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Enumeration of standard Young tableaux of shifted strips with constant width

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Abstract Let g_{n_1,n_2} be the number of standard Young tableau of truncated shifted shape with n_1 rows and n_2 boxes in each row. By using of the integral method this paper derives the recurrence relations of $g_{3,n}$, $g_{n,4}$ and $g_{n,5}$ respectively. Specially, $g_{n,4}$ is the $(2n-1)$-st Pell number.

MSC: 05A15, 05E15

Keywords: Standard Young tableau; multiple integrals; recurrences; Pell numbers

1 Introduction

A shifted diagram of shape $\lambda = (\lambda_1, \cdots, \lambda_d)$ $(\lambda_1 > \cdots > \lambda_d)$ is an array of $|\lambda|$ boxes, where row i (from top to bottom) containing λ_i boxes starts with its leftmost box in position (i, i). A standard shifted Young tableau of shape λ is a labeling by $\{1, 2, \cdots, |\lambda|\}$ of the boxes in the shifted diagram such that each row and column is increasing (from left to right and from top to bottom respectively). Specially, a standard Young tableaux (SYT) of shifted staircase shape is $\delta_n = (n, n-1, \cdots, 1)$. The enumeration of SYT is an important problem in enumerative combinatorics. See R. P. Stanley's monograph [9] and R. M. Adin and Y. Roichman's recent survey paper [2].

The number of SYT of shifted shape λ is given by the well-known product formula[7, 11]:

$$g^{\lambda} = \frac{|\lambda|!}{\prod_{i=1}^{d} \lambda_i!} \prod_{i<j} \frac{\lambda_i - \lambda_j}{\lambda_i + \lambda_j}. \tag{1}$$

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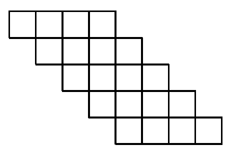
A SYT of truncated shape is the SYT with some boxes removed from the NE corner, which was recently discussed by some authors [1, 6, 10]. Adin et al. in [1] derived the formulas of δ_n truncated by a square or nearly a square by the method of pivoting theory. G. Panova independently obtained the product formulas of δ_n truncated by a box in term of Schur function [6].

This paper considers the SYT of shifted shape in case of $\lambda_1 = \cdots = \lambda_d$, which is the SYT of shifted shape $(n_1 + n_2 - 1, n_1 + n_2 - 2, \cdots, n_2)$ truncated by a staircase $\delta_{n_1 - 1}$, namely a SYT of truncated shifted shape with n_1 rows and n_2 boxes in each row, illustrated as follows



$n_1 = 5, n_2 = 4.$

Let $g_{n_1,n_2} (n_1, n_2 \geq 2)$ be the number of SYT of truncated shifted shape with n_1 rows and n_2 boxes in each row. It is clear that $g_{2,n}$ is the $(n-1)$-th Catalan number $\frac{1}{n}\binom{2n-2}{n-1}$, $g_{n,2} = 1$ and $g_{n,3} = 2^{n-1}$. This kind of SYT of shifted strips with constant width has many applications. J. B. Lewis conjectured that the number of alternating permutations of length $2n - 2$ avoiding the pattern 3412 is $g_{3,n}$, which summation representation was given by G. Pabova [5]. In fact, g_{n_1,n_2} is also the number of $n_1 \times n_2$ matrices containing a permutation of $[n_1 n_2]$ in increasing order rowwise, columnwise, diagonally and (downwards) antidiagonally, because the character in increasing order antidiagonally of matrix coincides with the shifted property of SYT.

Suppose $T(n, k)$ is the number of $n \times k$ matrices containing a permutation of $[nk]$ in increasing order rowwise, columnwise, diagonally and antidiagonally, R. H. Hardin gives several empirical formulas of $T(n, k)$ in case of $k \leq 7$ [8, A181196]:

Empirical Recurrence Relations (*R.H.Hardin*)

$$T(n,4): \ a_n = 6a_{n-1} - a_{n-2}. \tag{2}$$

$$T(n,5): \ a_n = 24a_{n-1} - 40a_{n-2} - 8a_{n-3}. \tag{3}$$

$$T(n,6): \ a_n = 120a_{n-1} - 1672a_{n-2} + 544a_{n-3} - 6672a_{n-4} + 256a_{n-5}. \tag{4}$$

$$\begin{aligned} T(n,7): \ a_n = {} & 720a_{n-1} - 84448a_{n-2} + 1503360a_{n-3} - 17912224a_{n-4} - 318223104a_{n-5} \\ & + 564996096a_{n-6} + 270471168a_{n-7} - 11373824a_{n-8} + 65536a_{n-9}. \end{aligned} \tag{5}$$

In addition, Ping Sun in [10] shows that g_{n_1,n_2} is involved in the nested distribution of n_1 groups of independent order statistics with n_2 samples from uniform distribution on interval $(0,1)$. Generally, the order statistics model of SYT in [10] implies

Proposition 1. *For $n_1, n_2 \geq 2$,*

$$g_{n_1,n_2} = (n_1 n_2)! \cdot J(n_1, n_2) = (n_1 n_2)! \int \cdots \int_{D(n_1,n_2)} dx_{i,j}, \tag{6}$$

where $D(n_1, n_2)$ is the following SYT-type integral domain (the variables are increasing from left to right and from top to bottom)

$$\begin{array}{cccccc} 0 < x_{1,1} < & x_{1,2} & < \cdots & < x_{1,n_2} & & \\ & \wedge & & \wedge & & \\ & x_{2,1} & < \cdots < & x_{2,n_2-1} & < x_{2,n_2} & \\ & & \ddots & \ddots & \ddots & \\ & & & x_{n_1,1} < \cdots & \cdots & < x_{n_1,n_2} < 1. \end{array}$$

We derive the recurrence formulas of $g_{3,n}$, $g_{n,4}$ and $g_{n,5}$ by using of the integral method of [10] in this paper. In Section 2 we evaluate the nested distribution of *three* groups of independent order statistics with n samples from uniform distribution on interval $(0,1)$, which implies a new summation representation of $g_{3,n}$. So that a non-homogeneous linear recurrence relation of $g_{3,n}$ is given. In Section 3 we compute the corresponding multiple integrals and prove the recurrence relations (2) and (3) respectively. In particular, $g_{n,4}$ is shown to be the $(2n-1)$-st Pell number which implies the empirical formula (2).

2 New recurrence relation of $g_{3,n}$

There are two results of $g_{3,n}$ in the literature. Considering the enumeration of SYT of shifted shape $(n, n-1, i), 0 \leq i \leq n-2$, G. Panova gives the following summation representation.

Proposition 2. [5] *For $n \geq 2$,*

$$g_{3,n} = \sum_{i=0}^{n-2} \frac{(2n+i-1)!(n-i)(n-i-1)}{n!\,(n-1)!\,i!\,(2n-1)(n+i)(n+i-1)}. \tag{7}$$

For the number of $3 \times n$ matrices containing a permutation of $[3n]$ in increasing order rowwise, columnwise, diagonally and antidiagonally, V. Kotesovec gives the complicated order 2 recurrence relation.

Proposition 3. [8, A181197] *For $n \geq 3$,*

$$(2n-1)(7n-13)n^2 \cdot g_{3,n} = 2(182n^4 - 1185n^3 + 2722n^2 - 2625n + 900) \cdot g_{3,n-1} + 3(2n-5)(3n-5)(3n-4)(7n-6) \cdot g_{3,n-2}, \qquad g_{3,1} = 1,\ g_{3,2} = 1. \tag{8}$$

For $0 < t_1 < t_2 < t_3 < 1$, we consider the following integral

$$J_{n-1}(t_1, t_2, t_3) = \int \cdots \int_{D(t)} dx_{i,j}, n \geq 2,$$

where $D(t)$ is the following SYT-type integral domain

$$0 < x_{i,1} < x_{i,2} < \cdots < x_{i,n-1} < t_i, 1 \leq i \leq 3; \quad x_{1,2} < x_{2,1}, x_{2,n-1} < x_{3,n-2};$$
$$t_1 < x_{2,n-1}, t_2 < x_{3,n-1}; \quad x_{1,j+2} < x_{2,j+1} < x_{3,j}, 1 \leq j \leq n-3.$$

From Proposition 1, there is

$$g_{3,n} = (3n)! \iiint\limits_{0<t_1<t_2<t_3<1} J_{n-1}(t_1, t_2, t_3) dt_i, \quad n \geq 2. \tag{9}$$

Lemma 1. *For $n \geq 1$, $0 < t_1 < t_2 < t_3 < 1$,*

$$J_n(t_1, t_2, t_3) = \frac{t_1^n t_2^n t_3^n}{n!^3} + \frac{2}{n!(2n)!} \sum_{i=0}^{n-1} (-1)^{n-i} \binom{2n}{i} \left[t_1^n t_2^{2n-i} t_3^i - t_1^{2n-i} t_2^n t_3^i + t_1^{2n-i} t_2^i t_3^n \right]. \tag{10}$$

Proof. It is clear that

$$J_1(t_1, t_2, t_3) = t_1(t_2 - t_1)(t_3 - t_2) = t_1 t_2 t_3 - (t_1 t_2^2 - t_1^2 t_2 + t_1^2 t_3).$$

Suppose (10) is true for $n-1$, then

$$J_n(t_1, t_2, t_3) = \iiint\limits_{0<x<t_1<y<t_2<z<t_3} J_{n-1}(x, y, z) dx dy dz$$
$$= \frac{t_1^n (t_2^n - t_1^n)(t_3^n - t_2^n)}{n!^3} + \frac{2}{(n-1)!(2n-2)!} \sum_{i=0}^{n-2} (-1)^{n-1-i} \binom{2n-2}{i} \frac{1}{n(i+1)(2n-i-1)} \times$$
$$\left[t_1^n (t_2^{2n-i-1} - t_1^{2n-i-1})(t_3^{i+1} - t_2^{i+1}) - t_1^{2n-i-1}(t_2^n - t_1^n)(t_3^{i+1} - t_2^{i+1}) + t_1^{2n-i-1}(t_2^{i+1} - t_1^{i+1})(t_3^n - t_2^n) \right]$$
$$= \frac{t_1^n t_2^n t_3^n}{n!^3} + \frac{2}{n!(2n)!} \sum_{i=0}^{n-1} (-1)^{n-i} \binom{2n}{i} \left[t_1^n t_2^{2n-i} t_3^i - t_1^{2n-i} t_2^n t_3^i + t_1^{2n-i} t_2^i t_3^n \right] + R_n(t_1, t_2, t_3),$$

where

$$R_n(t_1,t_2,t_3) = (t_1^{2n}t_2^n - t_1^n t_2^{2n} - t_1^{2n}t_3^n)\left[\frac{1}{n!^3} + \frac{2}{n!(2n)!}\sum_{i=0}^{n-1}(-1)^{n-i}\binom{2n}{i}\right] = 0$$

follows from the known identity [4, 1.86]

$$\sum_{i=0}^{n}(-1)^i\binom{2n}{i} = (-1)^n\binom{2n-1}{n}. \tag{11}$$

The proof of lemma 1 is complete by induction. □

It should be noted that $J_{n-1}(t_1,t_2,t_3)$ is the shifted nested conditional distribution of *three* groups of independent order statistics with n samples from uniform distribution on interval $(0,1)$. The following result gives a non-homogeneous linear recurrence of $g_{3,n}$.

Theorem 1. *For $n \geq 1$, the number $g_{3,n}$ of SYT of truncated shifted shape with* 3 *rows and n boxes in each row satisfies*

$$g_{3,n+1} = -g_{3,n} + \frac{7n+1}{n^2(n+1)^2}\binom{2n-2}{n-1}\binom{3n}{n-1}, \quad g_{3,1} = 1. \tag{12}$$

Proof. For $n \geq 2$, combining (9) and (10) we have the summation representation

$$g_{3,n} = \frac{(3n)!}{6 \cdot n!^3} + \sum_{i=0}^{n-2}(-1)^{n-1-i}\frac{(3n-1)!(5n-3i-3)}{n!\; i!\; (2n-i-1)!\; (3n-i-1)}. \tag{13}$$

Decomposing $5n-3i-3 = 3(3n-i-1)-4n$ in above summation, from (11) and Frisch's identity [4, 4.2]:

$$\sum_{i=0}^{2n-1}(-1)^i\binom{2n-1}{i}\frac{1}{n+i} = \frac{(n-1)!(2n-1)!}{(3n-1)!},$$

the summation representation (13) of $g_{3,n}$ is simplified to be

$$g_{3,n} = -\frac{4n-5}{6(2n-1)}\cdot\frac{(3n)!}{n!^3} + 4(-1)^{n-1} + (-1)^n 4n\binom{3n-1}{n}\cdot A_n, \tag{14}$$

where

$$A_n = \sum_{i=0}^{n}(-1)^i\binom{2n-1}{i}\frac{1}{n+i}.$$

The recurrence of A_n is not difficult to derive by using of the equality (11).

$$\begin{aligned} A_{n+1} &= \frac{1}{3n+2}\sum_{i=0}^{n+1}(-1)^i \frac{(2n+1)!}{i!\,(2n-i)!}\left[\frac{1}{2n+1-i}+\frac{1}{n+1+i}\right] \\ &= \frac{(-1)^{n+1}\binom{2n}{n+1}}{3n+2} + \frac{2n(2n+1)}{(3n+1)(3n+2)}\sum_{i=0}^{n+1}(-1)^i\binom{2n-1}{i}\left[\frac{1}{2n-i}+\frac{1}{n+1+i}\right] \\ &= \frac{(-1)^{n+1}(10n^3+8n^2-n-1)\binom{2n}{n+1}}{2n(n+1)(3n+1)(3n+2)} + \frac{2n(2n+1)}{(3n+1)(3n+2)}\sum_{i=0}^{n}(-1)^i\frac{\binom{2n-1}{i}}{n+1+i}, \end{aligned}$$

and the summation in last equality is equal to

$$\begin{aligned} &\sum_{i=0}^{n}(-1)^{i+1}\frac{1}{n+1+i}\left[\binom{2n-1}{i+1}-\binom{2n}{i+1}\right] \\ &= \frac{(-1)^{n+1}\left[\binom{2n-1}{n+1}-\binom{2n}{n+1}\right]}{2n+1} + A_n - \frac{1}{3n}\sum_{i=0}^{n}(-1)^i\frac{(2n)!}{i!\,(2n-1-i)!}\left[\frac{1}{n+i}+\frac{1}{2n-i}\right] \\ &= \frac{1}{3}A_n - \frac{n^2-1}{6n^2(2n+1)}(-1)^{n+1}\binom{2n}{n+1}, \end{aligned}$$

therefore the recurrence of A_n is

$$A_{n+1} = \frac{2n(2n+1)}{3(3n+1)(3n+2)}A_n + (-1)^{n+1}\frac{28n^3+22n^2-n-1}{6n(n+1)(3n+1)(3n+2)}\binom{2n}{n+1},\ A_1 = \frac{1}{2}.$$

So that the recurrence relation (12) of $g_{3,n}$ follows from

$$A_n = \frac{(-1)^n}{4n\binom{3n-1}{n}}g_{3,n} + \frac{1}{n\binom{3n-1}{n}} + (-1)^n\frac{4n-5}{8n(2n-1)}\binom{2n-1}{n}.$$

The proof of theorem 1 is complete. $\square$

3 Recurrence relations of $g_{n,4}$ and $g_{n,5}$

It is well-known the Pell numbers P_n are defined by the recurrence relation [8, A000129]

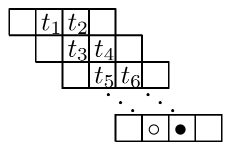
$$P_n = 2P_{n-1} + P_{n-2}, \quad P_0 = 0, P_1 = 1. \tag{15}$$

The Pell numbers arise historically in the rational approximation to $\sqrt{2}$, which are used to enumerate the numbers of certain pattern-avoiding permutations recently[3].

Theorem 2. *For $n \geq 1$, the number $g_{n,4}$ of SYT of truncated shifted shape with n rows and 4 boxes in each row is the $(2n-1)$-st Pell number P_{2n-1}, which satisfies*

$$g_{n,4} = 6g_{n-1,4} - g_{n-2,4}, \quad g_{1,4} = 1, g_{2,4} = 5. \tag{16}$$

Proof. For $1 \leq i \leq n$, write the variable t_{2i-1} corresponding to the box $(i, i+1)$, t_{2i} corresponding to the box $(i, i+2)$ in the SYT of shifted strip with width 4 respectively:



$\circ = t_{2n-1}, \quad \bullet = t_{2n}.$

Proposition 1 implies that

$$\begin{aligned} g_{n,4} &= (4n)! \int\limits_{0<t_1<t_2<t_3<\cdots<t_{2n}<1} \cdots \int t_1(1-t_{2n}) \prod_{i=2}^{n} (t_{2i-1} - t_{2i-3})(t_{2i} - t_{2i-2}) dt_1 \cdots t_{2n} \\ &= (4n)! \iint\limits_{0<t_{2n-1}<t_{2n}<1} (1-t_{2n}) J_n(t_{2n-1}, t_{2n}) dt_{2n-1} dt_{2n}, \quad n \geq 2. \end{aligned}$$

We shall now use the method of induction to prove the following

$$\begin{aligned} J_n(t_{2n-1}, t_{2n}) &= \int\limits_{0<t_1<t_2<t_3<\cdots<t_{2n-2}<t_{2n-1}} \cdots \int t_1 \prod_{i=2}^{n} (t_{2i-1} - t_{2i-3})(t_{2i} - t_{2i-2}) dt_1 \cdots t_{2n-2} \\ &= \frac{t_{2n-1}^{4n-4}}{(4n-3)!} \left\{ (4n-3) P_{2n-2} t_{2n} - [(4n-4) P_{2n-2} - P_{2n-3}] t_{2n-1} \right\}, \end{aligned} \tag{17}$$

where P_i is the Pell number.

It is clear that

$$J_2(t_3, t_4) = \iint\limits_{0<t_1<t_2<t_3} t_1(t_3 - t_1)(t_4 - t_2) dt_1 dt_2 = \frac{t_3^4}{5!}(10t_4 - 7t_3),$$

which agrees with (17) because $P_1 = 1, P_2 = 2$.

Furthermore, from the recurrence relation (15) of Pell numbers,

$$\begin{aligned}
J_{n+1}(t_{2n+1}, t_{2n+2}) &= \iint\limits_{0<t_{2n-1}<t_{2n}<t_{2n+1}} J_n(t_{2n-1}, t_{2n})(t_{2n+1} - t_{2n-1})(t_{2n+2} - t_{2n})dt_{2n-1}dt_{2n} \\
&= \int_0^{t_{2n+1}} \frac{t_{2n}^{4n-2}}{(4n-1)!}\{(4n-1)P_{2n-1}t_{2n+1} - [(4n-2)P_{2n-1} - P_{2n-2}]t_{2n}\}(t_{2n+2} - t_{2n})dt_{2n} \\
&= \frac{t_{2n+1}^{4n}}{(4n+1)!}\left[(4n+1)P_{2n}t_{2n+2} - (4nP_{2n} - P_{2n-1})t_{2n+1}\right],
\end{aligned}$$

which shows (17) is true. Therefore,

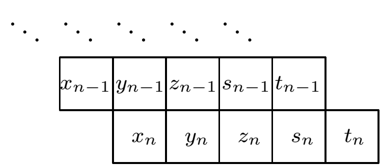
$$\begin{aligned}
g_{n,4} &= (4n)! \iint\limits_{0<t_{2n-1}<t_{2n}<1} (1 - t_{2n})J_n(t_{2n-1}, t_{2n})dt_{2n-1}dt_{2n} \\
&= \frac{(4n)!}{(4n-3)!}\int_0^1 \left[P_{2n-2} - \frac{(4n-4)P_{2n-2} - P_{2n-3}}{4n-2}\right] t_{2n}^{4n-2}(1 - t_{2n})dt_{2n} \\
&= 4n(4n-1)\int_0^1 P_{2n-1}t_{2n}^{4n-2}(1 - t_{2n})dt_{2n} = P_{2n-1}.
\end{aligned}$$

It is clear (16) follows from the recurrence relation of Pell number. □

Theorem 3. *For $n \geq 4$, the numbers $g_{n,5}$ of SYT of truncated shifted shape with n rows and 5 boxes in each row satisfy the following recurrence relation*

$$g_{n,5} = 24g_{n-1,5} - 40g_{n-2,5} - 8g_{n-3,5}, \quad g_{1,5} = 1, g_{2,5} = 14, g_{3,5} = 290. \tag{18}$$

Proof. We shall derive the recurrence of $g_{n,5}$ from the relations of certain integrals. For convenient, write the variables $0 < x_i < y_i < z_i < s_i < t_i < 1$ corresponding to the five boxes in row i $(1 \leq i \leq n)$ in the SYT of shifted strip with width 5 respectively.



From Proposition 1, we have

$$g_{n,5} = (5n)! \int \cdots \int_{D(n,5)} dx_i dy_i dz_i ds_i dt_i = (5n)! \cdot J_n(5).$$

Denote $D_1(x_1, y_1, z_1, s_1) = 1$, consider the following integral

$$D_n(x_n, y_n, z_n, s_n) = \int\limits_{\substack{D_{n-1,5},\, y_{n-1}<x_n, \\ z_{n-1}<y_n,\, s_{n-1}<z_n,\, t_{n-1}<s_n}} \cdots \int dx_1 y_1 z_1 s_1 t_1 \cdots dx_{n-1} y_{n-1} z_{n-1} s_{n-1} t_{n-1},$$

$J_n(5)$ can be written to be

$$\begin{aligned} J_n(5) = & \int\limits_{\substack{0<x_{n-1}<y_{n-1}<z_{n-1}<s_{n-1}<t_{n-1}<s_n, \\ s_{n-1}<z_n,\, z_{n-1}<y_n<z_n<s_n<t_n<1}} \cdots \int (y_n - y_{n-1}) D_{n-1}(x_{n-1}, \cdots, s_{n-1}) dx_{n-1} \cdots t_{n-1} dy_n z_n s_n t_n \\ = & \int\limits_{\substack{0<z_{n-1}<s_{n-1}<t_{n-1}<s_n, \\ s_{n-1}<z_n, z_{n-1}<y_n<z_n<s_n<t_n<1}} \cdots \int (A_n y_n - B_n) dz_{n-1} s_{n-1} t_{n-1} dy_n z_n s_n t_n, \end{aligned} \tag{19}$$

where

$$\begin{aligned} A_n = & \iint\limits_{0<x_{n-1}<y_{n-1}<z_{n-1}} D_{n-1}(x_{n-1}, y_{n-1}, z_{n-1}, s_{n-1}) dx_{n-1} dy_{n-1} \\ = & C_1(n) \frac{z_{n-1}^{5n-9}}{(5n-9)!} s_{n-1} - C_2(n) \frac{z_{n-1}^{5n-8}}{(5n-8)!}, \\ B_n = & \iint\limits_{0<x_{n-1}<y_{n-1}<z_{n-1}} y_{n-1} D_{n-1}(x_{n-1}, y_{n-1}, z_{n-1}, s_{n-1}) dx_{n-1} dy_{n-1} \\ = & C_3(n) \frac{z_{n-1}^{5n-8}}{(5n-8)!} s_{n-1} - C_4(n) \frac{z_{n-1}^{5n-7}}{(5n-7)!}. \end{aligned}$$

Notice that the definition of $D_n(x_n, y_n, z_n, s_n)$ implies

$$\begin{aligned} A_{n+1} = & \iint\limits_{0<x_n<y_n<z_n} D_n(x_n, y_n, z_n, s_n) dx_n dy_n \\ = & \int\limits_{\substack{0<z_{n-1}<s_{n-1}<t_{n-1}<s_n, \\ s_{n-1}<z_n, z_{n-1}<y_n<z_n<s_n}} \cdots \int (A_n y_n - B_n) dz_{n-1} s_{n-1} t_{n-1} dy_n, \end{aligned} \tag{20}$$

then

$$\begin{aligned} A_{n+1} = & [10(n-1)(5n-7)C_1(n) - (10n-11)C_2(n) - (10n-11)C_3(n) + 2C_4(n)] \times \\ & \frac{z_n^{5n-4}}{(5n-4)!} s_n - [(5n-4)(5n-7)(10n-11)C_1(n) - 2(5n-4)(5n-6)C_2(n) \\ & - 50(n-1)^2 C_3(n) + (10n-9)C_4(n)] \frac{z_n^{5n-3}}{(5n-3)!}. \end{aligned}$$

By the similar arguments,

$$
\begin{aligned}
B_{n+1} =& \iint\limits_{0<x_n<y_n<z_n} y_n D_n(x_n,\cdots,s_n)dx_ndy_n \\
=& \int\limits_{\substack{0<z_{n-1}<s_{n-1}<t_{n-1}<s_n,\\ s_{n-1}<z_n,z_{n-1}<y_n<z_n<s_n}}\cdots\int (A_ny_n^2-B_ny_n)dz_{n-1}s_{n-1}t_{n-1}y_n \\
=&[(5n-4)(5n-7)(10n-11)C_1(n)-50(n-1)^2C_2(n)-2(5n-4)(5n-6)C_3(n) \\
&+(10n-9)C_4(n)]\frac{z_n^{5n-3}}{(5n-3)!}s_n-(5n-3)[2(5n-4)(5n-6)(5n-7)C_1(n) \\
&-(5n-6)(10n-9)C_2(n)-(5n-6)(10n-9)C_3(n)+10(n-1)C_4(n)]\frac{z_n^{5n-2}}{(5n-2)!}.
\end{aligned}
$$

Therefore, For $n \geq 2$, the recurrence relations of $C_i(n)$ $(1 \leq i \leq 4)$ are

$$
\begin{aligned}
C_1(n+1) =&10(n-1)(5n-7)C_1(n)-(10n-11)C_2(n)-(10n-11)C_3(n)+2C_4(n), \\
C_2(n+1) =&(5n-4)(5n-7)(10n-11)C_1(n)-2(5n-4)(5n-6)C_2(n) \\
&-50(n-1)^2C_3(n)+(10n-9)C_4(n), \\
C_3(n+1) =&(5n-4)(5n-7)(10n-11)C_1(n)-50(n-1)^2C_2(n) \\
&-2(5n-4)(5n-6)C_3(n)+(10n-9)C_4(n), \\
C_4(n+1) =&2(5n-3)(5n-4)(5n-6)(5n-7)C_1(n)-(5n-3)(5n-6)(10n-9)C_2(n) \\
&-(5n-3)(5n-6)(10n-9)C_3(n)+10(n-1)(5n-3)C_4(n),
\end{aligned}
$$

with the initial values $C_1(2)=0,\ C_2(2)=-1,\ C_3(2)=0,\ C_4(2)=-2.$

On the other hand, combining (19) and (20), we have

$$
\begin{aligned}
J_n(5) =& \iiint\limits_{0<z_n<s_n<t_n<1} A_{n+1}dz_ns_nt_n \\
=& \iiint\limits_{0<z_n<s_n<t_n<1} [C_1(n+1)\frac{z_n^{5n-4}}{(5n-4)!}s_n-C_2(n+1)\frac{z_n^{5n-3}}{(5n-3)!}]dz_ns_nt_n \\
=& \frac{(5n-2)C_1(n+1)-C_2(n+1)}{(5n)!},
\end{aligned}
$$

then,

$$
g_{n,5}=(5n-2)C_1(n+1)-C_2(n+1), \tag{21}
$$

and

$$g_{n,5} = (5n-7)(25n-24)C_1(n) - (25n-26)C_2(n) - (25n-28)C_3(n) + 5C_4(n), \quad (22)$$

which follows from (21) and the recurrences of $C_i(n)$.

Furthermore, by using of the recurrence relations of $C_i(n)$ again, (22) implies

$$\begin{aligned} g_{n+1,5} =& (5n-2)(25n+1)C_1(n+1) - (25n-1)C_2(n+1) \\ & - (25n-3)C_3(n+1) + 5C_4(n+1) \\ =& (5n-7)(550n-524)C_1(n) - (550n-590)C_2(n) \\ & - (550n-594)C_3(n) + 110C_4(n). \end{aligned} \quad (23)$$

So that from (21)-(23) we have

$$\begin{aligned} g_{n+1,5} &= 22g_{n,5} + 4(5n-7)C_1(n) + 18C_2(n) - 22C_3(n) \\ &= 22g_{n,5} + 4g_{n-1,5} + 22[C_2(n) - C_3(n)], \quad n \geq 2. \end{aligned}$$

Finally, the recurrence relation of $C_i(n)$ shows

$$C_2(n+1) - C_3(n+1) = 2[C_2(n) - C_3(n)], \quad n \geq 2,$$

which implies

$$g_{n+1,5} - 22g_{n,5} - 4g_{n-1,5} = 2(g_{n,5} - 22g_{n-1,5} - 4g_{n-2,5}), \quad n \geq 3.$$

The initial values $g_{1,5} = 1, g_{2,5} = 14$ and $g_{3,5} = 290$ are obvious, then the proof of theorem 3 is complete. $\square$

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