More on Rotations as Spin Matrix Polynomials

Thomas L. Curtright

curtright@miami.edu

Department of Physics, University of Miami
Coral Gables, FL 33124-8046, USA

Abstract

Any nonsingular function of spin j matrices always reduces to a matrix polynomial of order $2j$. The challenge is to find a convenient form for the coefficients of the matrix polynomial. The theory of biorthogonal systems is a useful framework to meet this challenge. Central factorial numbers play a key role in the theoretical development. Explicit polynomial coefficients for rotations expressed either as exponentials or as rational Cayley transforms are considered here. Structural features of the results are discussed and compared, and large j limits of the coefficients are examined.

1   Introduction

As a consequence of the Cayley-Hamilton theorem [1, 2] any nonsingular function [3] of an $n \times n$ matrix always reduces to a polynomial of order $n-1$ in powers of the matrix, since the nth and higher powers of the matrix can be reduced to a linear combination of lower powers (see [5] and earlier literature cited therein). The challenge for anyone wishing to take advantage of this fact is to find a convenient form for the coefficients of the matrix polynomial.

In particular, a nonsingular matrix function of any one component, $\hat{\boldsymbol{n}} \cdot \boldsymbol{J}$, for a unitary, irreducible angular momentum representation of spin j, always reduces to a matrix polynomial of order $2j$. Explicit polynomial coefficients for rotations expressed either as exponentials or as rational Cayley transforms [1, 4] are considered here, to augment some recent studies [6, 7, 8]. Structural features of the results are discussed and compared, including the behavior of the coefficients for large j.

The theory of biorthogonal systems [9, 10, 11] is a useful framework to find the coefficients of the matrix polynomial, even in the most general case. For analytic spin functions of $\hat{\boldsymbol{n}} \cdot \boldsymbol{J}$, central factorial numbers [12, 13] play a key role in the construction of the biorthogonal system.

In prior work, Curtright, Fairlie, and Zachos (CFZ) obtained explicit and intuitive results [6] expressing the standard, exponential rotation matrix for *any* quantized angular momentum j as a polynomial of order $2j$ in the corresponding $(2j+1)\times(2j+1)$ spin matrices $\hat{\boldsymbol{n}} \cdot \boldsymbol{J}$ that generate rotations about axis $\hat{\boldsymbol{n}}$. While many previous studies of this or closely related problems can be found in the literature [14, 15, 16, 17, 18, 19], none of these other studies succeeded to find such simple, compact expressions for the coefficients in the spin matrix polynomial, as elementary functions of the rotation angle, as those obtained by CFZ. For each angle-dependent coefficient in the polynomial, the explicit formula found by CFZ involves nothing more complicated than a truncated series expansion for a power of the arcsin function. Although a detailed proof of the CFZ result is not exhibited in [6], the essential ingredients needed to provide such a proof are in that paper, and indeed, the details of two elementary derivations were subsequently given in [7].

More recently, Van Kortryk [8] discovered the corresponding polynomial result for the Cayley rational form of an irreducible, unitary $SU(2)$ representation. At first glance the Cayley transform for a spin representation would seem to follow immediately from the CFZ result just by changing variables from θ to a function $\theta(\alpha)$, where α parameterizes the transform. For any given numerical value of $\hat{\boldsymbol{n}} \cdot \boldsymbol{J}$ this would be so, obviously, but it is not obviously so for matrix-valued $\hat{\boldsymbol{n}} \cdot \boldsymbol{J}$. Nevertheless, as it turns out, the result for the Cayley transform is actually much simpler than the CFZ result for the exponential. For the Cayley transform the explicit coefficients involve nothing more complicated than truncated series expansions of already finite polynomials in a real parameter α. Remarkably, this is true for the Cayley transform of *any* matrix, not just spin matrices [20, 21, 22].

In this paper I compare the CFZ and Van Kortryk results, and I provide a more complete discussion of the latter, including a detailed derivation of the polynomial coefficients for the $SU(2)$ Cayley transforms. I show explicitly how the CFZ results are connected to Cayley transforms by Laplace transformations.

2 Methodology

The theory that underlies construction of spin matrix polynomials is discussed in several places in the literature (see [6, 7] and references therein). Here I condense and summarize the key results.

2.1 A fundamental identity

The fundamental identity for spin $j \in \left\{ 0, \frac{1}{2}, 1, \frac{3}{2}, \cdots \right\}$ is [7]

$$(\hat{\boldsymbol{n}} \cdot \boldsymbol{J})^{2j+1} = - \sum_{m=0}^{2j} t\,(2 + 2j, 1 + m) \times (\hat{\boldsymbol{n}} \cdot \boldsymbol{J})^m \ , \tag{1}$$

where the coefficients $t\,(n, k)$ are central factorial numbers [12, 13]. (For details, see Appendices A and B.) This identity ensures that all powers of $(\hat{\boldsymbol{n}} \cdot \boldsymbol{J})$ higher than $2j$ can be reduced to linear combinations of lower powers. This is a specific illustration of the Cayley-Hamilton theorem, applied to the monomial on the LHS of (1).

2.2 Vandermondeum

The Vandermonde matrix [23] for spin j is

$$V\,[j] = \begin{bmatrix} 1 & 2j & (2j)^2 & \cdots & (2j)^{2j} \\ 1 & 2j - 2 & (2j - 2)^2 & \cdots & (2j - 2)^{2j} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & -2j & (-2j)^2 & \cdots & (-2j)^{2j} \end{bmatrix} \ , \tag{2}$$

while its inverse [24] is given by the "Fin du monde" matrix elements

$$\left(V^{-1}\,[j] \right)_{kl} = \frac{2^{1-k}\ (-1)^{1-l}\ N_k\,(l, j)}{(2j + 1 - l)!\ (l - 1)!} \ , \tag{3}$$

where $k, l \in \{1, 2, \cdots, 2j + 1\}$, and where the numerator factors $N_k\,(l, j)$ are given by nested sums subject to some exclusions.

$$N_k\,(l, j) = (-1)^{k-2j-1} \begin{cases} 1 & \text{for} \quad k = 2j + 1 \\ \sum_{\substack{1 \le m_1 < m_2 < \cdots < m_{2j+1-k} \le 2j+1 \\ m_1, m_2, \cdots, m_{2j+1-k} \ne l}} (j + 1 - m_1) \cdots (j + 1 - m_{2j+1-k}) & \text{for} \quad 1 \le k < 2j + 1 \end{cases} \ . \tag{4}$$

These are unfamiliar polynomials [25] in l and j. For $k = 2j + 1$ the numerator is always 1, as it says above, and for $k = 2j$ it reduces to a single sum giving a result only first order in both j and l. Thus

$$N_{2j+1}\,(l, j) = 1 \ , \quad N_{2j}\,(l, j) = 1 - l + j \ . \tag{5}$$

But for other values of k these numerator polynomials quickly get out of hand. For example, for $k = 2j - 1$ the resulting double sum yields a polynomial 3rd order in j and 2nd order in l.

$$N_{2j-1}\,(l, j) = (1 - l)^2 + 2\,(1 - l)\,j + \frac{1}{6}\,(1 - j)\,(2j - 1)\,j \ . \tag{6}$$

For reasonable values of j — up to 100, say — it is easiest to just use numerical machine computation to obtain $V^{-1}\,[j]$ directly from $V\,[j]$, with no need of the analytic closed-form expressions for the $N_k\,(l, j)$.

2.3 Biorthogonal matrices

It is useful to have in hand the *dual matrices* which are trace orthonormalized with respect to powers of the spin matrix, $S \equiv 2\,\hat{\boldsymbol{n}} \cdot \boldsymbol{J}$. Without loss of generality, choose $S = 2J_3$, since any other choice for $\hat{\boldsymbol{n}}$ merely requires

selecting a different basis to diagonalize the spin matrix, thereby obtaining the same eigenvalues as $2J_3$. Thus the powers are

$$
S^m = \begin{bmatrix}
(2j)^m & 0 & \cdots & 0 & 0 \\
0 & (2j-2)^m & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & (-2j+2)^m & 0 \\
0 & 0 & \cdots & 0 & (-2j)^m
\end{bmatrix} .
\tag{7}
$$

It is now straightforward to construct orthonormalized dual matrices T_n such that

$$
\delta_{n,m} = \mathrm{Trace}\,(T_n\ S^m)\ ,\quad n, m = 0, 1, \cdots, 2j\ .
\tag{8}
$$

The T_n may also be chosen to be diagonal $(2j+1) \times (2j+1)$ matrices in the basis that diagonalizes S. In fact, for any spin j the required entries on the diagonal of T_n are just the entries in the $(n+1)$st row of the inverted Vandermonde matrix, $V^{-1}[j]$. (Note that here, unlike the conventions in [6], both rows and columns of the Vandermonde matrix and its inverse are indexed as $1, 2, \cdots, 2j+1$.) That is,

$$
(T_{n-1})_{kk} = \left(V^{-1}[j]\right)_{n,k}\ ,\quad n, k = 1, \cdots, 2j+1\ .
\tag{9}
$$

This result follows immediately from the fact that the diagonal entries for S^m are just the entries in the corresponding column (i.e. the $(m+1)$st column) of the Vandermonde matrix, $V[j]$.

Explicit examples of spin matrix powers and their duals, and the corresponding Vandermonde matrix inverse $V^{-1}[j]$, are given in Appendix C for $j = 1/2,\ 1,\ 3/2$, and 2.

2.4 Projecting coefficients

Suppose f is a nonsingular matrix function for spin j. In a basis where $S = 2J_3$, the matrix $f(S)$ is diagonal and so is its reduction to a polynomial in S:

$$
\begin{bmatrix}
f(2j) & 0 & \cdots & 0 & 0 \\
0 & f(2j-2) & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & f(-2j+2) & 0 \\
0 & 0 & \cdots & 0 & f(-2j)
\end{bmatrix} = f(S)
$$

$$
= \sum_{m=0}^{2j} f_m\, S^m = \sum_{m=0}^{2j} f_m
\begin{bmatrix}
(2j)^m & 0 & \cdots & 0 & 0 \\
0 & (2j-2)^m & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & (-2j+2)^m & 0 \\
0 & 0 & \cdots & 0 & (-2j)^m
\end{bmatrix} .
$$

Trace orthogonality of the dual matrices then gives the coefficients as

$$
f_m = \mathrm{Trace}\,(T_m\ f(S))\ .
\tag{10}
$$

Assembling the coefficients into a column and using (9) gives

$$
\begin{bmatrix}
f_0 \\
f_1 \\
\vdots \\
f_{2j-1} \\
f_{2j}
\end{bmatrix}
=
\begin{bmatrix}
\sum_{k=1}^{2j+1} \left(V^{-1}[j]\right)_{1,k} f(2j+2-2k) \\
\sum_{k=1}^{2j+1} \left(V^{-1}[j]\right)_{2,k} f(2j+2-2k) \\
\vdots \\
\sum_{k=1}^{2j+1} \left(V^{-1}[j]\right)_{2j,k} f(2j+2-2k) \\
\sum_{k=1}^{2j+1} \left(V^{-1}[j]\right)_{2j+1,k} f(2j+2-2k)
\end{bmatrix}
= V^{-1}[j]
\begin{bmatrix}
f(2j) \\
f(2j-2) \\
\vdots \\
f(2-2j) \\
f(-2j)
\end{bmatrix} .
\tag{11}
$$

So the spin matrix polynomial coefficients for $f(S)$ are determined by straightforward matrix multiplication using the inverse Vandermonde matrix.

3

In terms of the explicit form in (3), for $k = 1, 2, \cdots, 2j + 1$,

$$f_{k-1} = \sum_{l=1}^{2j+1} \left(V^{-1}[j]\right)_{kl} \ f(2j+2-2l) = 2^{1-k} \sum_{l=1}^{2j+1} \frac{(-1)^{1-l} \ N_k(l,j) \ f(2j+2-2l)}{(2j+1-l)! \ (l-1)!}$$

$$= 2^{1-k} \sum_{l=1}^{2j+1} \frac{(-1)^{k-l-2j} \ f(2j+2-2l)}{(2j+1-l)! \ (l-1)!} \sum_{\substack{1 \le m_1 < \cdots < m_{2j+1-k} \le 2j+1 \\ m_1, m_2, \cdots, m_{2j+1-k} \ne l}} (j+1-m_1) \cdots (j+1-m_{2j+1-k}) \ . \quad (12)$$

The coefficients of the highest powers appearing in the spin matrix polynomial are often worked out readily from the general closed form expressions for the $N_k(l,j)$, as given in (5) and (6). For the highest three powers in the polynomial,

$$f_{2j} = 2^{-2j} \sum_{l=1}^{2j+1} \frac{(-1)^{1-l} \ f(2j+2-2l)}{(2j+1-l)! \ (l-1)!} \ , \quad (13)$$

$$f_{2j-1} = 2^{1-2j} \sum_{l=1}^{2j+1} \frac{(-1)^{1-l} \ (1-l+j) \ f(2j+2-2l)}{(2j+1-l)! \ (l-1)!} \ , \quad (14)$$

$$f_{2j-2} = 2^{2-2j} \sum_{l=1}^{2j+1} \frac{(-1)^{1-l} \ \left((1-l)^2 + 2(1-l)j + \frac{1}{6}(1-j)(2j-1)j\right) \ f(2j+2-2l)}{(2j+1-l)! \ (l-1)!} \ , \quad (15)$$

etc. But again, for reasonable values of j it is probably easiest to just use numerical machine computation to first obtain $V^{-1}[j]$ directly from $V[j]$, and then perform the matrix multiplication in (11).

Indeed, by taking several low values of j and their explicit $V^{-1}[j]$, Van Kortryk [8] was able to deduce the general formula for the coefficients in the Cayley transform polynomial, which he then confirmed for specific higher values of j. I provide a proof of the formula in Section 4 of this paper.

2.5   Lagrange-Sylvester expansions

I note in passing another useful method, albeit equivalent to that involving the inverse Vandermonde matrix.

Reduction of nonsingular matrix functions for spin j to polynomials can be efficiently carried out in specific cases using Lagrange-Sylvester expansions [26]. Consider functions of an $N \times N$ diagonalizable matrix $\mathbb{M}$ with non-degenerate eigenvalues λ_i, $i = 1, \cdots, N$. On the span of the eigenvectors, there is an obviously correct Lagrange formula, as extended to diagonalizable matrices by Sylvester,

$$f(\mathbb{M}) = \sum_{i=1}^{N} f(\lambda_i) \, \mathbb{P}_i \ , \quad (16)$$

where the projection operators — the so-called Frobenius covariants — are given by products,

$$\mathbb{P}_i = \prod_{\substack{j=1 \\ j \ne i}}^{N} \frac{\mathbb{M} - \lambda_j}{\lambda_i - \lambda_j} \ . \quad (17)$$

From expanding each such product it is evident that any $f(\mathbb{M})$ reduces to a polynomial of order $N-1$ in powers of $\mathbb{M}$,

$$f(\mathbb{M}) = \sum_{m=0}^{N-1} C_m[f] \ \mathbb{M}^m \ , \quad (18)$$

and the function-dependent coefficients can be expressed in terms of the eigenvalues of $\mathbb{M}$ by expanding the projection operators (17) as polynomials in $\mathbb{M}$.

CFZ used the methods described above to obtain the coefficients for the spin matrix polynomial reduction of the exponential form for rotations for several low values of j. They then re-expressed the results in a form that surprisingly suggested an immediate generalization, after which they developed the theory to confirm their ansatz, thereby raising it to the level of a theorem with explicit proofs [7]. I review and illustrate the CFZ formula in the next section.

3 Rotations as exponentials

For comparison to the results of the next section, we first recall the CFZ formula for a rotation through an angle θ about an axis $\hat{\boldsymbol{n}}$, valid for any spin $j \in \left\{ 0, \frac{1}{2}, 1, \frac{3}{2}, \cdots \right\}$. The formula reduces the *manifestly* nonsingular exponential function of any spin matrix to an explicit polynomial:

$$\exp\left(i\,\theta\,\hat{\boldsymbol{n}} \cdot \boldsymbol{J}\right) = \sum_{k=0}^{2j} \frac{1}{k!} A_k^{[j]}\left(\theta\right) \left(2i\,\hat{\boldsymbol{n}} \cdot \boldsymbol{J}\right)^k \ , \tag{19}$$

where the angle-dependent coefficients of the various spin matrix powers are given by

$$A_k^{[j]}\left(\theta\right) = \sin^k\left(\theta/2\right)\,\left(\cos\left(\theta/2\right)\right)^{\epsilon(j,k)} \underset{\lfloor j-k/2\rfloor}{\text{Trunc}} \left[\frac{1}{(\sqrt{1-x})^{\epsilon(j,k)}} \left(\frac{\arcsin\sqrt{x}}{\sqrt{x}} \right)^k \right]_{x=\sin^2(\theta/2)} . \tag{20}$$

Here, $\lfloor \cdots \rfloor$ is the integer-valued floor function and $\underset{n}{\text{Trunc}}\left[f\left(x\right)\right]$ is the nth-order Taylor polynomial truncation for any $f\left(x\right)$ admitting a power series representation:

$$f\left(x\right) = \sum_{m=0}^{\infty} f_m x^m \ , \quad \underset{n}{\text{Trunc}}\left[f\left(x\right)\right] \equiv \sum_{m=0}^{n} f_m x^m \ . \tag{21}$$

In addition, $\epsilon\left(j,k\right)$ is a binary-valued function of $2j - k$ that distinguishes even and odd integers: $\epsilon\left(j,k\right) = 0$ for even $2j - k$, and $\epsilon\left(j,k\right) = 1$ for odd $2j - k$.

The simplest two nontrivial cases of (19) are well-known: $j = 1/2$ and $j = 1$. Explicitly,

$$\begin{aligned} \exp\left(i\,\theta\,\hat{\boldsymbol{n}} \cdot \boldsymbol{J}\right)|_{j=1/2} &= A_0^{[1/2]}\left(\theta\right)\,\boldsymbol{I}_{2\times 2} + 2i A_1^{[1/2]}\left(\theta\right)\,\left(\hat{\boldsymbol{n}} \cdot \boldsymbol{J}\right)_{2\times 2} \\ &= \cos\left(\theta/2\right)\,\boldsymbol{I}_{2\times 2} + 2i\sin\left(\theta/2\right)\,\left(\hat{\boldsymbol{n}} \cdot \boldsymbol{J}\right)_{2\times 2} \ , \end{aligned} \tag{22}$$

$$\begin{aligned} \exp\left(i\,\theta\,\hat{\boldsymbol{n}} \cdot \boldsymbol{J}\right)|_{j=1} &= A_0^{[1]}\left(\theta\right)\,\boldsymbol{I}_{3\times 3} + 2i A_1^{[1]}\left(\theta\right)\,\left(\hat{\boldsymbol{n}} \cdot \boldsymbol{J}\right)_{3\times 3} - 2A_2^{[1]}\left(\theta\right)\,\left(\hat{\boldsymbol{n}} \cdot \boldsymbol{J}\right)_{3\times 3}^2 \\ &= \boldsymbol{I}_{3\times 3} + i\sin\left(\theta\right)\left(\hat{\boldsymbol{n}} \cdot \boldsymbol{J}\right)_{3\times 3} + \left(\cos\theta - 1\right)\left(\hat{\boldsymbol{n}} \cdot \boldsymbol{J}\right)_{3\times 3}^2 \ . \end{aligned} \tag{23}$$

The former involves the Pauli matrices, $\boldsymbol{J} = \boldsymbol{\sigma}/2$, while the latter is sometimes known as the Euler-Rodrigues formula. Several other explicit cases may be found in [17] and [6].

In practice, for finite j of reasonable size, the truncations needed to evaluate (20) are easily obtained as a matter of course by machine computation, for example by using either $Maple^{\circledR}$ or $Mathematica^{\circledR}$. Nevertheless, it is interesting and useful for a systematic analysis that Taylor series for powers of cyclometric functions can be expressed in terms of $t\left(m,n\right)$, the so-called *central factorial numbers of the first kind* [12, 13]. Thus for $|z| \leq 1$ and non-negative integer n (cf. Theorem (4.1.2) in [13]),

$$\left(\arcsin\left(z\right)\right)^n = \frac{n!}{2^n} \sum_{m=n}^{\infty} \frac{|t\left(m,n\right)|}{m!} \left(2z\right)^m \ . \tag{24}$$

Note that the coefficients in these Taylor series are all non-negative. In general, the values of $t\left(m,n\right)$ are defined by and obtained from simple polynomials, as described in Appendix A.

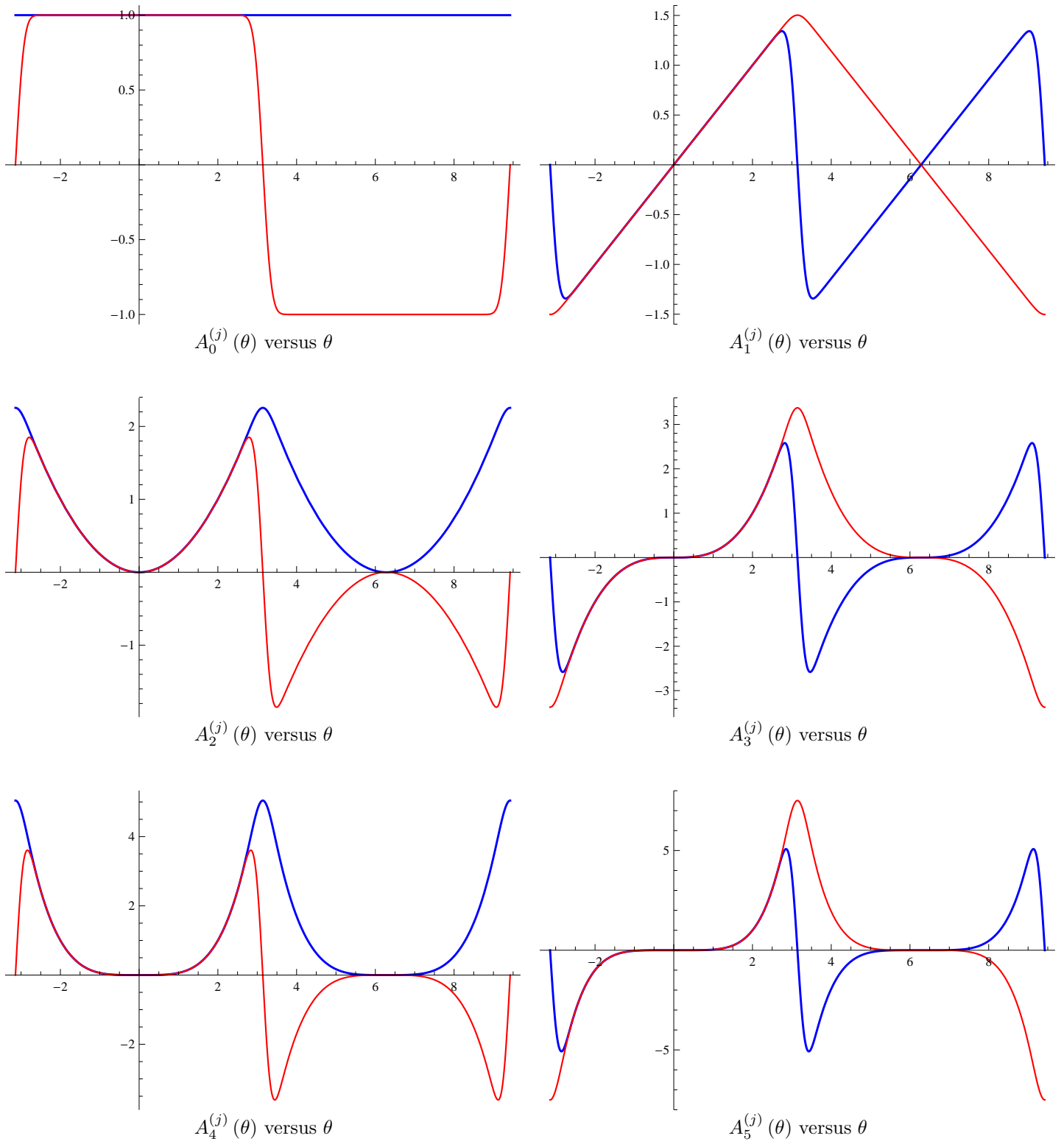Incorporating (24) into the expression for the coefficients (20) gives

$$A_k^{[j]}\left(\theta\right) = \frac{k!}{2^k} \sum_{m=k}^{2j} \frac{2^m}{m!} |t\left(m,k\right)| \sin^m\left(\theta/2\right) \quad \text{for even } 2j - k \ . \tag{25}$$

As firmly established in [17, 6], the remaining coefficients in (19) may then be obtained from

$$A_{k-1}^{[j]}\left(\theta\right) = \frac{2}{k} \frac{d}{d\theta}\,A_k^{[j]}\left(\theta\right) \quad \text{for odd } 2j - k + 1 \ . \tag{26}$$

Also, as observed in [6], the results (20) display the limit $j \to \infty$ for fixed k in a beautifully intuitive way. In that limit, the truncation is lifted to obtain trigonometrical series for the *periodicized* θ^k monomials. But even as $j \to \infty$, integer j (bosonic) and semi-integer j (fermionic) coefficients are clearly distinguished by a relative

sign flip for $\theta \in [\pi, 3\pi] \bmod (4\pi)$. This is evident upon plotting the first few coefficients for very large spins. For example, $A_{0,1,\cdots,5}^{[j]}(\theta)$ are plotted here for $j = 69$ (darker curves, in blue) and $j = 137/2$ (lighter curves, in red).

$A_0^{(j)}(\theta)$ versus θ

$A_1^{(j)}(\theta)$ versus θ

$A_2^{(j)}(\theta)$ versus θ

$A_3^{(j)}(\theta)$ versus θ

$A_4^{(j)}(\theta)$ versus θ

$A_5^{(j)}(\theta)$ versus θ

As stated in (26), the slope of each red (fermionic) curve plotted on the right is given exactly by the red curve plotted to its immediate left. Similarly, the slope of any blue (bosonic) curve on the left is given by the blue curve to its right, but in the row above.

4 Rotations as Cayley transforms

The Cayley rational form of a unitary $SU(2)$ group element is also nonsingular for any irreducible representation with spin j. Therefore it can also be reduced to a spin matrix polynomial of order $2j$. Thus

$$\frac{1 + 2i\alpha\ \hat{\boldsymbol{n}} \cdot \boldsymbol{J}}{1 - 2i\alpha\ \hat{\boldsymbol{n}} \cdot \boldsymbol{J}} = \sum_{k=0}^{2j} \mathfrak{A}_k^{[j]}(\alpha)\,(2i\ \hat{\boldsymbol{n}} \cdot \boldsymbol{J})^k\ . \tag{27}$$

Here α is a *real* parameter, to avoid singularities for *any* $j \in \left\{0, \frac{1}{2}, 1, \frac{3}{2}, 2, \cdots\right\}$. The coefficients $\mathfrak{A}_k^{[j]}(\alpha)$ are to be determined as functions of α. These coefficients can be obtained by rewriting the geometric series $1/(1 - 2i\alpha\ \hat{\boldsymbol{n}} \cdot \boldsymbol{J})$ for spin j as a polynomial in $\hat{\boldsymbol{n}} \cdot \boldsymbol{J}$. Thus define the auxiliary polynomial

$$\frac{1}{1 - 2i\alpha\ \hat{\boldsymbol{n}} \cdot \boldsymbol{J}} = \sum_{k=0}^{2j} \mathfrak{B}_k^{[j]}(\alpha)\,(2i\ \hat{\boldsymbol{n}} \cdot \boldsymbol{J})^k\ . \tag{28}$$

Then clearly

$$\mathfrak{A}_0^{[j]} = 2\mathfrak{B}_0^{[j]} - 1\ , \quad \mathfrak{A}_{k \geq 1}^{[j]} = 2\mathfrak{B}_{k \geq 1}^{[j]}\ . \tag{29}$$

The simplest two nontrivial cases are again $j = 1/2$ and $j = 1$. Explicitly,

$$\left.\frac{1}{1 - 2i\alpha\ \hat{\boldsymbol{n}} \cdot \boldsymbol{J}}\right|_{j=1/2} = \mathfrak{B}_0^{[1/2]}(\alpha)\,\boldsymbol{I}_{2\times 2} + 2i\,\mathfrak{B}_1^{[1/2]}(\alpha)\,(\hat{\boldsymbol{n}} \cdot \boldsymbol{J})_{2\times 2}$$

$$= \frac{1}{1 + \alpha^2}\left(\boldsymbol{I}_{2\times 2} + 2i\alpha\ (\hat{\boldsymbol{n}} \cdot \boldsymbol{J})_{2\times 2}\right)\ , \tag{30}$$

$$\left.\frac{1 + 2i\alpha\ \hat{\boldsymbol{n}} \cdot \boldsymbol{J}}{1 - 2i\alpha\ \hat{\boldsymbol{n}} \cdot \boldsymbol{J}}\right|_{j=1/2} = \mathfrak{A}_0^{[1/2]}(\alpha)\,\boldsymbol{I}_{2\times 2} + 2i\,\mathfrak{A}_1^{[1/2]}(\alpha)\,(\hat{\boldsymbol{n}} \cdot \boldsymbol{J})_{2\times 2}$$

$$= \frac{1}{1 + \alpha^2}\left(\left(1 - \alpha^2\right)\boldsymbol{I}_{2\times 2} + 4i\alpha\ (\hat{\boldsymbol{n}} \cdot \boldsymbol{J})_{2\times 2}\right)\ , \tag{31}$$

$$\left.\frac{1}{1 - 2i\alpha\ \hat{\boldsymbol{n}} \cdot \boldsymbol{J}}\right|_{j=1} = \mathfrak{B}_0^{[1]}(\alpha)\,\boldsymbol{I}_{3\times 3} + 2i\,\mathfrak{B}_1^{[1]}(\alpha)\,(\hat{\boldsymbol{n}} \cdot \boldsymbol{J})_{3\times 3} - 4\,\mathfrak{B}_2^{[1]}(\alpha)\,(\hat{\boldsymbol{n}} \cdot \boldsymbol{J})_{3\times 3}^2$$

$$= \frac{1}{1 + 4\alpha^2}\left(\left(1 + 4\alpha^2\right)\boldsymbol{I}_{3\times 3} + 2i\alpha\ (\hat{\boldsymbol{n}} \cdot \boldsymbol{J})_{3\times 3} - 4\alpha^2\,(\hat{\boldsymbol{n}} \cdot \boldsymbol{J})_{3\times 3}^2\right)\ , \tag{32}$$

$$\left.\frac{1 + 2i\alpha\ \hat{\boldsymbol{n}} \cdot \boldsymbol{J}}{1 - 2i\alpha\ \hat{\boldsymbol{n}} \cdot \boldsymbol{J}}\right|_{j=1} = \mathfrak{A}_0^{[1]}(\alpha)\,\boldsymbol{I}_{3\times 3} + 2i\,\mathfrak{A}_1^{[1]}(\alpha)\,(\hat{\boldsymbol{n}} \cdot \boldsymbol{J})_{3\times 3} - 4\,\mathfrak{A}_2^{[1]}(\alpha)\,(\hat{\boldsymbol{n}} \cdot \boldsymbol{J})_{3\times 3}^2$$

$$= \frac{1}{1 + 4\alpha^2}\left(\left(1 + 4\alpha^2\right)\boldsymbol{I}_{3\times 3} + 4i\alpha\ (\hat{\boldsymbol{n}} \cdot \boldsymbol{J})_{3\times 3} - 8\alpha^2\,(\hat{\boldsymbol{n}} \cdot \boldsymbol{J})_{3\times 3}^2\right)\ . \tag{33}$$

Yet again, the first two of these polynomials involve the Pauli matrices, $\boldsymbol{J} = \boldsymbol{\sigma}/2$, while the last is related to the Cayley transform of the Euler-Rodrigues formula. Appendix D repeats these two examples and also gives the Cayley transforms for $j = 3/2$, 2, 5/2, and 3.

The coefficients in the geometric series expansion for arbitrary integer or semi-integer j follow directly from the methods in [6, 7]. The result is [8]

$$\mathfrak{B}_k^{[j]}(\alpha) = \frac{\alpha^k}{\det(1 - 2i\alpha\ \hat{\boldsymbol{n}} \cdot \boldsymbol{J})}\ \underset{2j-k}{\mathrm{Trunc}}\left[\det(1 - 2i\alpha\ \hat{\boldsymbol{n}} \cdot \boldsymbol{J})\right]\ , \tag{34}$$

where the truncation is in powers of α, and where the determinant for spin j is

$$\det(1 - 2i\alpha\ \hat{\boldsymbol{n}} \cdot \boldsymbol{J}) = \prod_{m=0}^{2j}(1 - 2i\alpha(j - m)) = \prod_{n=1}^{\lfloor j+1/2 \rfloor}\left(1 + 4\alpha^2(j + 1 - n)^2\right)\ . \tag{35}$$

These results are readily checked for small values of j upon using explicit matrices, say $\hat{\boldsymbol{n}} \cdot \boldsymbol{J} = J_3$. I provide a detailed proof of (34) in the following subsection.

But first, note for integer spins it is always true that $\mathfrak{B}_0^{[j]}(\alpha) = 1$, and that the higher coefficients are paired,

$$\mathfrak{B}_{2k+2}^{[j]}(\alpha) = \alpha\, \mathfrak{B}_{2k+1}^{[j]}(\alpha) \quad \text{for} \quad 0 \le k \le j-1 \quad \text{and integer } j , \tag{36}$$

while *all* the coefficients are paired for semi-integer spins. In this case

$$\mathfrak{B}_{2k+1}^{[j]}(\alpha) = \alpha\, \mathfrak{B}_{2k}^{[j]}(\alpha) \quad \text{for} \quad 0 \le k \le j - \frac{1}{2} \quad \text{and semi-integer } j . \tag{37}$$

So, the coefficients in the spin matrix polynomials for bosonic (integer j) and fermionic (semi-integer j) Cayley transforms are easily distinguished, both quantitatively and qualitatively. The coefficients are paired as indicated because, for either bosonic or fermionic spins, only *even* powers of α are produced by the determinant factors in (34). Hence the parities: $\mathfrak{A}_k^{[j]}(-\alpha) = (-1)^k \mathfrak{A}_k^{[j]}(\alpha)$ and $\mathfrak{B}_k^{[j]}(-\alpha) = (-1)^k \mathfrak{B}_k^{[j]}(\alpha)$.

Moreover, for any allowed j all the coefficients of α^{2k} in (35) are positive. As a consequence, for any allowed j the $\mathfrak{A}_k^{[j]}(\alpha)$ and $\mathfrak{B}_k^{[j]}(\alpha)$ coefficients have no singularities for real α. In particular, for any j the highest two coefficients reduce to

$$\frac{1}{\alpha^{2j}}\, \mathfrak{B}_{2j}^{[j]}(\alpha) = \frac{1}{\alpha^{2j-1}}\, \mathfrak{B}_{2j-1}^{[j]}(\alpha) = \frac{1}{\det\left(1 - 2i\alpha\, \hat{\boldsymbol{n}} \cdot \boldsymbol{J}\right)} . \tag{38}$$

And as j increases, the domain where $1/\det\left(1 - 2i\alpha\, \hat{\boldsymbol{n}} \cdot \boldsymbol{J}\right)$ achieves significant values collapses towards the origin, at which point it always has unit value. Here is a plot for $j = 1/2,\ 1,\ 3/2,\ 2,\ 5/2,\ \&\ 3$.
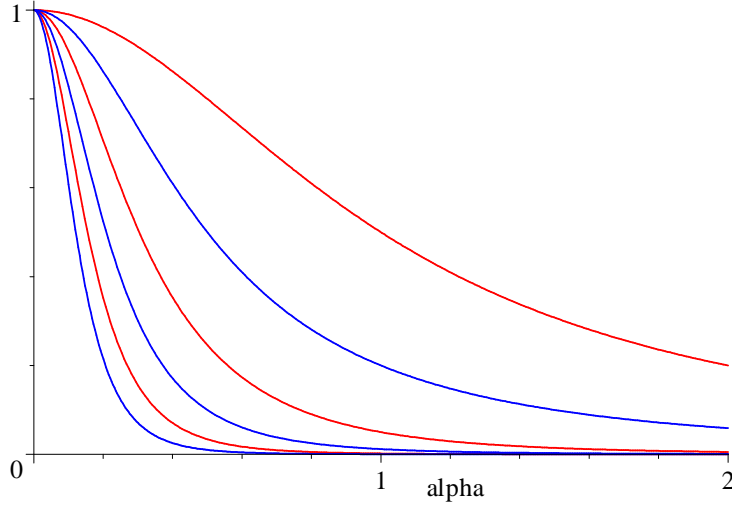
$1/\det\left(1 - 2i\alpha\, \hat{\boldsymbol{n}} \cdot \boldsymbol{J}\right)$ plotted for $j = \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}$, and 3, for upper to lower curves, respectively.

Two additional comments are warranted [8]. First, the determinants in (34) are essentially generating functions of the central factorial numbers $t(m,n)$ (see Appendix A), a fact previously exploited in [6, 7] and already emphasized for the Cayley transforms in [8]. For either integer or semi-integer j,

$$\operatorname*{Trunc}_{2j-k}\left[\det\left(1 - 2i\alpha\, \hat{\boldsymbol{n}} \cdot \boldsymbol{J}\right)\right] = \sum_{m=0}^{\lfloor j-k/2 \rfloor} 4^m \alpha^{2m}\ |t(2j+2, 2j+2-2m)| , \tag{39}$$

with the full determinant obtained for $k = 0$ if j is integer. For semi-integer j, the same expression is true for truncations, only now the full determinant includes an additional term $4^{j+\frac{1}{2}}\alpha^{2j+1}\ |t(2j+2,1)|$. Thus the general formula for the determinant, for either integer or semi-integer j, is

$$\det\left(1 - 2i\alpha\, \hat{\boldsymbol{n}} \cdot \boldsymbol{J}\right) = \sum_{k=0}^{\lfloor j+\frac{1}{2} \rfloor} 4^k \alpha^{2k}\ |t(2j+2, 2j+2-2k)| . \tag{40}$$

Second, as $j \to \infty$ for any fixed k the truncation in (34) is lifted — but with some subtleties to be discussed below — to obtain for small α, $\lim_{j\to\infty} \mathfrak{B}_k^{[j]}(\alpha) \sim \alpha^k$. But in contrast to the periodicized θ-monomials found in [6], the large j behavior here does not make the periodicity of rotations manifest. To exhibit periodicity even for finite values of j, θ must be expressed, on a case-by-case basis, as cyclometric functions of α, and then the branch structure of those cyclometric functions must be invoked [8].

4.1 A derivation for general matrix resolvents

In this subsection I obtain the coefficients in the Cayley transform spin matrix polynomial by deriving the expansion coefficients for the resolvent of a general matrix. This is a well-known result in resolvent theory (e.g. see [20]) and has been established many times in the literature (e.g. see [21, 22]). While there is hardly any need to plow again a field that has been tilled as much as this one, I include here a variant of the standard proof just to make the discussion self-contained. (Also see Appendix D.)

For any finite $N \times N$ matrix M consider the resolvent written as a matrix polynomial,

$$\frac{1}{1 - \alpha M} = \sum_{m=0}^{N-1} r_m(\alpha) \ M^m . \tag{41}$$

This is much simpler than the exponential case [6, 7] in that one *need not* differentiate, e.g. to find

$$(1 - \alpha M) \frac{d}{d\alpha} \left(\frac{1}{1 - \alpha M} \right) = \frac{M}{1 - \alpha M} , \tag{42}$$

so as to obtain an expression linear in M multiplying the original polynomial, and subsequently a set of first-order equations for the coefficients r_m. Rather, it suffices here just to multiply through by $1 - \alpha M$. Thus

$$1 = (1 - \alpha M) \frac{1}{1 - \alpha M} = (1 - \alpha M) \sum_{m=0}^{N-1} M^m r_m . \tag{43}$$

On the other hand, the Cayley-Hamilton theorem gives

$$\sum_{m=0}^{N} M^m d_m = 0 , \quad M^N = -\frac{1}{d_N} \sum_{m=0}^{N-1} M^m d_m , \tag{44}$$

where the polynomial coefficients d_n are defined by the determinant

$$\det(1 - \alpha M) = \sum_{m=0}^{N} \alpha^m d_m , \tag{45}$$

with $d_0 = 1$, $d_N = (-1)^N \det M$, etc. (see [27]). Therefore (43) becomes

$$1 = r_0 + \frac{\alpha r_{N-1}}{d_N} d_0 + \sum_{m=1}^{N-1} M^m \left(r_m - \alpha r_{m-1} + \frac{\alpha r_{N-1}}{d_N} d_m \right) . \tag{46}$$

If the powers of M^m for $0 \leq m \leq N-1$ are all linearly independent, as is the case for the spin matrices $\hat{\boldsymbol{n}} \cdot \boldsymbol{J}$, then

$$r_0 + \frac{\alpha r_{N-1}}{d_N} d_0 = 1 \quad \text{and} \quad r_m - \alpha r_{m-1} + \frac{\alpha r_{N-1}}{d_N} d_m = 0 \quad \text{for } 1 \leq m \leq N-1 . \tag{47}$$

It is now straightforward to show the solution to these recursion relations is given by

$$r_n = \frac{\alpha^n}{\det(1 - \alpha M)} \ \underset{N-1-n}{\text{Trunc}} \left[\det(1 - \alpha M) \right] . \tag{48}$$

For $M = 2i\hat{\boldsymbol{n}} \cdot \boldsymbol{J}$ with $N = 2j+1$, the r_n are essentially just renamed $\mathfrak{B}_n^{[j]}$. Thus the result (34) is proven.

4.2 Additional exact results and asymptotic behavior for large j

The determinants can also be written as Pochhammer symbols: $x^{(n)} \equiv x(x+1)\cdots(x+n-1) = \frac{\Gamma(x+n)}{\Gamma(x)}$. For example, for integer j,

$$\prod_{m=0}^{j} (1 - 2i\alpha(j-m)) = \prod_{n=1}^{j} (1 - 2i\alpha n) = (-2i\alpha)^{j+1} \frac{\Gamma\left(j+1-\frac{1}{2i\alpha}\right)}{\Gamma\left(-\frac{1}{2i\alpha}\right)} , \tag{49}$$

$$\prod_{m=j+1}^{2j} (1 - 2i\alpha(j-m)) = \prod_{n=1}^{j} (1 + 2i\alpha n) = (2i\alpha)^{j+1} \frac{\Gamma\left(j+1+\frac{1}{2i\alpha}\right)}{\Gamma\left(\frac{1}{2i\alpha}\right)} , \tag{50}$$

which leads to a nice form for the determinant expressed in terms of analytic functions of both j and α. Again, for integer j,

$$\det\left(1 - 2i\alpha\ \hat{\boldsymbol{n}} \cdot \boldsymbol{J}\right) = \left(4\alpha^2\right)^{j+1} \frac{\Gamma\left(j + 1 - \frac{1}{2i\alpha}\right)}{\Gamma\left(-\frac{1}{2i\alpha}\right)} \frac{\Gamma\left(j + 1 + \frac{1}{2i\alpha}\right)}{\Gamma\left(\frac{1}{2i\alpha}\right)}$$

$$= \frac{1}{\pi}\left(2\alpha\right)^{2j+1}\sinh\left(\frac{\pi}{2\alpha}\right) \times \left|\Gamma\left(j + 1 + \frac{1}{2i\alpha}\right)\right|^2 \ , \tag{51}$$

upon using $\Gamma\left(\frac{1}{2i\alpha}\right)\Gamma\left(-\frac{1}{2i\alpha}\right) = 2\pi\alpha/\sinh\frac{\pi}{2\alpha}$. On the other hand, for semi-integer j,

$$\det\left(1 - 2i\alpha\ \hat{\boldsymbol{n}} \cdot \boldsymbol{J}\right) = \frac{1}{\pi}\left(2\alpha\right)^{2j+1}\cosh\left(\frac{\pi}{2\alpha}\right)\left|\Gamma\left(j + 1 + \frac{1}{2i\alpha}\right)\right|^2 \ . \tag{52}$$

In the limit $j \to \infty$, these are the well-known infinite product representations of sinh and cosh. That is to say,

$$\lim_{j\to\infty} \frac{\det\left(1 - 2i\alpha\ \hat{\boldsymbol{n}} \cdot \boldsymbol{J}\right)}{\left(4\alpha^2\right)^{\lfloor j + \frac{1}{2}\rfloor}\left(\Gamma\left(1 + j\right)\right)^2} = \begin{cases} \frac{2\alpha}{\pi}\sinh\left(\frac{\pi}{2\alpha}\right) & \text{for integer } j \\[2mm] \frac{1}{\pi}\cosh\left(\frac{\pi}{2\alpha}\right) & \text{for semi-integer } j \end{cases} \ . \tag{53}$$

This shows that there is a distinguishable difference between Cayley transforms for bosonic and fermionic spins, even as $j \to \infty$, much as there were distinguishable differences in the behavior of the coefficients for the exponential (19), also as $j \to \infty$, as illustrated in the Figures of Section 3.

Now consider the truncations for $k = 1$ or 2, say for integer spin. This choice of coefficients removes from $\det\left(1 - 2i\alpha\ \hat{\boldsymbol{n}} \cdot \boldsymbol{J}\right)$ the highest power of α, namely the α^{2j} term. But this term is easily computed:

$$\prod_{n=1}^{j}\left(4\alpha^2\left(j + 1 - n\right)^2\right) = \prod_{k=1}^{j}\left(4\alpha^2 k^2\right) = 4^j\alpha^{2j}\left(j!\right)^2 = 4^j\alpha^{2j}\left|\Gamma\left(j + 1\right)\right|^2 \ . \tag{54}$$

Therefore for integer j

$$\underset{2j-1}{\mathrm{Trunc}}\left[\det\left(1 - 2i\alpha\ \hat{\boldsymbol{n}} \cdot \boldsymbol{J}\right)\right] = \underset{2j-2}{\mathrm{Trunc}}\left[\det\left(1 - 2i\alpha\ \hat{\boldsymbol{n}} \cdot \boldsymbol{J}\right)\right] = \det\left(1 - 2i\alpha\ \hat{\boldsymbol{n}} \cdot \boldsymbol{J}\right) - 4^j\alpha^{2j}\left(j!\right)^2$$

$$= 4^j\alpha^{2j}\left(\frac{2\alpha}{\pi}\sinh\left(\frac{\pi}{2\alpha}\right) \times \left|\Gamma\left(j + 1 + \frac{1}{2i\alpha}\right)\right|^2 - \left|\Gamma\left(j + 1\right)\right|^2\right) \ . \tag{55}$$

The corresponding coefficients in the Cayley rational form are

$$\mathfrak{B}_1^{[j]}\left(\alpha\right) = \frac{\alpha}{\det\left(1 - 2i\alpha\ \hat{\boldsymbol{n}} \cdot \boldsymbol{J}\right)}\left(\det\left(1 - 2i\alpha\ \hat{\boldsymbol{n}} \cdot \boldsymbol{J}\right) - 4^j\alpha^{2j}\left(j!\right)^2\right)$$

$$= \alpha\left(1 - \frac{4^j\alpha^{2j}\left|\Gamma\left(j + 1\right)\right|^2}{\frac{1}{\pi}\left(2\alpha\right)^{2j+1}\sinh\left(\frac{\pi}{2\alpha}\right) \times \left|\Gamma\left(j + 1 + \frac{1}{2i\alpha}\right)\right|^2}\right) \ . \tag{56}$$

Similarly for $\mathfrak{B}_2^{[j]}\left(\alpha\right) = \alpha\ \mathfrak{B}_1^{[j]}\left(\alpha\right)$. The exact results for these two coefficients, for all integer spins, are then given by

$$\frac{1}{\alpha}\mathfrak{B}_1^{[j]}\left(\alpha\right) = \frac{1}{\alpha^2}\mathfrak{B}_2^{[j]}\left(\alpha\right) = 1 - \frac{\pi\ \left|\Gamma\left(j + 1\right)\right|^2}{2\alpha\sinh\left(\frac{\pi}{2\alpha}\right) \times \left|\Gamma\left(j + 1 + \frac{1}{2i\alpha}\right)\right|^2} \ . \tag{57}$$

For fixed α, the large j limits of these follow immediately from

$$\lim_{j\to\infty} \frac{\left|\Gamma\left(j + 1\right)\right|^2}{\left|\Gamma\left(j + 1 + \frac{1}{2i\alpha}\right)\right|^2} = 1 \ . \tag{58}$$

In fact, the limit is just 1 to all orders in $\frac{1}{\alpha}$. This should be obvious, but in case it is not, as a check the reader may consider the following expansion:

$$\frac{\left(\Gamma\left(j + 1\right)\right)^2}{\Gamma\left(j + 1 + z\right)\Gamma\left(j + 1 - z\right)} = 1 - \mathrm{Psi}\left(1, j + 1\right)z^2 + \frac{1}{12}\left(6\left(\mathrm{Psi}\left(1, j + 1\right)\right)^2 - \mathrm{Psi}\left(3, j + 1\right)\right)z^4 \tag{59}$$

$$- \frac{1}{360}\left(\mathrm{Psi}\left(5, j + 1\right) - 30\,\mathrm{Psi}\left(3, j + 1\right)\mathrm{Psi}\left(1, j + 1\right) + 60\left(\mathrm{Psi}\left(1, j + 1\right)\right)^3\right)z^6$$

$$+ \frac{1}{20\,160}\left(\begin{array}{c} 70\left(\mathrm{Psi}\left(3, j + 1\right)\right)^2 + 840\left(\mathrm{Psi}\left(1, j + 1\right)\right)^4 - \mathrm{Psi}\left(7, j + 1\right) \\ +56\,\mathrm{Psi}\left(5, j + 1\right)\mathrm{Psi}\left(1, j + 1\right) - 840\,\mathrm{Psi}\left(3, j + 1\right)\left(\mathrm{Psi}\left(1, j + 1\right)\right)^2 \end{array}\right)z^8$$

$$+ O\left(z^{10}\right) \ ,$$

10

where $\mathrm{Psi}\,(n,z)$ is the nth derivative of the digamma function $\mathrm{Psi}\,(z)$ (also known as PolyGamma in *Mathematica*®). Except for the $O\left(z^0\right)$ term, all of the $O\left(z^k\right)$ coefficients in this last expansion vanish as $j \to \infty$.

Therefore it follows for integer j that

$$\frac{1}{\alpha}\,\mathfrak{B}_1^{[j]}(\alpha) = \frac{1}{\alpha^2}\,\mathfrak{B}_2^{[j]}(\alpha) \xrightarrow[j\to\infty]{} 1 - \frac{\pi}{2\alpha \sinh\left(\frac{\pi}{2\alpha}\right)}\;. \tag{60}$$

So, as α approaches zero, the RHS approaches unity faster than any power of α due to the essential singularity in the exponential,

$$\frac{\pi}{2\alpha \sinh\left(\frac{\pi}{2\alpha}\right)} \underset{\alpha\to 0}{\sim} \frac{\pi}{|\alpha|}\, e^{-\frac{\pi}{2|\alpha|}}\;. \tag{61}$$

This is the precise meaning of the statement in [8] that

$$\lim_{j\to\infty} \mathfrak{B}_{1,2}^{[j]}(\alpha) \underset{\alpha\to 0}{\sim} \alpha^{1,2}\;. \tag{62}$$

In fact, the asymptotic form is rapidly approached as j is increased.

A similar calculation, again for integer j, gives

$$\frac{1}{\alpha^3}\,\mathfrak{B}_3^{[j]}(\alpha) = \frac{1}{\alpha^4}\,\mathfrak{B}_4^{[j]}(\alpha) = 1 - \frac{\pi\,|\Gamma(j+1)|^2}{2\alpha \sinh\left(\frac{\pi}{2\alpha}\right) \times \left|\Gamma\left(j+1+\frac{1}{2i\alpha}\right)\right|^2}\left(1 + \frac{1}{24\alpha^2}\left(\pi^2 - 6\,\mathrm{Psi}\,(1,1+j)\right)\right) \tag{63}$$

$$\xrightarrow[j\to\infty]{} 1 - \frac{\pi}{2\alpha \sinh\left(\frac{\pi}{2\alpha}\right)}\left(1 + \frac{\pi^2}{24\alpha^2}\right)\;. \tag{64}$$

For the case at hand,

$$\mathrm{Psi}\,(1,1+j) = \zeta(2) - \sum_{k=1}^{j} \frac{1}{k^2}\;. \tag{65}$$

Here are some other facts that were used to get exact expressions as well as asymptotic behaviors for the $\mathfrak{B}_{1-4}^{[j]}(\alpha)$ coefficients:

$$\prod_{l=0}^{j}\left(z+l^2\right) = \frac{\Gamma\left(j+1-\sqrt{-z}\right)\Gamma\left(j+1+\sqrt{-z}\right)}{\Gamma\left(-\sqrt{-z}\right)\Gamma\left(\sqrt{-z}\right)}\;, \tag{66}$$

$$|t(2j+2,2)| = \left.\frac{d}{dz}\prod_{l=0}^{j}\left(z+l^2\right)\right|_{z=0} = (\Gamma(j+1))^2\;, \tag{67}$$

$$|t(2j+2,4)| = \left.\frac{1}{2}\frac{d^2}{dz^2}\prod_{l=0}^{j}\left(z+l^2\right)\right|_{z=0} = \frac{1}{6}(\Gamma(j+1))^2\,\pi^2 - (\Gamma(j+1))^2\,\mathrm{Psi}\,(1,1+j)\;, \tag{68}$$

$$\frac{\mathrm{Trunc}_{2j-1}\left[\det(1-2ix\,\hat{\boldsymbol{n}}\cdot\boldsymbol{J})\right]}{\det(1-2ix\,\hat{\boldsymbol{n}}\cdot\boldsymbol{J})} = \frac{\mathrm{Trunc}_{2j-2}\left[\det(1-2ix\,\hat{\boldsymbol{n}}\cdot\boldsymbol{J})\right]}{\det(1-2ix\,\hat{\boldsymbol{n}}\cdot\boldsymbol{J})} = 1 - \frac{4^j x^{2j}\,(\Gamma(j+1))^2}{\det(1-2ix\,\hat{\boldsymbol{n}}\cdot\boldsymbol{J})}$$

$$= 1 - \frac{(\Gamma(j+1))^2}{\frac{2x}{\pi}\left(\sinh\frac{\pi}{2x}\right)\left|\Gamma\left(j+1+\frac{i}{2x}\right)\right|^2}\;, \tag{69}$$
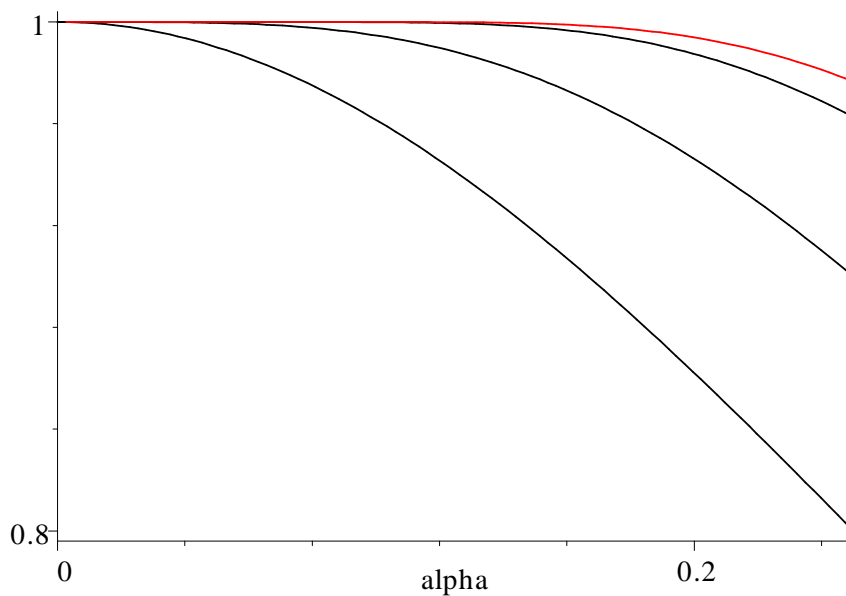
$$\frac{\mathrm{Trunc}_{2j-3}\left[\det(1-2ix\,\hat{\boldsymbol{n}}\cdot\boldsymbol{J})\right]}{\det(1-2ix\,\hat{\boldsymbol{n}}\cdot\boldsymbol{J})} = \frac{\mathrm{Trunc}_{2j-4}\left[\det(1-2ix\,\hat{\boldsymbol{n}}\cdot\boldsymbol{J})\right]}{\det(1-2ix\,\hat{\boldsymbol{n}}\cdot\boldsymbol{J})} = 1 - \frac{4^j x^{2j}\,(\Gamma(j+1))^2}{\det(1-2ix\,\hat{\boldsymbol{n}}\cdot\boldsymbol{J})}$$

$$- \frac{4^{j-1}x^{2j-2}\left(\frac{1}{6}(\Gamma(j+1))^2\,\pi^2 - (\Gamma(j+1))^2\,\mathrm{Psi}\,(1,1+j)\right)}{\det(1-2ix\,\hat{\boldsymbol{n}}\cdot\boldsymbol{J})}$$

$$= 1 - \frac{\pi\,|\Gamma(j+1)|^2}{2\alpha \sinh\left(\frac{\pi}{2\alpha}\right) \times \left|\Gamma\left(j+1+\frac{1}{2i\alpha}\right)\right|^2}\left(1 + \frac{1}{24\alpha^2}\left(\pi^2 - 6\,\mathrm{Psi}\,(1,1+j)\right)\right)\;. \tag{70}$$

11

I plot a few examples to illustrate these features, beginning with $\mathfrak{B}_1^{[j]}(\alpha)$ and $\mathfrak{B}_2^{[j]}(\alpha)$.
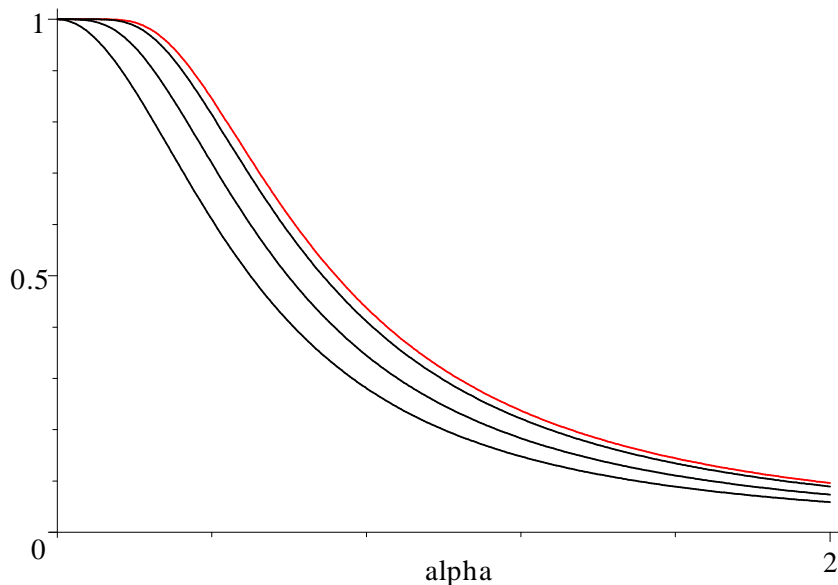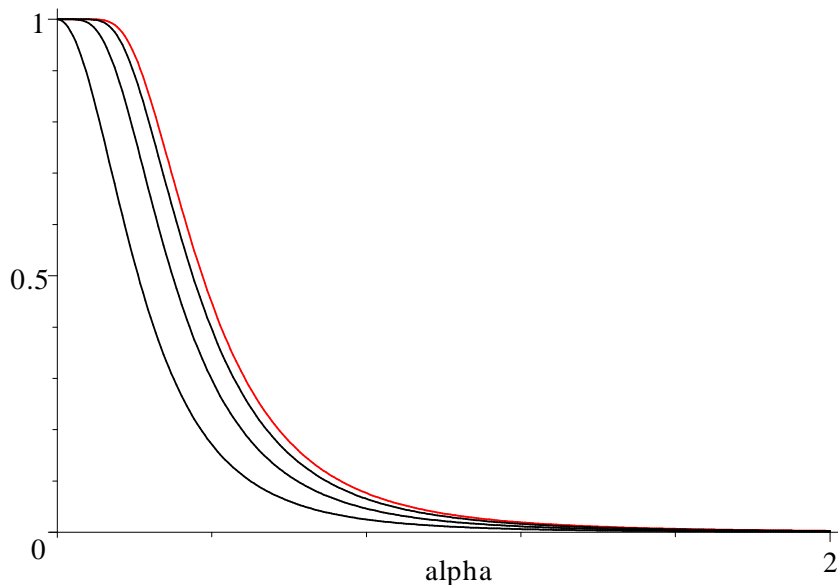

$\mathfrak{B}_1^{[j]}(\alpha)/\alpha = \mathfrak{B}_2^{[j]}(\alpha)/\alpha^2$ plotted for $j = 1, 2,$ & 8 (lower to upper black curves) and the $j \to \infty$ limit (uppermost curve, in red).

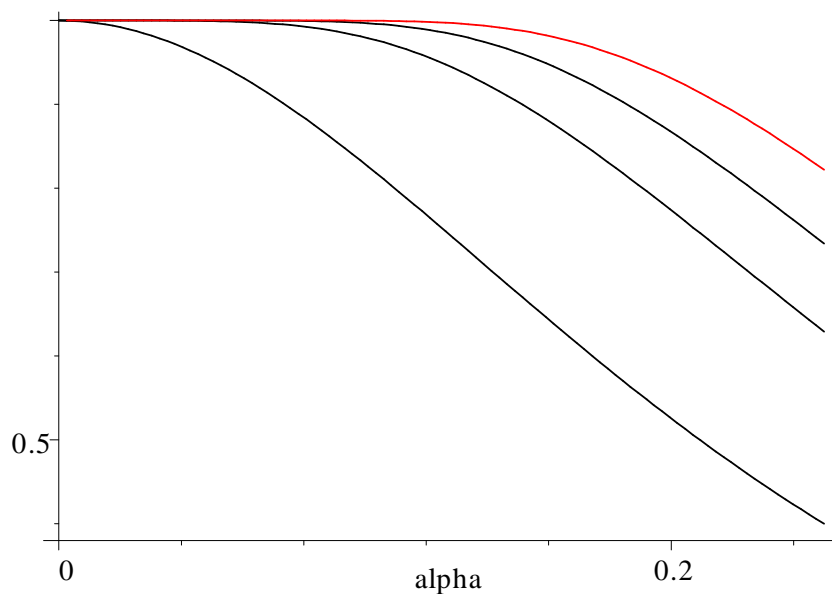A closer look near $\alpha = 0$ is provided by the next graph.

$\mathfrak{B}_1^{[j]}(\alpha)/\alpha = \mathfrak{B}_2^{[j]}(\alpha)/\alpha^2$ for $j = 1, 2,$ & 8 (lower to upper black curves) and the $j \to \infty$ limit (uppermost curve, in red).

I also plot the corresponding results for $\mathfrak{B}_3^{[j]}(\alpha)$ and $\mathfrak{B}_4^{[j]}(\alpha)$.

$\mathfrak{B}_3^{[j]}(\alpha)/\alpha^3 = \mathfrak{B}_4^{[j]}(\alpha)/\alpha^4$ for $j = 2, 4,$ & 12 (lower to upper black curves) and the $j \to \infty$ limit (uppermost curve, in red).

Again, a closer look near $\alpha = 0$ is given in the next graph.

$\mathfrak{B}_3^{[j]}(\alpha)/\alpha^3 = \mathfrak{B}_4^{[j]}(\alpha)/\alpha^4$ for $j = 2, 4,$ & 12 (lower to upper black curves) and the $j \to \infty$ limit (uppermost curve, in red).

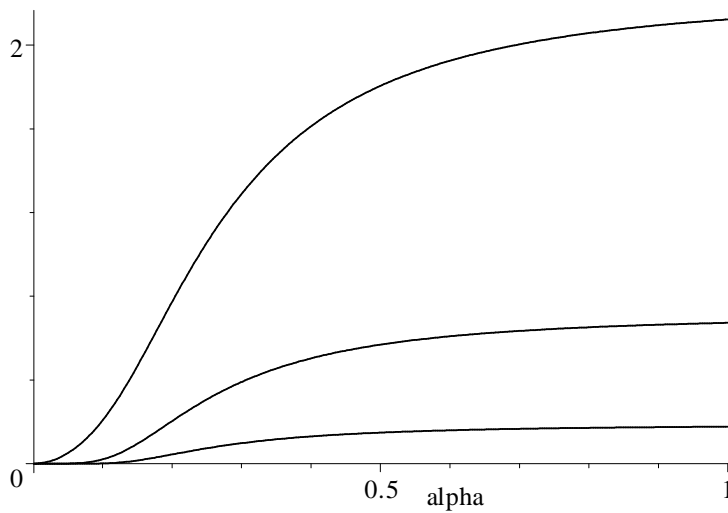Similar behavior is exhibited by the higher coefficients.

These plots should be compared to the large j behavior of the coefficients for the exponential, given in (20). As the Figures in Section 3 attest, the asymptotic $j \to \infty$ behavior is not closely approached by the $A_k^{[j]}(\theta)$ for fixed k until rather large values of j are considered, and for some θ not even then. This is especially significant for θ near $\pm \pi$, where $\lim_{j \to \infty} A_k^{[j]}(\theta)$ has a discontinuity in θ for every other value of k. By contrast, for fixed k as $j \to \infty$ the $\mathfrak{A}_k^{[j]}(\alpha)$ converge uniformly and monotonically to $\mathfrak{A}_k^{[\infty]}(\alpha)$ on any finite interval in α, and indeed, the large j limit of the coefficients is closely approached for moderate values of j. These last points are clearly displayed in plots of the relative error between $\mathfrak{A}_k^{[j]}(\alpha)$ and $\mathfrak{A}_k^{[\infty]}(\alpha)$ for a few values of j. Further numerical studies suggest that, for any fixed k, the relative error

$$\Delta_k^{[j]}(\alpha) \equiv \frac{\mathfrak{A}_k^{[\infty]}(\alpha) - \mathfrak{A}_k^{[j]}(\alpha)}{\mathfrak{A}_k^{[j]}(\alpha)} \tag{71}$$

is always ≥ 0 and goes monotonically to zero as $j \to \infty$, for all α.

Here are some examples.
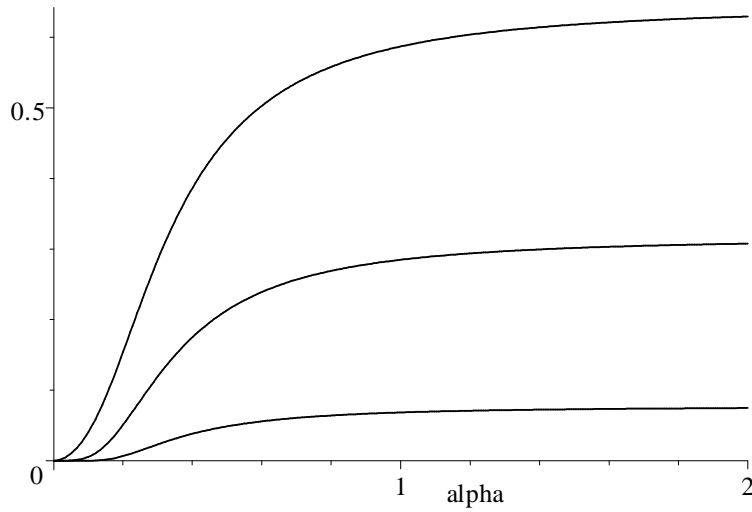
The relative error $\Delta_1^{[j]}(\alpha) = \Delta_2^{[j]}(\alpha)$ plotted for $j = 1, 2,$ & 8, as upper to lower curves, respectively.

The relative error $\Delta_3^{[j]}(\alpha) = \Delta_4^{[j]}(\alpha)$ plotted for $j = 1, 2,$ & 12, as upper to lower curves, respectively.

Again, similar behavior is exhibited by the higher coefficients.

Large j conjectures and proofs for all the coefficients

Here are the first four requisite asymptotic $j \to \infty$ results, for integer j, as obtained by explicit calculation:

$$\mathrm{asymp}_1(\alpha) = 1 - \frac{1}{\frac{2\alpha}{\pi} \sinh \frac{\pi}{2\alpha}} , \tag{72}$$

$$\mathrm{asymp}_2(\alpha) = 1 - \frac{1}{\frac{2\alpha}{\pi} \sinh \frac{\pi}{2\alpha}} \left(1 + \frac{1}{3!}\left(\frac{\pi}{2\alpha}\right)^2\right) , \tag{73}$$

$$\mathrm{asymp}_3(\alpha) = 1 - \frac{1}{\frac{2\alpha}{\pi} \sinh \frac{\pi}{2\alpha}} \left(1 + \frac{1}{3!}\left(\frac{\pi}{2\alpha}\right)^2 + \frac{1}{5!}\left(\frac{\pi}{2\alpha}\right)^4\right) , \tag{74}$$

$$\mathrm{asymp}_4(\alpha) = 1 - \frac{1}{\frac{2\alpha}{\pi} \sinh \frac{\pi}{2\alpha}} \left(1 + \frac{1}{3!}\left(\frac{\pi}{2\alpha}\right)^2 + \frac{1}{5!}\left(\frac{\pi}{2\alpha}\right)^4 + \frac{1}{7!}\left(\frac{\pi}{2\alpha}\right)^6\right) . \tag{75}$$

(Semi-integer j are different. See below.) Note that the numerators are just truncations of the series for the denominator, $\frac{2\alpha}{\pi} \sinh \frac{\pi}{2\alpha} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!}\left(\frac{\pi}{2\alpha}\right)^{2n}$. Hence the obvious conjecture for integer j is

$$\mathrm{asymp}_k(\alpha) = 1 - \frac{1}{\frac{2\alpha}{\pi} \sinh \frac{\pi}{2\alpha}} \left(\sum_{n=0}^{k-1} \frac{1}{(2n+1)!}\left(\frac{\pi}{2\alpha}\right)^{2n}\right) = \frac{\left(\frac{\pi}{2\alpha}\right)^{1+2k}}{(1+2k)!\sinh \frac{\pi}{2\alpha}} \mathrm{hypergeom}\left([1], \left[1+k, \frac{3}{2}+k\right], \frac{\pi^2}{16\alpha^2}\right) . \tag{76}$$

Given this result, the coefficients of the spin matrix powers in the reduction (28) are, as $j \to \infty$ *for integer j*,

$$\lim_{j\to\infty} \frac{1}{\alpha^{2k-1}} \mathfrak{B}_{2k-1}^{[j]}(\alpha) = \lim_{j\to\infty} \frac{1}{\alpha^{2k}} \mathfrak{B}_{2k}^{[j]}(\alpha) = 1 - \frac{1}{\frac{2\alpha}{\pi} \sinh \frac{\pi}{2\alpha}} \left(\sum_{n=0}^{k-1} \frac{1}{(2n+1)!}\left(\frac{\pi}{2\alpha}\right)^{2n}\right) , \quad \text{for } k \geq 1 . \tag{77}$$

As an immediate consequence, $\lim_{k\to\infty} \mathrm{asymp}_k(\alpha) = 0$ for all $\alpha \neq 0$, with the limit approached monotonically from above, while at $\alpha = 0$, $\mathrm{asymp}_k(0) = 1$ for all k.

Also, to put it briefly, explicit calculation of the lowest few coefficients reveals the corresponding conjecture for semi-integer j: The asymptotic expressions may be obtained from the bosonic spin case by substituting $\frac{2\alpha}{\pi} \sinh \frac{\pi}{2\alpha} \Longrightarrow \cosh \frac{\pi}{2\alpha}$. Thus for the fermionic case, as $j \to \infty$ *for semi-integer j*,

$$\lim_{j\to\infty} \frac{1}{\alpha^{2k}} \mathfrak{B}_{2k}^{[j]}(\alpha) = \lim_{j\to\infty} \frac{1}{\alpha^{2k+1}} \mathfrak{B}_{2k+1}^{[j]}(\alpha) = 1 - \frac{1}{\cosh \frac{\pi}{2\alpha}} \left(\sum_{n=0}^{k} \frac{1}{(2n)!}\left(\frac{\pi}{2\alpha}\right)^{2n}\right) , \quad \text{for } k \geq 0 . \tag{78}$$

While plots of the various functions of α in (77) and (78) are qualitatively similar, nevertheless, the bosonic and fermionic Cayley transforms are still easily distinguished, even as $j \to \infty$.

Proofs of these conjectures are straightforward using (34), (39), (53), and Proposition 2.7 in [13]. For example, for integer j, use the result (39) to rewrite (34) for the even-indexed coefficients as

$$1 - \frac{1}{\alpha^{2k}} \mathfrak{B}_{2k}^{[j]}(\alpha) = \frac{1}{\det(1 - 2i\alpha\, \hat{\boldsymbol{n}} \cdot \boldsymbol{J})} \sum_{l=1}^{k} (2\alpha)^{2(j+1-l)} \ |t(2j+2, 2l)| . \tag{79}$$

The limit $j \to \infty$ now follows from the integer j result in (53), written as

$$\lim_{j\to\infty} \left(\frac{(2\alpha)^{2j} (\Gamma(1+j))^2}{\det(1 - 2i\alpha\, \hat{\boldsymbol{n}} \cdot \boldsymbol{J})}\right) = \frac{\pi}{2\alpha \sinh\left(\frac{\pi}{2\alpha}\right)} , \tag{80}$$

and from the asymptotic form of $|t(2j+2, 2m)|$ as given in [13], Proposition 2.7 (xxvi), written as

$$\lim_{j\to\infty} \frac{(2\alpha)^{2(1-l)}}{(\Gamma(1+j))^2} \ |t(2j+2, 2l)| = \frac{1}{(2l-1)!}\left(\frac{\pi}{2\alpha}\right)^{2(l-1)} . \tag{81}$$

The net result is an expression whose numerator is a truncation of the infinite series for the denominator, as conjectured above.

$$\lim_{j\to\infty} \left(1 - \frac{1}{\alpha^{2k}} \mathfrak{B}_{2k}^{[j]}(\alpha)\right) = \frac{1}{\frac{2\alpha}{\pi} \sinh \frac{\pi}{2\alpha}} \left(\sum_{n=0}^{k-1} \frac{1}{(2n+1)!}\left(\frac{\pi}{2\alpha}\right)^{2n}\right) . \tag{82}$$

The odd-indexed coefficients for the bosonic spin case are given by this same expression according to the elementary pairing (36). Thus for integer j the conjectured $j \to \infty$ form of the coefficients is established.

A corresponding proof for semi-integer j follows from (53) and Proposition 2.7 (xxvii) in [13].

15

The result for semi-integer j is as conjectured in (78). Alternatively, it may be expressed as a hypergeometric function similar to that in (76).

$$1 - \frac{1}{\cosh\frac{\pi}{2\alpha}}\left(\sum_{n=0}^{k}\frac{1}{(2n)!}\left(\frac{\pi}{2\alpha}\right)^{2n}\right) = \frac{\left(\frac{\pi}{2\alpha}\right)^{2+2k}}{(2+2k)!\cosh\frac{\pi}{2\alpha}} \text{ hypergeom}\left([1],\left[\frac{3}{2}+k,2+k\right],\frac{\pi^2}{16\alpha^2}\right) . \quad (83)$$

5 Relating exponentials and rational forms by variable changes

To finish the comparison of rotations as exponentials and Cayley transforms, for arbitrary j, it is instructive to return to an issue briefly mentioned in the Introduction. Namely, why does a map from one form to the other **not** succeed through the use of a numerical-valued change of variable $\alpha(\theta)$ or its inverse $\theta(\alpha)$? After all, a comparison of the exponential and Cayley results for $j = 1/2$, as well as for $j = 1$, as given in (22), (23), (31), and (33), would give the *misleading* impression that this can succeed.

To see the difficulty, take a basis where $\hat{\boldsymbol{n}}\cdot\boldsymbol{J}$ is diagonal, with eigenvalues $M \in \{-j, -j+1, \cdots, j-1, j\}$, and equate the exponential and Cayley forms acting on a specific eigenstate $|M\rangle$. The result of course is

$$\exp(i\,\theta\,M) = \frac{1+2i\alpha\,M}{1-2i\alpha\,M} . \quad (84)$$

This is easily solved either for $\alpha(\theta)$ or for the inverse map $\theta(\alpha)$:

$$\alpha(\theta) = \frac{1}{2M}\tan\left(\frac{M\theta}{2}\right) , \quad \theta(\alpha) = \frac{2}{M}\arctan(2M\alpha) . \quad (85)$$

So the relation is M-dependent, although in this particular instance it *is in*dependent of the sign of M. In general, nontrivial M-dependence would be a feature that would arise upon comparing *any* two functions of $\hat{\boldsymbol{n}}\cdot\boldsymbol{J}$ when acting on $|M\rangle$.

Now it just so happens that a unique map works for *all* the M-eigenstates for $j = 1/2$, and another unique map works for all the M-eigenstates states for $j = 1$. (Note that for $M = 0$ the relation (84) always reduces to $1 = 1$ *no matter what* the relation between α and θ.) But this is fortuitous and peculiar to those two spins due to $|M|$ having only one nontrivial value in those special cases. For all $j \geq 3/2$ the map must change with $|M|$. There is *no consistent* $\alpha \rightleftarrows \theta$ *relation* that works on all $\hat{\boldsymbol{n}}\cdot\boldsymbol{J}$ eigenstates for $j > 1$. In a sense made precise by (85), there is a "parameter shear" in the relation between α and θ as the $\hat{\boldsymbol{n}}\cdot\boldsymbol{J}$ spectrum is traversed.

Similarly, if one were to equate the coefficients of any given power $(\hat{\boldsymbol{n}}\cdot\boldsymbol{J})^k$ in the spin matrix polynomial reductions of the exponential and the Cayley transform, the result would be an $\alpha \rightleftarrows \theta$ relation that depends on k, and this result would be *in*consistent (except in the two special cases $j = 1/2$ and $j = 1$) with that obtained from equating the coefficients of some other power of $\hat{\boldsymbol{n}}\cdot\boldsymbol{J}$.

6 Relating exponentials and rational forms by Laplace transforms

Some standard results from resolvent theory provide a direct relation between (19) and (28). Consider the Laplace transform and its inverse,

$$F(s) = \mathcal{L}[f] = \int_0^\infty e^{-st}f(t)\,dt , \quad f(t) = \mathcal{L}^{-1}[F] = \frac{1}{2\pi i}\lim_{T\to\infty}\int_{\gamma-iT}^{\gamma+iT}e^{st}F(s)\,ds . \quad (86)$$

For any hermitian $N \times N$ matrix, a Laplace transform yields the resolvent from the exponential, and vice versa.

$$\frac{1}{s-iM} = \mathcal{L}\left[e^{itM}\right] , \quad e^{itM} = \mathcal{L}^{-1}\left[\frac{1}{s-iM}\right] . \quad (87)$$

This furnishes another route to derive the CFZ formula starting from the known matrix polynomial expression for the resolvent. Given the ease by which the polynomial coefficients for the resolvent are obtained, this provides perhaps the simplest proof of the CFZ result.

For linearly independent powers M^k, $0 \leq k < N$, as is the case for spin matrices, the matrix polynomial expansion coefficients for the exponential and the resolvent are distinctly related order-by-order by the Laplace transform.

$$\exp(itM) = \sum_{n=0}^{N-1}A_n(t)(iM)^n , \quad \frac{1}{s-iM} = \sum_{n=0}^{N-1}B_n(s)(iM)^n , \quad B_n(s) = \mathcal{L}[A_n] , \quad A_n(t) = \mathcal{L}^{-1}[B_n] . \quad (88)$$

16

If the first k powers of M are not independent, for $k < N - 1$, correct relations between the exponential and the resolvent are still obtained by taking $\mathcal{L}$ or $\mathcal{L}^{-1}$, as in (87) and (88), although the final results may admit further simplification. Fortunately, it is not necessary to worry about that situation in the following.

Rescaling variables produces a form of the Laplace transform that is more useful for the direct comparison of (20) and (34). Thus

$$\frac{1}{1 - 2i\ \alpha\ \hat{\boldsymbol{n}} \cdot \boldsymbol{J}} = \int_0^\infty e^{-t} \exp\left(2i\ \alpha t\ \hat{\boldsymbol{n}} \cdot \boldsymbol{J}\right)\ dt\ , \tag{89}$$

$$\mathfrak{B}_k^{[j]}(\alpha) = \frac{1}{k!} \int_0^\infty e^{-t}\ A_k^{[j]}(2\alpha t)\ dt\ . \tag{90}$$

It suffices here to consider only the direct transform. The inverse relation is automatic, given the large α behavior of the $\mathfrak{B}_k^{[j]}(\alpha)$. From (25), along with (26), the required calculation is

$$I_m^{[j]}(\alpha) = \frac{1}{m!} \int_0^\infty e^{-t} \sin^m(\alpha t)\, dt \quad \text{for even } 2j - k\ . \tag{91}$$

These are elementary integrals, of course, resulting in

$$I_m^{[j]}(\alpha) = \alpha^m \prod_{l=1}^{m/2} \frac{1}{1 + 4l^2\alpha^2} \quad \text{for even } m\ , \quad I_m^{[j]}(\alpha) = \alpha^m \prod_{l=1}^{(m+1)/2} \frac{1}{1 + (2l-1)^2\alpha^2} \quad \text{for odd } m\ . \tag{92}$$

After combining these results with (25) and (35), and some elementary manipulations of the central factorial numbers, the transform (90) yields the expected result:

$$\mathfrak{B}_k^{[j]}(\alpha) = \frac{\alpha^k}{\det\left(1 - 2i\alpha\ \hat{\boldsymbol{n}} \cdot \boldsymbol{J}\right)} \sum_{m=0}^{\lfloor j - k/2 \rfloor} 4^m \alpha^{2m}\ |t\left(2j+2, 2j+2-2m\right)|\ . \tag{93}$$

For example, for spin $j = 1/2$, with the usual 2×2 matrices,

$$\exp\left(i\ \theta\ \hat{\boldsymbol{n}} \cdot \boldsymbol{J}\right) = \cos(\theta/2)\ \boldsymbol{I} + 2i\sin(\theta/2)\ (\hat{\boldsymbol{n}} \cdot \boldsymbol{J})\ , \tag{94}$$

$$\int_0^\infty e^{-t} \cos(\alpha t)\, dt = \frac{1}{1 + \alpha^2}\ , \quad \int_0^\infty e^{-t} \sin(\alpha t)\, dt = \frac{\alpha}{1 + \alpha^2}\ , \tag{95}$$

$$\det\left(1 - 2i\alpha\ \hat{\boldsymbol{n}} \cdot \boldsymbol{J}\right) = 1 + \alpha^2\ , \tag{96}$$

$$\frac{1}{1 - 2i\alpha\ \hat{\boldsymbol{n}} \cdot \boldsymbol{J}} = \frac{1}{1 + \alpha^2}\left(\boldsymbol{I} + 2i\alpha\ \hat{\boldsymbol{n}} \cdot \boldsymbol{J}\right)\ , \tag{97}$$

and for spin $j = 1$, with 3×3 matrices,

$$\exp\left(i\ \theta\ \hat{\boldsymbol{n}} \cdot \boldsymbol{J}\right) = \boldsymbol{I} + i\sin(\theta)\ (\hat{\boldsymbol{n}} \cdot \boldsymbol{J}) + (\cos\theta - 1)(\hat{\boldsymbol{n}} \cdot \boldsymbol{J})^2\ , \tag{98}$$

$$\int_0^\infty e^{-t} dt = 1\ , \quad \int_0^\infty e^{-t} \cos(2\alpha t)\, dt = \frac{1}{1 + 4\alpha^2}\ , \quad \int_0^\infty e^{-t} \sin(2\alpha t)\, dt = \frac{2\alpha}{1 + 4\alpha^2} \tag{99}$$

$$\det\left(1 - 2i\alpha\ \hat{\boldsymbol{n}} \cdot \boldsymbol{J}\right) = 1 + 4\alpha^2\ , \tag{100}$$

$$\frac{1}{1 - 2i\alpha\ \hat{\boldsymbol{n}} \cdot \boldsymbol{J}} = \frac{1}{1 + 4\alpha^2}\left(\left(1 + 4\alpha^2\right)I + 2i\alpha\ \hat{\boldsymbol{n}} \cdot \boldsymbol{J} - 4\alpha^2\left(\hat{\boldsymbol{n}} \cdot \boldsymbol{J}\right)^2\right)\ . \tag{101}$$

7 Concluding remarks

Polynomial reductions of spin matrix exponentials have long been recognized as important and useful in many different contexts [16]-[19], [28]-[31]. Perhaps the results presented in [6, 7, 8] and discussed further in this paper can help to facilitate these and other applications.

Moreover, Cayley transforms have numerous applications, many of a practical nature. For example, consider [32] and [33]. In the first of these two papers, previous results on attitude representations were generalized using Cayley transforms, for application to guidance and control problems, while in the second paper, the theory of time stepping methods were developed in a systematic manner based on the Cayley transform, for application to discretised differential equations that describe evolution in Lie groups. Both of these papers stress that Cayley transform methods are easier to implement than the evaluation of matrix exponentials.

This relative ease of implementation is borne out in the present paper by direct comparison of (20) and (34), and more indirectly by the relative simplicity of the derivation for the Cayley transform coefficients, as given here, when compared to the proofs for the coefficients in the exponential case, as given in [7].

References

[1] *The Collected Mathematical Papers of Arthur Cayley*, Cambridge University Press (1889). See Vol. I. pp 28-35 for the Cayley transform, and see Vol. II. pp 475-496 for the Cayley-Hamilton theorem.

[2] A Cayley "A Memoir on the Theory of Matrices" Phil.Trans. 148 (1858) 17-37. Also see http://en.wikipedia.org/wiki/Cayley-Hamilton_theorem.

[3] When I say that $f(M)$ is a *nonsingular function* of a matrix M, I mean that all matrix elements of $f(M)$ are finite in a basis where all matrix elements of M are finite.

[4] A Cayley, "On the Motion of Rotation of a Solid Body" Camb.Math.J. 3 (1843) 224-232. Also see http://en.wikipedia.org/wiki/Cayley_transform. For a recent paper which provides a brief guide to more current literature and some applications, see Y Hardy, A Fošner, and W-H Steeb, "Cayley transform and the Kronecker product of Hermitian matrices" Linear Algebra and its Applications 439 (2013) 4023–4031.

[5] Y Lehrer, "On functions of matrices" Rendiconti del Circolo Matematico di Palermo 6 (1957) 103-108.

[6] T L Curtright, D B Fairlie, and C K Zachos, "A Compact Formula for Rotations as Spin Matrix Polynomials" SIGMA 10 (2014) 084. e-Print: arXiv:1402.3541 [math-ph]

[7] T L Curtright and T S Van Kortryk, "On rotations as spin matrix polynomials" J.Phys.A: Math.Theor. 48 (2015) 025202. e-Print: arXiv:1408.0767 [math-ph]

[8] T S Van Kortryk, "Cayley transforms of $su(2)$ representations" eprint: arXiv:1506.00500 [math-ph]

[9] D C Brody, "Biorthogonal Quantum Mechanics" J.Phys.A: Math.Theor. 47 (2014) 035305.

[10] T Curtright and L Mezincescu, "Biorthogonal quantum systems" J.Math.Phys. 48 (2007) 092106.

[11] T Curtright, L Mezincescu, and D Schuster, "Supersymmetric biorthogonal quantum systems" J.Math.Phys. 48 (2007) 092108.

[12] J Riordan, *Combinatorial Identities*, Wiley (1968). (Also see *http://oeis.org/A008955* and additional references therein.)

[13] P L Butzer, M Schmidt, E L Stark, and L Vogt, "Central Factorial Numbers: Their main properties and some applications" Numerical Functional Analysis and Optimization 10 (1989) 419-488.

[14] E P Wigner, *Group Theory and its Application to the Quantum Mechanics of Atomic Spectra,* Academic Press (1959).

[15] E P Wigner, "On Matrices which Reduce Kroenecker Products of Representations of S. R. Groups" (1938), pp 608-654 in *The Collected Works of Eugene Paul Wigner* (Volume 1), B R Judd (Editor), G W Mackey (Editor), and J Mehra (Contributor), Springer-Verlag softcover edition (2013).

[16] Y Lehrer–Ilamed,. "On the direct calculations of the representations of the three-dimensional pure rotation group" Proc.Camb.Phil.Soc. 60 (1964) 61–66.

[17] R van Wageningen, "Explicit polynomial expressions for finite rotation operators" Nuc.Phys. 60 (1964) 250–263.

[18] T A Weber and S A Williams, "Spin-matrix polynomials and the rotation operator for arbitrary spin" J.Math.Phys. 6 (1965) 1980-1983.

[19] A J Torruella, "Representations of the three-dimensional rotation group by the direct method" J.Math.Phys. 16 (1975) 1637-1642.

[20] M X He and P E Ricci, "On Taylor's formula for the resolvent of a complex matrix" Computers and Mathematics with Applications 56 (2008) 2285–2288.

[21] A S Householder, *The Theory of Matrices in Numerical Analysis*, Dover (1975). See Section 6.7, and also the references cited therein, where the history of the identity is traced backed to U J J Leverrier, "Sur les variations seculaires des elements des orbites" J. Math. 5 (1840) 220-254.

[22] D K Faddeev and V N Faddeeva, *Computational Methods of Linear Algebra*, Freeman (1963). See Section 47. Also see https://de.wikipedia.org/wiki/Algorithmus_von_Faddejew-Leverrier.

[23] http://en.wikipedia.org/wiki/Vandermonde_matrix

[24] N Macon and A Spitzbart, "Inverses of Vandermonde matrices" Amer. Math. Monthly 65 (1958), 95-100. Also see https://proofwiki.org/wiki/Inverse_of_Vandermonde's_Matrix.

[25] At least they were unfamiliar to me when I first started work on spin matrix expansions. Also, they are unnamed polynomials, as far as I can tell.

[26] https://en.wikipedia.org/wiki/Sylvester's_formula

[27] T L Curtright and D B Fairlie, "A Galileon Primer" eprint: arXiv:1212.6972 [hep-th]. Especially see Section 3.1.

[28] S Weinberg, "Feynman Rules for Any Spin" Phys.Rev. 133 (1964) B1318-B1332 (especially see Appendix A)

[29] C Zemach, "Use of Angular-Momentum Tensors" Phys.Rev. 140 (1965) B97-B108 (especially see Section IV.3)

[30] S A Williams, J P Draayer, and T A Weber, "Spin-matrix polynomial development of the Hamiltonian for a free particle of arbitrary spin and mass" Phys. Rev. 152 (1966) 1207-1212.

[31] T J Nelson, "Spin-matrix polynomials and the Veneziano formula" Phys. Rev. 184 (1969) 1954-1955.

[32] P Tsiotras, J L Junkins, and H Schaub, "Higher Order Cayley Transforms with Applications to Attitude Representations" Journal of Guidance, Control, and Dynamics 20 (1997) 528-536.

[33] A Iserles, "On Cayley-transform methods for the discretization of Lie-group equations" Found.Comp.Math. 1 (2001), 129–160.

[34] Apologies to Jackie Gleason.

Appendix A: Central factorial numbers

For historical reasons, central factorial numbers are defined as the coefficients in simple polynomials [12, 13]. They can be either positive or negative, but only their absolute values are needed for the coefficients of the spin matrix expansions in the main text. Moreover, $t\,(even, even)$ are integers, but $t\,(odd, odd)$ are not integers, and $t\,(odd, even) = 0 = t\,(even, odd)$. So the even and odd cases are best handled separately.

By definition and as elementary consequences thereof (cf. *http://oeis.org/A182867*),

$$\prod_{l=0}^{m-1} \left(x^2 - l^2\right) = \sum_{k=1}^{m} t\,(2m, 2k)\, x^{2k} \,, \tag{A1}$$

$$t\,(2m, 2k) = (-1)^{m-k}\, |t\,(2m, 2k)| = (-1)^{m-k}\, \frac{1}{k!}\, \frac{d^k}{dz^k} \prod_{l=0}^{m-1} \left(z + l^2\right)\Bigg|_{z=0} \,, \tag{A2}$$

as well as (cf. *http://oeis.org/A008956*)

$$x \prod_{l=0}^{m-1} \left(x^2 - \left(l + \frac{1}{2}\right)^2\right) = \sum_{k=0}^{m} t\,(2m+1, 2k+1)\, x^{2k+1} \,, \tag{A3}$$

$$t\,(2m+1, 2k+1) = (-1)^{m-k}\, |t\,(2m+1, 2k+1)| = (-1)^{m-k}\, \frac{1}{k!}\, \frac{d^k}{dz^k} \prod_{l=0}^{m-1} \left(z + \left(l + \frac{1}{2}\right)^2\right)\Bigg|_{z=0} \,. \tag{A4}$$

Appendix B: Proof of the fundamental identity

Here I provide a demonstration that

$$(2\hat{\boldsymbol{n}} \cdot \boldsymbol{J})^{2j+1} = -\sum_{k=0}^{2j} 2^{1+2j-k} \times t\,(2+2j, 1+k) \times (2\hat{\boldsymbol{n}} \cdot \boldsymbol{J})^{k} \,, \tag{B1}$$

where $t\,(m, n)$ are the central factorial numbers, defined in Appendix A. In a basis where $2\hat{\boldsymbol{n}} \cdot \boldsymbol{J}$ is diagonal, (B1) reduces to a matrix equation,

$$\begin{bmatrix} (2j)^{2j+1} \\ (2j-2)^{2j+1} \\ \vdots \\ (-2j+2)^{2j+1} \\ (-2j)^{2j+1} \end{bmatrix} = -2^{1+2j} \times V\,[j] \begin{bmatrix} t\,(2+2j, 1) \\ \frac{1}{2}\, t\,(2+2j, 2) \\ \vdots \\ \frac{1}{2^{2j-1}}\, t\,(2+2j, 2j) \\ \frac{1}{2^{2j}}\, t\,(2+2j, 1+2j) \end{bmatrix} \,, \tag{B2}$$

where the Vandermonde matrix for spin j is defined as in (2). So, consider the kth row on the RHS of (B2):

$$-\frac{2^{1+2j}}{(j+1-k)} \times \left(\begin{array}{c} t\,(2+2j, 1)\,(j+1-k) + t\,(2+2j, 2)\,(j+1-k)^2 + t\,(2+2j, 3)\,(j+1-k)^3 + \cdots \\ +t\,(2+2j, 2j)\,(j+1-k)^{2j-1} + t\,(2+2j, 1+2j)\,(j+1-k)^{2j} \end{array} \right) \,. \tag{B3}$$

If j is semi-integer, say $j = n + 1/2$ for integer n, then $t\,(2+2j, even) = 0$, and this kth row becomes

$$-\frac{2^{1+2j}}{(j+1-k)} \times \left(\begin{array}{c} t\,(2+2j, 1)\,(j+1-k) + t\,(2+2j, 3)\,(j+1-k)^3 + \cdots \\ +t\,(2+2j, 2j-2)\,(j+1-k)^{2j-2} + t\,(2+2j, 2j)\,(j+1-k)^{2j} \\ +t\,(2+2j, 2j+2)\,(j+1-k)^{2j+2} - t\,(2+2j, 2j+2)\,(j+1-k)^{2j+2} \end{array} \right)$$

$$= -\frac{2^{1+2j}}{(j+1-k)} \times \left((j+1-k) \prod_{l=0}^{n} \left((j+1-k)^2 - \left(l + \frac{1}{2}\right)^2 \right) - t\,(2+2j, 2j+2)\,(j+1-k)^{2j+2} \right) \,, \tag{B4}$$

where the $t\,(2+2j, 2j+2)$ term was added and subtracted to obtain the complete sum on the RHS of (A3), for $m = j + 1/2 = n+1$ and $x = j+1-k$. The sum was then replaced with the product on the LHS of (A3). But the product evaluates to zero because one of the terms in the product always vanishes for $k \geq 1$, and therefore the kth row on the RHS of (B2) is

$$t\,(2+2j, 2+2j) \times (2j+2-2k)^{2j+1} = (2j+2-2k)^{2j+1} \,, \tag{B5}$$

since $t\,(2+2j, 2j+2) = 1$. Thus the kth row on the LHS of (B2) is obtained, and the identity is established for semi-integer j. If j is integer, a similar proof goes through upon using (A1). (See [7], Appendix B.)

Appendix C: Biorthogonal matrix examples

Here are more details about the biorthogonal systems of spin matrices described in the main text, for $j = 1/2$, 1, 3/2, and 2.

Spin $j = 1/2$ is deceptively simple. The independent powers of the spin matrix are

$$ S^0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad S^1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \tag{C1} $$

and the corresponding trace-orthonormal dual matrices are the same up to a normalization factor. (NB This is *not* true for any other j.)

$$ T_0 = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad T_1 = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \text{i.e.} \quad V^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}. \tag{C2} $$

Spin $j = 1$ is a more interesting example. The independent powers of the spin matrix are given by

$$ S^0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad S^1 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \quad S^2 = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \end{bmatrix}, \tag{C3} $$

and the corresponding trace-orthonormal dual matrices, as well as V^{-1}, are given by

$$ T_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad T_1 = \frac{1}{4} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad T_3 = \frac{1}{8} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad V^{-1} = \frac{1}{8} \begin{bmatrix} 0 & 8 & 0 \\ 2 & 0 & -2 \\ 1 & -2 & 1 \end{bmatrix}. \tag{C4} $$

Spin $j = 3/2$ is also interesting. The independent powers of the spin matrix are given by

$$ S^0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad S^1 = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}, \quad S^2 = \begin{bmatrix} 9 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 9 \end{bmatrix}, \quad S^3 = \begin{bmatrix} 27 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -27 \end{bmatrix}, $$

$$ \tag{C5} $$

and the corresponding trace-orthonormal dual matrices, as well as V^{-1}, are given by

$$ T_0 = \frac{1}{16} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 9 & 0 & 0 \\ 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad T_1 = \frac{1}{48} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 27 & 0 & 0 \\ 0 & 0 & -27 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad T_2 = \frac{1}{16} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{C6} $$

$$ T_3 = \frac{1}{48} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad V^{-1} = \frac{1}{48} \begin{bmatrix} -3 & 27 & 27 & -3 \\ -1 & 27 & -27 & 1 \\ 3 & -3 & -3 & 3 \\ 1 & -3 & 3 & -1 \end{bmatrix}. \tag{C7} $$

Spin $j = 2$ helps to establish the general pattern.

$$ S^m = \begin{bmatrix} 4^m & 0 & 0 & 0 & 0 \\ 0 & 2^m & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & (-2)^m & 0 \\ 0 & 0 & 0 & 0 & (-4)^m \end{bmatrix}, \quad V^{-1} = \frac{1}{384} \begin{bmatrix} 0 & 0 & 384 & 0 & 0 \\ -16 & 128 & 0 & -128 & 16 \\ -4 & 64 & -120 & 64 & -4 \\ 4 & -8 & 0 & 8 & -4 \\ 1 & -4 & 6 & -4 & 1 \end{bmatrix}. \tag{C8} $$

From the 1st through 5th rows of V^{-1} one extracts, respectively, the diagonal dual matrices T_0 through T_4.

Appendix D: A derivation using differential equations

In this Appendix I derive the coefficients in the Cayley transform spin matrix polynomial by solving differential equations. This parallels the first derivation of the CFZ results, as given in [7]. Although the resolvent analysis for general matrices, as given in the main text, is much simpler, perhaps the discussion here has some pedagogical value in that it emphasizes how much more effort is needed to pursue the same differential equation approach as was used for the exponential case.

For a given spin j let $M = 2i\hat{\boldsymbol{n}} \cdot \boldsymbol{J}$ and write

$$\frac{1}{1 - \alpha M} = \sum_{m=0}^{2j} b_m(\alpha) \ M^m \ . \tag{D1}$$

Differentiate with respect to α to obtain

$$(1 - \alpha M) \frac{d}{d\alpha}\left(\frac{1}{1 - \alpha M}\right) = \frac{M}{1 - \alpha M} \ . \tag{D2}$$

That is to say,

$$(1 - \alpha M) \sum_{m=0}^{2j} M^m \frac{db_m}{d\alpha} = \sum_{m=0}^{2j} M^{m+1} b_m \ , \tag{D3}$$

or upon rearranging the terms,

$$\sum_{m=0}^{2j} M^m \frac{db_m}{d\alpha} - \sum_{m=0}^{2j-1} M^{m+1} \frac{d}{d\alpha}(\alpha b_m) = M^{2j+1} \frac{d}{d\alpha}(\alpha b_{2j}) \ . \tag{D4}$$

Now apply the fundamental identity (1) to reexpress the RHS of (D4) to obtain

$$\frac{db_0}{d\alpha} + \sum_{m=1}^{2j} M^m \frac{db_m}{d\alpha} - \sum_{m=1}^{2j} M^m \frac{d}{d\alpha}(\alpha b_{m-1}) = - \sum_{m=0}^{2j} M^m (2i)^{2j+1-m} \, t(2+2j, 1+m) \ \frac{d}{d\alpha}(\alpha b_{2j}) \ . \tag{D5}$$

Trace projections using the dual matrices (i.e. the linear independence of the M^m for $0 \le m \le 2j$) gives the set of 1st order equations

$$\frac{db_0}{d\alpha} = -(2i)^{2j+1} \, t(2+2j, 1) \frac{d}{d\alpha}(\alpha b_{2j}) \ , \tag{D6}$$

$$\frac{db_m}{d\alpha} - \frac{d}{d\alpha}(\alpha b_{m-1}) = -(2i)^{2j+1-m} \, t(2+2j, 1+m) \frac{d}{d\alpha}(\alpha b_{2j}) \quad \text{for } m \ge 1 \ . \tag{D7}$$

These are immediately integrated, with the constants of integration fixed by the requirement that all the b_n behave properly as $\alpha \to 0$, namely, $b_0(\alpha = 0) = 1$ and $b_m(\alpha = 0) = 0$ for $m \ge 1$. The only additional ingredients needed to establish

$$b_n = \frac{\alpha^n}{\det(1 - 2i\alpha \ \hat{\boldsymbol{n}} \cdot \boldsymbol{J})} \ \mathop{\text{Trunc}}_{2j-n}\left[\det(1 - 2i\alpha \ \hat{\boldsymbol{n}} \cdot \boldsymbol{J})\right] \tag{D8}$$

are $t(even, odd) = 0 = t(odd, even)$ [12, 13] and the relations (39) and (40). The details are spelled out more fully in the following few paragraphs.

For *integer spin* (D6) and (D7) become

$$\frac{db_0}{d\alpha} = 0 \ , \tag{D9}$$

$$\frac{db_{2k}}{d\alpha} = \frac{d}{d\alpha}(\alpha b_{2k-1}) \quad \text{for even } m = 2k \ , \tag{D10}$$

$$\frac{db_{2k+1}}{d\alpha} = \frac{d}{d\alpha}(\alpha b_{2k}) - 4^{j-k} |t(2+2j, 2+2k)| \frac{d}{d\alpha}(\alpha b_{2j}) \quad \text{for odd } m = 2k+1 \ , \tag{D11}$$

while for *semi-integer spin* the equations become

$$\frac{db_0}{d\alpha} = -4^{j+\frac{1}{2}} |t(2+2j, 1)| \frac{d}{d\alpha}(\alpha b_{2j}) \ , \tag{D12}$$

$$\frac{db_{2k}}{d\alpha} = \frac{d}{d\alpha}(\alpha b_{2k-1}) - 4^{j+\frac{1}{2}-k} |t(2+2j, 1+2k)| \frac{d}{d\alpha}(\alpha b_{2j}) \quad \text{for even } m = 2k \ , \tag{D13}$$

$$\frac{db_{2k+1}}{d\alpha} = \frac{d}{d\alpha}(\alpha b_{2k}) \quad \text{for odd } m = 2k+1 \ . \tag{D14}$$

Here I have incorporated the phases of the central factorial numbers (see Appendix A) to write

$$(-1)^{j-k} t(2+2j, 2+2k) = |t(2+2j, 2+2k)| \quad \text{for integer } j , \tag{D15}$$

$$(i)^{2j+1-2k} t(2+2j, 1+2k) = |t(2+2j, 1+2k)| \quad \text{for semi-integer } j . \tag{D16}$$

Upon integration of (D10) and (D14), with the $\alpha = 0$ initial conditions, it follows that

$$b_{2k}(\alpha) = \alpha b_{2k-1}(\alpha) \quad \text{for } 1 \le k \le j \text{ and integer } j , \tag{D17}$$

$$b_{2k+1}(\alpha) = \alpha b_{2k}(\alpha) \quad \text{for } 0 \le k \le j - \frac{1}{2} \text{ and semi-integer } j . \tag{D18}$$

Moreover, for integer j

$$b_0(\alpha) = 1 \tag{D19}$$

solves (D9) with the correct value at $\alpha = 0$.

Equations (D11) and (D13) also integrate easily to give two difference equations.

$$b_{2k+1} - \alpha b_{2k} = -4^{j-k} |t(2+2j, 2+2k)| \, \alpha b_{2j} \quad \text{for integer } j , \tag{D20}$$

$$b_{2k} - \alpha b_{2k-1} = -4^{j+\frac{1}{2}-k} |t(2+2j, 1+2k)| \, \alpha b_{2j} \quad \text{for semi-integer } j . \tag{D21}$$

Note the second of these is obtained from the first by $k \to k - \frac{1}{2}$. The first of these, along with (D17), may be solved by upward recursion starting from $k = 0$, using (D19), with consistency fixing $b_{2j} = b_{2j-1}$. The result for integer j is

$$b_m = \frac{\alpha^m}{\sum_{m=0}^{j} 4^m \alpha^{2m} \, |t(2j+2, 2j+2-2m)|} \sum_{k=0}^{\lfloor j-m/2 \rfloor} 4^k \alpha^{2k} \, |t(2j+2, 2j+2-2k)| . \tag{D22}$$

Note that $b_0 = 1$ when j is an integer, and that $d^m b_m / d\alpha^m |_{\alpha=0} = m!$ for all m, a property that is clear from the original rational form (D1).

Similarly, (D21) may be solved by recursion to obtain for semi-integer j,

$$b_m = \frac{\alpha^m}{\sum_{m=0}^{\lfloor j+\frac{1}{2} \rfloor} 4^m \alpha^{2m} \, |t(2j+2, 2j+2-2m)|} \sum_{k=0}^{\lfloor j-m/2 \rfloor} 4^k \alpha^{2k} \, |t(2j+2, 2j+2-2k)| . \tag{D23}$$

Again note that $d^m b_m / d\alpha^m |_{\alpha=0} = m!$ for all m. But in this case $b_0 \ne 1$ since the $O(\alpha^{2j+1})$ term in the denominator is never present in the numerator. Nevertheless, as previously announced [8], in view of (39) and (40) it follows that the *same final determinantal form* holds for both integer and semi-integer j, namely (D8). Technically [34], how sweet it is!

Alternatively, one may simply substitute (D22) into (D20), and (D23) into (D21), to verify that those difference equations are both satisfied, along with (D17) and (D18), with the proper behavior at $\alpha = 0$.

Since for a given j the b_n are just renamed $\mathfrak{B}_n^{[j]}$, the result (34) is proven.

Appendix E: Examples of Cayley transforms for $j = \frac{1}{2},\ 1,\ \frac{3}{2},\ 2,\ \frac{5}{2},\ \text{and } 3$

$$\det\left(1 - 2i\alpha\ \hat{\boldsymbol{n}} \cdot \boldsymbol{J}\right)\big|_{j=1/2} = 1 + \alpha^2 \tag{E1}$$

$$\left.\frac{1 + 2i\alpha\ \hat{\boldsymbol{n}} \cdot \boldsymbol{J}}{1 - 2i\alpha\ \hat{\boldsymbol{n}} \cdot \boldsymbol{J}}\right|_{j=1/2} = \frac{1}{1 + \alpha^2}\left(\left(1 - \alpha^2\right)\boldsymbol{I}_{2\times 2} + 4i\alpha\left(\hat{\boldsymbol{n}} \cdot \boldsymbol{J}\right)_{2\times 2}\right) \tag{E2}$$

$$\det\left(1 - 2i\alpha\ \hat{\boldsymbol{n}} \cdot \boldsymbol{J}\right)\big|_{j=1} = 1 + 4\alpha^2 \tag{E3}$$

$$\left.\frac{1 + 2i\alpha\ \hat{\boldsymbol{n}} \cdot \boldsymbol{J}}{1 - 2i\alpha\ \hat{\boldsymbol{n}} \cdot \boldsymbol{J}}\right|_{j=1} = \boldsymbol{I}_{3\times 3} + \frac{4i\alpha}{1 + 4\alpha^2}\left(\left(\hat{\boldsymbol{n}} \cdot \boldsymbol{J}\right)_{3\times 3} + 2i\alpha\left(\hat{\boldsymbol{n}} \cdot \boldsymbol{J}\right)_{3\times 3}^2\right) \tag{E4}$$

$$\det\left(1 - 2i\alpha\ \hat{\boldsymbol{n}} \cdot \boldsymbol{J}\right)\big|_{j=3/2} = \left(1 + \alpha^2\right)\left(1 + 9\alpha^2\right) = 1 + 10\alpha^2 + 9\alpha^4 \tag{E5}$$

$$\left.\frac{1 + 2i\alpha\ \hat{\boldsymbol{n}} \cdot \boldsymbol{J}}{1 - 2i\alpha\ \hat{\boldsymbol{n}} \cdot \boldsymbol{J}}\right|_{j=3/2} = \frac{1}{\left(1 + \alpha^2\right)\left(1 + 9\alpha^2\right)}\left(\left(1 + 10\alpha^2 - 9\alpha^4\right)\boldsymbol{I}_{4\times 4} + 4i\alpha\left(1 + 10\alpha^2\right)\left(\hat{\boldsymbol{n}} \cdot \boldsymbol{J}\right)_{4\times 4}\right)$$
$$+ \frac{-8\alpha^2}{\left(1 + \alpha^2\right)\left(1 + 9\alpha^2\right)}\left(\left(\hat{\boldsymbol{n}} \cdot \boldsymbol{J}\right)_{4\times 4}^2 + 2i\alpha\left(\hat{\boldsymbol{n}} \cdot \boldsymbol{J}\right)_{4\times 4}^3\right) \tag{E6}$$

$$\det\left(1 - 2i\alpha\ \hat{\boldsymbol{n}} \cdot \boldsymbol{J}\right)\big|_{j=2} = \left(1 + 4\alpha^2\right)\left(1 + 16\alpha^2\right) = 1 + 20\alpha^2 + 64\alpha^4 \tag{E7}$$

$$\left.\frac{1 + 2i\alpha\ \hat{\boldsymbol{n}} \cdot \boldsymbol{J}}{1 - 2i\alpha\ \hat{\boldsymbol{n}} \cdot \boldsymbol{J}}\right|_{j=2} = \boldsymbol{I}_{5\times 5} + \frac{4i\alpha\left(1 + 20\alpha^2\right)}{\left(1 + 4\alpha^2\right)\left(1 + 16\alpha^2\right)}\left(\left(\hat{\boldsymbol{n}} \cdot \boldsymbol{J}\right)_{5\times 5} + 2i\alpha\left(\hat{\boldsymbol{n}} \cdot \boldsymbol{J}\right)_{5\times 5}^2\right)$$
$$+ \frac{-16i\alpha^3}{\left(1 + 4\alpha^2\right)\left(1 + 16\alpha^2\right)}\left(\left(\hat{\boldsymbol{n}} \cdot \boldsymbol{J}\right)_{5\times 5}^3 + 2i\alpha\left(\hat{\boldsymbol{n}} \cdot \boldsymbol{J}\right)_{5\times 5}^4\right) \tag{E8}$$

$$\det\left(1 - 2i\alpha\ \hat{\boldsymbol{n}} \cdot \boldsymbol{J}\right)\big|_{j=5/2} = \left(1 + \alpha^2\right)\left(1 + 9\alpha^2\right)\left(1 + 25\alpha^2\right) = 1 + 35\alpha^2 + 259\alpha^4 + 225\alpha^6 \tag{E9}$$

$$\left.\frac{1 + 2i\alpha\ \hat{\boldsymbol{n}} \cdot \boldsymbol{J}}{1 - 2i\alpha\ \hat{\boldsymbol{n}} \cdot \boldsymbol{J}}\right|_{j=5/2} = \frac{1}{\left(1 + \alpha^2\right)\left(1 + 9\alpha^2\right)\left(1 + 25\alpha^2\right)}\left(\begin{array}{c}\left(1 + 35\alpha^2 + 259\alpha^4 - 225\alpha^6\right)\boldsymbol{I}_{6\times 6} \\[2mm] +4i\alpha\left(1 + 35\alpha^2 + 259\alpha^4\right)\left(\hat{\boldsymbol{n}} \cdot \boldsymbol{J}\right)_{6\times 6}\end{array}\right)$$
$$+ \frac{-8\alpha^2\left(1 + 35\alpha^2\right)}{\left(1 + \alpha^2\right)\left(1 + 9\alpha^2\right)\left(1 + 25\alpha^2\right)}\left(\left(\hat{\boldsymbol{n}} \cdot \boldsymbol{J}\right)_{6\times 6}^2 + 2i\alpha\left(\hat{\boldsymbol{n}} \cdot \boldsymbol{J}\right)_{6\times 6}^3\right)$$
$$+ \frac{32\alpha^4}{\left(1 + \alpha^2\right)\left(1 + 9\alpha^2\right)\left(1 + 25\alpha^2\right)}\left(\left(\hat{\boldsymbol{n}} \cdot \boldsymbol{J}\right)_{6\times 6}^4 + 2i\alpha\left(\hat{\boldsymbol{n}} \cdot \boldsymbol{J}\right)_{6\times 6}^5\right) \tag{E10}$$

$$\det\left(1 - 2i\alpha\ \hat{\boldsymbol{n}} \cdot \boldsymbol{J}\right)\big|_{j=3} = \left(1 + 4\alpha^2\right)\left(1 + 16\alpha^2\right)\left(1 + 36\alpha^2\right) = 1 + 56\alpha^2 + 784\alpha^4 + 2304\alpha^6 \tag{E11}$$

$$\left.\frac{1 + 2i\alpha\ \hat{\boldsymbol{n}} \cdot \boldsymbol{J}}{1 - 2i\alpha\ \hat{\boldsymbol{n}} \cdot \boldsymbol{J}}\right|_{j=3} = \boldsymbol{I}_{7\times 7} + \frac{4i\alpha\left(1 + 56\alpha^2 + 784\alpha^4\right)}{\left(1 + 4\alpha^2\right)\left(1 + 16\alpha^2\right)\left(1 + 36\alpha^2\right)}\left(\left(\hat{\boldsymbol{n}} \cdot \boldsymbol{J}\right)_{7\times 7} + 2i\alpha\left(\hat{\boldsymbol{n}} \cdot \boldsymbol{J}\right)_{7\times 7}^2\right)$$
$$+ \frac{-16i\alpha^3\left(1 + 56\alpha^2\right)}{\left(1 + 4\alpha^2\right)\left(1 + 16\alpha^2\right)\left(1 + 36\alpha^2\right)}\left(\left(\hat{\boldsymbol{n}} \cdot \boldsymbol{J}\right)_{7\times 7}^3 + 2i\alpha\left(\hat{\boldsymbol{n}} \cdot \boldsymbol{J}\right)_{7\times 7}^4\right)$$
$$+ \frac{64i\alpha^5}{\left(1 + 4\alpha^2\right)\left(1 + 16\alpha^2\right)\left(1 + 36\alpha^2\right)}\left(\left(\hat{\boldsymbol{n}} \cdot \boldsymbol{J}\right)_{7\times 7}^5 + 2i\alpha\left(\hat{\boldsymbol{n}} \cdot \boldsymbol{J}\right)_{7\times 7}^6\right) \tag{E12}$$