Semi-canonical binary matrices

Krasimir Yordzhev

Faculty of Mathematics and Natural Sciences South-West University, Blagoevgrad, Bulgaria E-mail: yordzhev@swu.bg

Abstract: In this paper, we define the concepts of semi-canonical and canonical binary matrix. Strictly mathematical, we prove the correctness of these definitions. We describe and we implement an algorithm for finding all $n \times n$ semi-canonical binary matrices taking into account the number of 1 in each of them. This problem relates to the combinatorial problem of finding all pairs of disjoint $n^2 \times n^2$ S-permutation matrices. In the described algorithm, the bitwise operations are substantially used.

Keywords: binary matrix, ordered n-tuple, semi-canonical and canonical binary matrix, disjoint S-permutation matrices, bitwise operations **MSC[2010]:** 05B20, 68N15

1.INTRODUCTION

Binary (or *boolean*, or (0,1)-matrix) is called a matrix whose elements belong to the set $\mathcal{B} = \{0,1\}$.

Let *n* and *m* be positive integers. With $\mathcal{B}_{n\times m}$ we will denote the set of all $n \times m$ binary matrices, while with $\mathcal{B}_n = \mathcal{B}_{n\times n}$ we will denote the set of all square $n \times n$ binary matrices.

A square binary matrix is called a *permutation matrix*, if there is just one 1 in every row and every column. Let us denote by \mathcal{P}_n the group of all $n \times n$ permutation matrices, and by \mathcal{S}_n the symmetric group of order n, i.e. the group of all one-to-one mappings of the set $[n] = \{1, 2, ..., n\}$ in itself. In effect is the isomorphism $\mathcal{P}_n \cong \mathcal{S}_n$.

As it is well known [4,5] the multiplication of an arbitrary real or complex matrix A from the left with a permutation matrix (if the multiplication is possible) leads to dislocation of the rows of the matrix A, while the multiplication of A from the right with a permutation matrix leads to the dislocation of the columns of A.

Let *n* be a positive integer and let $A \in \mathcal{B}_n$ be a $n^2 \times n^2$ binary matrix. With the help of n-1 horizontal lines and n-1 vertical lines *A* has been separated into n^2 of number non-intersecting $n \times n$ square sub-matrices A_{kl} , $1 \le k, l \le n$, e.i.

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix}.$$

A matrix $A \in \mathcal{B}_{n^2}$ is called an *S-permutation* if in each row, each column, and each sub-matrice A_{kl} , $1 \le k, l \le n$ of A there is exactly one 1. Two Spermutation matrices A and B will be called *disjoint*, if there are not $i, j \in [n^2] = \{1, 2, ..., n^2\}$ such that for the elements $a_{ij} \in A$ and $b_{ij} \in B$ the condition $a_{ii} = b_{ii} = 1$ is satisfied.

The concept of S-permutation matrix was introduced by Geir Dahl [1] in relation to the popular Sudoku puzzle. Obviously a square $n^2 \times n^2$ matrix M with elements of $[n^2] = \{1, 2, ..., n^2\}$ is a Sudoku matrix if and only if there are S-permutation matrices $A_1, A_2, ..., A_{n^2}$, each two of them are disjoint and such that M can be given in the following way:

(1)
$$M = 1 \cdot A_1 + 2 \cdot A_2 + \dots + n^2 \cdot A_{n^2}.$$

In [2] Roberto Fontana offers an algorithm which randomly gets a family of $n^2 \times n^2$ mutually disjoint S-permutation matrices, where n = 2,3. In n = 3 he ran the algorithm 1000 times and found 105 different families of nine mutually disjoint S-permutation matrices. Then using (1) he obtained 9!105 = 38102400 Sudoku matrices.

Bipartite graph is the ordered triplet $g = \langle R_g, C_g, E_g \rangle$, where R_g and C_g are non-empty sets such that $R_g \cap C_g = \emptyset$, the elements of which will be called *vertices*. $E_g \subseteq R_g \times C_g = \{\langle r, c \rangle \mid r \in R_g, c \in C_g\}$ - the set of *edges*. Repeated edges are not allowed in our considerations.

If $x \in \{1, 2, ..., n\}$, $\rho \in S_n$, then the image of the element x in the mapping ρ we denote by $\rho(x)$. Let $g' = \langle R_{g'}, C_{g'}, E_{g'} \rangle$ and

 $g'' = \langle R_{g''}, C_{g''}, E_{g''} \rangle$. We will say that the graphs g' and g'' are *isomorphic* and we will write $g' \cong g''$, if $R_{g'} \cong R_{g''}$, $C_{g'} \cong C_{g''}$, $|R_{g'}| = |R_{g''}| = m$, $|C_{g'}| = |C_{g''}| = n$ and there exist $\rho \in S_m$ and $\sigma \in S_n$ such that $\langle r, c \rangle \in E_{g'} \Leftrightarrow \langle \rho(r), \sigma(c) \rangle \in E_{g''}$. In this paper we consider only bipartite graphs up to isomorphism.

Analyzing the works of G. Dahl [1] and R. Fontana [2], the question of finding a general formula for counting disjoint pairs of $n^2 \times n^2$ S-permutation matrices as a function of the integer *n* naturally arises. This is an interesting combinatorial problem that deserves its consideration. The work [7] solves this problem. To do that, the graph theory techniques have been used. It has been shown that to count the number of disjoint pairs of $n^2 \times n^2$ S-permutation matrices, it is sufficient to obtain some numerical characteristics of the set of all bipartite graphs considered to within isomorphism of the type $g = \langle R_g, C_g, E_g \rangle$, where $V = R_g \cup C_g$ is the set of vertices, and E_g is the set of edges of the graph g, $R_g \cap C_g = \emptyset$, $|R_g| = |C_g| = n$ $|E_g| = k$, $k = 0, 1, ..., n^2$.

Let $g = \langle R_g, C_g, E_g \rangle$ be a bipartite graph, where $R_g = \{r_1, r_2, ..., r_n\}$ and $C_g = \{c_1, c_2, ..., c_n\}$. Then we build the matrix $A = [a_{ij}] \in \mathcal{B}_n$, such that $a_{ij} = 1$ if and only if $\langle r_i, c_j \rangle \in E_g$. Inversely, let $A = [a_{ij}] \in \mathcal{B}_n$. We denote the *i*-th row of *A* with r_i , while the *j*-th column of *A* with c_j . Then we build the bipartite graph $g = \langle R_g, C_g, E_g \rangle$, where $R_g = \{r_1, r_2, ..., r_n\}$, $C_g = \{c_1, c_2, ..., c_n\}$ and there exists an edge from the vertex r_i to the vertex c_j if and only if $a_{ij} = 1$. It is easy to see that if *g* and *h* are two isomorphic graphs and *A* and *B* are the corresponding matrices, then *A* is obtained from *B* by a permutation of columns and/or rows.

Thus, the combinatorial problem to obtain and enumerate all of $n \times n$ binary matrices up to a permutation of columns or rows having exactly k units naturally arises. The present work is devoted to this problem.

2. SEMI-CANONICAL AND CANONICAL BINARY MATRICES

Definition 1. Let $A \in \mathcal{B}_{n \times m}$. With r(A) we will denote the ordered *n*-tuple

$$r(A) = \langle x_1, x_2, \dots, x_n \rangle,$$

where $0 \le x_i \le 2^m - 1$, i = 1, 2, ..., n and x_i is a natural number written in binary notation with the help of the *i*-th row of *A*.

Similarly with c(A) we will denote the ordered *m*-tuple

$$c(A) = \langle y_1, y_2, \dots, y_m \rangle,$$

where $0 \le y_j \le 2^n - 1$, j = 1, 2, ...m and y_j is a natural number written in binary notation with the help of the *j*-th column of *A*.

We consider the sets:

$$\mathcal{R}_{n \times m} = \{ \langle x_1, x_2, \dots, x_n \rangle \mid 0 \le x_i \le 2^m - 1, i = 1, 2, \dots n \}$$
$$= \{ r(A) \mid A \in \mathcal{B}_{n \times m} \}$$

and

$$\mathcal{C}_{n \times m} = \{ \langle y_1, y_2, \dots, y_m \rangle | 0 \le y_j \le 2^n - 1, j = 1, 2, \dots m \} \\ = \{ c(A) | A \in \mathcal{B}_{n \times m} \}$$

With "<" we will denote the lexicographic orders in $\mathcal{R}_{n \times m}$ and in $\mathcal{C}_{n \times m}$ It is easy to see that Definition 1 describes two mappings:

$$r:\mathcal{B}_{n\times m}\to\mathcal{R}_{n\times m}$$

and

$$c:\mathcal{B}_{n\times m}\to \mathcal{C}_{n\times m},$$

which are bijective and therefore

$$\mathcal{R}_{n \times m} \cong \mathcal{B}_{n \times m} \cong \mathcal{C}_{n \times m}.$$

Definition 2. Let $A \in \mathcal{B}_{n \times m}$,

$$r(A) = \langle x_1, x_2, \dots, x_n \rangle,$$
$$c(A) = \langle y_1, y_2, \dots, y_m \rangle.$$

We will call the matrix A semi-canonical, if

 $x_1 \le x_2 \le \dots \le x_n$

and

$$y_1 \le y_2 \le \dots \le y_m.$$

Proposition 1. Let $A = [a_{ij}] \in \mathcal{B}_{n \times m}$ be a semi-canonical matrix. Then there exist integers i, j, such that $1 \le i \le n$, $1 \le j \le m$ and

(2)
$$a_{11} = a_{12} = \dots = a_{1j} = 0, \quad a_{1j+1} = a_{1j+2} = \dots = a_{1m} = 1,$$

(3)
$$a_{11} = a_{21} = \dots = a_{i1} = 0, \quad a_{i+11} = a_{i+21} = \dots = a_{n1} = 1.$$

Proof. Let $r(A) = \langle x_1, x_2, ..., x_n \rangle$ and $c(A) = \langle y_1, y_2, ..., y_m \rangle$. We assume that there exist integers p and q, such that $1 \le p < q \le m$, $a_{1p} = 1$ and $a_{1q} = 0$. In this case $y_p > y_q$, which contradicts the condition for semi-canonicity of the matrix A. We have proven (2). Similarly, we prove (3) as well.

Corollary 1. Let $A = [a_{ij}] \in \mathcal{B}_{n \times m}$ be a semi-canonical matrix. Then there exist integers s, t, such that $0 \le s \le m$, $0 \le t \le n$, $x_1 = 2^s - 1$ and $y_1 = 2^t - 1$

Definition 3. Let $A, B \in \mathcal{B}_{n \times m}$. We will say that the matrices A and B are equivalent and we will write

if there exist permutation matrices $X \in \mathcal{P}_n$ and $Y \in \mathcal{P}_m$, such that

$$(5) A = XBY.$$

In other words $A \sim B$ if A is received from B after dislocation of some of the rows and the columns of B.

Obviously, the introduced relation is an equivalence relation.

Definition 4. We will call the matrix $A \in \mathcal{B}_{n \times m}$ canonical matrix, if r(A) is a minimal element about the lexicographic order in the set $\{r(B) | B \sim A\}$.

If the matrix $A \in \mathcal{B}_{n \times m}$ is canonical and $r(A) = \langle x_1, x_2, ..., x_n \rangle$, then obviously

$$(6) x_1 \le x_2 \le \dots \le x_n.$$

From definition 4 immediately follows that in every equivalence class about the relation " \sim " (definition 3) there exists only one canonical matrix. Therefore, to find all bipartite graphs of type $g = \langle R_g, C_g, E_g \rangle$, where

 $V = R_g \cup C_g$ is the set of vertices, and E_g is the set of edges of the graph $g, R_g \cap C_g = \emptyset, |R_g| = |C_g| = n, |E_g| = k$, up to isomorphism, it suffices to find all canonical matrices with k 1's from the set $\mathcal{B}_{n \times n}$.

With $\mathcal{T}_n \subset \mathcal{P}_n$ we denote the set of all *transpositions* in \mathcal{P}_n , i.e. the set of all $n \times n$ permutation matrices, which multiplying from the left an arbitrary $n \times m$ matrix swaps the places of exactly two rows, while multiplying from the right an arbitrary $k \times n$ matrix swaps the places of exactly two columns.

Theorem 1. Let *A* be an arbitrary matrix from $\mathcal{B}_{n \times m}$. Then: a) If $X_1, X_2, \dots, X_s \in \mathcal{T}_n$ are such that

$$r(X_1X_2...X_sA) < r(X_2X_3...X_sA) < \cdots < r(X_sA) < r(A),$$

then

$$c(X_1X_2\ldots X_sA) < c(A).$$

b) If $Y_1, Y_2, \dots, Y_t \in \mathcal{T}_m$ are such that

$$c(AY_1Y_2\ldots Y_t) < c(AY_2Y_3\ldots Y_t) < \cdots < c(AX_t) < r(A),$$

then

$$r(AY_1Y_2\ldots Y_t) < r(A).$$

Proof. a) Induction by *s*. Let s = 1 and let $X \in T_n$ be a transposition which multiplying an arbitrary matrix $A = [a_{ij}] \in \mathcal{B}_{n \times m}$ from the left swaps the places of the rows of *A* with numbers *u* and *v* $(1 \le u < v \le n)$, while the remaining rows stay in their places. In other words if

	a_{11}	a_{12}		a_{1r}		a_{1m}
	<i>a</i> ₂₁	<i>a</i> ₂₂		a_{2r}		a_{2m}
	:	÷		÷		:
A =	a_{u1}	a_{u2}		a_{ur}		a_{um}
	÷	:		÷		÷
	a_{v1}	a_{v2}	•••	a_{vr}	•••	a_{vm}
	:	:		•		:
	a_{n1}	a_{n2}	•••	a_{nr}	•••	a_{nm}

then

$$XA = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1r} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2r} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{v1} & a_{v2} & \cdots & a_{vr} & \cdots & a_{vm} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{u1} & a_{u2} & \cdots & a_{ur} & \cdots & a_{um} \\ \vdots & \vdots & & & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nr} & \cdots & a_{nm} \end{bmatrix},$$

where $a_{ii} \in \{0,1\}, 1 \le i \le n, 1 \le j \le m$.

Let $r(A) = \langle x_1, x_2, ..., x_u, ..., x_v, ..., x_n \rangle$. Then $r(XA) = \langle x_1, x_2, ..., x_v, ..., x_u, ..., x_n \rangle$. Since r(XA) < r(A), then according to the properties of the lexicographic order $x_v < x_u$. According to Definition 1 the representation of x_u and x_v in binary notation with an eventual addition if necessary with unessential zeros in the beginning is respectively as follows:

$$x_u = a_{u1}a_{u2}\cdots a_{um},$$
$$x_v = a_{v1}a_{v2}\cdots a_{vm}.$$

Since $x_v < x_u$, then there exists an integer $r \in \{1, 2, ..., m\}$, such that $a_{uj} = a_{vj}$ when j < r, $a_{ur} = 1$ and $a_{vr} = 0$.

Hence if $c(A) = \langle y_1, y_2, ..., y_m \rangle$, $c(XA) = \langle z_1, z_2, ..., z_m \rangle$, then $y_j = z_j$ when j < r, while the representation of y_r and z_r in binary notation with an eventual addition if necessary with unessential zeroes in the beginning is respectively as follows:

$$y_r = a_{1r}a_{2r}\cdots a_{u-1r}a_{ur}\cdots a_{vr}\cdots a_{nr},$$

$$z_r = a_{1r}a_{2r}\cdots a_{u-1r}a_{vr}\cdots a_{ur}\cdots a_{nr}.$$

Since $a_{ur} = 1$, $a_{vr} = 0$, then $z_r < y_r$, whence it follows that c(XA) < c(A).

We assume that for every *s*-tuple of transpositions $X_1, X_2, ..., X_s \in T_n$ and for every matrix $A \in \mathcal{B}_{n \times m}$ from

$$r(X_1X_2\ldots X_sA) < r(X_2\cdots X_sA) < \cdots < r(X_sA) < r(A)$$

it follows that

$$c(X_1X_2\ldots X_sA) < c(A)$$

and let $X_{s+1} \in \mathcal{T}_n$ be such that

$$r(X_1X_2...X_sX_{s+1}A) < r(X_2...X_{s+1}A) < \dots < r(X_{s+1}A) < r(A).$$

According to the induction assumption $c(X_{s+1}A) < c(A)$. We put

$$A_1 = X_{s+1}A.$$

According to the induction assumption from

$$r(X_1X_2...X_sA_1) < r(X_2...X_sA_1) < \cdots < r(X_sA_1) < r(A_1)$$

it follows that

$$c(X_1X_2\cdots X_sX_{s+1}A) = c(X_1X_2\cdots X_sA_1) < c(A_1) = c(X_{s+1}A) < c(A),$$

 \square

 \square

with which we have proven a).

b) is proven similarly to a).

Obviously in effect is also the dual to Theorem 1 statement, in which everywhere instead of the sign "<" we put the sign ">".

Corollary 2. If the matrix $A \in \mathcal{B}_{n \times m}$ is a canonical matrix, then it is a semicanonical matrix.

Proof. Let $A \in \mathcal{B}_{n \times m}$ be a canonical matrix and $r(A) = \langle x_1, x_2, ..., x_n \rangle$. Then from (6) it follows that $x_1 \leq x_2 \leq \cdots \leq x_n$. Let $c(A) = \langle y_1, y_2, ..., y_m \rangle$. We assume that there are *s* and *t* such that $s \leq t$ and $y_s > y_t$. Then we swap the columns of numbers *s* and *t*. Thus we obtain the matrix $A' \in \mathcal{B}_{n \times m}$, $A' \neq A$. Obviously c(A') < c(A). From Theorem 1 it follows that r(A') < r(A), which contradicts the minimality of r(A).

In the next example, we will see that the opposite statement of Corollary 2 is not always true.

Example 1. We consider the matrices:

<i>A</i> =	0	0	1	1	and B =	$\left\lceil 0 \right\rceil$	0	0	1]	
	0	0	1	1		0	1	1	0	
	0	1	0	0		0	1	1	0	•
	1	0	0	0		1	0	0	0	

After immediate verification, we find that $A \sim B$. Furthermore $r(A) = \langle 3, 3, 4, 8 \rangle$, $c(A) = \langle 1, 2, 12, 12 \rangle$, $r(B) = \langle 1, 6, 6, 8 \rangle$, $c(B) = \langle 1, 6, 6, 8 \rangle$. So A and B are two equivalent to each other semi-canonical matrices, but they are not canonical. Canonical matrix in this equivalence class is the matrix

$$C = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix},$$

where $r(C) = \langle 1, 2, 12, 12 \rangle$, $c(C) = \langle 3, 3, 4, 8 \rangle$.

From example 1 immediately follows that in a given equivalence class it is possible to exist more than one semi-canonical element.

3. ROGRAMME CODE OF AN ALGORITHM FOR FINDING ALL SEMI-CANONICAL MATRICES

Corollary 2 is useful that it is enough to seek canonical matrices from among the semi-canonical.

In this section, we are going to suggest an algorithm (Algorithm 1) for finding the semi-canonical matrices without checking all elements of the set $\mathcal{B}_{n\times m}$, described with the help of programming language C++. In the described algorithm, bitwise operations are substantially used. In [3] and [6] we prove that the representation of the elements of \mathcal{B}_n using ordered n-tuples of natural numbers and bitwise operations leads to making a fast and saving memory algorithms. Similar techniques we used in the article [8], where we describe an algorithm for solving the combinatorial problem for finding the semi-canonical matrices in the set consisting of all $n \times n$ binary matrices having exactly k 1's in every row and every column. The results of this work are given in the Encyclopedia of Integer Sequences [10], respectively under the numbers A229161, A229162, A229163 and A229164. N. J. A. Sloane, who cites the work [9], presents all of them.

Algorithm 1. Receives all $n \times n$ semi-canonical binary matrices.

```
#define n ...
    /*
    The function check(int[], int) verifies whether obtained n-tuple represents
a semi-canonical matrix and returns the number of 1's in the matrix.
    */
    int check(int x[], int n)
    {
        int k=0; // The number of 1's in the matrix. If the matrix is not
                // semi-canonical, the function returns -1.
                // The number represents the (n-j)-th column of the matrix
       int vi:
        int y0=-1; // The number before yi
        for (int j=n-1; j>=0; j--)
        {
          yj=0;
          for (int i=0; i<n; i++)
          {
             if (1<<j & x[i])
            yj |= 1 << (n-1-i);
            k++;
           }
          }
          if (yj<y0) return -1;
          y0 = yj;
        }
        // This n-tuple represents a semi-canonical matrix. We print it.
        for (int i=0; i<n; i++) cout<<x[i]<<" ":
        cout<<"k="<<k<<'\n':
        return k;
    }
    int main(int argc, char *argv[])
    {
    int x[n]; // x[n]-ordered n-tuple of integers that represent the rows
              // of the matrix
    int k[n*n+1]; //k[i] - Number of semi-canonical matrices with
                  // exactly i 1's, 0<= i<=n*n
    int m=n*n:
        for (int i=0; i<=m; i++) k[i]=0;
        int xmax = (1 < n) - 1;
        int p,c;
        for (int s=0; s<=n; s++)
        {
```

```
for (int i=0; i<n; i++) x[i] = (1 << s)-1;
     c=check(x,n);
     k[s*n]++;
     p=n-1;
     while (p>0 && x[p]<xmax)
     {
        x[p]++;
        for (int i=p+1; i<n; i++) x[i]=x[p];
        c = check(x,n);
        if (c>=0) k[c]++;
        p=n-1;
        while (x[p] == xmax) p--;
   }
 }
   for (int i=0; i<=m; i++)
       cout<<"k("<<n<<","<<i<<") = "<<k[i]<<endl;
}
```

4. RESULTS

Let us denote with $\kappa(n,i)$ the number of all $n \times n$ semi-canonical binary matrices with exactly *i* 1's, where $0 \le i \le n^2$. Using Algorithm 1, we received the following integer sequences:

$$\{\kappa(2,i)\}_{i=0}^{4} = \{1,1,3,1,1\}$$

$$\{\kappa(3,i)\}_{i=0}^{9} = \{1,1,3,8,10,9,8,3,1,1\}$$

$$\{\kappa(4,i)\}_{i=0}^{16} = \{1,1,3,8,25,49,84,107,121,101,72,41,24,8,3,1,1\}$$

$$\{\kappa(5,i)\}_{i=0}^{25} = \{1,1,3,8,25,80,220,524,1057,1806,2671,3365,3680,3468,2865, 2072,1314,723,362,166,72,24,8,3,1,1\}$$

 $\left\{ \kappa(6,i) \right\}_{i=0}^{36} = \{1,1,3,8,25,80,283,925,2839,7721,18590,39522,74677,125449, \\ 188290,252954,305561,332402,326650,290171,233656,170704,113448,68677, \\ 37996,19188,8910,3847,1588,613,299,72,24,8,3,1,1 \}$

5. REFERENCES

[1] Dahl, G. (2009) Permutation matrices related to sudoku, Linear Algebra and its Applications 430 (8–9), 2457–2463.

- [2] Fontana, R. (2011) Fractions of permutations an application to sudoku, Journal of Statistical Planning and Inference 141 (12), 3697– 3704.
- [3] Kostadinova, H., Yordzhev, K. (2010) A Representation of Binary Matrices. Mathematics and education in mathematics, 39, 198-206.
- [4] Sachkov, V. N., Tarakanov, V. E. (1975) Combinatorics of Nonnegative Matrices of Nonnegative Matrices. Amer. Math. Soc.
- [5] Tarakanov, V. E. (1985) Combinatorial problems and (0,1)-matrices. Moscow, Nauka, (in Russian).
- [6] Yordzhev, K. (2009) An example for the use of bitwise operations in programming. Mathematics and education in mathematics, v. 38, 196-202.
- [7] Yordzhev, K. (2013) On the Number of Disjoint Pairs of Spermutation Matrices. Discrete Applied Mathematics 161, 3072–3079.
- [8] Yordzhev, K. (2013) On an Algorithm for Isomorphism-Free Generations of Combinatorial Objects. International Journal of Emerging Trends & Technology in Computer Science, 2 (6), 215-220.
- [9] Yordzhev, K. (2014) Fibonacci sequence related to a combinatorial problem on binary matrices, American Journal Mathematics and Sciences (AJMS), ISSN 2250 3102, Vol. 3, No. 1 (2014), 79–83; arXiv preprint: 1305.6790v2.
- [10] The On-Line Encyclopedia of Integer Sequences (OEIS). http://oeis.org/