , and we assume that  $f$  has odd degree, so there is a rational point at infinity on  $C$ , which eliminates possible shortcuts that can be used to show that a curve does not have any rational points.

1. Search for rational points on  $C$ .

This can be done reasonably efficiently for  $x$ -coordinates whose numerator and denominator are at most  $10^5$ , say. Rational points on curves of genus  $\geq 2$  are expected to be fairly small (in relation to the coefficients), so the result very likely is  $C(\mathbb{Q})$ . It remains to show that we have not overlooked any points.

2. Compute the 2-Selmer group  $\text{Sel}_2 J$  [Sto01].

The 'global' part of this computation requires arithmetic information related to class group and unit group data for the number fields generated by the roots of  $f$ . If the degrees of the irreducible factors of  $f$  are not too large (and the coefficients are of moderate size), then this computation is feasible in many cases, possibly assuming the Generalized Riemann Hypothesis to speed up the class group computation. The 'local' part of the computation is fairly easy for the infinite place and the odd finite places, but it can be quite involved to find a basis of  $J(\mathbb{Q}_2)/2J(\mathbb{Q}_2)$ .

To proceed further, we need the resulting bound  $r$  for the rank of  $J(\mathbb{Q})$ ,

$$r = \dim_{\mathbb{F}_2} \text{Sel}_2 J - \dim_{\mathbb{F}_2} J(\mathbb{Q})[2],$$

to be strictly less than the genus  $g$  of  $C$ . By work of Bhargava and Gross [BG13] it is known that the Selmer group is small on average, independent of the genus, so when  $g$  is not very small, this condition is likely to be satisfied.

---

*Date:* December 7, 2017.

3. Find  $r$  independent points in  $J(\mathbb{Q})$ .

We can use the points on  $C$  we have found in Step 1 to get some points in  $J(\mathbb{Q})$ . However, it can be quite hard to find further points if the points we get from the curve generate a subgroup of rank  $< r$ . There are two potential problems. The first is of theoretical nature: the rank of  $J(\mathbb{Q})$  can be strictly smaller than  $r$ , in which case it is obviously impossible to find  $r$  independent points. Standard conjectures imply that the difference between  $r$  and the rank is even, so we will not be in this situation when we are missing just one point. In any case, if we suspect our bound is not tight, we can try to use visualization techniques [BF06] to improve the bound. The second problem is practical: some of the generators of  $J(\mathbb{Q})$  can have fairly large height and are therefore likely to fall outside our search space. When the genus  $g$  is moderately large, then we also have the very basic problem that the dimension of our search space is large.

To proceed further, we need to know generators of a finite-index subgroup  $G$  of  $J(\mathbb{Q})$ .

4. Fix some (preferably small) prime  $p$  (preferably of good reduction) and use the knowledge of  $G$  to compute a basis of the space  $V$  of  $\mathbb{Q}_p$ -defined regular differentials on  $C$  that kill the Mordell-Weil group  $J(\mathbb{Q})$  under the Chabauty-Coleman pairing (see for example [Sto06]). This requires evaluating a bunch of  $p$ -adic abelian integrals on  $C$ , which (in the case of good reduction with  $p$  odd) can be done by an algorithm due to Bradshaw and Kedlaya and made practical by Balakrishnan [BBK10].
5. Find the common zeros of the functions  $P \mapsto \int_{\infty}^P \omega$  on  $C(\mathbb{Q}_p)$ , where  $\omega$  runs through a basis of  $V$ .

The rational points are among this set. If there are additional zeros, then they can usually be excluded by an application of the Mordell-Weil sieve [BS10].

The most serious stumbling block is Step 3, in particular when the genus  $g$  is of ‘medium’ size (say between 5 and 15), so that Step 2 is feasible, but we are likely to run into problems when trying to find sufficiently many independent points in the Mordell-Weil group.

In this paper we propose an approach that circumvents this problem. Its great advantage is that it uses only the 2-Selmer group and data that can be obtained by a purely 2-adic computation. Its disadvantage is that it may fail: for it to work, several conditions have to be satisfied, which, however, are likely to hold in particular when the genus gets large.

Generally speaking, the method tries to use the ideas of [PS14] (where it is shown that many curves as above have the point at infinity as their only rational point) to deal with given concrete curves. Section 2 gives a slightly more flexible version of one of the relevant results of this paper. In Section 4, we formulate the algorithm for hyperelliptic curves of odd degree that is based on this key result. The method will apply in other situations as well (whenever we are able to compute a suitable Selmer group), and we plan to work this out in more detail in a follow-up paper for the case of general hyperelliptic curves and also for the setting of ‘Elliptic Curve Chabauty’, where one wants to find the set of  $k$ -points  $P$  on an elliptic curve  $E$  defined over a number field  $k$  such that  $f(P) \in \mathbb{P}^1(\mathbb{Q})$ , where  $f: E \rightarrow \mathbb{P}^1$  is a non-constant  $k$ -morphism. One application in the latter setting is given at the end of this paper. The approach has also already been applied in [FNS16] to complete the resolution of the Generalized Fermat Equation  $x^2 + y^3 = z^{11}$ .

One ingredient of the algorithm is the computation of ‘halves’ of points in the group  $J(\mathbb{Q}_2)$ . In Section 5 we give a general procedure for doing this in  $J(k)$ , when  $J$  is the Jacobian of an odd degree hyperelliptic curve and  $k$  is any field not of characteristic 2. In Section 6, we demonstrate the usefulness of our approach by showing that the only integral solutions of  $y^2 - y = x^{21} - x$  are the obvious ones.

In Section 7, we show how our method leads to a fairly simple criterion that implies the validity of Fermat’s Last Theorem for a given prime exponent. This does not lead to any new results, of course, but it gives a nice illustration of the power of the method. In Section 8, we then apply our approach to the curves  $5y^2 = 4x^p + 1$ . Carrying out the computations, we can show that the only rational points on these curves are the three obvious ones, namely  $\infty$ ,  $(1, 1)$  and  $(1, -1)$ , when  $p$  is a prime  $\leq 53$  (assuming GRH for  $p \geq 23$ ). A result due to Dahmen and Siksek [DS14] then implies that the only coprime integer solutions of the Generalized Fermat Equation

$$x^5 + y^5 = z^p$$

are the trivial ones (where  $xyz = 0$ ).

As already mentioned, we end with another type of example, which uses the method in the context of ‘Elliptic Curve Chabauty’ to show that a certain hyperelliptic curve of genus 4 over  $\mathbb{Q}$  has only the obvious pair of rational points. The Mordell-Weil rank is 4 in this case, so no variant of Chabauty’s method applies directly to the curve.

**Acknowledgments.** I would like to thank Bjorn Poonen for useful discussions and MIT for its hospitality during a visit of two weeks in May 2015, when these discussions took place. All computations were done using the computer algebra system Magma [BCP97].

## 2. THE ALGORITHM

In this section we formulate and prove a variant of [PS14, Proposition 6.2]. We then use it to give an algorithm that can show that the set of known rational points in some subset  $X$  of the  $p$ -adic points of a curve already consists of all rational points contained in  $X$ , using as input only the  $p$ -Selmer group of the Jacobian of the curve. The idea behind this goes back to McCallum’s paper [McC94].

Let  $k$  be a number field, let  $C/k$  be a nice (meaning smooth, projective and geometrically irreducible) curve of genus  $g \geq 2$  and let  $A/k$  be an abelian variety, together with a map  $i: C \rightarrow A$  such that  $A$  is generated by the image of  $C$  (for example,  $A$  could be the Jacobian of  $C$  and  $i$  the embedding given by taking some  $k$ -rational point  $P_0 \in C(k)$  as basepoint). Fix a prime number  $p$ . We write  $\text{Sel}_p A$  for the  $p$ -Selmer group of  $A$ . Recall that this is defined as the kernel of the diagonal homomorphism in the commuting diagram with exact

rows

$$\begin{array}{ccccccc}
0 & \longrightarrow & \frac{A(k)}{pA(k)} & \xrightarrow{\delta} & H^1(k, A[p]) & \longrightarrow & H^1(k, A)[p] \longrightarrow 0 \\
& & \downarrow & & \downarrow & \searrow & \downarrow \\
0 & \longrightarrow & \prod_v \frac{A(k_v)}{pA(k_v)} & \xrightarrow{\delta} & \prod_v H^1(k_v, A[p]) & \longrightarrow & \prod_v H^1(k_v, A)[p] \longrightarrow 0
\end{array}$$

that is induced by applying Galois cohomology to the short exact sequence

$$0 \longrightarrow A[p] \longrightarrow A \xrightarrow{p} A \longrightarrow 0$$

of Galois modules over  $k$  and over all completions  $k_v$  of  $k$ , so the products in the second row run over all places  $v$  of  $k$ . The vertical maps are induced by  $k \hookrightarrow k_v$ . In particular, for each place  $v$  there is a canonical map  $\text{Sel}_p(A) \rightarrow A(k_v)/pA(k_v)$ .

We write  $k_p = k \otimes_{\mathbb{Q}} \mathbb{Q}_p$ ; this is the product of the various completions of  $k$  at places above  $p$ . The set  $C(k_p)$  and the group  $A(k_p)$  can similarly be understood as products of the sets or groups of  $k_v$ -points, for the various  $v \mid p$ . The inclusion  $k \hookrightarrow k_p$  induces natural maps  $C(k) \hookrightarrow C(k_p)$  and  $A(k) \hookrightarrow A(k_p)$ . Let  $X \subseteq C(k_p)$  be a subset (for example, the points in a product of  $v$ -adic residue disks). We then have the following commutative diagram of maps.

$$\begin{array}{ccccccc}
C(k) \cap X & \hookrightarrow & C(k) & \xrightarrow{i} & A(k) & \xrightarrow{\pi} & \frac{A(k)}{pA(k)} \xrightarrow{\delta} \text{Sel}_p A \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \swarrow \sigma \\
X & \hookrightarrow & C(k_p) & \xrightarrow{i} & A(k_p) & \xrightarrow{\pi_p} & \frac{A(k_p)}{pA(k_p)}
\end{array}$$

We introduce some more notation. For  $P \in A(k_p)$ , we set

$$(2.1) \quad q(P) := \{ \pi_p(Q) : Q \in A(k_p), \exists n \geq 0 : p^n Q = P \} \subseteq \frac{A(k_p)}{pA(k_p)},$$

and for a subset  $S \subseteq A(k_p)$ , we set  $q(S) = \bigcup_{P \in S} q(P)$ . We further define

$$\nu(P) := \sup\{n : n \geq 0, P \in p^n A(k_p)\} \in \mathbb{Z}_{\geq 0} \cup \{\infty\}.$$

Note that  $\nu(P) = \infty$  is equivalent to  $P$  having finite order prime to  $p$ ; on the complement of the finite set consisting of such  $P$ ,  $\nu$  and  $q$  are locally constant.

With a view toward further applications, we first state a more general version of our result, which we will then specialize (see Theorem 2.6 below). We remark that  $C$  could also be a variety of higher dimension here.

**Theorem 2.1.** *In the situation described above, fix some subgroup  $\Gamma \subseteq A(k)$  and assume that*

- (1)  $\ker \sigma \subseteq \delta(\pi(\Gamma))$ , and that
- (2)  $q(i(X) + \Gamma) \cap \text{im}(\sigma) \subseteq \pi_p(\Gamma)$ .

Then  $i(X \cap C(k)) \subseteq \bar{\Gamma} := \{Q \in A(k) : \exists n \geq 1 : nQ \in \Gamma\}$ .

*Proof.* Let  $P \in X \cap C(k)$ . We show by induction on  $n$  that for each  $n \geq 0$ , there are  $T_n \in \Gamma$  and  $Q_n \in A(k)$  such that  $i(P) = T_n + p^n Q_n$ . This is clear for  $n = 0$  (take  $T_0 = 0$  and  $Q_0 = i(P)$ ). Now assume that  $T_n$  and  $Q_n$  exist. Note that  $\pi_p(Q_n) \in q(i(P) - T_n)$ , so

$$\pi_p(Q_n) = \sigma(\delta(\pi(Q_n))) \in q(i(X) + \Gamma) \cap \text{im}(\sigma);$$

by (2) this implies  $\sigma(\delta(\pi(Q_n))) \in \pi_p(\Gamma) = \sigma(\delta(\pi(\Gamma)))$ . This shows that  $Q_n \in \Gamma + \ker(\sigma \circ \delta \circ \pi)$ . By (1) and since  $\delta$  is injective, we have

$$\ker(\sigma \circ \delta \circ \pi) = \pi^{-1}(\delta^{-1}(\ker \sigma)) \subseteq \pi^{-1}(\pi(\Gamma)) = \Gamma + \ker \pi = \Gamma + pA(k),$$

which implies that  $Q_n \in \Gamma + pA(k)$ . So there are  $T' \in \Gamma$  and  $Q_{n+1} \in A(k)$  such that  $Q_n = T' + pQ_{n+1}$ . We set  $T_{n+1} = T_n + p^n T' \in \Gamma$ ; then

$$i(P) = T_n + p^n Q_n = T_n + p^n (T' + pQ_{n+1}) = T_{n+1} + p^{n+1} Q_{n+1}.$$

Now consider the quotient map  $\psi: A(k) \twoheadrightarrow A(k)/\bar{\Gamma}$ . Since  $\bar{\Gamma}$  is saturated in the finitely generated group  $A(k)$ , the quotient group is torsion free and hence free. Observe that for every  $n \geq 0$ ,

$$\psi(i(P)) = \psi(T_n + p^n Q_n) = \psi(T_n) + p^n \psi(Q_n) = p^n \psi(Q_n) \in p^n (A(k)/\bar{\Gamma}),$$

which implies that  $\psi(i(P)) = 0$  and so  $i(P) \in \bar{\Gamma}$ .  $\square$

The point of formulating the statement in this way (as compared to [PS14]) is that we avoid the use of  $p$ -adic abelian logarithms, which would require us to compute  $p$ -adic abelian integrals, usually with  $p = 2$  and in a situation when the curve has bad reduction at 2. Instead, we need to be able to compute  $q(P)$  for a given point  $P$ , which comes down to finding its  $p$ -division points. At least in some cases of interest, this approach seems to be computationally preferable.

*Remark 2.2.* Instead of considering multiplication by  $p$ , we could use an endomorphism  $\psi$  of  $A$  that is an isogeny of degree a power of  $p$  and such that some power of  $\psi$  is divisible by  $p$  in the endomorphism ring of  $A$ . We then consider  $A(k)/\psi A(k)$ ,  $A(k_p)/\psi A(k_p)$  and the  $\psi$ -Selmer group  $\text{Sel}_\psi A$ . Note that when  $\psi: A \rightarrow A'$  is any isogeny whose kernel has order a power of  $p$  and with dual isogeny  $\hat{\psi}$ , then we can consider  $A \times A'$  with the endomorphism  $\tilde{\psi}: (P, P') \mapsto (\hat{\psi}(P'), \psi(P))$ , which satisfies  $\tilde{\psi}^2 = \deg \psi = p^m$ , together with the morphism  $\tilde{i}: X \rightarrow A \times A'$ ,  $P \mapsto (i(P), 0)$ . Taking  $\Gamma \times \{0\}$  in place of  $\Gamma$  and writing the relevant maps as

$$A(k) \xrightarrow{\pi} \frac{A(k)}{\hat{\psi}(A'(k))} \xrightarrow{\subset \delta} \text{Sel}_{\hat{\psi}} A' \xrightarrow{\sigma} \frac{A(k_p)}{\hat{\psi}(A'(k_p))}$$

and

$$A'(k) \xrightarrow{\pi'} \frac{A'(k)}{\psi(A(k))} \xrightarrow{\subset \delta'} \text{Sel}_\psi A \xrightarrow{\sigma'} \frac{A'(k_p)}{\psi(A(k_p))},$$

the second condition in Theorem 2.1 translates into

$$q_A(i(X) + \Gamma) \cap \text{im}(\sigma) \subseteq \sigma(\delta(\pi(\Gamma)))$$

and

$$q_{A'}(\hat{\psi}^{-1}(i(X)) + A'(k)_{\text{tors}}) \cap \text{im}(\sigma') \subseteq \sigma'(\delta'(\pi'(A'(k)_{\text{tors}}))).$$

*Remark 2.3.* The set  $X \cap i^{-1}(\bar{\Gamma})$  that contains  $C(k) \cap X$  when Theorem 2.1 applies can in many cases be determined by the usual Chabauty-Coleman techniques; see for example [Sto06]. Of course, if  $\Gamma$  is finite (and so  $\bar{\Gamma} = A(k)_{\text{tors}}$  is finite as well), which is usually the case in applications, then determining  $X \cap i^{-1}(\bar{\Gamma})$  is essentially trivial.

We give some indication of how one can compute a set such as  $q(P + \Gamma)$ , where  $P \in A(k_p)$ . We assume that, given  $P \in A(k_p)$ , we can find all  $Q \in A(k_p)$  such that  $pQ = P$ .

**Lemma 2.4.** *With the notations used in Theorem 2.1, fix a complete set of representatives  $R \subseteq \Gamma$  for  $\Gamma/p\Gamma$ . Let  $P \in A(k_p)$  and set  $\mathcal{Q} = \{Q \in A(k_p) : \exists T \in R : pQ = P + T\}$ . Define an equivalence relation on  $\mathcal{Q}$  via  $Q \sim Q' \iff Q - Q' \in \Gamma$ , and let  $\mathcal{Q}'$  be a complete set of representatives for  $\mathcal{Q}/\sim$ . Then*

$$q(P + \Gamma) = \{\pi_p(P + T) : T \in R\} \cup \bigcup_{Q \in \mathcal{Q}'} q(Q + \Gamma).$$

*Proof.* Since  $Q + \Gamma = Q' + \Gamma$  whenever  $Q \sim Q'$ , it is sufficient to prove the equality with  $\mathcal{Q}$  in place of  $\mathcal{Q}'$ . We first show that the set on the right is contained in the set on the left. This is clear for the elements  $\pi_p(P + T)$ , taking  $n = 0$  in (2.1). So let now  $Q \in \mathcal{Q}$  and  $\xi \in q(Q + \Gamma)$ . Then there are  $n \geq 0$ ,  $T' \in \Gamma$  and  $Q' \in A(k_p)$  such that  $p^n Q' = Q + T'$  and  $\pi_p(Q') = \xi$ . There is also  $T \in \Gamma$  such that  $pQ = P + T$ . We then have

$$p^{n+1}Q' = p(Q + T') = P + (T + pT') \in P + \Gamma$$

and so  $\xi = \pi_p(Q') \in q(P + \Gamma)$ .

Now we show the reverse inclusion. Let  $\xi \in q(P + \Gamma)$ , so there are  $n \geq 0$ ,  $T' \in \Gamma$ ,  $Q' \in A(k_p)$  such that  $p^n Q' = P + T'$  and  $\pi_p(Q') = \xi$ . There is also some  $T \in R$  such that  $T - T' = pT''$  with  $T'' \in \Gamma$ . If  $n = 0$ , then  $\xi = \pi_p(P + T') = \pi_p(P + T)$ . If  $n > 0$ , we can write

$$P + T = (P + T') + pT'' = p(p^{n-1}Q' + T'') = pQ$$

with  $Q = p^{n-1}Q' + T'' \in \mathcal{Q}$ , and  $\xi = \pi_p(Q') \in q(Q - T'') \subseteq q(Q + \Gamma)$ .  $\square$

Whenever  $\sup \nu(P + \Gamma) < \infty$ , the recursion implied by Lemma 2.4 will terminate, and so the lemma translates into an algorithm for computing  $q(P + \Gamma)$ . We make this condition more explicit.

**Lemma 2.5.** *We write  $\text{cl}(\Gamma)$  for the topological closure of  $\Gamma$  in  $A(k_p)$ . Let  $P \in A(k_p)$ . Then  $\sup \nu(P + \Gamma) = \infty$  if and only if there is a point  $T \in A(k_p)$  of finite order prime to  $p$  such that  $P \in T + \text{cl}(\Gamma)$ .*

*Proof.* Let  $A(k_p)_1$  be the kernel of reduction (i.e., the product of the kernels of reduction of the various  $A(k_v)$  with  $v$  a place above  $p$ ) and let  $m$  denote the exponent of the finite group  $A(k_p)/A(k_p)_1$ . Then for all  $P \in A(k_p)$ ,  $p^n m P$  tends to the origin as  $n \rightarrow \infty$ . If  $\sup \nu(P + \Gamma) = \infty$ , then there are arbitrarily large  $n$  such there exist  $\gamma_n \in \Gamma$  and  $Q_n \in A(k_p)$  with  $P = \gamma_n + p^n Q_n$ , so  $mP - m\gamma_n$  tends to the origin as  $n$  gets large. Then  $P - \gamma_n$  must be close to a point of order  $m$ ; by restricting to a sub-sequence, we find that  $P - \gamma_n$  approaches a point  $T \in A(k_p)[m]$ . Since  $T$  is close to  $P - \gamma_n = p^n Q$  for arbitrarily large  $n$ ,  $T$  must be infinitely divisible by  $p$ , so the order of  $T$  is prime to  $p$ . We clearly have  $P \in T + \text{cl}(\Gamma)$ .

For the converse, it suffices to consider  $P$  in the closure of  $\Gamma$ , since the points of finite order prime to  $p$  in  $A(k_p)$  are infinitely  $p$ -divisible. Since any point sufficiently close to the origin is highly  $p$ -divisible, this implies that for each  $n \geq 0$  we can find  $\gamma_n \in \Gamma$  and  $Q_n \in A(k_p)$  such that  $P - \gamma_n = p^n Q_n$ . This is equivalent to  $\sup \nu(P + \Gamma) = \infty$ .  $\square$

We specialize Theorem 2.1 to the case that  $k = \mathbb{Q}$  and  $i$  embeds the curve into its Jacobian. Let  $\mathcal{C}$  be a proper regular model of  $C$  over  $\mathbb{Z}_p$ . Then the reduction map sends  $C(\mathbb{Q}_p) = \mathcal{C}(\mathbb{Z}_p)$  to the set of smooth  $\mathbb{F}_p$ -points on the special fiber of  $\mathcal{C}$ . The preimage  $D$  of a smooth  $\mathbb{F}_p$ -point on the special fiber of  $\mathcal{C}$  under the reduction map is called a *residue disk* in  $C(\mathbb{Q}_p)$ ; see [PS14]. It follows from Hensel's Lemma that there is an analytic map  $\varphi$  from the open  $p$ -adic unit disk to  $C$  such that  $D = \varphi(p\mathbb{Z}_p)$ . If  $p = 2$ , then we call the subsets  $\varphi(4\mathbb{Z}_2)$  and  $\varphi(2 + 4\mathbb{Z}_2)$  *half residue disks*.

**Theorem 2.6.** *Let  $C$  be a nice curve over  $\mathbb{Q}$ , with Jacobian  $J$ . Let  $P_0 \in C(\mathbb{Q})$  and take  $X \subseteq C(\mathbb{Q}_p)$  to be contained in a residue disk or, when  $p = 2$  and  $J(\mathbb{Q})[2] \neq 0$ , in a half residue disk, and to contain  $P_0$ . Let  $i: C \rightarrow J$  be the embedding sending  $P_0$  to zero. With the notation introduced above, assume that*

- (1)  $\ker \sigma \subseteq \delta(\pi(J(\mathbb{Q})_{\text{tors}}))$ , and that
- (2)  $q(i(X) + J(\mathbb{Q})_{\text{tors}}) \cap \text{im}(\sigma) \subseteq \pi_p(J(\mathbb{Q})_{\text{tors}})$ .

Then  $C(\mathbb{Q}) \cap X = \{P_0\}$ .

*Proof.* We apply Theorem 2.1 with  $k = \mathbb{Q}$ ,  $C$  our curve,  $A = J$ ,  $i$  as given in the statement and  $\Gamma = \bar{\Gamma} = J(\mathbb{Q})_{\text{tors}}$ . This tells us that  $i(C(\mathbb{Q}) \cap X) \subseteq J(\mathbb{Q})_{\text{tors}}$ . If  $p > 2$  or  $p = 2$  and  $J(\mathbb{Q})[2] = 0$ , then the only rational torsion point in the kernel of reduction of  $J(\mathbb{Q}_p)$  is the origin, which implies that there cannot be two distinct points in  $X$  both mapping to torsion under  $i$ . If  $p = 2$  and  $J(\mathbb{Q})[2] \neq 0$ , then the corresponding statement is true if  $X$  is a half residue disk, which means that  $i(X)$  is contained in  $K_2$ , the second kernel of reduction; see Section 3 below. In both cases, we find that there is at most one rational point in  $X$ ; since  $P_0$  is one such point, it must be the only one.  $\square$

This leads to the following algorithm. It either returns FAIL or it returns the set of rational points on the curve  $C$ . We refer to [Sto07] for the definition of the  $p$ -Selmer set  $\text{Sel}_p(C)$  of the curve  $C$ . Given an embedding  $i$  of  $C$  into its Jacobian  $J$ , it can be interpreted as the subset of  $\text{Sel}_p(J)$  consisting of elements that locally come from points on the curve.

**Algorithm 2.7.**

**Input:** A nice curve  $C$ , defined over  $\mathbb{Q}$ , with Jacobian  $J$ .  
A point  $P_0 \in C(\mathbb{Q})$ , defining an embedding  $i: C \rightarrow J$ .  
A prime number  $p$ .

**Output:** The set of rational points on  $C$ , or FAIL.

1. Compute the  $p$ -Selmer group  $\text{Sel}_p J$  and the  $p$ -Selmer set  $\text{Sel}_p C$ ;  
 $i$  induces a map  $i_*: \text{Sel}_p C \hookrightarrow \text{Sel}_p J$ .
2. Search for rational points on  $C$  and collect them in a set  $C(\mathbb{Q})_{\text{known}}$ .
3. Let  $\sigma: \text{Sel}_p J \rightarrow J(\mathbb{Q}_p)/pJ(\mathbb{Q}_p)$  be the canonical map.  
If  $\ker \sigma \not\subseteq \delta(\pi(J(\mathbb{Q})_{\text{tors}}))$ , then return FAIL.

4. Let  $R$  be the image of  $J(\mathbb{Q})_{\text{tors}}$  in  $J(\mathbb{Q}_p)/pJ(\mathbb{Q}_p)$ .
5. Let  $\mathcal{X}$  be a partition of  $C(\mathbb{Q}_p)$  into residue disks whose image in  $J(\mathbb{Q}_p)/pJ(\mathbb{Q}_p)$  consists of one element and that are contained in half residue disks when  $p = 2$  and  $J(\mathbb{Q})[2] \neq 0$ .
6. For each  $X \in \mathcal{X}$  do the following:
  - a. If  $X \cap C(\mathbb{Q})_{\text{known}} = \emptyset$ :  
If  $\pi_p(X) \subseteq \text{im}(\sigma \circ i_*)$ , then return FAIL;  
otherwise continue with the next  $X$ .
  - b. Pick some  $P_1 \in C(\mathbb{Q})_{\text{known}} \cap X$ .
  - c. Compute  $Y = \bigcup_{P \in X, T \in J(\mathbb{Q})_{\text{tors}}} q([P - P_1] + T) \subseteq J(\mathbb{Q}_p)/pJ(\mathbb{Q}_p)$ .
  - d. If  $Y \cap \text{im}(\sigma) \not\subseteq R$ , then return FAIL.
7. Return  $C(\mathbb{Q})_{\text{known}}$ .

**Proposition 2.8.** *The algorithm is correct: if it does not return FAIL, then it returns the set of rational points on  $C$ .*

*Proof.* First note that Step 3 verifies the first assumption of Theorem 2.6; it returns FAIL when the assumption does not hold. It is also clear that if the algorithm does not return FAIL, then the set it returns is a subset of  $C(\mathbb{Q})$ . We show the reverse inclusion. So let  $P \in C(\mathbb{Q})$  be some rational point. There will be some  $X \in \mathcal{X}$  such that  $P \in X$ . Then  $\pi_p(X)$  is contained in  $\text{im}(\sigma \circ i_*)$ , so since the algorithm did not return FAIL, by Step 6a. it follows that  $X \cap C(\mathbb{Q})_{\text{known}} \neq \emptyset$ ; let  $P_1 \in X \cap C(\mathbb{Q})_{\text{known}}$  as in Step 6b. Now by Step 6d. the second assumption of Theorem 2.6 is satisfied, taking the embedding with base-point  $P_1$ . So the theorem applies, and it shows that there is only one rational point in  $X$ , so  $P = P_1 \in C(\mathbb{Q})_{\text{known}}$ .  $\square$

*Remark 2.9.* We note that in Step 6, the set  $X$  can be further partitioned if necessary. If there are several points in  $C(\mathbb{Q})_{\text{known}}$  that end up in the same set  $X$ , then the second assumption of Theorem 2.6 cannot be satisfied. But it is still possible that the theorem can be applied to smaller disks that separate the points. (If the points are too close  $p$ -adically, this will not work, though. In this case, one could try to use  $\Gamma + J(\mathbb{Q})_{\text{tors}}$  in the more general version of the theorem, where  $\Gamma$  is the subgroup generated by the difference of the two points.)

There are also cases when it helps to combine several sets  $X$  into one. One such situation is when there are points in  $C(\mathbb{Q}_p)$  that differ by a torsion point of order prime to  $p$  and such that only one of the corresponding sets  $X$  contains a (known) rational point.

A particularly useful case is when  $C$  is hyperelliptic,  $A = J$  is the Jacobian of  $C$ , and we consider  $p = 2$ . There is an algorithm that computes 2-Selmer group  $\text{Sel}_2 J$ , which is feasible in many cases, compare [Sto01]. We discuss this further in Section 4 below in the case when the curve has a rational Weierstrass point at infinity.

Another useful case (using a slightly more general setting) is related to ‘‘Elliptic curve Chabauty’’. Here  $A$  is the Weil restriction of an elliptic curve  $E$  over some number field  $k$  such that there is a non-constant morphism  $C \rightarrow E$  defined over  $k$ . We give an example of this in Section 9.



### 3. COMPUTING THE IMAGE UNDER $q$ OF A DISK

In this section, we discuss in some detail how to find the image under  $q$  of (the image in  $J$  of) a residue disk of  $C(k_p)$ . The basic idea is that  $q$  is locally constant on the curve even near points where  $\nu$  becomes infinite (a variant of this was already used in [PS14]). To get a practical algorithm out of this idea, we have to produce an explicit neighborhood on which  $q$  is constant. We will do this first away from the points where  $\nu$  becomes infinite and then also on residue disks centered at a point where  $\nu$  becomes infinite.

Since objects over  $k_p$  are products of objects over the various completions  $k_v$  at places  $v$  above  $p$ , we will now work over a fixed such completion. We fix a non-constant morphism  $i: C \rightarrow J$ , where  $J$  can be any abelian variety that is spanned by  $i(C)$ . To ease notation, we write  $\pi$  instead of  $\pi_v$  for the map  $J(k_v) \rightarrow J(k_v)/pJ(k_v)$ .

We assume that we can compute  $q(P)$  for any given point  $P \in J(k_v)$  that is not (too close to) a point of finite order prime to  $p$ . When  $p = 2$  and  $C$  is hyperelliptic of odd degree and  $J$  is the Jacobian, this can be done by using the halving algorithm of Section 5 below: we compute the image of  $P$  in  $L_2^\square$  and record it; if the image is trivial, then we compute all halves of  $P$  and apply the same procedure to them. Since by assumption  $P$  is not infinitely 2-divisible, the recursion will eventually stop with an empty set of points still to be considered.

The following is essentially immediate from the definitions.

**Lemma 3.1.** *Let  $P_1, P_2 \in J(k_v)$  and assume that  $P_1 \equiv P_2 \not\equiv 0 \pmod{p^{m+1}J(k_v)}$ . Then  $\nu(P_1) = \nu(P_2)$  and  $q(P_1) = q(P_2)$ .*

*Proof.* The assumptions imply that  $P_1, P_2 \notin p^{m+1}J(k_v)$ , so whenever there are  $Q \in J(k_v)$  and  $n \geq 0$  such that  $p^n Q = P_1$  or  $P_2$ , then  $n \leq m$ . Let  $P' \in J(k_v)$  such that  $P_2 = P_1 + p^{m+1}P'$ . Then  $p^n Q = P_1$  implies  $p^n(Q + p^{m+1-n}P') = P_2$ , so that  $\nu(P_2) \geq \nu(P_1)$ , and by symmetry, we obtain equality.

Let  $\xi \in q(P_1)$ ; then  $\xi = \pi(Q)$  for some  $Q$  such that  $p^n Q = P_1$  as above. Then  $n \leq m$  and so  $\xi = \pi(Q) = \pi(Q + p^{m+1-n}P') \in q(P_2)$  as well. This shows that  $q(P_1) \subseteq q(P_2)$ ; the reverse inclusion follows again by symmetry.  $\square$

We write  $\mathcal{O}_v$  for the ring of integers in  $k_v$  and  $\varpi$  for a uniformizer. We abuse notation and write  $v: k_v^\times \rightarrow \mathbb{Z}$  for the additive valuation, normalized such that  $v(\varpi) = 1$ . Then  $e = v(p)$  is the absolute ramification index of  $K_v$ . We fix a proper regular model  $\mathcal{C}$  of  $C$  over  $\mathcal{O}_v$ . Let  $\mathcal{J}$  be the Néron model of  $J$  over  $\mathcal{O}_v$ . For  $n \geq 1$ , we denote by

$$K_n := \ker(J(k_v) = \mathcal{J}(\mathcal{O}_v) \rightarrow \mathcal{J}(\mathcal{O}_v/\varpi^n \mathcal{O}_v))$$

the ‘higher kernels of reduction’;  $K_n$  is also the group of  $\varpi^n \mathcal{O}_v$ -points of the formal group associated to  $\mathcal{J}$ .

We now fix a residue disk  $D \subseteq C(K_v)$  with respect to  $\mathcal{C}$ ; we will denote an analytic parameterization  $D_0 \rightarrow D$  by  $\varphi$ , where  $D_0$  is the open unit disk. Since  $i$  induces a morphism from the smooth part of  $\mathcal{C}$  to  $\mathcal{J}$ , it follows that

$$(3.1) \quad t, t' \in \varpi \mathcal{O}_v, \quad v(t - t') \geq m \quad \implies \quad i(\varphi(t)) - i(\varphi(t')) \in K_m.$$

The formal logarithm converges on  $K_1$  and gives a homomorphism  $K_1 \rightarrow k_v^{\dim J}$ . Restricted to  $K_m$  with  $m > e/(p-1)$ , the formal exponential provides an inverse, so that the formal logarithm gives an isomorphism  $K_m \rightarrow (\varpi^m \mathcal{O}_v)^{\dim J}$ . It follows that  $pK_m = K_{m+e}$ ; in particular,

$$(3.2) \quad K_{ne+m} = p^n K_m \subseteq p^n J(k_v) \quad \text{for all } n \geq 0.$$

This implies together with (3.1) that for  $m$  as above and  $n \geq 0$ ,

$$(3.3) \quad t, t' \in \varpi \mathcal{O}_v, \quad v(t - t') \geq ne + m \implies i(\varphi(t)) \equiv i(\varphi(t')) \pmod{p^n J(k_v)}.$$

In the following we write  $\mu$  for  $\lfloor e/(p-1) \rfloor + 1$ ; this is the smallest choice of  $m$  in the considerations above. If  $k_v = \mathbb{Q}_p$  (or, more generally, an unramified extension of  $\mathbb{Q}_p$ , so that  $e = 1$ ), then  $\mu = 1$  when  $p$  is odd, and  $\mu = 2$  when  $p = 2$ .

**Corollary 3.2.** *Consider  $\varphi: D_0 \rightarrow D \subseteq C(k_v)$  as above, and let  $t_0 \in \varpi \mathcal{O}_v$  be such that  $\nu(i(\varphi(t_0))) \leq n$ . Then for all  $t$  with  $v(t - t_0) \geq e(n+1) + \mu$ , we have*

$$\nu(i(\varphi(t))) = \nu(i(\varphi(t_0))) \quad \text{and} \quad q(i(\varphi(t))) = q(i(\varphi(t_0))).$$

More generally, let  $\Gamma \subseteq J(\mathbb{Q}_2)$  be a subgroup. If  $\max \nu(i(\varphi(t_0)) + \Gamma) \leq n$ , then for all  $t$  with  $v(t - t_0) \geq e(n+1) + \mu$ , we have

$$\max \nu(i(\varphi(t)) + \Gamma) = \max \nu(i(\varphi(t_0)) + \Gamma) \quad \text{and} \quad q(i(\varphi(t)) + \Gamma) = q(i(\varphi(t_0)) + \Gamma).$$

*Proof.* By (3.3), we have  $i(\varphi(t)) \equiv i(\varphi(t_0)) \pmod{p^{n+1} J(k_v)}$ . The first claim now follows from Lemma 3.1. The second claim follows from the first by considering  $i(\varphi(t)) + \gamma$  for each  $\gamma \in \Gamma$  separately, and applying the first claim to the shifted embedding  $P \mapsto i(P) + \gamma$ .  $\square$

If the image of the disk  $D$  in  $J$  does not contain a point of finite order prime to  $p$ , then  $\nu$  will be bounded on  $D$ . Corollary 3.2 then provides a partition of  $D$  into finitely many sub-disks such that  $q \circ i$  is constant on each of them. In this way, we can compute  $q(i(D))$ . In a similar way, this allows us to compute  $q(i(D) + \Gamma)$  if  $i(D)$  does not meet  $\text{cl}(\Gamma) + J(k_v)[p']$ , where  $G[p']$  denotes the subgroup of an abelian group  $G$  consisting of elements of finite order prime to  $p$ ; compare Lemma 2.5.

We now consider the situation when  $D$  contains a point  $P_0$  such that  $i(P_0) \in J(k_v)[p']$ . In this case, the result above will not produce a finite partition into sub-disks, so we need to have an explicit estimate for the size of the pointed disk around  $P_0$  on which  $q \circ i$  is constant. Without loss of generality,  $i(P_0) = 0$ . We also assume that  $\varphi(0) = P_0$ , so that  $i(\varphi(0)) = 0 \in J$ .

In the following, we write  $n_{\text{tors}}$  for the smallest  $n \geq 0$  such that  $J(k_v)[p^\infty] \subseteq J[p^n]$ . In other words,  $p^{n_{\text{tors}}}$  is the exponent of the  $p$ -power torsion subgroup  $J(k_v)[p^\infty]$ .

**Lemma 3.3.** *Let  $P \in J(k_v)$ .*

- (1) *If  $n_{\text{tors}} = 0$ , then  $\nu(pP) = \nu(P) + 1$  and  $q(P) \subseteq q(pP) \subseteq q(P) \cup \{0\}$ .*
- (2)  *$\nu(P) > n_{\text{tors}}$ , then  $\nu(pP) = \nu(P) + 1$  and  $q(pP) = q(P)$ .*

*Proof.* Since  $p^n Q = P$  implies  $p^{n+1} Q = pP$ , the inclusion  $q(P) \subseteq q(pP)$  is clear, as is the inequality  $\nu(pP) \geq \nu(P) + 1$ , for arbitrary  $P$ .

First assume that  $n_{\text{tors}} = 0$ . Consider  $\xi \in q(pP)$ , so there are  $Q \in J(k_v)$  and  $n \geq 0$  such that  $p^n Q = pP$  and  $\pi(Q) = \xi$ . If  $n = 0$ , then  $\xi = \pi(Q) = \pi(pP) = 0$ . If  $n \geq 1$ , then we must have  $p^{n-1} Q = P$  (since there is no nontrivial  $p$ -torsion), so  $\xi = \pi(Q) \in q(P)$ . Taking  $n = \nu(pP)$  shows that  $\nu(P) \geq \nu(pP) - 1$ .

Now assume that  $\nu(P) > n_{\text{tors}}$  and write  $P = p^{n_{\text{tors}}+1} P_0$  for  $P_0 \in J(k_v)$ . We first show that  $\pi(J(k_v)[p^\infty]) \subseteq q(P)$ . For this, let  $T \in J(k_v)[p^\infty] = J(k_v)[p^{n_{\text{tors}}}]$ . Then  $p^{n_{\text{tors}}}(T + pP_0) = P$ , so  $\pi(T) = \pi(T + pP_0) \in q(P)$  by (2.1).

To show that  $q(pP) \subseteq q(P)$ , let  $\xi \in q(pP)$ , so there are some  $Q \in J(k_v)$  and  $n \geq 0$  with  $\pi(Q) = \xi$  such that  $p^n Q = pP = p^{n_{\text{tors}}+2} P_0$ . If  $n \leq n_{\text{tors}} + 1$ , then it follows that  $Q = p^{n_{\text{tors}}+2-n} P_0 + T$  with  $T \in J(k_v)[p^\infty]$ , so  $\xi = \pi(Q) = \pi(T) \in q(P)$  by the argument above. If  $n \geq n_{\text{tors}} + 2$ , then  $p^{n-n_{\text{tors}}-2} Q = P_0 + T$  with  $T \in J(k_v)[p^{n_{\text{tors}}}]$ , and therefore  $p^{n-1} Q = p^{n_{\text{tors}}+1} P_0 = P$ , so  $\xi = \pi(Q) \in q(P)$ . Carrying out this argument with  $n = \nu(pP)$  and a suitable  $Q$ , we also get that  $\nu(P) \geq \nu(pP) - 1$ .  $\square$

For  $m \geq 1$  we define

$$N(m) = 1 + \min \left\{ \left\lfloor \frac{km - v(k) - \mu}{e} \right\rfloor : k \geq 2 \right\}.$$

Then  $N(m)e \geq 2m - a$  for some constant  $a$ .

**Lemma 3.4.** *Assume that  $v(t) = m \geq 1$ . Then*

$$p \cdot i(\varphi(t)) \equiv i(\varphi(pt)) \pmod{p^{N(m)} J(k_v)}.$$

*Proof.* In terms of formal group coordinates, we can write

$$\log_J i(\varphi(t)) = c_1 t + \frac{c_2}{2} t^2 + \frac{c_3}{3} t^3 + \dots$$

with  $c_1, c_2, c_3, \dots \in \mathcal{O}_v^{\dim J}$ . We find that

$$\begin{aligned} \log_J (pi(\varphi(t)) - i(\varphi(pt))) &= p \log_J i(\varphi(t)) - \log_J i(\varphi(pt)) \\ &= c_2 \frac{p - p^2}{2} t^2 + c_3 \frac{p - p^3}{3} t^3 + c_4 \frac{p - p^4}{4} t^4 + \dots \in (\varpi^{N(m)e+\mu} \mathcal{O}_v)^{\dim J}, \end{aligned}$$

by the definition of  $N(m)$ . We have that

$$p \cdot i(\varphi(t)) - i(\varphi(pt)) \in pK_m + K_{m+e} = K_{m+e} \subseteq K_\mu,$$

so we are in the domain of the isomorphism induced by the formal logarithm, which allows us to conclude that  $pi(\varphi(t)) - i(\varphi(pt)) \in K_{N(m)e+\mu}$ . The claim then follows from (3.2).  $\square$

**Corollary 3.5.** *If we have  $p = 2$  and  $e = 1$  (which is the case when  $k_v = \mathbb{Q}_2$ ) in the situation of Lemma 3.4, then*

$$2i(\varphi(t)) \equiv i(\varphi(2t)) \pmod{2^{2m-2} J(\mathbb{Q}_2)}.$$

*If in addition  $C$  is hyperelliptic,  $\varphi(0)$  is a Weierstrass point and  $\varphi(-t) = \iota(\varphi(t))$ , where  $\iota$  is the hyperelliptic involution, then*

$$2i(\varphi(t)) \equiv i(\varphi(2t)) \pmod{2^{3m-1} J(\mathbb{Q}_2)}.$$

*Proof.* If  $p = 2$  and  $e = 1$ , then  $\mu = 2$  and so  $N(m) = 2m - 2$  in Lemma 3.4 (the minimum is attained for  $k = 2$ ).

Under the additional assumptions on  $C$  and  $\varphi$ , it follows that  $\log_J \circ i \circ \varphi$  is odd, so that  $c_{2n} = 0$  for all  $n \geq 1$  in the proof of Lemma 3.4. We then obtain the better bound in the same way as in that proof, noting that we can restrict to odd  $k$  (which have  $v(k) = 0$ ).  $\square$

For our fixed  $\varphi$  and  $i$ , we define, for  $m \geq 1$ ,

$$n_m := \max\{\nu(i(\varphi(t))) : t \in \varpi\mathcal{O}_v, v(t) = m\}.$$

**Lemma 3.6.** *There is some  $b \in \mathbb{Z}$  such that  $n_m e \leq m + b$  for all  $m \geq 1$ .*

*Proof.* First note that  $i(\varphi(t)) \in K_{m+a} \setminus K_{m+a+1}$  for some fixed  $a$  when  $m$  is sufficiently large, where  $a$  is the valuation of  $c_1$  in the proof of Lemma 3.4 above.

Next, let  $a'$  denote the  $p$ -adic valuation of the exponent of the (finite) quotient group  $J(k_v)/K_\mu$ . Then for  $n \geq \mu$  and  $P \in K_n \setminus K_{n+1}$ , we have  $\nu(P)e \leq n - \mu + a'e$ . To see this, write  $P = p^{\nu(P)}Q$  for some  $Q \in J(k_v)$ ; assume  $\nu(P)e > n - \mu + a'e$ . Then  $p^{a'}Q$  maps to an element of order prime to  $p$  in  $J(k_v)/K_\mu$ , and since  $P = p^{\nu(P)-a'}(p^{a'}Q) \in K_\mu$ , it follows that  $p^{a'}Q \in K_\mu$  (its class in  $J(k_v)/K_\mu$  has order prime to  $p$  and a power of  $p$  at the same time, so it must be zero). This in turn implies, using (3.2),

$$P = p^{\nu(P)}Q = p^{\nu(P)-a'} \cdot (p^{a'}Q) \in K_{\mu+(\nu(P)-a')e} \subseteq K_{n+1},$$

a contradiction. So  $\nu(P)e \leq n - \mu + a'e$  as claimed.

Finally, combining these arguments, we see that  $n_m e \leq m + (a + a'e - \mu)$  for large  $m$ , which implies the claim.  $\square$

**Lemma 3.7.** *Let  $m_0 = 1$  if  $n_{\text{tors}} = 0$  and  $m_0 = n_{\text{tors}}e + \mu + e$  otherwise. There is some  $m \geq m_0$  such that  $N(m) \geq n_m + 1$ . For any such  $m$ , we have that*

$$q(i(\varphi(\{t : m \leq v(t) < \infty\}))) = q(i(\varphi(\{t : v(t) = m\}))) \cup \{0\}.$$

*Proof.* By Lemma 3.6,  $n_m e \leq m + b$  for some  $b$ ; on the other hand,  $N(m)e \geq 2m - a$  for some  $a$ , so whenever  $m \geq a + b + e$ , the inequality  $N(m) \geq n_m + 1$  holds. Fix such an  $m$  that also satisfies  $m \geq m_0$ . We now show that if  $v(t) = m$ , then

$$\nu(i(\varphi(p^n t))) = \nu(i(\varphi(t))) + n \quad \text{and} \quad q(i(\varphi(p^n t))) \subseteq q(i(\varphi(t))) \cup \{0\}$$

for all  $n \geq 0$ , which implies the claim (note that  $0 \in q(i(P))$  if  $P$  is sufficiently close to  $P_0$ ). Note that  $m \geq n_{\text{tors}}e + \mu + e$  implies  $n_m \geq n_{\text{tors}} + 1$  by (3.3) (taking  $t' = 0$ ). We proceed by induction on  $n$ , the case  $n = 0$  being trivial. So consider  $n \geq 1$ . By the inductive assumption, we have

$$\nu(i(\varphi(p^{n-1}t))) = \nu(i(\varphi(t))) + n - 1 \leq n_m + n - 1 \quad \text{and} \quad q(i(\varphi(p^{n-1}t))) \subseteq q(i(\varphi(t))) \cup \{0\}.$$

By Lemma 3.4, this implies  $p \cdot i(\varphi(p^{n-1}t)) \equiv i(\varphi(p^n t)) \pmod{p^{N(m+(n-1)e)}J(k_v)}$ , and since

$$N(m + (n - 1)e) \geq N(m) + 2n - 2 \geq n_m + n \geq \nu(i(\varphi(p^{n-1}t))) + 1$$

and  $n_m \geq n_{\text{tors}} + 1$  in case  $n_{\text{tors}} > 0$ , by Lemmas 3.1 and 3.3 it follows that

$$\nu(i(\varphi(p^n t))) = \nu(p \cdot i(\varphi(p^{n-1}t))) = \nu(i(\varphi(p^{n-1}t))) + 1 = \nu(i(\varphi(t))) + n$$

and

$$q(i(\varphi(p^n t))) = q(p \cdot i(\varphi(p^{n-1} t))) \subseteq q(i(\varphi(p^{n-1} t))) \cup \{0\} \subseteq q(i(\varphi(t))) \cup \{0\}. \quad \square$$

**Corollary 3.8.** *If  $p = 2$  and  $e = 1$  in the situation of Lemma 3.7, then we take  $m_0 = 1$  if  $n_{\text{tors}} = 0$  and  $m_0 = n_{\text{tors}} + 3$  otherwise. There is then some  $m \geq m_0$  such that  $2m - 3 \geq n_m$ . For any such  $m$ , we have have that*

$$q(i(\varphi(\{t : m \leq v(t) < \infty\}))) = q(i(\varphi(\{t : v(t) = m\}))) \cup \{0\}.$$

*If the curve is hyperelliptic,  $P_0 = \varphi(0)$  is a Weierstrass point and  $\varphi(-t) = \iota(\varphi(t))$ , where  $\iota$  is the hyperelliptic involution, then the condition above can be replaced by  $3m - 2 \geq n_m$ .*

*Proof.* This follows again from  $\mu = 2$  and  $N(m) \geq 2m - 2$ . The improved statement under the additional assumptions follows in the same way as for Corollary 3.5.  $\square$

This now allows us to find  $q(i(D))$  when  $0 \in i(D)$ . First we use Corollary 3.2 to determine  $q(i(\varphi(\{t : 1 \leq v(t) \leq m_0 - 1\})))$ . Then for  $m = m_0, m_0 + 1, \dots$ , we find in a similar way  $n_m$  and  $q(i(\varphi(\{t : v(t) = m\})))$ . As soon as  $n_m + 1 \leq N(m)$ , we can stop the computation; we then have  $q(i(D \setminus \{P_0\})) = q(i(\varphi(\{t : 1 \leq v(t) \leq m\}))) \cup \{0\}$ .

We state a special case for later use.

**Corollary 3.9.** *Assume that  $C$  is hyperelliptic, of good reduction mod 2, and satisfies  $J(\mathbb{Q}_2)[2] = 0$  and  $J(\mathbb{F}_2)[2] = 0$ . Let  $P_0 \in C(\mathbb{Q}_2)$ , choose a parameterization  $\varphi$  of a residue disk  $D$  centered at  $P_0$  and let  $i_{P_0}$  denote the embedding of  $C$  into  $J$  sending  $P_0$  to 0. Then*

- (1)  $q(i_{P_0}(D)) = q(i_{P_0}(\varphi(2\mathbb{Z}_2^\times \cup 4\mathbb{Z}_2^\times))) \cup \{0\}$ , and
- (2) *if  $P_0$  is a Weierstrass point and  $\varphi$  satisfies  $\varphi(-t) = \iota(\varphi(t))$ , then*  
 $q(i_{P_0}(D)) = q(i_{P_0}(\varphi(2\mathbb{Z}_2^\times))) \cup \{0\}$ .

*Proof.* Since  $k_v = \mathbb{Q}_2$ , we are in the case  $p = 2$  and  $e = 1$ . The assumptions on 2-torsion over  $\mathbb{Q}_2$  and over  $\mathbb{F}_2$  imply that  $n_{\text{tors}} = 0$ , which in turn implies that for  $P \in K_m \setminus K_{m+1}$ , we have  $\nu(P) \in \{m - 2, m - 1\}$ , for all  $m \geq 1$ , compare [PS14, Lemma 10.1] and its proof. Also,  $K_1$  has odd index in  $J(\mathbb{Q}_2)$ . We can therefore take  $b = -1$  in Lemma 3.6. Then  $m = 2$  is a suitable value in Corollary 3.8. When  $P_0$  is a Weierstrass point, then by Corollary 3.8 again even  $m = 1$  is sufficient.  $\square$

We now give a version of Lemma 3.7 that applies when we work with a subgroup  $\Gamma$  that does not consist of torsion points only. We restrict here to the case  $k_v = \mathbb{Q}_2$ ; a general statement can be obtained and proved along the same lines, with changes similar to the statement and proof of Lemma 3.7.

We let  $\Gamma \subseteq J(\mathbb{Q}_2)$  be a subgroup such that  $\Gamma \cap 2J(\mathbb{Q}_2) = 2\Gamma$  and such that  $\text{cl}(\Gamma)$  is not of finite index in  $J(\mathbb{Q}_2)$ . We define

$$n_{m,\Gamma} := \sup\{\nu(i(\varphi(t)) + \gamma) : \gamma \in \Gamma, t \in 2\mathbb{Z}_2, v(t) = m\}.$$

**Lemma 3.10.** *Let  $m_0 = 2$  if  $n_{\text{tors}} = 0$  and  $m_0 = n_{\text{tors}} + 3$  otherwise. Assume that there is  $m \geq m_0$  such that  $2m - 3 \geq n_{m,\Gamma}$ . For any such  $m$ , we have have that*

$$q(i(\varphi(\{t : m \leq v(t) < \infty\})) + \Gamma) = q(i(\varphi(\{t : v(t) = m\})) + \Gamma) \cup q(\Gamma).$$

If the curve is hyperelliptic,  $P_0 = \varphi(0)$  is a Weierstrass point and  $\varphi(-t) = \iota(\varphi(t))$ , where  $\iota$  is the hyperelliptic involution, then the condition above can be replaced by  $3m - 2 \geq n_{m,\Gamma}$ .

By standard Chabauty-Coleman, the intersection of  $i(D)$  with  $\text{cl}(\Gamma)$  is finite. So for  $m$  sufficiently large,  $i(\varphi(2^m\mathbb{Z}_2))$  will meet  $\text{cl}(\Gamma)$  only in  $P_0$ , hence  $n_{m,\Gamma} < \infty$ . So we can hope to find an  $m$  as in the lemma. It is conceivable, however, that the image of the curve meets  $\text{cl}(\Gamma)$  at  $i(P_0)$  with higher multiplicity, in which case  $n_{m,\Gamma}$  may grow too fast with  $m$ .

*Proof.* We show again inductively that if  $v(t) = m$ , then

$$\max \nu(i(\varphi(2^n t)) + \Gamma) = \max \nu(i(\varphi(t)) + \Gamma) + n \quad \text{and} \quad q(i(\varphi(2^n t)) + \Gamma) \subseteq q(i(\varphi(t)) + \Gamma) \cup q(\Gamma)$$

for all  $n \geq 0$  (note that  $q(\Gamma) \subseteq q(i(P) + \Gamma)$  if  $P$  is sufficiently close to  $P_0$ ). The case  $n = 0$  is trivial. So consider  $n \geq 1$ . By the inductive assumption, we have

$$\max \nu(i(\varphi(2^{n-1}t)) + \Gamma) = \max \nu(i(\varphi(t)) + \Gamma) + n - 1 \leq n_{m,\Gamma} + n - 1$$

and

$$q(i(\varphi(2^{n-1}t)) + \Gamma) \subseteq q(i(\varphi(t)) + \Gamma) \cup q(\Gamma).$$

By Corollary 3.5, we have  $2i(\varphi(2^{n-1}t)) \equiv i(\varphi(2^n t)) \pmod{2^{2m+2n-4}J(\mathbb{Q}_2)}$ . So for every  $\gamma \in \Gamma$ , we have

$$2(i(\varphi(2^{n-1}t)) + \gamma) \equiv i(\varphi(2^n t)) + 2\gamma \pmod{2^{2m+2n-4}J(\mathbb{Q}_2)}.$$

Since

$$2m + 2n - 4 \geq n_{m,\Gamma} + n \geq \nu(i(\varphi(2^{n-1}t)) + \gamma) + 1$$

and  $n_{m,\Gamma} \geq n_{\text{tors}} + 1$  in case  $n_{\text{tors}} > 0$ , by Lemmas 3.1 and 3.3 it follows that

$$\nu(i(\varphi(2^n t)) + 2\gamma) = \nu(2(i(\varphi(2^{n-1}t)) + \gamma)) = \nu(i(\varphi(2^{n-1}t)) + \gamma) + 1$$

and

$$q(i(\varphi(2^n t)) + 2\gamma) = q(2(i(\varphi(2^{n-1}t)) + \gamma)) \subseteq q(i(\varphi(2^{n-1}t)) + \gamma) \cup \{0\} \subseteq q(i(\varphi(t)) + \Gamma) \cup q(\Gamma).$$

Now consider  $\gamma \in \Gamma \setminus 2\Gamma$ . Since  $i(\varphi(2^n t)) \in 2^{n+m-2}J(\mathbb{Q}_2)$  and  $n + m - 2 \geq 1$ , we get

$$\nu(i(\varphi(2^n t)) + \gamma) = 0 \quad \text{and} \quad q(i(\varphi(2^n t)) + \gamma) = \{\pi(\gamma)\} \subseteq q(\Gamma).$$

(We use here that  $\gamma \notin 2J(\mathbb{Q}_2)$ .) Together, these relations imply that

$$\max \nu(i(\varphi(2^n t)) + \Gamma) = \max \nu(i(\varphi(t)) + \Gamma) + n$$

and

$$q(i(\varphi(2^n t)) + \Gamma) \subseteq q(i(\varphi(t)) + \Gamma) \cup q(\Gamma)$$

as claimed.

The improved statement under the additional assumptions follows again in the same way.  $\square$

#### 4. DETERMINING THE SET OF RATIONAL POINTS ON ODD HYPERELLIPTIC CURVES

In this section, we specialize the algorithm formulated in Section 2 to hyperelliptic curves of odd degree over  $\mathbb{Q}$ . So let

$$C: y^2 = f(x)$$

be a hyperelliptic curve, given by a squarefree polynomial  $f \in \mathbb{Z}[x]$  of odd degree  $2g + 1$  (then  $g$  is the genus of  $C$ ). We understand  $C$  to be the smooth projective model of the affine curve given by the equation; then  $C$  is a nice curve. We write  $J$  for the Jacobian of  $C$ . For a point  $P_0 \in C(\mathbb{Q})$  (or  $C(\mathbb{Q}_2)$ ), we let  $i_{P_0}: C \rightarrow J$  denote the embedding that sends  $P_0$  to the origin of  $J$ .

To carry out one of the relevant steps, we must compute  $q(P)$  for points  $P \in J(\mathbb{Q}_2)$  (where  $q(P)$  is defined as above with  $p = 2$ ). The basic strategy for this was explained in Section 3. To implement it, we need to be able to divide by 2 in  $J(\mathbb{Q}_2)$ . We consider this problem in Section 5 below.

We recall the algorithm for computing the 2-Selmer group of  $J$ , compare [Sch95, Sto01]. Let  $C$  be given by the affine equation  $y^2 = f(x)$  with  $f \in \mathbb{Z}[x]$  squarefree and of odd degree  $2g + 1$ , where  $g$  is the genus of  $C$ . Let  $L = \mathbb{Q}[x]/\langle f \rangle$  be the associated étale algebra and write  $\theta$  for the image of  $x$  in  $L$ . If  $A$  is any commutative ring, then we write  $A^\square$  for the group  $A^\times / (A^\times)^2$  of square classes in the multiplicative group  $A^\times$  of  $A$ .

For any field extension  $k$  of  $\mathbb{Q}$ , there is an isomorphism

$$(4.1) \quad H^1(k, J[2]) \xrightarrow{\simeq} \ker(N_{(L \otimes_{\mathbb{Q}} k)/k}: (L \otimes_{\mathbb{Q}} k)^\square \rightarrow k^\square)$$

realizing the Galois cohomology group on the left in a concrete way, and there is the ‘Cassels map’ or ‘ $x - T$ ’ map

$$\mu_k: J(k) \longrightarrow J(k)/2J(k) \hookrightarrow (L \otimes_{\mathbb{Q}} k)^\square$$

that is induced by evaluating  $x - \theta$  (multiplicatively) on divisors whose support is disjoint from the set of Weierstrass points of  $C$ . The image of  $\mu_k$  is contained in the kernel of the norm map above;  $\mu_k$  is the composition of the connecting map  $\delta_k: J(k) \rightarrow H^1(k, J[2])$  induced by the exact sequence of Galois modules

$$0 \longrightarrow J[2] \longrightarrow J(\bar{k}) \xrightarrow{-2} J(\bar{k}) \longrightarrow 0$$

with the isomorphism (4.1). We write  $\mu = \mu_{\mathbb{Q}}$ , and for  $v$  a place of  $\mathbb{Q}$ , we write  $L_v = L \otimes_{\mathbb{Q}} \mathbb{Q}_v$  (with  $\mathbb{Q}_\infty = \mathbb{R}$  as usual) and set  $\mu_v = \mu_{\mathbb{Q}_v}$ .

Let  $\Sigma$  be the set of places of  $\mathbb{Q}$  consisting of 2 and the finite places  $v$  such that the Tamagawa number of  $J$  at  $v$  is even. The subgroup  $L(\Sigma, 2)$  of  $L^\square$  consists of the elements represented by  $\alpha \in L^\times$  such that the fractional ideal generated by  $\alpha$  has the form  $I_1^2 I_2$  with  $I_2$  supported on the primes above primes in  $\Sigma$ . Then the isomorphic image of  $\text{Sel}_2 J$  in  $L^\square$ , which we will identify with  $\text{Sel}_2 J$ , is given by

$$\text{Sel}_2 J = \{\xi \in L(\Sigma, 2) : N_{L/\mathbb{Q}}(\xi) = \square, \forall v \in \Sigma \cup \{\infty\} : \rho_v(\xi) \in \text{im}(\mu_v)\},$$

where  $\rho_v: L^\square \rightarrow L_v^\square$  is the canonical map. There is also the 2-Selmer set of  $C$ , given by

$$\text{Sel}_2 C = \{\xi \in L(\Sigma, 2) : N_{L/\mathbb{Q}}(\xi) = \square, \forall v : \rho_v(\xi) \in \mu_v(i_\infty(C(\mathbb{Q}_v)))\}.$$

It is a subset of the 2-Selmer group. The set of places  $v$  in the condition can be restricted to the set  $\Sigma \cup \{\infty\}$  together with all ‘small’ primes, where ‘small’ in practice can be rather large; see [BS09].

Algorithm 2.7, combined with the representation of  $J(k)/2J(k)$  as a subgroup of  $(L \otimes_{\mathbb{Q}} k)^{\square}$ , then leads to the following.

**Algorithm 4.1.**

**Input:** A polynomial  $f \in \mathbb{Z}[x]$ , squarefree and of odd degree  $2g + 1$ .

**Output:** The set of rational points on  $C: y^2 = f(x)$ , or FAIL.

1. Let  $J$  denote the Jacobian of  $C$ . Set  $L = \mathbb{Q}[x]/\langle f \rangle$ .
2. Compute  $\text{Sel}_2 J$  and  $\text{Sel}_2 C$  as a subgroup and a subset of  $L^{\square}$ .
3. Let  $L_2 = L \otimes_{\mathbb{Q}} \mathbb{Q}_2$ ; let  $r: L^{\square} \rightarrow L_2^{\square}$  be the map induced by  $\mathbb{Q} \rightarrow \mathbb{Q}_2$ .  
If  $\ker r \cap \text{Sel}_2 J \not\subseteq \delta(\pi(J(\mathbb{Q})[2^{\infty}]))$ , then return FAIL.
4. Search for rational points on  $C$  and collect them in a set  $C(\mathbb{Q})_{\text{known}}$ .
5. Let  $\mathcal{X}$  be a partition of  $C(\mathbb{Q}_2)$  into residue disks whose image in  $L_2^{\square}$  consists of one element and that are contained in half residue disks when  $J(\mathbb{Q})[2] \neq 0$ .
6. Let  $R$  denote the image of  $J(\mathbb{Q})[2^{\infty}]$  in  $L_2^{\square}$ .
7. For each  $X \in \mathcal{X}$  do the following:
  - a. If  $X \cap C(\mathbb{Q})_{\text{known}} = \emptyset$ :  
If  $\mu_2(X) \subseteq \text{Sel}_2 C$ , then return FAIL;  
otherwise continue with the next  $X$ .
  - b. Pick some  $P_0 \in C(\mathbb{Q})_{\text{known}} \cap X$ .
  - c. Compute  $Y = \mu_2(q(i_{P_0}(X) + J(\mathbb{Q})[2^{\infty}])) \subseteq L_2^{\square}$ .
  - d. If  $Y \cap r(\text{Sel}_2 J) \not\subseteq R$ , then return FAIL.
8. Return  $C(\mathbb{Q})_{\text{known}}$ .

That the algorithm is correct is a special case of Proposition 2.8, taking into account that torsion points of odd order are infinitely 2-divisible, which allows us to replace  $J(\mathbb{Q})_{\text{tors}}$  with  $J(\mathbb{Q})[2^{\infty}]$  at the places where the latter occurs.

Remark 2.9 applies in the same way as to the general algorithm.

*Remark 4.2.* We note that the (image of the) Selmer group in  $L^{\square}$  that is used in the algorithm can be replaced by any subgroup  $S$  of  $L^{\square}$  that contains it (and similarly for the Selmer set). For example, we can take

$$S = \{\xi \in L(\Sigma, 2) : N_{L/\mathbb{Q}}(\xi) = \square, \forall v \in \Sigma \cup \{\infty\} \setminus \{2\} : \text{res}_v(\xi) \in \text{im}(\mu_v)\},$$

where  $\Sigma$  is the set of ‘bad primes’ for 2-descent on  $J$ . This leaves out the 2-adic Selmer condition. Taking it into account requires the computation of  $\mu_2(J(\mathbb{Q}_2))$ , which is usually the most time-consuming step in the local part of the computation of  $\text{Sel}_2 J$ . We can do without it, since using  $S$  in the algorithm is actually equivalent to using  $\text{Sel}_2 J$ . To see this, first consider Step 3. Since all elements in the kernel of  $r$  satisfy the 2-adic Selmer condition trivially, it follows that  $\ker r \cap S = \ker r \cap \text{Sel}_2 J$ , so that the outcome of Step 3 is the same in both cases. Now consider Step 7a. This does not involve  $\text{Sel}_2 J$ , so its outcome is trivially



the same in both cases. Finally consider Step 7d. If  $Y \cap r(S) \not\subseteq R$ , then there is some  $s \in S$  such that  $r(s) \notin R$  and  $r(s) \in Y$ . But everything in  $Y$  is of the form  $\mu_2(Q)$  for some  $Q \in J(\mathbb{Q}_2)$ , so  $Y \subseteq \text{im}(\mu_2)$ , which means that  $s$  satisfies the 2-adic Selmer condition. This shows that  $s \in \text{Sel}_2 J$  and then implies that  $Y \cap r(\text{Sel}_2 J) \ni r(s) \notin R$ , so that the outcome of this step is again the same in both cases. The preceding arguments show that the algorithm fails on  $S$  if and only if it fails on  $\text{Sel}_2 J$ . Finally, it is clear that the result will be the same, namely  $C(\mathbb{Q})_{\text{known}}$ , in both cases when the algorithm does not output FAIL

If  $\Sigma \subseteq \{2, p\}$  with  $p \not\equiv \pm 1 \pmod{8}$ , then we can also leave out the condition  $N_{L/\mathbb{Q}}(\xi) = \square$ , since then  $\mathbb{Q}(\Sigma, 2)$  injects into  $\mathbb{Q}_2^\square$ , so the norm condition is implied by the image under  $r$  being in  $Y$ .

Of course, we can also use a subset of  $L^\square$  that is possibly larger than  $\text{Sel}_2 C$  instead of the 2-Selmer set. In fact, this is what we have to do in practice, since the computation of the exact 2-Selmer set usually requires taking into account the local conditions for all primes up to some bound that is exponential in the genus of  $C$ ; compare [BS09].

If we assume that  $C(\mathbb{Q})_{\text{known}}$  meets every set in  $\mathcal{X}$ , then the other conditions required to avoid failure of the algorithm are likely to be satisfied. This follows from work of Bhargava and Gross [BG13], which we use in a similar way as in [PS14]: the ‘probability’ that the map  $\text{Sel}_2 J \rightarrow J(\mathbb{Q}_2)/2J(\mathbb{Q}_2)$  is injective is at least  $1 - 2^{1-g-\dim_{\mathbb{F}_2} J(\mathbb{Q}_2)[2]}$ , and the ‘probability’ that the image has intersection with  $Y$  contained in  $R$  is at least  $1 - (\#(Y/R) - 1)2^{1-g}$ . Since by the results of [PS14]  $Y$  is usually small and by [Sto15] the size of  $Y$  modulo  $R$  is uniformly bounded by some constant times  $g^2$ , there is a very good chance that both conditions are satisfied when  $g$  is large.

## 5. HALVING POINTS ON ODD HYPERELLIPTIC JACOBIANS

In this section we describe an algorithm that computes one ‘half’ or all ‘halves’ of a point  $P \in 2J(k)$ , where  $J$  is the Jacobian of a hyperelliptic curve  $C$  of odd degree over the field  $k$ . We assume that  $\text{char}(k) \neq 2$ , so that  $C$  can be given by an equation  $y^2 = f(x)$  with  $f \in k[x]$  squarefree and of odd degree  $2g + 1$ .

Recall that each point in  $J(k)$  is uniquely represented in the form  $[D - d\infty]$ , where  $D$  is an effective divisor in general position defined over  $k$  and  $d = \deg D \leq g$ . An effective divisor  $D$  is said to be *in general position* if its support does not contain  $\infty$  and  $D \not\geq P + \iota(P)$  for any point  $P \in C$ , where  $\iota: C \rightarrow C$  is the hyperelliptic involution.

Any effective divisor  $D$  in general position can be described by its *Mumford representation*  $(a, b)$ . Here  $a \in k[x]$  is a monic polynomial of degree  $d = \deg D$  whose roots are the  $x$ -coordinates of the points in the support of  $D$ , with appropriate multiplicity (so that  $a$  corresponds to the image of  $D$  under the hyperelliptic quotient map to  $\mathbb{P}^1$ ), and  $b \in k[x]$  is another polynomial such that  $b(\xi) = \eta$  for any point  $P = (\xi, \eta)$  in the support of  $D$  and satisfying  $a \mid f - b^2$ . This polynomial  $b$  is uniquely determined modulo  $a$ ; in particular, we obtain a unique representation if we require  $\deg(b) < d$ . However, it is sometimes useful to allow additional flexibility, so we will not always insist on this normalization. In fact, we may also want to allow polynomials  $a$  of larger degree (this leads to even more non-unique representations, but can be useful in certain situations).

We will use the notation  $(a, b)$  to denote the divisor  $D$ , and we will write  $[a, b] = [(a, b) - d\infty]$  for the point on  $J$  corresponding to it.

Let  $c$  be the leading coefficient of  $f$ . Then in terms of the Mumford representation, the descent map  $\mu: J(k) \rightarrow L^\square$  is given by

$$[a, b] \mapsto (-c)^{\deg(a)} a(\theta) \cdot (L^\times)^2$$

if  $a$  and  $f$  are coprime. In the general case, write  $a_1$  and  $f_1$  for  $a$  and  $f$  divided by their (monic) gcd; then

$$\mu([a, b]) = \tilde{\mu}(a) := (-c)^{\deg(a)} (a(\theta) - a_1(\theta)f_1(\theta)) \cdot (L^\times)^2;$$

compare [Sch95].

Since the kernel of  $\mu$  is  $2J(k)$ , this gives us a way of deciding whether a point  $P \in J(k)$  is divisible by 2 in  $J(k)$ : this is equivalent to the existence of a polynomial  $s \in k[x]$  such that

$$s(\theta)^2 = (-c)^{\deg(a)} (a(\theta) - a_1(\theta)f_1(\theta));$$

equivalently,

$$s^2 \equiv (-c)^{\deg(a)} (a - a_1 f_1) \pmod{f}.$$

We will now state a result that shows how to compute a point  $Q \in J(k)$  such that  $2Q = P$ , given such a polynomial  $s$ .

Note that when  $a = a_1^2 a_2$ , then  $P = [a, b]$  is divisible by 2 if and only if  $P_2 = [a_2, b]$  is, and each point  $Q$  such that  $2Q = P$  has the form  $Q = Q_1 + Q_2$  where  $Q_2$  satisfies  $2Q_2 = P_2$  and  $Q_1 = [a_1, b]$ . So we can assume that  $a$  is squarefree.

**Proposition 5.1.** *Let  $a \in k[x]$  be monic and squarefree, of degree  $\leq 2g + 1$ . Let  $d$  denote  $\gcd(a, f)$ , so that  $a = da_1$  and  $f = df_1$  as above. Suppose we have  $b, s \in k[x]$  with*

$$f \equiv b^2 \pmod{a} \quad \text{and} \quad (-c)^{\deg(a)} (a - a_1 f_1) \equiv s^2 \pmod{f},$$

so that  $[a, b] \in 2J(k)$ . For polynomials  $u, v$  and  $w$ , consider the following system of congruences:

$$(5.1) \quad vd \equiv ws \pmod{f_1}, \quad vd \equiv ub \pmod{a_1}, \quad uf_1 \equiv ws \pmod{d}.$$

Then this system has a nontrivial solution  $(u, v, w)$  with  $w$  monic such that

$$(5.2) \quad \deg(u) < \deg(a)/2, \quad \deg(v) \leq g + \deg(a)/2 - \deg(d) \quad \text{and} \quad \deg(w) \leq g.$$

Each such solution satisfies the relation

$$(5.3) \quad u^2 f_1 = dv^2 - (-c)^{\deg(a)} a_1 w^2.$$

Now assume that  $(u, v, w)$  is a solution such that  $w$  has minimal degree. Let  $d_1 = \gcd(u, w)$ ; then  $d_1$  divides  $fa_1$  and  $v$ . Write  $d_1 = d_f d_a$  with  $d_f = \gcd(d_1, f)$  and  $d_a = \gcd(d_1, a_1)$ . Set  $w_1 = w/d_1$ ,  $u_1 = u/d_1$ ,  $v_1 = v/d_1$  and let  $r \in k[x]$  be such that

$$ru_1 \equiv -v_1 d \pmod{w_1 d_a} \quad \text{and} \quad r \equiv 0 \pmod{d_f}.$$

Then  $Q = [w, r]$  satisfies  $P = 2Q$ .

If  $Q$  and  $Q'$  are computed starting from  $s$  and  $s'$  such that  $s' \not\equiv \pm s \pmod{f}$ , then  $Q$  and  $Q'$  are distinct.

*Proof.* First note that, since  $f$  is squarefree, we have that  $d$  and  $f_1$  are coprime. Also,  $d$  and  $a_1$  are coprime, since a divisor in general position contains no ramification point with multiplicity 2 or more. So  $f_1$ ,  $a_1$  and  $d$  are coprime in pairs and squarefree. The fact that  $a$  divides  $f - b^2$  implies that  $d$  divides  $b$  and that  $d$  is also the gcd of  $a$  and  $b$ .

The first claim is that the system of congruences has a nontrivial solution when the degrees of the polynomials are bounded as stated. To see this, note that the conditions are linear in (the coefficients of)  $u$ ,  $v$  and  $w$ , and that the total number of coefficients of  $u$ ,  $v$  and  $w$  is

$$\begin{aligned} \lceil \deg(a)/2 \rceil + (g + \lfloor \deg(a)/2 \rfloor - \deg(d) + 1) + (g + 1) &= 2g + \deg(a) - \deg(d) + 2 \\ &= \deg(f_1) + \deg(a_1) + \deg(d) + 1. \end{aligned}$$

On the other hand, the number of linear constraints is  $\deg(f_1) + \deg(a_1) + \deg(d)$ . So there are more variables than constraints, hence nontrivial solutions exist.

We claim that  $w$  cannot be zero in such a solution. Otherwise, the first congruence would imply that  $f_1$  divides  $v$  (since  $f_1$ ,  $a_1$  and  $d$  are coprime in pairs), which for degree reasons (recall that  $\deg(a) \leq 2g + 1$ ) is only possible when  $v = 0$ . In a similar way, the second congruence would then imply that  $a_1$  divides  $u$  (since  $a_1$  is coprime to  $b$ ), whereas the third congruence implies that  $d$  divides  $u$ , so  $a$  divides  $u$ , which is only possible when  $u = 0$ . But then our solution is trivial, a contradiction. So  $w \neq 0$ , and without loss of generality,  $w$  can be taken to be monic.

We show that every solution as above satisfies relation (5.3). Namely, by the first congruence and since  $s^2 \equiv (-c)^{\deg(a)} a \pmod{f_1}$ ,

$$d^2 v^2 = (dv)^2 \equiv (sw)^2 = s^2 w^2 \equiv (-c)^{\deg(a)} a w^2 = (-c)^{\deg(a)} d a_1 w^2 \pmod{f_1},$$

so (since  $d$  and  $f_1$  are coprime), the relation holds mod  $f_1$ . Next, by the second congruence,

$$d^2 v^2 = (dv)^2 \equiv (bu)^2 = b^2 u^2 \equiv f u^2 = d f_1 u^2 \pmod{a_1},$$

so (since  $d$  and  $a_1$  are coprime), the relation holds mod  $a_1$ . Finally, by the last congruence,

$$u^2 f_1^2 = (u f_1)^2 \equiv (s w)^2 = s^2 w^2 \equiv -(-c)^{\deg(a)} a_1 f_1 w^2 \pmod{d},$$

so (since  $d$  and  $f_1$  are coprime again), the relation holds also mod  $d$ . It follows that it holds mod  $f_1 a_1 d$ . Since the degrees of all terms are strictly less than the degree of  $f_1 a_1 d$ , equality follows, and (5.3) is verified.

We note that the fact shown above that a nontrivial solution has  $w \neq 0$  implies that  $w$  determines the solution uniquely. It follows that there is in fact a *unique* solution with  $w$  monic and  $\deg(w)$  minimal.

Since  $d$  is squarefree, (5.3) implies that the gcd  $d_1$  of  $w$  and  $u$  also divides  $v$ . We can therefore divide all three by this gcd, obtaining  $u_1$ ,  $v_1$  and  $w_1$ ; they satisfy

$$u_1^2 f_1 = d v_1^2 - (-c)^{\deg(a)} a_1 w_1^2.$$

If some irreducible factor  $p$  of  $d_1$  does not divide  $f a_1$ , then  $(u/p, v/p, w/p)$  also satisfy the system of congruences, contradicting the minimality of  $\deg(w)$ . Now assume that  $p^2$  divides  $d_1$  for some irreducible polynomial  $p$ . Then  $p$  divides  $f_1$ ,  $a_1$  or  $d$ , say  $p \mid a_1$  (the other cases are analogous). Since  $a_1$  is squarefree, the congruence  $vd \equiv ub \pmod{a_1}$  implies  $(v/p)d \equiv (u/p)b \pmod{a_1}$ , and so again  $(u/p, v/p, w/p)$  satisfy the system of congruences,

contradiction. So  $d_1$  is squarefree and must therefore divide  $a_1 f_1 d$ . In particular, we can write  $d_1 = d_f d_a$  as claimed.

Note that  $(u_1 b)^2 \equiv u_1^2 f \equiv (v_1 d)^2 \pmod{a_1}$ , so that  $a_1$  divides  $(u_1 b - v_1 d)(u_1 b + v_1 d)$ . We claim that  $d_a = \gcd(u_1 b + v_1 d, a_1)$ . For this, consider an irreducible factor  $p$  of  $d_a$ . If  $p$  divides  $u_1 b - v_1 d$ , then  $(u/p, v/p, w/p)$  are a solution, a contradiction. So  $p$  must divide  $u_1 b + v_1 d$ . Conversely, if  $p$  is any irreducible factor of  $a_1$  that divides  $u_1 b + v_1 d$ , then (noticing that  $b$  is invertible mod  $a_1$ ) for  $p$  to divide  $ub - vd$ , it must necessarily divide  $u$  and  $v$ , so  $p \mid d_a$ .

$u_1$  is invertible mod  $w_1$ , but also mod  $d_a$  (since  $u_1$  and  $v_1$  are coprime as well —  $a_1$  is squarefree — and  $d_a$  is coprime with  $f_1$  and  $d$ ). Furthermore,  $d_f$  is coprime with  $w_1$  (and of course also with  $d_a$ ), for essentially the same reason. Therefore a polynomial  $r$  exists such that  $u_1 r \equiv -v_1 d \pmod{w_1 d_a}$  and  $r \equiv 0 \pmod{d_f}$ .

Now we consider the function

$$\phi = u(x)y - v(x)d(x) = d_f(x)d_a(x) (u_1(x)y - v_1(x)d(x))$$

on  $C$ . Its divisor of zeros is

$$\begin{aligned} & 2(d_f, 0) + ((d_a, b) + (d_a, -b)) + ((d, 0) + (d_a, -b) + (a_1/d_a, b) + 2(w_1, -r)) \\ &= (a_1, b) + (d, 0) + 2((d_f, 0) + (d_a, -b) + (w_1, -r)) \\ &= (a, b) + 2(w, -r). \end{aligned}$$

To see this, note that the norm in  $k[x]$  of the last factor of  $\phi$  is  $u_1^2 f - v_1^2 d^2 = (-c)^{\deg(a)} d a_1 w_1^2$  and that  $u_1 b \equiv v_1 d \pmod{a_1/d_a}$  and  $u_1 b \equiv -v_1 d \pmod{d_a}$  (and so also  $r \equiv b \pmod{d_a}$ ). Setting  $Q = [w, r]$ , we therefore obtain  $2Q = P$ .

We now show that  $Q$  determines  $s \pmod{f}$  up to sign. Given  $Q = [w, r]$  such that  $2Q = P$ , there is a unique function (up to scaling) on  $C$  whose divisor is  $(a, b) + 2(w, -r) - n\infty$  (where  $n = \deg(a) + 2 \deg(w)$ ); this function must then be  $\phi$ , which gives us  $u$  and  $v$  up to scaling; the relation  $u^2 f_1 = dv^2 - (-c)^{\deg(a)} a w^2$  then fixes them up to a common sign. Write  $d_f = d_{f_1} d_d$  with  $d_{f_1} = \gcd(d_f, f_1)$  and  $d_d = \gcd(d_f, d)$ . In a similar way as above for  $d_a$ , one shows that  $d_{f_1} = \gcd(w_1 s + v_1 d, f_1)$  and  $d_d = \gcd(u_1 f_1 + w_1 s, d)$ . Since  $w_1$  is coprime with  $f$ , this determines  $s \pmod{f}$  via the congruences

$$\begin{aligned} w_1 s &\equiv v_1 d \pmod{f_1/d_{f_1}}, & w_1 s &\equiv -v_1 d \pmod{d_{f_1}}, \\ w_1 s &\equiv u_1 f_1 \pmod{d/d_d}, & w_1 s &\equiv -u_1 f_1 \pmod{d_d}. \end{aligned}$$

A common sign change of  $u$  and  $v$  (which is the only ambiguity here) results in a sign change of  $s$ .  $\square$

We can try to use the algorithm implied by Proposition 5.1 over a  $p$ -adic field. It will possibly run into precision problems when some of the roots of  $a$  get close to roots of  $f$  (but with the resultant of  $a$  and  $f$  still being nonzero, albeit  $p$ -adically small) or when the resulting point is represented by a divisor of lower degree or such that some points are close to the point at infinity. In practice, however, these problems occur fairly rarely. A possible remedy in such a case is to replace  $(a, b)$  by another representation  $(a', b')$  such that  $[a', b'] = [a, b]$  and  $\deg(a) > g$ . Writing  $f - b^2 = ac$ , we have  $[c - 2hb - h^2 a, -b - ha] = [a, b]$  for all polynomials  $h$ . Taking  $h$  to be constant already allows us to replace  $a$  by a polynomial  $a'$  that is coprime with  $f$  (and probably we can also arrange  $a'$  to be squarefree) and satisfies  $\deg(a') \leq g + 1$

if  $\deg(a) = g$ . Another possibility is to consider points in a residue disk given by suitable Laurent series, perform the computation on the Laurent series and then specialize.

*Remark 5.2.* In the context of computing  $q(P)$ , the following observation can be useful. Given  $P = [a, b]$  with  $\deg(a) \leq g + 1$  and  $T = [h, 0] \in J(k)[2]$  with  $h \mid f$  and  $\deg(h) \leq g$ , we can use the method described in Proposition 5.1 to compute halves of  $P + T$  without first computing a representation of the sum. For simplicity assume  $\gcd(a, f) = 1$  (this can be arranged, see above). Then  $P + T = [ah, b'h]$  where  $b'h \equiv b \pmod{a}$ . There will be  $s_1$  and  $s_2$  such that  $s_1^2 \equiv (-c)^{\deg(a)} ah \pmod{f/h}$  and  $s_2^2 \equiv -(-c)^{\deg(a)} a(f/h) \pmod{h}$ . We obtain the congruences

$$vh \equiv ws_1 \pmod{f/h}, \quad vh \equiv ub \pmod{a}, \quad u(f/h) \equiv ws_2 \pmod{h}$$

with  $\deg(u) < (\deg(a) + \deg(h))/2$ ,  $\deg(v) \leq g + (\deg(a) - \deg(h))/2$  and  $\deg(w) \leq g$ .

In a similar way, we can divide  $P + P'$  by 2: let  $P = [a, b]$ ,  $P' = [a', b']$  and assume that  $\deg(a) + \deg(a') \leq 2g + 1$  and that  $a, a'$  and  $f$  are coprime in pairs. Given a polynomial  $s$  such that  $s^2 \equiv (-c)^{\deg(a) + \deg(a')} aa' \pmod{f}$ , the system to be solved is

$$v \equiv ws \pmod{f}, \quad v \equiv ub \pmod{a}, \quad v \equiv ub' \pmod{a'}$$

with  $\deg(u) < (\deg(a) + \deg(a'))/2$ ,  $\deg(v) \leq g + (\deg(a) + \deg(a'))/2$  and  $\deg(w) \leq g$ .

We mention one implication that can be helpful in applications.

**Corollary 5.3.** *Let  $[a', b]$  be the Mumford representation of a point  $P \in J(k)$ , write  $a' = a_0^2 a$  with  $a$  squarefree and monic and fix a polynomial  $s$  such that  $s^2 \equiv (-c)^{\deg(a)} (a - a_1 f_1) \pmod{f}$  as above. Let  $(u, v, w)$  be the solution with  $w$  monic and of smallest degree of the system (5.1) with the restrictions in (5.2), and let  $Q \in J(k)$  be the associated point such that  $2Q = P$ . Then  $\mu(Q) = \tilde{\mu}(a_0) \tilde{\mu}(w)$ .*

*Proof.* This is because according to Proposition 5.1,  $Q = [a_0, b] + [w, r]$  for some  $r \in k[x]$ .  $\square$

**Corollary 5.4.** *In the situation of Corollary 5.3, we have the following special cases.*

(1) *If  $P = [(\xi, \eta) - \infty] \in 2J(k)$  with  $\eta \neq 0$ , fix  $s \in k[x]$  such that  $s^2 \equiv c(\xi - x) \pmod{f}$ . Let  $w$  be the monic polynomial of smallest degree such that the residue of smallest degree of  $ws$  modulo  $f$  has degree  $\leq g$ . Then the point  $Q \in J(k)$  with  $2Q = P$  that is associated to  $s$  satisfies*

$$\mu(Q) = \tilde{\mu}(w).$$

(2) *If  $P = [(\xi_1, \eta_1) - (\xi_2, \eta_2)] \in 2J(k)$  with  $\xi_1 \neq \xi_2$  and  $\eta_j \neq 0$  for  $j \in \{1, 2\}$ , fix  $s \in k[x]$  such that  $s^2 \equiv (x - \xi_1)(x - \xi_2) \pmod{f}$ . Let  $w$  be the monic polynomial of smallest degree such that the residue  $v$  of smallest degree of  $ws$  modulo  $f$  has degree  $\leq g + 1$  and satisfies  $\eta_2 v(\xi_1) + \eta_1 v(\xi_2) = 0$ . Then the point  $Q \in J(k)$  with  $2Q = P$  that is associated to  $s$  satisfies*

$$\mu(Q) = \tilde{\mu}(w).$$

*Proof.* This follows directly from Corollary 5.3, using that  $d = 1$  (in the notation of Proposition 5.1) in both cases and that  $u$  has to be constant. In the first case, the congruence  $v \equiv ub \pmod{a}$  is redundant, and the system reduces to just  $v \equiv ws \pmod{f}$ . In the second case, the congruence  $v \equiv ub \pmod{a}$  is equivalent to the condition  $\eta_2 v(\xi_1) + \eta_1 v(\xi_2) = 0$ .  $\square$

## 6. A CONCRETE EXAMPLE

In this section we use the approach described above to show the following result.

**Theorem 6.1.** *Assuming GRH, the only integral solutions of the equation*

$$y^2 - y = x^{21} - x$$

*have  $x \in \{-1, 0, 1\}$ .*

We remark that  $l = 21$  is the smallest odd exponent such that our method can be successfully applied to determine the set of integral points on the curve  $y^2 - y = x^l - x$ . One can check that for  $l \in \{5, 7, 9, 11, 13, 17\}$  the 2-Selmer rank of the Jacobian is  $\geq g = (l - 1)/2$ , and for  $l \in \{15, 19\}$ , the map from  $\text{Sel}_2 J$  to  $J(\mathbb{Q}_2)/2J(\mathbb{Q}_2)$  is not injective.

We also note that all these curves have a pair of rational points with  $x = 1/4$ ; these points are of the form  $\varphi(2u)$  for a parameterization  $\varphi$  of the residue disk at infinity, where  $u \in \mathbb{Z}_2^\times$ . For such a point  $P$ ,  $[P - \infty]$  has nontrivial image in  $J(\mathbb{Q}_2)/2J(\mathbb{Q}_2)$ , and this image is contained in the image of the Selmer group. On the other hand, by Corollary 3.9, the value of  $q$  on the residue disk of  $\infty$  is given by the values at points of the form  $\varphi(2u)$ , so  $q(i_\infty(D))$  will meet the image of the Selmer group non-trivially for every disk  $D$  around infinity, no matter how small. This implies that our approach cannot be used to show that  $\infty$  is the only rational point 2-adically close to  $\infty$ . This is why we restrict to integral points in the statement of Theorem 6.1. The result is in fact stronger: it covers all rational solutions whose  $x$ -coordinate has odd denominator.

In principle, one could try to deal with the residue disk at infinity using  $\Gamma = \langle \gamma \rangle$  where  $\gamma = [(\frac{1}{4}, \frac{1}{2} + \frac{1}{2^{21}}) - \infty]$ , since the three (known) rational points in the disk map into this group. Unfortunately, it turns out that  $q(i_\infty(P_4))$  meets the image of the Selmer group outside the image of  $\Gamma$ , so that we cannot conclude. Here  $P_4 = \varphi(4)$  denotes a point with  $x$ -coordinate  $1/4^2$  (we can use a parameterization of the disk at infinity whose  $x$ -coordinate is given by  $t^{-2}$ ):  $i_\infty(P_4) + 6\gamma = 2^3Q$  with  $\pi_2(Q) \in \sigma(\text{Sel}_2 J) \setminus \pi_2(\Gamma)$ .

*Proof.* Let  $C$  denote the curve defined by the equation  $y^2 - y = x^{21} - x$ , and let  $J$  be its Jacobian. Note that  $C$  is isomorphic to the curve given by  $y^2 = 4x^{21} - 4x + 1 =: f(x)$ ; let  $L = \mathbb{Q}[x]/\langle f \rangle$ . We compute a group  $S \subset L^\square$  containing  $\text{Sel}_2 J$  using the algorithm described in [Sto01]. The discriminant of  $f$  is  $-2^{40}$  times the product of six distinct odd primes. This implies that 2 is the only ‘bad’ prime for 2-descent, so that the image of the Selmer group is contained in  $L(\{2\}, 2)$ . Since  $L$  is totally ramified at 2, we can reduce this to  $S = L(\emptyset, 2)$  (if  $\xi$  represents an element of  $L(\{2\}, 2)$  and  $N_{L/\mathbb{Q}}(\xi)$  is a square, then the ideal generated by  $\xi$  must be a square). The class group of  $L$  turns out to be trivial, so that  $L(\emptyset, 2) = \mathcal{O}_L^\square$ , but we do not need this fact. We do need to compute  $L(\emptyset, 2)$  and explicit generators of it, though. This is where we use GRH to make the computation feasible in reasonable time. We check that the map  $S \rightarrow L_2^\square$  is injective.

The curve has good reduction mod 2, and  $J(\mathbb{F}_2)$  and  $J(\mathbb{Q}_2)$  both have no elements of order 2. Up to the action of the hyperelliptic involution, there are two residue disks with 2-adically integral  $x$ -coordinates; we can center them at the rational points  $(0, 0)$  and  $(1, 0)$ , respectively. By [Sto01, Lemma 6.3], it follows that the image in  $L_2^\square$  of a point  $P \in C(\mathbb{Q}_2)$  with  $x(P) \in \mathbb{Z}_2$  depends only on  $x \bmod 4$ . We check that the image in  $L_2^\square$  of the points with  $x(P) \equiv 2 \bmod 4$

is not in the image of  $S$ . This shows that any (2-adically) integral point  $P \in C(\mathbb{Q})$  must have  $x(P) \equiv -1, 0$  or  $1 \pmod{4}$ . We consider each of the corresponding (pairs of) half residue disks separately. Let  $P_0$  be one of the points  $(-1, 0)$ ,  $(0, 0)$  or  $(1, 0)$  on  $C$  and let  $D$  be the disk around  $P_0$  consisting of points  $P$  with  $x(P) \equiv x(P_0) \pmod{4}$  and  $y(P) \equiv 0 \pmod{2}$ . By Corollary 3.9 (note that the disk  $D$  corresponds to  $m \geq 2$  in terms of the maximal residue disk around  $P_0$ ), we have

$$q(i_{P_0}(D)) = q(i_{P_0}(\varphi(4\mathbb{Z}_2^\times))),$$

where  $\varphi$  is a parameterization of the residue disk containing  $P_0$  such that  $\varphi(0) = P_0$  and  $D = \varphi(4\mathbb{Z}_2)$ . By Lemma 3.1 and since  $\nu(i_{P_0}(\varphi(4u))) = 1$  for some  $u \in \mathbb{Z}_2^\times$  (as becomes apparent in the course of the computation), it is sufficient to consider  $\varphi(4)$  and  $\varphi(-4)$ . So we compute the (unique) half of  $i_{P_0}(P)$  for each point  $P \in D$  such that  $x(P) = x(P_0) \pm 4$ ; we find that its image in  $L_2^\square$  is nontrivial (and does not depend on the sign) and is not contained in the image of  $S$ . By Theorem 2.6 this now implies that  $D \cap C(\mathbb{Q}) = \{P_0\}$ , for each of the three points. So we obtain the result that

$$C(\mathbb{Q}) \cap C(\mathbb{Z}_2) = \{(-1, 0), (-1, 1), (0, 0), (0, 1), (1, 0), (1, 1)\}$$

as claimed. □

## 7. AN APPLICATION TO FERMAT'S LAST THEOREM

In this section we apply the criterion that is given by the algorithm in Section 4 to a certain family of hyperelliptic curves that are related to Fermat curves. This leads to a criterion for Fermat's Last Theorem to hold for a given prime  $p$ . Of course, FLT has been proved in general by Wiles [Wil95, TW95], so this will not produce a new result. On the other hand, it shows that the method does work in practice. In the next section, we will deal with a similar family of curves that are related to certain generalized Fermat equations; our method applies again and does indeed solve some new cases of generalized Fermat equations.

Consider

$$C_l: y^2 = f(x) := 4x^l + 1$$

with  $l = 2g + 1$ . This curve has good reduction at 2, since it is isomorphic to  $y^2 + y = x^l$ . The reduction has three  $\mathbb{F}_2$ -points, so there are three residue classes in  $C_l(\mathbb{Q}_2)$ . We also note that  $C_l$  has the three obvious rational points  $\infty$ ,  $(0, 1)$  and  $(0, -1)$  and that  $[(0, \pm 1) - \infty] \in J_l(\mathbb{Q})$ , where  $J_l$  denotes the Jacobian of  $C_l$ , is a point of odd order  $l$ . We note that  $J_l(\mathbb{Q}_2)$  and  $J_l(\mathbb{F}_2)$  contain no points of order 2.

**Corollary 7.1.** *Let  $\varphi: D_0 \rightarrow D \subseteq C_l(\mathbb{Q}_2)$  be a parameterization of one of the three residue disks of  $C_l(\mathbb{Q}_2)$ , with  $\varphi(0)$  being  $\infty$  or  $(0, \pm 1)$ . Then*

$$q(i_\infty(D)) = \begin{cases} q(i_\infty(\varphi(2\mathbb{Z}_2^\times \cup 4\mathbb{Z}_2^\times))) \cup \{0\} & \text{if } \varphi(0) = (0, \pm 1); \\ q(i_\infty(\varphi(2\mathbb{Z}_2^\times))) \cup \{0\} & \text{if } \varphi(0) = \infty. \end{cases}$$

*Proof.* This is simply Corollary 3.9 specialized to the case at hand. □

We now want to find  $q(i_\infty(C_l(\mathbb{Q}_2)))$  in terms of its image in  $L_2^\square$  as in Algorithm 4.1. To do this, we need a basis for the latter group. We first note that  $f$  is irreducible over  $\mathbb{Q}_2$ , so  $L_2$  is a field. Let  $\lambda = 2^{1/l}$ , then  $L_2 = \mathbb{Q}_2(\lambda)$  is totally and tamely ramified and  $\theta = -\lambda^{-2}$  is

a root of  $f$ . Clearly,  $2 = \lambda^l$ . Note that an element of the form  $1 + 4\alpha\lambda = 1 + \alpha\lambda^{2l+1}$  with  $\alpha \in \mathcal{O}_{L_2}$  is always a square in  $L_2$  (the power series for  $\sqrt{1+x}$  converges when the valuation of  $x$  exceeds that of 4). Furthermore,

$$(1 + \lambda^n)^2 = 1 + \lambda^{2n} + \lambda^{n+l} = (1 + \lambda^{2n})(1 + \lambda^{n+l} + \dots),$$

which allows us to eliminate factors of the form  $1 + \lambda^{2n}$  for  $n \leq l-1$  when working modulo squares. In this way, we find that the following elements represent an  $\mathbb{F}_2$ -basis for  $L_2^\square$ :

$$\lambda, 1 + \lambda, 1 + \lambda^3, \dots, 1 + \lambda^{2n+1}, \dots, 1 + \lambda^{2l-3}, 1 + \lambda^{2l-1}, 1 + \lambda^{2l}.$$

**Lemma 7.2.** *The image of  $q(i_\infty(C_l(\mathbb{Q}_2)))$  in  $L_2^\square$  consists of the classes of*

$$1, \quad 1 + \lambda^{l+2}, \quad 1 + \lambda^{2l-1}, \quad \prod_{k \geq 1} (1 + \lambda^{l+2^k}).$$

We let  $Z$  denote the set consisting of the three nontrivial classes in this image.

*Proof.* We first consider the residue disk  $D_\infty$  around  $\infty$ . By Corollary 7.1, it is sufficient to find  $\mu_2(q(i_\infty(\varphi(t))))$  for  $t = 2u$  with  $u \in \mathbb{Z}_2^\times$ . One choice of  $\varphi$  is

$$\varphi(t) = (t^{-2}, 2t^{-l}(1 + 2^{-3}t^{2l} - 2^{-7}t^{4l} \pm \dots)).$$

Then  $\mu_2(i_\infty(\varphi(2u)))$  is the class of  $(2u)^{-2} + \lambda^{-2}$  in  $L_2^\square$ . We have

$$(2u)^{-2} + \lambda^{-2} = (2u)^{-2}(1 + u^2\lambda^{2l-2}) \sim 1 + \lambda^{2l-2} \sim 1 + \lambda^{2l-1},$$

where  $\sim$  denotes equivalence mod squares, by the relation

$$1 \sim (1 + \lambda^{l-1})^2 = 1 + \lambda^{2l-2} + \lambda^{2l-1} \sim (1 + \lambda^{2l-2})(1 + \lambda^{2l-1}).$$

We conclude that (specifying elements of  $L_2^\square$  using representatives in  $L_2^\times$ )

$$\mu_2(q(i_\infty(D_\infty))) = \{1, 1 + \lambda^{2l-1}\}.$$

(Compare [PS14, Lemma 10.2], which says that the image of the residue disk at infinity under the  $\rho \log$  map has just one element.)

Now we consider the residue disk  $D_{(0,1)}$  around  $(0,1)$ . If  $P = (\xi, \eta) \in C_l(\mathbb{Q}_2)$  has integral  $x$ -coordinate, then we must have  $\xi \in 2\mathbb{Z}_2$  (otherwise the right hand side is 5 mod 8 and therefore not a square). We can parameterize  $D_{(0,1)}$  by

$$\varphi(t) = \left(t, \sqrt{1 + 4t^l} = 1 + 2t^l - 2t^{2l} + \dots\right).$$

Then  $\mu_2(i_\infty(\varphi(2u)))$  is the class of

$$2u + \lambda^{-2} = \lambda^{-2}(1 + \lambda^{l+2}u) \sim 1 + \lambda^{l+2};$$

the latter relation holds when  $u$  is a unit. By Corollary 7.1, we also need to find the image under  $q$  of points given by  $t \in 4\mathbb{Z}_2^\times$ , so  $t = 4u$  with  $u \in \mathbb{Z}_2^\times$ . In this case (recall that  $\theta = -\lambda^{-2}$ )

$$4u - \theta = (2\theta^{g+1})^2(1 - 4u/\theta) = s(\theta)^2$$

where  $s \in \mathbb{Q}_2[x]$  is a polynomial of degree  $\leq l-1$  such that

$$\begin{aligned} s(\theta) &= 2\theta^{g+1}\sqrt{1 - 4u/\theta} \\ &= 2(\theta^{g+1} - 2u\theta^g - 2u^2\theta^{g-1} - 4u^3\theta^{g-2} - \dots). \end{aligned}$$



The coefficients of

$$\sqrt{1-4x} = \sum_{n=0}^{\infty} 2^{2n} (-1)^n \binom{1/2}{n} x^n = 1 - \sum_{n=1}^{\infty} 2^n \frac{1 \cdot 3 \cdot 5 \cdots (2n-3)}{n!} x^n$$

(except for the constant term) all have 2-adic valuation at least 1 (since  $v_2(n!) \leq n-1$ ) and  $\theta^{-1} = -4\theta^{2g}$ , so

$$\frac{1}{2}s(x) \equiv x^{g+1} - 2ux^g - 2u^2x^{g-1} - \dots - c_{g+1}u^{g+1} \pmod{8\mathbb{Z}_2[x]},$$

where  $c_{g+1}$  denotes the coefficient of  $x^{g+1}$  in  $-\sqrt{1-4x}$ . Let  $w_0(x)$  denote the partial sum of the power series of  $(1-4ux)^{-1/2}$  up to and including the term with  $x^g$ , and set

$$\tilde{w}(x) = x^g w_0(1/x) = x^g + 2ux^{g-1} + 6u^2x^{g-2} - \dots$$

Then

$$\tilde{w}(x)s(x) \equiv 2x^{2g+1} + (\text{terms up to } x^g) \pmod{8\mathbb{Z}_2[x]},$$

so  $\tilde{w}(x) \equiv w(x) \pmod{8\mathbb{Z}_2[x]}$ , where  $w(x)$  is the monic polynomial of degree  $g$  such that  $w(\theta)s(\theta) \in \mathbb{Q}_2 + \mathbb{Q}_2\theta + \dots + \mathbb{Q}_2\theta^g$ . Let  $Q \in J_l(\mathbb{Q}_2)$  denote the point such that  $2Q = [(4u, *) - \infty]$ . By Corollary 5.4, the image of  $Q$  in  $L_2^\square$  is given by the class of

$$\begin{aligned} (-1)^g w(\theta) &\sim (-1)^g \tilde{w}(\theta) \\ &= (-\theta)^g (1 + 2u\lambda^2 + 6u^2\lambda^4 + 20u^3\lambda^6 + 70u^4\lambda^8 + \dots) \\ &\sim 1 + 2\lambda^2 + 6\lambda^4 + 20\lambda^6 + 70\lambda^8 + \dots \\ &\sim 1 + \sum_{k=1}^{\infty} \lambda^{l+2^k} \\ &\sim (1 + \lambda^{l+2})(1 + \lambda^{l+4})(1 + \lambda^{l+8}) \cdots (1 + \lambda^{l+2^k}) \cdots, \end{aligned}$$

where the product can be truncated as soon as  $2^k > l$ . (We have used that the valuation of the coefficient of  $x^n$  in  $(1-4x)^{-1/2}$  is 1 precisely when  $n$  is a power of 2.)  $\square$

We can generalize this result to certain curves of the form  $y^2 = 4x^l + A$ . Let  $A \in \mathbb{Z}$  with  $A \equiv 1 \pmod{8}$  and consider

$$C_{l,A}: y^2 = 4x^l + A.$$

Then  $C_{l,A}$  is  $\mathbb{Q}_2$ -isomorphic to  $C_l = C_{l,1}$ , since  $A$  is a square and an  $l$ th power in  $\mathbb{Q}_2$ . In particular, we still have  $L_2 = \mathbb{Q}_2(\lambda)$ , where now  $L = \mathbb{Q}[x]/\langle 4x^l + A \rangle$ , and the image of  $q(i_\infty(C_{l,A}(\mathbb{Q}_2)))$  in  $L_2^\square$  is the same as for  $C_l$ , namely  $Z \cup \{1\}$ .

**Proposition 7.3.** *Let  $A \in \mathbb{Z}$  satisfy  $A \equiv 1 \pmod{8}$ ; consider the curve  $C_{l,A}: y^2 = 4x^l + A$  over  $\mathbb{Q}$  with  $l = 2g + 1 \geq 5$ , with Jacobian  $J_{l,A}$ . Let  $L = \mathbb{Q}[x]/\langle 4x^l + A \rangle$  and  $L_2 = L \otimes_{\mathbb{Q}} \mathbb{Q}_2 = \mathbb{Q}_2(\lambda)$  with  $\lambda = 2^{1/l}$ . If*

- (1) *the canonical map  $\text{Sel}_2 J_{l,A} \hookrightarrow L^\square \rightarrow L_2^\square$  is injective and*
- (2) *its image does not meet  $Z$ ,*

*then  $C_{l,A}(\mathbb{Q}) = \{\infty\}$  if  $A$  is not a square, and  $C_{l,A}(\mathbb{Q}) = \{\infty, (0, a), (0, -a)\}$  if  $A = a^2$ .*

*Proof.* We apply Theorem 2.1 with  $A = J_{l,A}$ ,  $i = i_\infty$ ,  $\Gamma = \{0\}$  and  $X = C_{l,A}(\mathbb{Q}_2)$ . By Lemma 7.2,  $Z$  is the set of nontrivial images in  $L_2^\square$  of elements in  $q(i_\infty(C_{l,A}(\mathbb{Q}_2)))$ . So the assumptions here match the assumptions of Theorem 2.1, and we conclude that  $i_\infty(C_{l,A}(\mathbb{Q})) \subseteq \overline{\{0\}} = J_{l,A}(\mathbb{Q})_{\text{tors}}$ . Since  $C_{l,A}$  has good reduction at 2 and  $J_{l,A}(\mathbb{Q})[2]$  is trivial, we find that  $J_{l,A}(\mathbb{Q})_{\text{tors}}$  injects into  $J_{l,A}(\mathbb{F}_2)$ ; in particular,  $C_{l,A}(\mathbb{Q})$  will inject into  $C_{l,A}(\mathbb{F}_2)$ , which has three elements. Since each residue class in  $C_{l,A}(\mathbb{Q}_2)$  contains exactly one torsion point (namely,  $\infty$ ,  $(0, a)$  and  $(0, -a)$ , respectively, where  $a$  is a square root of  $A$  in  $\mathbb{Q}_2$ ), the claim follows.  $\square$

It is known that Fermat's Last Theorem holds for a prime  $p \geq 3$  if (and only if) the curve  $y^2 = 4x^p + 1$  has only the obvious three rational points. So Proposition 7.3 gives a criterion for FLT for exponent  $p$  to hold, in terms of the 2-Selmer group of the Jacobian of this curve. We can deduce the following criterion.

**Proposition 7.4.** *Let  $p \geq 5$  be a prime and set  $L = \mathbb{Q}(2^{1/p})$  and  $L_2 = \mathbb{Q}_2(2^{1/p})$ . Let  $r: \mathcal{O}_L^\square \rightarrow \mathcal{O}_{L_2}^\square$  denote the canonical map. If*

- (1)  $p^2 \nmid 2^{p-1} - 1$ ,
- (2) *the class number of  $L$  is odd, and*
- (3)  $\text{im}(r) \cap Z = \emptyset$  (where  $Z$  is as above),

*then Fermat's Last Theorem holds for the exponent  $p$ .*

*Proof.* Let  $f(x) = x^p + 1/4$ . Then  $f(x - 1/4) \equiv x^p \pmod{p\mathbb{Z}_p[x]}$ , and the first assumption  $p^2 \nmid 2^{p-1} - 1$  implies that the constant term is not divisible by  $p^2$ . This in turn implies that  $C_p: y^2 = 4x^p + 1$  is regular over  $\mathbb{Z}_p$  and the component group of the Néron model of the Jacobian  $J$  of  $C$  over  $\mathbb{Z}_p$  is trivial. By [Sto01, Lemma 4.5] or [SS04, Proposition 3.2] (which applies equally to abelian varieties), the only 'bad prime' for the computation of  $\text{Sel}_2 J_p$  is 2. By the second assumption, the class group of  $L$  has odd order and therefore trivial 2-torsion. Together, the previous two sentences imply that the isomorphic image of  $\text{Sel}_2 J_p$  in  $L^\square$  is contained in the subgroup generated by  $\mathcal{O}_L^\square$  and the image of  $2^{1/p}$ . The map to  $L_2^\square$  decomposes as a direct sum of the map  $r$  and an isomorphism of 1-dimensional  $\mathbb{F}_2$ -vector spaces (since the class of  $\lambda = 2^{1/p}$  is not contained in the image of the (global or 2-adic) units). We note that  $r$  is injective: assume that  $u \in \mathcal{O}_L^\times$  is a square in  $\mathcal{O}_{L_2}$ . Since  $u$  is a unit, the extension  $L(\sqrt{u})/L$  is unramified at all places not dividing 2 or  $\infty$ . The extension is unramified at  $\infty$ , since  $N_{L/\mathbb{Q}}(u)$  must be 1 (it is a 2-adic square by assumption), so the image of  $u$  under the unique real embedding of  $L$  is positive. Finally, it is unramified (and even split) at the prime above 2. Since the class number is odd, there are no nontrivial everywhere unramified quadratic extensions of  $L$ , hence  $u$  must be a square. This implies that  $\text{Sel}_2 J \rightarrow L_2^\square$  is injective. Since  $Z$  is contained in  $\mathcal{O}_{L_2}^\square$ , assumption (3) implies that  $r(\text{Sel}_2 J) \cap Z = \emptyset$  as well. We can now apply Proposition 7.3 and conclude that  $C_p(\mathbb{Q}) = \{\infty, (0, 1), (0, -1)\}$ .

Now let  $F_p: u^p + v^p + w^p = 0$  denote the projective Fermat curve of exponent  $p$ . Then there is a non-constant morphism

$$\psi: F_p \longrightarrow C_p, \quad (u : v : w) \longmapsto (x, y) = \left( -\frac{uv}{w^2}, 2\frac{u^p}{w^p} + 1 \right).$$

So if  $P = (u : v : w) \in F_p(\mathbb{Q})$ , then either  $w = 0$  (if  $\psi(P) = \infty$ ) or  $uv = 0$  (if  $\psi(P) = (0, \pm 1)$ ), so  $P$  is a trivial point.  $\square$

Note that by Remark 4.2, the criterion formulated in the proposition above is equivalent to what we would obtain when using the 2-Selmer group  $\text{Sel}_2 J_p$  instead of  $\mathcal{O}_L^\square$ .

We can improve on Proposition 7.4 a bit. Note that if  $u, u' \in \mathcal{O}_L$  are units with  $u$  positive (in the unique real embedding of  $L$ ), then the Hilbert symbol  $(u, u')_v$  is 1 for all places  $v$  distinct from the place  $\lambda$  above 2. The product formula for the Hilbert symbol implies that  $(u, u')_\lambda = 1$  as well. There are the two positive global units  $\lambda - 1$  and  $(1 - \lambda + \lambda^2)/(1 + \lambda)$ . Multiplying the latter by the square  $(1 + \lambda)^2$ , we obtain  $1 + \lambda^3$ . So if  $u \in \mathcal{O}_{L_2}^\times$  and we can show that  $(\lambda - 1, u)_\lambda = -1$  or  $(1 + \lambda^3, u)_\lambda = -1$ , then  $u$  cannot be in the image of  $\mathcal{O}_L^\times$ .

**Lemma 7.5.** *We work in  $L_2 = \mathbb{Q}_2(\lambda)$  with  $\lambda^l = 2$  as before. If  $1 \leq m < l$ , then we have*

$$(\lambda - 1, 1 + \lambda^{2l-m})_\lambda = -1 \quad \text{and} \quad (1 + \lambda^3, 1 + \lambda^{2l-m})_\lambda = \begin{cases} 1 & \text{if } 3 \nmid m, \\ -1 & \text{if } 3 \mid m. \end{cases}$$

*Proof.* We first consider  $\lambda - 1$ . Note that  $(-1, 1 + \lambda^{2l-m})_\lambda = (-1, 1 + 2^{2l-m})_2 = 1$ , so we can as well work with  $(1 - \lambda, 1 + \lambda^{2l-m})_\lambda$ . We have for  $n \geq (l - 1)/2$  that

$$(1 + \lambda^n)^2 - (1 - \lambda)(\lambda^n)^2 = 1 + \lambda^{2n+1} + \lambda^{l+n} \sim (1 + \lambda^{2n+1})(1 + \lambda^{l+n})$$

is a norm from  $L_2(\sqrt{1 - \lambda})$ , which implies that

$$(1 - \lambda, 1 + \lambda^{2l-m})_\lambda = (1 - \lambda, 1 + \lambda^{2l-(m+1)/2})_\lambda$$

when  $1 \leq m < l$  is odd. For even  $m$ , we have

$$1 \sim (1 + \lambda^{l-m/2})^2 = 1 + \lambda^{2l-m} + \lambda^{2l-m/2} \sim (1 + \lambda^{2l-m})(1 + \lambda^{2l-m/2}),$$

which implies that

$$(1 - \lambda, 1 + \lambda^{2l-m})_\lambda = (1 - \lambda, 1 + \lambda^{2l-m/2})_\lambda$$

when  $1 \leq m < l$  is even. An easy induction then shows that

$$(1 - \lambda, 1 + \lambda^{2l-m})_\lambda = (1 - \lambda, 1 + \lambda^{2l-1})_\lambda$$

for all  $1 \leq m < l$ . Finally, this last symbol is  $-1$ : an element is a norm from  $L_2(\sqrt{1 + \lambda^{2l-1}})$  if and only if it has the form  $x^2 - (1 + \lambda^{2l-1})y^2$ . Substituting  $(x, y) \leftarrow (\lambda^{l-1}x + y, y)$  and dividing by  $\lambda^{2l-2}$ , we see that norms have the form  $x^2 + \lambda xy - \lambda y^2$ . If the norm is integral, then  $x$  and  $y$  must be in  $\mathcal{O}_{L_2}$  as well. Considering the equation

$$1 - \lambda = x^2 + \lambda xy - \lambda y^2$$

modulo  $\lambda^2$ , we see that it has no solution.

Now we consider  $1 + \lambda^3$ . For even  $1 \leq m < l$  we have in the same way as above that

$$(1 + \lambda^3, 1 + \lambda^{2l-m})_\lambda = (1 + \lambda^3, 1 + \lambda^{2l-m/2})_\lambda.$$

For  $n \geq (l - 1)/2$ , we have the norms

$$(1 + \lambda^n)^2 - (1 + \lambda^3)(\lambda^n)^2 = 1 - \lambda^{2n+3} + \lambda^{l+n} \sim (1 + \lambda^{2n+3})(1 + \lambda^{l+n}),$$

leading to

$$(1 + \lambda^3, 1 + \lambda^{2l-m})_\lambda = (1 - \lambda, 1 + \lambda^{2l-(m+3)/2})_\lambda$$

when  $1 \leq m < l$  is odd. By induction again, we see that

$$(1 + \lambda^3, 1 + \lambda^{2l-m})_\lambda = \begin{cases} (1 + \lambda^3, 1 + \lambda^{2l-1})_\lambda & \text{if } 3 \nmid m, \\ (1 + \lambda^3, 1 + \lambda^{2l-3})_\lambda & \text{if } 3 \mid m. \end{cases}$$

Let  $a \in L_2$  satisfy  $a^2 - a + \lambda^2 = 0$  (such  $a$  exist by Hensel's Lemma). Then  $1^2 + \lambda \cdot 1 \cdot a - \lambda \cdot a^2 = 1 + \lambda^3$  is a norm from  $L_2(\sqrt{1 + \lambda^{2l-1}})$ , so the first symbol is 1. In a similar way as before, we see that norms from  $L_2(\sqrt{1 + \lambda^{2l-3}})$  are of the form  $x^2 + \lambda^2 xy - \lambda y^2$ . A consideration modulo  $\lambda^4$  shows that this can never equal  $1 + \lambda^3$ , so the second symbol is  $-1$ .  $\square$

**Corollary 7.6.** *Let  $p \geq 5$  be a prime and set  $L = \mathbb{Q}(2^{1/p})$  and  $L_2 = \mathbb{Q}_2(2^{1/p})$ . As before,  $r: \mathcal{O}_L^\square \rightarrow \mathcal{O}_{L_2}^\square$  denotes the canonical map. If*

- (1)  $p^2 \nmid 2^{p-1} - 1$ ,
- (2) *the class number of  $L$  is odd, and*
- (3)  $4 \nmid \lfloor \log_2 p \rfloor$  *or*  $z \notin \text{im}(r)$ , *where  $z$  is the last element listed in Lemma 7.2,*

*then Fermat's Last Theorem holds for the exponent  $p$ .*

*Proof.* We only have to show that the third condition here implies that  $\text{im}(r) \cap Z = \emptyset$ . By Lemma 7.5, we have (with  $l = p$ )

$$(\lambda - 1, 1 + \lambda^{p+2})_\lambda = (\lambda - 1, 1 + \lambda^{2p-1})_\lambda = -1,$$

which implies that the first two elements of  $Z$  can never be images of global units. We also have  $(\lambda - 1, z)_\lambda = (-1)^{\lfloor \log_2 p \rfloor}$ , so we can also rule out  $z$  when  $\lfloor \log_2 p \rfloor$  is odd. So we can now assume that  $\lfloor \log_2 p \rfloor \equiv 2 \pmod{4}$ . Then by Lemma 7.5 again, we find that  $(1 + \lambda^3, z)_\lambda = -1$  (note that every other term in the sequence  $(p - 2^k)_k$  is divisible by 3), and we can again rule out  $z$ .  $\square$

**Corollary 7.7.** *FLT holds for exponents 5, 7, 11, 13, 17, 19 and, assuming the Generalized Riemann Hypothesis, also for exponents 23, 29, 31, 37, 41, 43, 47, 53 and 59.*

*Proof.* We use Magma [BCP97] to check the assumptions (assuming GRH where indicated to speed up the computation of the class group). It turns out that the class group of  $\mathbb{Q}(2^{1/p})$  is trivial for all primes considered. We note that  $p = 17, 19, 23, 29, 31$  are the only primes  $p$  up to 59 that satisfy  $4 \mid \lfloor \log_2 p \rfloor$ , so we need a basis of  $\mathcal{O}_L^\square$  only for these primes; for the remaining ones it suffices to know that the class number is odd.  $\square$

*Remark 7.8.* Computations show that the class group of  $\mathbb{Q}(2^{1/n})$  is trivial for *all*  $n \leq 50$  (assuming GRH for  $n \geq 20$ ), regardless whether  $n$  is prime or not. According to class group heuristics [VE10, Section 4.1], the 2-torsion in the class group of a number field with unit rank  $u$  should behave like the cokernel of a random linear map  $\mathbb{F}_2^{n+u} \rightarrow \mathbb{F}_2^n$  for large  $n$  (at least in absence of special effects leading to systematically occurring elements of order 2). Such a map is surjective with probability  $> 1 - 2^{-u}$ , so noting that  $u = (p - 1)/2$  in the case of interest, the ‘probability’ that the class number of  $L$  is odd for all  $p$  is  $> 1 - 2^{-29}$  (assuming we know it for  $p \leq 59$ ). See also [HSV16].

We also remark that when the first condition  $p^2 \nmid 2^{p-1} - 1$  is not satisfied, the criterion does still work when we replace  $\mathcal{O}_L^\square$  by the larger subgroup  $L(\{p\}, 2)$  of  $L^\square$  represented by elements generating ideals of the form  $I_1^2 I_2$  with  $I_2$  supported on the ideals above  $p$ . In

this case, however, we also have to check that the map to  $L_2^\square$  is injective. A similar remark applies to the case when the class group does have even order.

## 8. AN APPLICATION TO CERTAIN GENERALIZED FERMAT EQUATIONS

Recall the following statement.

**Proposition 8.1** (Dahmen and Siksek, [DS14, Lemma 3.1 and Proposition 3.3]). *Let  $p$  be an odd prime. If the only rational points on the curve*

$$C'_p: 5y^2 = 4x^p + 1$$

*are the obvious three (namely  $\infty$ ,  $(1, 1)$  and  $(1, -1)$ ), then the only primitive integral solutions of the generalized Fermat equation  $x^5 + y^5 = z^p$  are the trivial ones:*

$$(x, y, z) = \pm(0, 1, 1), \pm(1, 0, 1), \pm(1, -1, 0).$$

Dahmen and Siksek show that this is true when  $p \in \{7, 19\}$  and also when  $p \in \{11, 13\}$ , assuming GRH. We will use our approach to extend the range of primes  $p$  for which it can be shown that  $C'_p(\mathbb{Q})$  has only the obvious three rational points.

So we now consider the curves  $C'_l$ , with  $l = 2g + 1$  odd, but not necessarily prime. The corresponding étale algebra is still  $L = \mathbb{Q}(\lambda)$  with  $\lambda = 2^{1/l}$  (since  $C'_l$  is the quadratic twist by 5 of  $y^2 = 4x^l + 1$ ), but the descent map is now given on a point on the Jacobian with Mumford representation  $[a, b]$  by the class of  $-5a(\theta)$  (instead of  $-a(\theta)$ ) if the degree of  $a$  is odd.

It is still the case that  $C'_l$  has good reduction mod 2 (replacing  $y$  by  $2y + 1$  and dividing by 4 gives  $5(y^2 + y) = x^l - 1$ ) and that there is no nontrivial 2-torsion in  $J'_l(\mathbb{Q}_2)$  nor in  $J'_l(\mathbb{F}_2)$ , where  $J'_l$  denotes the Jacobian of  $C'_l$ . We therefore have a statement similar to Corollary 7.1. Note that we have again three residue disks, centered at  $\infty$ ,  $(1, 1)$  and  $(1, -1)$ , respectively.

If  $P_0 \in C'_l(\mathbb{Q})$ , then we write  $D_{P_0}$  for the residue disk centered at  $P_0$ . We let  $\varphi_{P_0}: D_0 \rightarrow D_{P_0}$  be a parameterization of  $D_{P_0}$  such that  $\varphi(0) = P_0$  (and such that  $i_\infty \circ \varphi_\infty$  is odd).

**Corollary 8.2.** *We have*

$$\begin{aligned} q(i_\infty(D_\infty)) &= q(i_\infty(\varphi_\infty(2\mathbb{Z}_2^\times))) \cup \{0\} \\ q(i_{(1,1)}(D_{(1,1)})) &= q(i_{(1,1)}(\varphi_{(1,1)}(2\mathbb{Z}_2^\times \cup 4\mathbb{Z}_2^\times))) \cup \{0\} \end{aligned}$$

*Proof.* This again follows from Corollary 3.9. □

The main difference with the case discussed in the previous section is that, if  $l \geq 7$ , the two points  $(1, \pm 1)$  do not map to points of finite order in  $J'_l$  under the embedding that sends  $\infty$  to zero. So from now on, we assume that  $l \geq 7$ . Note that the rank of  $J'_5(\mathbb{Q})$  is zero (the 2-Selmer group is trivial), so it is almost immediate that  $C'_5(\mathbb{Q}) = \{\infty, (1, \pm 1)\}$ .

We first consider the image of  $C'_l(\mathbb{Q}_2)$  in  $J'_l(\mathbb{Q}_2)/2J'_l(\mathbb{Q}_2)$  under  $q \circ i_\infty$ .

**Lemma 8.3.** *In terms of representatives in  $L_2^\times$ , we have*

- (1)  $\mu_2(q(i_\infty(D_\infty))) = \{1, 1 + \lambda^{2l-1}\}$ .
- (2)  $\mu_2(q(i_\infty(D_{(1,1)}))) = \{5(1 + \lambda^2), 5(1 + \lambda^2 + \lambda^{l+2})\}$ .

*Proof.* By Corollary 8.2, we know that  $q(i_\infty(D_\infty)) = q(i_\infty(\varphi_\infty(2\mathbb{Z}_2^\times))) \cup \{0\}$ , where we can choose  $\varphi_\infty$  such that  $x(\varphi_\infty(t)) = 5t^{-2}$ . So let  $u \in \mathbb{Z}_2^\times$ , then  $\mu_2(i_\infty(\varphi_\infty(2u)))$  is represented by

$$5 \left( \frac{5}{4u^2} + \lambda^{-2} \right) = \left( \frac{5}{2u} \right)^2 \left( 1 + \frac{4u^2}{5} \lambda^{-2} \right) \sim 1 + \lambda^{2l-2} \sim 1 + \lambda^{2l-1}.$$

This proves (1).

Now let  $P \in D_{(1,1)}$ . We can choose  $\varphi_{(1,1)}$  such that  $x(\varphi_{(1,1)}(t)) = 1 + t$ . If  $u \in \mathbb{Z}_2^\times$ , then  $\mu_2(i_\infty(\varphi_{(1,1)}(2u)))$  is represented by

$$5(1 + 2u + \lambda^{-2}) \sim 5(1 + \lambda^2 + u\lambda^{l+2}) \sim 5(1 + \lambda^2 + \lambda^{l+2}),$$

and for any  $u \in \mathbb{Z}_2$ ,  $\mu_2(i_\infty(\varphi_{(1,1)}(4u)))$  is represented by

$$5(1 + 4u + \lambda^{-2}) \sim 5(1 + \lambda^2 + u\lambda^{2l+2}) \sim 5(1 + \lambda^2).$$

This proves (2). □

Now we consider the embedding  $i_{(1,1)}$ .

**Lemma 8.4.** *In terms of representatives in  $L_2^\times$ , we have*

$$\mu_2(q(i_{(1,1)}(D_{(1,1)}))) = \{1, 1 + \lambda^{l+2}/(1 + \lambda^2), \sigma, \sigma'\},$$

where  $\sigma = \mu_2(Q)$  for the point  $Q \in J_1(\mathbb{Q}_2)$  such that  $2Q = \varphi_{(1,1)}(4)$  and  $\sigma' = \mu_2(Q')$  where  $2Q' = \varphi_{(1,1)}(-4)$ .

*Proof.* We make use of Corollary 8.2 again, which tells us that it suffices to consider points  $P$  with  $x$ -coordinates  $1 + 2u$  or  $1 + 4u$ , where  $u \in \mathbb{Z}_2^\times$ . If  $x = 1 + 2u$ , then by the computation in the proof of Lemma 8.3, we have  $\pi_2(i_{(1,1)}(P)) = \pi_2(i_\infty(P)) - \pi_2(i_\infty((1,1)))$ , which is represented by

$$5(1 + \lambda^2) \cdot 5(1 + \lambda^2 + \lambda^{l+2}) \sim 1 + \frac{\lambda^{l+2}}{1 + \lambda^2}.$$

If  $x = 1 + 4u$ , then  $i_{(1,1)}(P)$  is divisible by 2 in  $J'_l(\mathbb{Q}_2)$ , so we have to look at  $\pi_2(Q)$  where  $2Q = P$ , for suitable values of  $u$ . Since  $i_{(1,1)}(P) \in K_2 \setminus K_3$ , we have  $\nu(i_{(1,1)}(P)) = 1$ , so by Corollary 3.2,  $\pi_2(Q)$  depends only on  $u \pmod{4}$ , so the two values  $u = 1$  and  $u = -1$  are sufficient. □

In practice, it appears that  $\sigma = \sigma'$  in all cases, which would be implied by the difference of the images of any pair chosen from the relevant points being divisible by 4. We know this difference is in  $K_3$ , but we did not exclude the possibility that it is only divisible by 2 and not by 4.

We can now formulate a criterion.

**Proposition 8.5.** *Consider  $C'_l: 5y^2 = 4x^l + 1$ , with Jacobian  $J'_l$ , where  $l = 2g + 1 \geq 7$  is odd. Recall that  $L = \mathbb{Q}(2^{1/l})$ ; let  $S \subseteq L^\square$  be a finite subgroup that contains the image of  $\text{Sel}_2 J'_l$ . Assume that*

(1) *the canonical map  $S \hookrightarrow L^\square \rightarrow L_2^\square$  is injective, and*

(2) its image does not meet the set  $Z'$  consisting of the classes of

$$1 + \lambda^{2l-1}, \quad 1 + \frac{\lambda^{l+2}}{1 + \lambda^2}, \quad \sigma, \quad \sigma'$$

in  $L_2^\square$ .

Then  $C'_l(\mathbb{Q}) = \{\infty, (1, 1), (1, -1)\}$ .

In particular, if  $l = p$  is a prime, then the generalized Fermat equation  $x^5 + y^5 = z^p$  has no nontrivial coprime integral solutions.

*Proof.* Note that Lemmas 8.3 and 8.4 imply that  $Z' \cup \{1\}$  is the union of the sets  $Y$  occurring in Algorithm 4.1 when it is applied to the curve  $C'_l$ , so the assumptions imply that the algorithm will not return FAIL. (There cannot be any elements in  $\text{Sel}_2 C$  other than the images of the known points, since this would lead to a non-trivial intersection of  $Z'$  with the image of  $S$ .) The set returned by the algorithm can contain at most one point in each 2-adic residue disk. Since there are only three such disks, the known points must account for all rational points on  $C'_l$ .  $\square$

Computing  $\sigma$  and  $\sigma'$  for many values of  $l$ , it appears that their images in  $L_2^\square$  are represented uniformly by an infinite product

$$(1 + \lambda^{l+2})(1 + \lambda^{l+6})(1 + \lambda^{l+8})(1 + \lambda^{l+10})(1 + \lambda^{l+14})(1 + \lambda^{l+18})(1 + \lambda^{l+22}) \cdots,$$

but it is not obvious which rule is behind the sequence  $(2, 6, 8, 10, 14, 18, 22, \dots)$ . However, extending it further and consulting the OEIS [OEIS] gives exactly one hit, namely A036554, the sequence of ‘numbers  $n$  whose binary representation ends in an odd number of zeros’, i.e., such that  $v_2(n)$  is odd. So we propose the following.

**Conjecture 8.6.**  $\mu_2(\sigma)$  (and also  $\mu_2(\sigma')$ ) is represented by

$$\prod_{n \geq 1, 2 \nmid v_2(n)} (1 + \lambda^{l+n}) \sim 1 + \frac{\lambda^l}{\prod_{k \geq 1} (1 + \lambda^{2^k})}.$$

We give a more concrete version of the criterion, following the considerations of Remark 4.2.

**Corollary 8.7.** *Assume that  $l$  is prime and that  $l^2 \nmid 2^{l-1} - 1$ . Then a possible choice of the subgroup  $S$  in Proposition 8.5 is the subgroup of  $L(\{5\}, 2)$  consisting of elements mapping into the image of  $J'_l(\mathbb{Q}_5)$  in  $L_5^\square$ . In fact, the resulting criterion is equivalent to what would be obtained by taking  $S$  to be the image of  $\text{Sel}_2 J'_l$ .*

*Proof.* As in the case discussed in the preceding section, the assumption  $l^2 \nmid 2^{l-1} - 1$  implies that the Tamagawa number at  $l$  is 1, so that we can reduce to  $\Sigma = \{2, 5\}$ . Furthermore, since 2 is totally ramified in  $L$  and  $L$  has odd degree, the norm of any element  $\alpha \in L^\times$  whose valuation with respect to the prime above 2 is odd will have odd 2-adic valuation and cannot be a square. This lets us reduce to  $L(\{5\}, 2)$ . Remark 4.2 now shows that using  $S$  is equivalent to using  $\text{Sel}_2 J'_l$  in the algorithm.  $\square$

We note that it is fairly easy to find  $S$ , given  $L(\{5\}, 2)$ , since the image of  $J'_l(\mathbb{Q}_5)$  in  $L_5^\square$  equals the image of  $J'_l(\mathbb{Q}_5)[2]$ , unless there are elements of order 4 in  $J'_l(\mathbb{Q}_5)$ . We can easily exclude this by checking that the images of an  $\mathbb{F}_2$ -basis of  $J'_l(\mathbb{Q}_5)[2]$  are independent.

We carried out the computations necessary to test the criterion of Proposition 8.5 in the version of Corollary 8.7. This results in the following.

**Theorem 8.8.** *For  $7 \leq p \leq 53$  prime, we have (assuming GRH when  $p \geq 23$ )*

$$C'_p(\mathbb{Q}) = \{\infty, (1, 1), (1, -1)\}.$$

*In particular, the generalized Fermat equation  $x^5 + y^5 = z^p$  has only the trivial coprime integral solutions.*

## 9. AN ‘ELLIPTIC CHABAUTY’ EXAMPLE

In this section, we apply our approach to ‘Elliptic Curve Chabauty’. The curve in the following result comes up in the course of trying to find all primitive integral solutions to the Generalized Fermat Equation  $x^2 + y^3 = z^{25}$ . It is a hyperelliptic curve over  $\mathbb{Q}$  of genus 4; it can be shown that the Mordell-Weil group of its Jacobian has rank 4 (generators of a finite-index subgroup can be found), so that Chabauty’s method does not apply directly to the curve.

**Theorem 9.1.** *Let  $C$  be the smooth projective curve given by the affine equation*

$$y^2 = 81x^{10} + 420x^9 + 1380x^8 + 1860x^7 + 3060x^6 - 66x^5 + 3240x^4 - 1740x^3 + 1320x^2 - 480x + 69.$$

*If GRH holds, then  $C(\mathbb{Q})$  consists of the two points at infinity only.*

*Proof.* As a first step, we compute the fake 2-Selmer set as in [BS09]. We obtain a one-element set (this requires local information only at the primes 2, 3 and 29). Using [Sto01, Lemma 6.3], we then show that the points in  $C(\mathbb{Q}_2)$  whose image in  $L_2^\square/\mathbb{Q}_2^\square$  is the image of the unique element of the fake 2-Selmer set are those whose  $x$ -coordinate has 2-adic valuation  $\leq -3$ . This set is the union of two half residue disks (the maximal residue disks contain the points  $P$  such that  $v_2(x(P)) \leq -2$ ) that are mapped to each other by the hyperelliptic involution, so it is sufficient to consider just one of them, say the disk that contains  $P_0 = \infty_9$ , the point at infinity such that  $(y/x^5)(P_0) = 9$ .

The splitting field of the polynomial  $f$  on the right hand side of the curve equation contains three pairwise non-conjugate subfields  $k$  of degree 10 over which  $f$  is divisible by a monic polynomial  $g \in k[x]$  of degree 4. If  $P$  is any rational point on  $C$ , then it follows that  $g(x(P))$  is a square in  $k$  (this is because the image of  $P$  in the fake 2-Selmer set is the same as that of  $P_0$ ), so we obtain a point  $(\xi, \eta) \in H(k)$  with  $\xi \in \mathbb{Q}$  (or  $\xi = \infty$ ) where  $H$  is the smooth projective curve given by  $y^2 = g(x)$ . We can parameterize the image of the residue disk around  $P_0$  by a pair of Laurent series  $((2t)^{-1}, \sqrt{g((2t)^{-1})})$  (where we can take the square root to have leading term  $t^{-2}/4$ ). Since  $H(k)$  contains the two points at infinity,  $H$  is isomorphic to an elliptic curve  $E$  over  $k$ ; we take the isomorphism so that it sends  $\infty_1 \in H(k)$  to the origin of  $E$ . We then obtain a Laurent series  $\xi(t) \in k((t))$  that gives the  $x$ -coordinate of the image on  $E$  of the point whose parameter is  $t$ .



For the following, we take  $k$  to be the field generated by a root of

$$x^{10} + 75x^6 - 50x^5 + 100x^3 + 625x^2 + 1250x + 645;$$

the polynomial  $g$  and the curves  $H$  and  $E$  are taken with respect to this field. We next compute the 2-Selmer group of  $E$  over  $k$ . There is exactly one point of order 2 in  $E(k)$ , which means that we have to work with a quadratic extension of  $k$ . This is where we use GRH, which allows us to find the relevant arithmetic information for this field of degree 20 faster. The Selmer group has  $\mathbb{F}_2$ -dimension 6 (so the bound for the rank of  $E(k)$  is 5). We check that it injects into  $E(k_2)/2E(k_2)$ , where  $k_2 = k \otimes_{\mathbb{Q}} \mathbb{Q}_2$ ; note that this splits as a product of two extensions of  $\mathbb{Q}_2$ , both of ramification index 2 and one of residue class degree 1, the other of residue class degree 4.

In the context of our method, we consider the curve that is the (desingularization of) the curve over  $\mathbb{Q}$  in  $A = R_{k/\mathbb{Q}}E$  (the latter denotes the Weil restriction of scalars) that corresponds to the set of points on  $H$  whose  $x$ -coordinate is rational. We have  $E(k_2) \cong A(\mathbb{Q}_2)$ , so we can use arithmetic on  $E$  over  $k$  and its completions for the computations. We check that  $n_{\text{tors}} = 1$  (the map from  $E(k_2)[2]$  to  $E(k_2)/2E(k_2)$  is injective). By Lemma 3.7, a suitable value of  $m$  is  $m = 4$ , provided  $5 \geq n_4$  in the notation of the lemma. Note that in this situation halving points is easy, since doubling a point corresponds to an explicit map of degree 4 on the  $x$ -coordinate. If  $P$  is in our half residue disk, then  $i(P) + T$  (where  $T$  is the point of order 2 in  $E(k) = A(\mathbb{Q})$ ) is not divisible by 2, and its image in  $A(\mathbb{Q}_2)/2A(\mathbb{Q}_2)$  is the same as that of  $T$ . So we only have to determine  $q(i(P))$  for a suitable selection of points  $P$  in our disk. We write  $P_\tau$  for the point corresponding to the parameter  $\tau \in 2\mathbb{Z}_2$ .

We compute  $Y_\tau = q(i(P_\tau))$  for  $\tau \in \{\pm 4, \pm 8, \pm 16\}$ . Note that this can be done solely in terms of the  $x$ -coordinate  $\xi(\tau)$ . We find that  $\nu(i(P_\tau)) = v_2(\tau) - 1$  for these values and that  $Y_{-\tau} = Y_\tau$ . This implies that  $n_m = m - 1$  for  $2 \leq m \leq 4$  and that  $q$  of the disk in question is the union  $Y_4 \cup Y_8 \cup Y_{16}$ . In fact, it turns out that this union is equal to  $Y_8$ , and we verify that  $Y_8$  meets the image of the 2-Selmer group only in the image of the global torsion. By Theorem 2.6, this then implies that there can be no other point than  $P_0$  in our (half) residue disk. The claim follows.  $\square$

We note that the two other possible choices of  $k$  also lead to elliptic curves with a 2-Selmer rank of 5 (this is unconditional for one of the choices, where the curve  $E$  happens to have full 2-torsion over  $k$ ), but for these other fields, the condition that the image of the disk under  $q$  meets the image of the Selmer group only in the image of the global torsion is not satisfied. We also remark that we have been unable to find five independent points in  $E(k)$  (for any of the three possible choices of  $k$  and  $E$ ), so that we could not apply the standard Elliptic Curve Chabauty method.

Another application of our ‘Selmer group Chabauty’ approach in the setting of Elliptic Curve Chabauty was made in [FNS16]. We use this to show that there are no unexpected points on the elliptic curve  $X_0(11)$  defined over certain number fields of degree 12 and such that the image under the  $j$ -map is in  $\mathbb{Q}$ . This is a vital step in the proof that the only nontrivial primitive integral solutions of the Generalized Fermat Equation  $x^2 + y^3 = z^{11}$  are  $(x, y, z) = (\pm 3, -2, 1)$ . The situation is similar to what happens for the example presented here: we can compute the 2-Selmer group of  $X_0(11)$  over the fields of interest, but we are

unable to produce enough independent points to meet the upper bound on the rank, so we cannot apply the standard method.

## REFERENCES

- [BBK10] Jennifer S. Balakrishnan, Robert W. Bradshaw, and Kiran S. Kedlaya, *Explicit Coleman integration for hyperelliptic curves*, Algorithmic number theory, Lecture Notes in Comput. Sci., vol. 6197, Springer, Berlin, 2010, pp. 16–31, DOI 10.1007/978-3-642-14518-6\_6, (to appear in print). MR2721410 (2012b:14048) ↑4
- [BG13] Manjul Bhargava and Benedict H. Gross, *The average size of the 2-Selmer group of Jacobians of hyperelliptic curves having a rational Weierstrass point*, Automorphic representations and L-functions, Tata Inst. Fundam. Res. Stud. Math., vol. 22, Tata Inst. Fund. Res., Mumbai, 2013, pp. 23–91. MR3156850 ↑2, 4
- [BCP97] Wieb Bosma, John Cannon, and Catherine Playoust, *The Magma algebra system. I. The user language*, J. Symbolic Comput. **24** (1997), no. 3-4, 235–265, DOI 10.1006/jsc.1996.0125. Computational algebra and number theory (London, 1993). MR1484478 ↑1, 7
- [BF06] N. Bruin and E. V. Flynn, *Exhibiting SHA[2] on hyperelliptic Jacobians*, J. Number Theory **118** (2006), no. 2, 266–291, DOI 10.1016/j.jnt.2005.10.007. MR2225283 (2006m:11091) ↑3
- [BS09] Nils Bruin and Michael Stoll, *Two-cover descent on hyperelliptic curves*, Math. Comp. **78** (2009), no. 268, 2347–2370, DOI 10.1090/S0025-5718-09-02255-8. MR2521292 (2010e:11059) ↑4, 4.2, 9
- [BS10] ———, *The Mordell-Weil sieve: proving non-existence of rational points on curves*, LMS J. Comput. Math. **13** (2010), 272–306, DOI 10.1112/S1461157009000187. MR2685127 (2011j:11118) ↑5
- [DS14] Sander R. Dahmen and Samir Siksek, *Perfect powers expressible as sums of two fifth or seventh powers*, Acta Arith. **164** (2014), no. 1, 65–100, DOI 10.4064/aa164-1-5. MR3223319 ↑1, 8.1
- [FNS16] Nuno Freitas, Bartosz Naskręcki, and Michael Stoll, *The generalized Fermat equation with exponents 2, 3, n*, May 17, 2016. Preprint, <http://www.mathe2.uni-bayreuth.de/stoll/schrift.html#AG57>. ↑1, 9
- [HSV16] Wei Ho, Arul Shankar, and Ila Varma, *Odd degree number fields with odd class number*, November 9, 2016. Preprint, arXiv:1603.06269. ↑7.8
- [McC94] William G. McCallum, *On the method of Coleman and Chabauty*, Math. Ann. **299** (1994), no. 3, 565–596, DOI 10.1007/BF01459799. MR1282232 (95c:11079) ↑2
- [OEIS] *The On-Line Encyclopedia of Integer Sequences*, 2010. Published electronically at <http://oeis.org>. ↑8
- [PS14] Bjorn Poonen and Michael Stoll, *Most odd degree hyperelliptic curves have only one rational point*, Ann. of Math. (2) **180** (2014), no. 3, 1137–1166, DOI 10.4007/annals.2014.180.3.7. MR3245014 ↑1, 2, 2, 2, 3, 3, 4, 7
- [Sch95] Edward F. Schaefer, *2-descent on the Jacobians of hyperelliptic curves*, J. Number Theory **51** (1995), no. 2, 219–232, DOI 10.1006/jnth.1995.1044. MR1326746 (96c:11066) ↑4, 5
- [SS04] Edward F. Schaefer and Michael Stoll, *How to do a p-descent on an elliptic curve*, Trans. Amer. Math. Soc. **356** (2004), no. 3, 1209–1231, DOI 10.1090/S0002-9947-03-03366-X. MR2021618 (2004g:11045) ↑7
- [Sto01] Michael Stoll, *Implementing 2-descent for Jacobians of hyperelliptic curves*, Acta Arith. **98** (2001), no. 3, 245–277, DOI 10.4064/aa98-3-4. MR1829626 (2002b:11089) ↑2, 2, 4, 6, 7, 9
- [Sto06] ———, *Independence of rational points on twists of a given curve*, Compos. Math. **142** (2006), no. 5, 1201–1214. MR2264661 ↑4, 2.3
- [Sto07] ———, *Finite descent obstructions and rational points on curves*, Algebra Number Theory **1** (2007), no. 4, 349–391, DOI 10.2140/ant.2007.1.349. MR2368954 ↑2
- [Sto15] ———, *Uniform bounds for the number of rational points on hyperelliptic curves of small Mordell-Weil rank*, March 15, 2015. Preprint, arXiv:1307.1773; to appear in J. Eur. Math. Soc. ↑4
- [TW95] Richard Taylor and Andrew Wiles, *Ring-theoretic properties of certain Hecke algebras*, Ann. of Math. (2) **141** (1995), no. 3, 553–572, DOI 10.2307/2118560. MR1333036 ↑7

- [VE10] Akshay Venkatesh and Jordan S. Ellenberg, *Statistics of number fields and function fields*, Proceedings of the International Congress of Mathematicians. Volume II, Hindustan Book Agency, New Delhi, 2010, pp. 383–402. MR2827801 (2012h:11160) ↑7.8
- [Wil95] Andrew Wiles, *Modular elliptic curves and Fermat's last theorem*, Ann. of Math. (2) **141** (1995), no. 3, 443–551, DOI 10.2307/2118559. MR1333035 ↑7

MATHEMATISCHES INSTITUT, UNIVERSITÄT BAYREUTH, 95440 BAYREUTH, GERMANY.

*E-mail address:* Michael.Stoll@uni-bayreuth.de

*URL:* <http://www.mathe2.uni-bayreuth.de/stoll/>