# Semi-canonical binary matrices 

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#### Abstract

In this paper, we define the concepts of semi-canonical and canonical binary matrix. Strictly mathematical, we prove the correctness of these definitions. We describe and we implement an algorithm for finding all $n \times n$ semi-canonical binary matrices taking into account the number of 1 in each of them. This problem relates to the combinatorial problem of finding all pairs of disjoint $n^{2} \times n^{2}$ S-permutation matrices. In the described algorithm, the bitwise operations are substantially used.

Keywords: binary matrix, ordered n-tuple, semi-canonical and canonical binary matrix, disjoint S-permutation matrices, bitwise operations

MSC[2010]: 05B20, 68N15


## 1. INTRODUCTION

Binary (or boolean, or (0,1)-matrix) is called a matrix whose elements belong to the set $\mathcal{B}=\{0,1\}$.

Let $n$ and $m$ be positive integers. With $\mathcal{B}_{n \times m}$ we will denote the set of all $n \times m$ binary matrices, while with $\mathcal{B}_{n}=\mathcal{B}_{n \times n}$ we will denote the set of all square $n \times n$ binary matrices.

A square binary matrix is called a permutation matrix, if there is just one 1 in every row and every column. Let us denote by $\mathcal{P}_{n}$ the group of all $n \times n$ permutation matrices, and by $\mathcal{S}_{n}$ the symmetric group of order $n$, i.e. the group of all one-to-one mappings of the set $[n]=\{1,2, \ldots n\}$ in itself. In effect is the isomorphism $\mathcal{P}_{n} \cong \mathcal{S}_{n}$.

As it is well known $[4,5]$ the multiplication of an arbitrary real or complex matrix $A$ from the left with a permutation matrix (if the multiplication is possible) leads to dislocation of the rows of the matrix $A$, while the multiplication of $A$ from the right with a permutation matrix leads to the dislocation of the columns of $A$.

Let $n$ be a positive integer and let $A \in \mathcal{B}_{n}$ be a $n^{2} \times n^{2}$ binary matrix. With the help of $n-1$ horizontal lines and $n-1$ vertical lines $A$ has been separated into $n^{2}$ of number non-intersecting $n \times n$ square sub-matrices $A_{k l}$, $1 \leq k, l \leq n$, e.i.

$$
A=\left[\begin{array}{cccc}
A_{11} & A_{12} & \cdots & A_{1 n} \\
A_{21} & A_{22} & \cdots & A_{2 n} \\
\vdots & \vdots & \ddots & \vdots \\
A_{n 1} & A_{n 2} & \cdots & A_{n n}
\end{array}\right] .
$$

A matrix $A \in \mathcal{B}_{n^{2}}$ is called an S-permutation if in each row, each column, and each sub-matrice $A_{k l}, 1 \leq k, l \leq n$ of $A$ there is exactly one 1 . Two Spermutation matrices $A$ and $B$ will be called disjoint, if there are not $i, j \in\left[n^{2}\right]=\left\{1,2, \ldots, n^{2}\right\}$ such that for the elements $a_{i j} \in A$ and $b_{i j} \in B$ the condition $a_{i j}=b_{i j}=1$ is satisfied.

The concept of S-permutation matrix was introduced by Geir Dahl [1] in relation to the popular Sudoku puzzle. Obviously a square $n^{2} \times n^{2}$ matrix $M$ with elements of $\left[n^{2}\right]=\left\{1,2, \ldots, n^{2}\right\}$ is a Sudoku matrix if and only if there are S-permutation matrices $A_{1}, A_{2}, \ldots, A_{n^{2}}$, each two of them are disjoint and such that $M$ can be given in the following way:

$$
\begin{equation*}
M=1 \cdot A_{1}+2 \cdot A_{2}+\cdots+n^{2} \cdot A_{n^{2}} \tag{1}
\end{equation*}
$$

In [2] Roberto Fontana offers an algorithm which randomly gets a family of $n^{2} \times n^{2}$ mutually disjoint S-permutation matrices, where $n=2,3$. In $n=3$ he ran the algorithm 1000 times and found 105 different families of nine mutually disjoint S-permutation matrices. Then using (1) he obtained $9!\cdot 105=38102400$ Sudoku matrices .

Bipartite graph is the ordered triplet $g=\left\langle R_{g}, C_{g}, E_{g}\right\rangle$, where $R_{g}$ and $C_{g}$ are non-empty sets such that $R_{g} \cap C_{g}=\varnothing$, the elements of which will be called vertices. $E_{g} \subseteq R_{g} \times C_{g}=\left\{\langle r, c\rangle \mid r \in R_{g}, c \in C_{g}\right\}$ - the set of edges. Repeated edges are not allowed in our considerations.

If $x \in\{1,2, \ldots, n\}, \rho \in \mathcal{S}_{n}$, then the image of the element $x$ in the mapping $\quad \rho$ we denote by $\rho(x)$. Let $g^{\prime}=\left\langle R_{g^{\prime}}, C_{g^{\prime}}, E_{g^{\prime}}\right\rangle$ and
$g^{\prime \prime}=\left\langle R_{g^{\prime \prime}}, C_{g^{\prime \prime}}, E_{g^{\prime \prime}}\right\rangle$. We will say that the graphs $g^{\prime}$ and $g^{\prime \prime}$ are isomorphic and we will write $g^{\prime} \cong g^{\prime \prime}$, if $R_{g^{\prime}} \cong R_{g^{\prime \prime}}, \quad C_{g^{\prime}} \cong C_{g^{\prime \prime}}, \quad\left|R_{g^{\prime}}\right|=\left|R_{g^{\prime \prime}}\right|=m$, $\left|C_{g^{\prime}}\right|=\left|C_{g^{\prime \prime}}\right|=n \quad$ and there exist $\rho \in \mathcal{S}_{m}$ and $\sigma \in \mathcal{S}_{n}$ such that $\langle r, c\rangle \in E_{g^{\prime}} \Leftrightarrow\langle\rho(r), \sigma(c)\rangle \in E_{g^{\prime \prime}}$. In this paper we consider only bipartite graphs up to isomorphism.

Analyzing the works of G. Dahl [1] and R. Fontana [2], the question of finding a general formula for counting disjoint pairs of $n^{2} \times n^{2}$ S-permutation matrices as a function of the integer $n$ naturally arises. This is an interesting combinatorial problem that deserves its consideration. The work [7] solves this problem. To do that, the graph theory techniques have been used. It has been shown that to count the number of disjoint pairs of $n^{2} \times n^{2} S$-permutation matrices, it is sufficient to obtain some numerical characteristics of the set of all bipartite graphs considered to within isomorphism of the type $g=\left\langle R_{g}, C_{g}, E_{g}\right\rangle$, where $V=R_{g} \cup C_{g}$ is the set of vertices, and $E_{g}$ is the set of edges of the graph $g, \quad R_{g} \cap C_{g}=\varnothing, \quad\left|R_{g}\right|=\left|C_{g}\right|=n \quad\left|E_{g}\right|=k$, $k=0,1, \ldots, n^{2}$.

Let $g=\left\langle R_{g}, C_{g}, E_{g}\right\rangle$ be a bipartite graph, where $R_{g}=\left\{r_{1}, r_{2}, \ldots, r_{n}\right\}$ and $C_{g}=\left\{c_{1}, c_{2}, \ldots, c_{n}\right\}$. Then we build the matrix $A=\left[a_{i j}\right] \in \mathcal{B}_{n}$, such that $a_{i j}=1$ if and only if $\left\langle r_{i}, c_{j}\right\rangle \in E_{g}$. Inversely, let $A=\left[a_{i j}\right] \in \mathcal{B}_{n}$. We denote the $i$-th row of $A$ with $r_{i}$, while the $j$-th column of $A$ with $c_{j}$. Then we build the bipartite graph $g=\left\langle R_{g}, C_{g}, E_{g}\right\rangle$, where $R_{g}=\left\{r_{1}, r_{2}, \ldots, r_{n}\right\}$, $C_{g}=\left\{c_{1}, c_{2}, \ldots, c_{n}\right\}$ and there exists an edge from the vertex $r_{i}$ to the vertex $c_{j}$ if and only if $a_{i j}=1$. It is easy to see that if $g$ and $h$ are two isomorphic graphs and $A$ and $B$ are the corresponding matrices, then $A$ is obtained from $B$ by a permutation of columns and/or rows.

Thus, the combinatorial problem to obtain and enumerate all of $n \times n$ binary matrices up to a permutation of columns or rows having exactly $k$ units naturally arises. The present work is devoted to this problem.

## 2. SEMI-CANONICAL AND CANONICAL BINARY MATRICES

Definition 1. Let $A \in \mathcal{B}_{n \times m}$. With $r(A)$ we will denote the ordered $n$-tuple

$$
r(A)=\left\langle x_{1}, x_{2}, \ldots, x_{n}\right\rangle,
$$

where $0 \leq x_{i} \leq 2^{m}-1, i=1,2, \ldots n$ and $x_{i}$ is a natural number written in binary notation with the help of the $i$-th row of $A$.

Similarly with $c(A)$ we will denote the ordered $m$-tuple

$$
c(A)=\left\langle y_{1}, y_{2}, \ldots, y_{m}\right\rangle
$$

where $0 \leq y_{j} \leq 2^{n}-1, j=1,2, \ldots m$ and $y_{j}$ is a natural number written in binary notation with the help of the $j$-th column of $A$.

We consider the sets:

$$
\begin{aligned}
\mathcal{R}_{n \times m} & =\left\{\left\langle x_{1}, x_{2}, \ldots, x_{n}\right\rangle \mid 0 \leq x_{i} \leq 2^{m}-1, i=1,2, \ldots n\right\} \\
& =\left\{r(A) \mid A \in \mathcal{B}_{n \times m}\right\}
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{C}_{n \times m} & =\left\{\left\langle y_{1}, y_{2}, \ldots, y_{m}\right\rangle \mid 0 \leq y_{j} \leq 2^{n}-1, j=1,2, \ldots m\right\} \\
& =\left\{c(A) \mid A \in \mathcal{B}_{n \times m}\right\}
\end{aligned}
$$

With "<" we will denote the lexicographic orders in $\mathcal{R}_{n \times m}$ and in $\mathcal{C}_{n \times m}$ It is easy to see that Definition 1 describes two mappings:

$$
r: \mathcal{B}_{n \times m} \rightarrow \mathcal{R}_{n \times m}
$$

and

$$
c: \mathcal{B}_{n \times m} \rightarrow \mathcal{C}_{n \times m}
$$

which are bijective and therefore

$$
\mathcal{R}_{n \times m} \cong \mathcal{B}_{n \times m} \cong \mathcal{C}_{n \times m} .
$$

Definition 2. Let $A \in \mathcal{B}_{n \times m}$,

$$
\begin{aligned}
& r(A)=\left\langle x_{1}, x_{2}, \ldots, x_{n}\right\rangle \\
& c(A)=\left\langle y_{1}, y_{2}, \ldots, y_{m}\right\rangle .
\end{aligned}
$$

We will call the matrix A semi-canonical, if

$$
x_{1} \leq x_{2} \leq \cdots \leq x_{n}
$$

and

$$
y_{1} \leq y_{2} \leq \cdots \leq y_{m} .
$$

Proposition 1. Let $A=\left[a_{i j}\right] \in \mathcal{B}_{n \times m}$ be a semi-canonical matrix. Then there exist integers $i, j$, such that $1 \leq i \leq n, 1 \leq j \leq m$ and

$$
\begin{array}{ll}
a_{11}=a_{12}=\cdots=a_{1 j}=0, & a_{1 j+1}=a_{1 j+2}=\cdots=a_{1 m}=1, \\
a_{11}=a_{21}=\cdots=a_{i 1}=0, & a_{i+11}=a_{i+21}=\cdots=a_{n 1}=1 . \tag{3}
\end{array}
$$

Proof. Let $r(A)=\left\langle x_{1}, x_{2}, \ldots x_{n}\right\rangle$ and $c(A)=\left\langle y_{1}, y_{2}, \ldots y_{m}\right\rangle$. We assume that there exist integers $p$ and $q$, such that $1 \leq p<q \leq m, a_{1 p}=1$ and $a_{1 q}=0$. In this case $y_{p}>y_{q}$, which contradicts the condition for semi-canonicity of the matrix $A$. We have proven (2). Similarly, we prove (3) as well.

Corollary 1. Let $A=\left[a_{i j}\right] \in \mathcal{B}_{n \times m}$ be a semi-canonical matrix. Then there exist integers s, $t$, such that $0 \leq s \leq m, 0 \leq t \leq n, x_{1}=2^{s}-1$ and $y_{1}=2^{t}-1$

Definition 3. Let $A, B \in \mathcal{B}_{n \times m}$. We will say that the matrices $A$ and $B$ are equivalent and we will write

$$
\begin{equation*}
A \sim B, \tag{4}
\end{equation*}
$$

if there exist permutation matrices $X \in \mathcal{P}_{n}$ and $Y \in \mathcal{P}_{m}$, such that

$$
\begin{equation*}
A=X B Y . \tag{5}
\end{equation*}
$$

In other words $A \sim B$ if $A$ is received from $B$ after dislocation of some of the rows and the columns of $B$.

Obviously, the introduced relation is an equivalence relation.
Definition 4. We will call the matrix $A \in \mathcal{B}_{n \times m}$ canonical matrix, if $r(A)$ is a minimal element about the lexicographic order in the set $\{r(B) \mid B \sim A\}$.

If the matrix $A \in \mathcal{B}_{n \times m}$ is canonical and $r(A)=\left\langle x_{1}, x_{2}, \ldots, x_{n}\right\rangle$, then obviously

$$
\begin{equation*}
x_{1} \leq x_{2} \leq \cdots \leq x_{n} . \tag{6}
\end{equation*}
$$

From definition 4 immediately follows that in every equivalence class about the relation " $\sim$ " (definition 3) there exists only one canonical matrix. Therefore, to find all bipartite graphs of type $g=\left\langle R_{g}, C_{g}, E_{g}\right\rangle$, where
$V=R_{g} \cup C_{g}$ is the set of vertices, and $E_{g}$ is the set of edges of the graph $g, R_{g} \cap C_{g}=\varnothing,\left|R_{g}\right|=\left|C_{g}\right|=n,\left|E_{g}\right|=k$, up to isomorphism, it suffices to find all canonical matrices with $k$ 1's from the set $\mathcal{B}_{n \times n}$.

With $\mathcal{T}_{n} \subset \mathcal{P}_{n}$ we denote the set of all transpositions in $\mathcal{P}_{n}$, i.e. the set of all $n \times n$ permutation matrices, which multiplying from the left an arbitrary $n \times m$ matrix swaps the places of exactly two rows, while multiplying from the right an arbitrary $k \times n$ matrix swaps the places of exactly two columns.

Theorem 1. Let $A$ be an arbitrary matrix from $\mathcal{B}_{n \times m}$. Then:
a) If $X_{1}, X_{2}, \cdots, X_{s} \in \mathcal{T}_{n}$ are such that

$$
r\left(X_{1} X_{2} \ldots X_{s} A\right)<r\left(X_{2} X_{3} \ldots X_{s} A\right)<\cdots<r\left(X_{s} A\right)<r(A),
$$

then

$$
c\left(X_{1} X_{2} \ldots X_{s} A\right)<c(A) .
$$

b) If $Y_{1}, Y_{2}, \cdots, Y_{t} \in \mathcal{T}_{m}$ are such that

$$
c\left(A Y_{1} Y_{2} \ldots Y_{t}\right)<c\left(A Y_{2} Y_{3} \ldots Y_{t}\right)<\cdots<c\left(A X_{t}\right)<r(A),
$$

then

$$
r\left(A Y_{1} Y_{2} \ldots Y_{t}\right)<r(A) .
$$

Proof. a) Induction by $s$. Let $s=1$ and let $X \in \mathcal{T}_{n}$ be a transposition which multiplying an arbitrary matrix $A=\left[a_{i j}\right] \in \mathcal{B}_{n \times m}$ from the left swaps the places of the rows of $A$ with numbers $u$ and $v(1 \leq u<v \leq n)$, while the remaining rows stay in their places. In other words if

$$
A=\left[\begin{array}{cccccc}
a_{11} & a_{12} & \cdots & a_{1 r} & \cdots & a_{1 m} \\
a_{21} & a_{22} & \cdots & a_{2 r} & \cdots & a_{2 m} \\
\vdots & \vdots & & \vdots & & \vdots \\
a_{u 1} & a_{u 2} & \cdots & a_{u r} & \cdots & a_{u m} \\
\vdots & \vdots & & \vdots & & \vdots \\
a_{v 1} & a_{v 2} & \cdots & a_{v r} & \cdots & a_{v m} \\
\vdots & \vdots & & \vdots & & \vdots \\
a_{n 1} & a_{n 2} & \cdots & a_{n r} & \cdots & a_{n m}
\end{array}\right]
$$

then

$$
X A=\left[\begin{array}{cccccc}
a_{11} & a_{12} & \cdots & a_{1 r} & \cdots & a_{1 m} \\
a_{21} & a_{22} & \cdots & a_{2 r} & \cdots & a_{2 m} \\
\vdots & \vdots & & \vdots & & \vdots \\
a_{v 1} & a_{v 2} & \cdots & a_{v r} & \cdots & a_{v m} \\
\vdots & \vdots & & \vdots & & \vdots \\
a_{u 1} & a_{u 2} & \cdots & a_{u r} & \cdots & a_{u m} \\
\vdots & \vdots & & \vdots & & \vdots \\
a_{n 1} & a_{n 2} & \cdots & a_{n r} & \cdots & a_{n m}
\end{array}\right],
$$

where $a_{i j} \in\{0,1\}, 1 \leq i \leq n, 1 \leq j \leq m$.
Let $r(A)=\left\langle x_{1}, x_{2}, \ldots, x_{u}, \ldots, x_{v}, \ldots, x_{n}\right\rangle$. Then $r(X A)=\left\langle x_{1}, x_{2}, \ldots, x_{v}, \ldots, x_{u}, \ldots, x_{n}\right\rangle$. Since $r(X A)<r(A)$, then according to the properties of the lexicographic order $x_{v}<x_{u}$. According to Definition 1 the representation of $x_{u}$ and $x_{v}$ in binary notation with an eventual addition if necessary with unessential zeros in the beginning is respectively as follows:

$$
\begin{aligned}
x_{u} & =a_{u 1} a_{u 2} \cdots a_{u m}, \\
x_{v} & =a_{v 1} a_{v 2} \cdots a_{v m} .
\end{aligned}
$$

Since $x_{v}<x_{u}$, then there exists an integer $r \in\{1,2, \ldots, m\}$, such that $a_{u j}=a_{v j}$ when $j<r, a_{u r}=1$ and $a_{v r}=0$.

Hence if $c(A)=\left\langle y_{1}, y_{2}, \ldots, y_{m}\right\rangle, c(X A)=\left\langle z_{1}, z_{2}, \ldots, z_{m}\right\rangle$, then $y_{j}=z_{j}$ when $j<r$, while the representation of $y_{r}$ and $z_{r}$ in binary notation with an eventual addition if necessary with unessential zeroes in the beginning is respectively as follows:

$$
\begin{aligned}
& y_{r}=a_{1 r} a_{2 r} \cdots a_{u-1 r} a_{u r} \cdots a_{v r} \cdots a_{n r}, \\
& z_{r}=a_{1 r} a_{2 r} \cdots a_{u-1 r} a_{v r} \cdots a_{u r} \cdots a_{n r} .
\end{aligned}
$$

Since $a_{u r}=1, a_{v r}=0$, then $z_{r}<y_{r}$, whence it follows that $c(X A)<c(A)$.
We assume that for every $s$-tuple of transpositions $X_{1}, X_{2}, \ldots, X_{s} \in \mathcal{T}_{n}$ and for every matrix $A \in \mathcal{B}_{n \times m}$ from

$$
r\left(X_{1} X_{2} \ldots X_{s} A\right)<r\left(X_{2} \cdots X_{s} A\right)<\cdots<r\left(X_{s} A\right)<r(A)
$$

it follows that

$$
c\left(X_{1} X_{2} \ldots X_{s} A\right)<c(A)
$$

and let $X_{s+1} \in \mathcal{T}_{n}$ be such that

$$
r\left(X_{1} X_{2} \ldots X_{s} X_{s+1} A\right)<r\left(X_{2} \cdots X_{s+1} A\right)<\cdots<r\left(X_{s+1} A\right)<r(A) .
$$

According to the induction assumption $c\left(X_{s+1} A\right)<c(A)$.
We put

$$
A_{1}=X_{s+1} A
$$

According to the induction assumption from

$$
r\left(X_{1} X_{2} \ldots X_{s} A_{1}\right)<r\left(X_{2} \cdots X_{s} A_{1}\right)<\cdots<r\left(X_{s} A_{1}\right)<r\left(A_{1}\right)
$$

it follows that

$$
c\left(X_{1} X_{2} \cdots X_{s} X_{s+1} A\right)=c\left(X_{1} X_{2} \cdots X_{s} A_{1}\right)<c\left(A_{1}\right)=c\left(X_{s+1} A\right)<c(A)
$$

with which we have proven a).
b) is proven similarly to a).

Obviously in effect is also the dual to Theorem 1 statement, in which everywhere instead of the sign "<" we put the sign ">".

Corollary 2. If the matrix $A \in \mathcal{B}_{n \times m}$ is a canonical matrix, then it is a semicanonical matrix.

Proof. Let $A \in \mathcal{B}_{n \times m}$ be a canonical matrix and $r(A)=\left\langle x_{1}, x_{2}, \ldots, x_{n}\right\rangle$. Then from (6) it follows that $x_{1} \leq x_{2} \leq \cdots \leq x_{n}$. Let $c(A)=\left\langle y_{1}, y_{2}, \ldots, y_{m}\right\rangle$. We assume that there are $s$ and $t$ such that $s \leq t$ and $y_{s}>y_{t}$. Then we swap the columns of numbers $s$ and $t$. Thus we obtain the matrix $A^{\prime} \in \mathcal{B}_{n \times m}$, $A^{\prime} \neq A$. Obviously $c\left(A^{\prime}\right)<c(A)$. From Theorem 1 it follows that $r\left(A^{\prime}\right)<r(A)$, which contradicts the minimality of $r(A)$.

In the next example, we will see that the opposite statement of Corollary 2 is not always true.

Example 1. We consider the matrices:

$$
A=\left[\begin{array}{llll}
0 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{array}\right] \text { and } B=\left[\begin{array}{llll}
0 & 0 & 0 & 1 \\
0 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 \\
1 & 0 & 0 & 0
\end{array}\right] .
$$

After immediate verification, we find that $A \sim B$. Furthermore $r(A)=\langle 3,3,4,8\rangle, c(A)=\langle 1,2,12,12\rangle, r(B)=\langle 1,6,6,8\rangle, c(B)=\langle 1,6,6,8\rangle$. So $A$ and $B$ are two equivalent to each other semi-canonical matrices, but they are not canonical. Canonical matrix in this equivalence class is the matrix

$$
C=\left[\begin{array}{llll}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0
\end{array}\right],
$$

where $r(C)=\langle 1,2,12,12\rangle, c(C)=\langle 3,3,4,8\rangle$.
From example 1 immediately follows that in a given equivalence class it is possible to exist more than one semi-canonical element.

## 3. ROGRAMME CODE OF AN ALGORITHM FOR FINDING ALL SEMI-CANONICAL MATRICES

Corollary 2 is useful that it is enough to seek canonical matrices from among the semi-canonical.

In this section, we are going to suggest an algorithm (Algorithm 1) for finding the semi-canonical matrices without checking all elements of the set $\mathcal{B}_{n \times m}$, described with the help of programming language $\mathrm{C}++$. In the described algorithm, bitwise operations are substantially used. In [3] and [6] we prove that the representation of the elements of $\mathcal{B}_{n}$ using ordered $n$-tuples of natural numbers and bitwise operations leads to making a fast and saving memory algorithms. Similar techniques we used in the article [8], where we describe an algorithm for solving the combinatorial problem for finding the semi-canonical matrices in the set consisting of all $n \times n$ binary matrices having exactly $k$ 1's in every row and every column. The results of this work are given in the Encyclopedia of Integer Sequences [10], respectively under the numbers A229161, A229162, A229163 and A229164. N. J. A. Sloane, who cites the work [9], presents all of them.

Algorithm 1. Receives all $n \times n$ semi-canonical binary matrices.
\#define n ...
/*
The function check(int[], int) verifies whether obtained n-tuple represents a semi-canonical matrix and returns the number of 1's in the matrix.
*/
int check(int x $\$, int n)
\{
int $\mathrm{k}=0$; // The number of 1 's in the matrix. If the matrix is not
// semi-canonical, the function returns -1.
int yj; // The number represents the ( $\mathrm{n}-\mathrm{j}$ )-th column of the matrix int $\mathrm{y} 0=-1$; // The number before yi
for (int j=n-1; j>=0; j--)
\{ yj=0; for (int $\mathrm{i}=0 ; \mathrm{i}<\mathrm{n} ; \mathrm{i}++$ ) \{ if $(1 \ll j \& x[i])$
\{ yj $\mid=1 \ll(n-1-i) ;$ k++;
\}
\}
if ( $\mathrm{yj}<\mathrm{y} 0$ ) return -1;
y0 = yj;
\}
// This n-tuple represents a semi-canonical matrix. We print it.
for (int i=0; i<n; i++) cout $\ll x[i] \ll "$ ";
cout<<"k="<<k<<'ln';
return k;
\}
int main(int argc, char *argv[])
\{
int $x[n] ; / / x[n]$-ordered n-tuple of integers that represent the rows
// of the matrix
int $k[n * n+1] ; / / k[i]-$ Number of semi-canonical matrices with // exactly i 1's, $0<=\mathrm{i}<=n * n$
int $m=n * n$;
for (int $i=0 ; i<=m ; i++$ ) $k[i]=0$;
int $x m a x=(1 \ll n)-1$;
int p,c;
for (int s=0; s<=n; s++)
\{

```
        for (int i=0; i<n; i++) x[i] = (1<<s)-1;
        c=check(x,n);
        k[s*n]++;
        p=n-1;
        while (p>0 && x[p]<xmax)
        {
            x[p]++;
            for (int i=p+1; i<n; i++) x[i]=x[p];
            c = check(x,n);
            if (c>=0) k[c]++;
            p=n-1;
        while (x[p] == xmax) p--;
    }
}
```

    for (int i=0; i<=m; i++)
    cout<<"k("<<n<<","<<i<<") = "<<k[i]<<endl;
    \}

## 4. RESULTS

Let us denote with $\kappa(n, i)$ the number of all $n \times n$ semi-canonical binary matrices with exactly $i 1$ 's, where $0 \leq i \leq n^{2}$. Using Algorithm 1, we received the following integer sequences:

$$
\begin{gathered}
\{\kappa(2, i)\}_{i=0}^{4}=\{1,1,3,1,1\} \\
\{\kappa(3, i)\}_{i=0}^{9}=\{1,1,3,8,10,9,8,3,1,1\} \\
\{\kappa(4, i)\}_{i=0}^{16}=\{1,1,3,8,25,49,84,107,121,101,72,41,24,8,3,1,1\} \\
\{\kappa(5, i)\}_{i=0}^{25}=\{1,1,3,8,25,80,220,524,1057,1806,2671,3365,3680,3468,2865, \\
2072,1314,723,362,166,72,24,8,3,1,1\} \\
\{\kappa(6, i)\}_{i=0}^{36}=\{1,1,3,8,25,80,283,925,2839,7721,18590,39522,74677,125449, \\
188290,252954,305561,332402,326650,290171,233656,170704,113448,68677, \\
37996,19188,8910,3847,1588,613,299,72,24,8,3,1,1\}
\end{gathered}
$$

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