

COMBINATORICS OF POINCARÉ'S AND SCHRÖDER'S EQUATIONS

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ABSTRACT. We investigate the combinatorial properties of the functional equation $\phi[h(z)] = h(qz)$ for the conjugation of a formal diffeomorphism ϕ of \mathbb{C} to its linear part $z \mapsto qz$. This is done by interpreting the functional equation in terms of symmetric functions, and then lifting it to noncommutative symmetric functions. We describe explicitly the expansion of the solution in terms of plane trees and prove that its expression on the ribbon basis has coefficients in $\mathbb{N}[q]$ after clearing the denominators $(q)_n$. We show that the conjugacy equation can be lifted to a quadratic fixed point equation in the free triduplicial algebra on one generator. This can be regarded as a q -deformation of the duplicial interpretation of the noncommutative Lagrange inversion formula. Finally, these calculations are interpreted in terms of the group of the operad of Stasheff polytopes, and are related to Ecalle's arborified expansion by means of morphisms between various Hopf algebras of trees.

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1. INTRODUCTION

Algebraic identities between generic formal power series can often be interpreted as identities between symmetric functions. This is the case, for example, with the Lagrange inversion formula (see, e.g., [34], Ex. 24 p. 35, Ex. 25 p. 132, [30] Section 2.4, and [31]). The problem can be stated as follows. Given

$$(1) \quad \varphi(z) = \sum_{n \geq 0} \varphi_n z^n \quad (\varphi_0 \neq 0)$$

find the coefficients g_n of the unique power series

$$(2) \quad g(z) = \sum_{n \geq 0} g_n z^{n+1} \quad \text{satisfying} \quad z = \frac{g(z)}{\varphi(g(z))}.$$

We can assume that $\varphi_0 = 1$ and that

$$(3) \quad \varphi(g) = \sum_{n \geq 0} h_n(X) g^n = \prod_{n \geq 1} (1 - g x_n)^{-1} =: \sigma_g(X)$$

is the generating series of the homogeneous symmetric functions of an infinite set of variables X . In λ -ring notation, the solution reads

$$(4) \quad g_n = \frac{1}{n+1} h_n((n+1)X)$$

(recall that $\sigma_t(nX) = \sigma_t(X)^n$, see, e.g., [34] p. 25). On this expression, it is clear that g_n is Schur positive, in fact, it is the Frobenius characteristic of the permutation representation of \mathfrak{S}_n on the set PF_n of parking functions of length n . These calculations can be lifted to the algebra of noncommutative symmetric functions, and the result is then interpreted in terms of representations of 0-Hecke algebras. This in turn leads to various combinatorial interpretations, to q -analogues, and to a new interpretation of the antipode of the Hopf algebra of noncommutative formal diffeomorphisms [39].

There is another functional equation which can be investigated in this setting. Given a formal diffeomorphism

$$(5) \quad \phi(z) = \sum_{n \geq 0} \phi_n z^{n+1} \quad \text{with } \phi_0 = q \neq 0,$$

one may look for a formal diffeomorphism tangent to identity

$$(6) \quad h(z) = \sum_{n \geq 0} g_n z^{n+1} = zg(z) \quad (g_0 = 1),$$

conjugating ϕ to its linear part

$$(7) \quad h^{-1} \circ \phi \circ h(z) = qz \quad \text{or equivalently } \phi[h(z)] = h(qz) = qzg(qz).$$

In terms of symmetric functions, we can assume that

$$(8) \quad \phi(z) = qz\sigma_z(X)$$

so that the conjugacy equation reads

$$(9) \quad \phi[h(z)] = qh(z)\sigma_{h(z)}(X) = qz \sum_{n \geq 0} g_n (qz)^n,$$

and interpreting g_n as symmetric functions $g_n(X)$, we can get rid of z by homogeneity (since $g_n(zX) = z^n g_n(X)$). Our functional equation reads now

$$(10) \quad g(X)\sigma_{g(X)} = g(qX).$$

We can lift this to noncommutative symmetric functions, for example as

$$(11) \quad g(qA) = \sum_{n \geq 0} S_n(A)g(A)^{n+1}.$$

For $q = 0$, this reduces to the functional equation for the antipode of the noncommutative Faà di Bruno Hopf algebra [2, 39], so that this problem can indeed be regarded as a generalisation of the noncommutative Lagrange inversion.

The conjugacy equation for h is often called Poincaré's equation, and the equivalent one for h^{-1} , Schröder's equation. Indeed, it has been first discussed by Schröder [43], who discovered a few explicit solutions, which are still essentially the only known ones. It is easy to show the existence and unicity of a formal solution when q is not a root of unity. The analyticity of the solution for $|q| \neq 1$ has been established by Koenigs [28]. It is interesting that this result can be easily proved by means of inequalities involving the Schröder numbers [33], defined by the same Schröder in a totally different context [42]. Much more difficult is Siegel's proof of convergence in the case $q = e^{2\pi i\theta}$ with θ satisfying a diophantine condition [44] (see also [16] for a modern proof under Bruno's condition). Again in this case, the Schröder numbers play a crucial role in the majorations.

We shall see that analyzing the conjugacy equation at the level of the noncommutative Faà di Bruno algebra provides a simple explanation of this fact, by letting Schröder trees appear naturally in the iterative solution of a q -difference equation. The resulting expressions turn out to be identical to those produced by Ecalle's arborification method [14, 15, 16]. This coincidence will be explained in Section 11,

where it will be proved that both methods can be interpreted in terms of calculations in the group of an operad and in related Hopf algebras.

Identifying the noncommutative Faà di Bruno algebra with noncommutative symmetric functions as in [39], we have several bases at our disposal. The solution g of the noncommutative Poincaré equation is naturally expressed in the complete basis S^I . After clearing out the denominators $(q; q)_n$, it turns out that its homogeneous components g_n are positive on the ribbon basis. This unexpected fact suggests that these should be the graded characteristics of some projective modules over 0-Hecke algebra, a conjecture that we expect to investigate in another paper. This positivity property will be proved in two different ways. We shall first recast the conjugacy equation as a quadratic fixed point problem, by means of the triduplicial operations introduced in [40]. On the ribbon basis, the quadratic map is manifestly positive. Next, comparing the binary tree expansion with the previous one based on reduced plane trees, we obtain a natural bijection between these trees and hypoplactic classes of parking functions (*aka* parking quasi-ribbons or segmented nondecreasing parking functions). This solves a problem which was left open in [38], and provides a bijection similar to the duplicial bijection of [40] between nondecreasing parking functions and binary trees.

In Section 6, we describe the expansion of g_n on the ribbon basis. The numerator of each coefficient is a q -analogue of $n!$, recording a statistic on permutations which is explicitly described.

In Section 7, we discuss Schröder's equation at the level of noncommutative symmetric functions. It leads to a different combinatorics. There is no natural expansion on trees, but instead, there is a rather explicit algebraic formula for the coefficients, which amounts to applying a simple linear transformation to a famous sequence of noncommutative symmetric functions, the q -Klyachko elements $K_n(q)$ [19, 29], which occur as well as Lie idempotents in descent algebras or as noncommutative Hall-Littlewood functions [24]. It is then shown in Section 8 that the same coefficients arise when the problem is considered from the point of view of mould calculus and differential operators. Thus, at least for this problem, the mould calculus approach can be seen to be dual to that relying on the noncommutative Faà di Bruno algebra.

The rest of the paper is devoted to the explanation of the coincidence between the coefficients of our first plane tree expansion, and Ecalle's arborified coefficients. The short story is that on the one hand, the paper [15] provides an interpretation of the arborification method as a lift of the original problem to an equation in the group of characters of a Connes-Kreimer algebra. On the other hand, our version with plane trees of the functional equation can be naturally interpreted in the group of a free operad. This group turns out to be isomorphic to the group of characters of a Hopf algebra of reduced plane forests, which admits a surjective morphism to the previous Connes-Kreimer algebra.

Section 9 provides some background on the operad of reduced plane trees. It is a free operad with one generator in each degree $n \geq 2$, also known as the operad of Stasheff polytopes, or as a free S -magmatic operad [26, 32]. We describe the

associated group, and prove that it is isomorphic to the group of characters of the Hopf algebra of reduced plane trees of [12].

In Section 10, we explain the encoding of the previous group by means of Polish codes of trees, and illustrate the method on the cases of Lagrange inversion and of the Poincaré equation.

In Section 11, we recall the Hopf algebraic interpretation of the arborification method [15, 16], and prove that the skeleton map already introduced in [39] induces a morphism of Hopf algebras between reduced plane forests and the \mathbb{N}^* -decorated Connes-Kreimer algebra.

In Section 12, we review briefly the interpretation of Lagrange inversion and of Cayley's formula for the solution of a generic differential equation in terms of an operad on (non-reduced) plane trees.

Finally, it is generally interesting to look at the images of formal series in combinatorial Hopf algebras under various characters. In the Appendix (Section 13), we review a few examples of explicit solutions of the conjugacy equation. Apart from the trivial case of linear fractional transformations $\phi(z) = qz/(1-z)$ (corresponding to the alphabet $A = \{1\}$), there is the already nontrivial case of the logistic map $\phi(z) = qz(1-z)$, corresponding to $A = \{-1\}$, for which explicit solutions (already given by Schröder) are known for $q = -2, 2, 4$. The case $A = \mathbb{E}$, corresponding to $\phi(z) = qze^z$ is not explicitly solved, but it leads to interesting statistics on pairs of permutations. These examples are investigated numerically in [7, 8, 9, 10].

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2. NOTATIONS

This paper is a continuation of [39, 40]. Our notations for ordinary symmetric functions are as in [34], and for noncommutative symmetric functions as in [19, 29].

The classical algebra of symmetric functions, denoted by Sym or $Sym(X)$, is a free associative and commutative graded algebra with one generator in each degree:

$$(12) \quad Sym = \mathbb{C}[h_1, h_2, \dots] = \mathbb{C}[e_1, e_2, \dots] = \mathbb{C}[p_1, p_2, \dots]$$

where the h_n are the complete homogeneous symmetric functions, the e_n the elementary symmetric functions, and the p_n the power sums.

Its usual bialgebra structure is defined by the coproduct

$$(13) \quad \Delta_0 h_n = \sum_{i=0}^n h_i \otimes h_{n-i} \quad (h_0 = 1)$$

which allows to interpret it as the algebra of polynomial functions on the multiplicative group

$$(14) \quad G_0 = \{a(z) = \sum_{n \geq 0} a_n z^n \mid a_0 = 1\}$$

of formal power series with constant term 1: h_n is the coordinate function

$$(15) \quad h_n : a(z) \longmapsto a_n.$$

Indeed, $h_n(a(z)b(z)) = (\Delta_0 h_n)(a(z) \otimes b(z))$.

But h_n can also be interpreted as a coordinate on the group

$$(16) \quad G_1 = \{A(z) = \sum_{n \geq 0} a_n z^{n+1} \ (a_0 = 1)\}$$

of formal diffeomorphisms tangent to identity, under functional composition. Again with $h_n(A(z)) = a_n$ and $h_n(A(z)B(z)) = \Delta_1(A(z) \otimes B(z))$, the coproduct is now

$$(17) \quad \Delta_1 h_n = \sum_{i=0}^n h_i \otimes h_{n-i}((i+1)X) \quad (h_0 = 1)$$

where $h_n(mX)$ is defined as the coefficient of t^n in $(\sum h_k t^k)^m$. The resulting bialgebra is known as the Faà di Bruno algebra [27].

These constructions can be repeated word for word with the algebra **Sym** of noncommutative symmetric functions. It is a free associative (and noncommutative) graded algebra with one generator S_n in each degree, which can be interpreted as above if the coefficients a_n belong to a noncommutative algebra. In this case, G_0 is still a group, but G_1 is not, as its composition is not anymore associative. However, the coproduct Δ_1

$$(18) \quad \Delta_1 S_n = \sum_{i=0}^n S_i \otimes S_{n-i}((i+1)A) \quad (S_0 = 1)$$

remains coassociative, and **Sym** endowed with this coproduct is a Hopf algebra, known as Noncommutative Formal Diffeomorphisms [2, 39], or as the noncommutative Faà di Bruno algebra [13].

The classical trick of regarding a generic series as a series of symmetric functions amounts to working in one of these Hopf algebras. The occurrence of trees in the solutions of certain problems can be traced back to the existence of Hopf algebras morphisms between these algebras and various Hopf algebras of trees.

Recall that bases of **Sym** _{n} are labelled by compositions I of n . The noncommutative complete and elementary functions are denoted by S_n and Λ_n , and the notation S^I means $S_{i_1} \cdots S_{i_r}$. The ribbon basis is denoted by R_I . The notation $I \vDash n$ means that I is a composition of n . The conjugate composition is denoted by I^\sim . The product formula for ribbons is

$$(19) \quad R_I R_J = R_{IJ} + R_{I \triangleright J}$$

where for $I = (i_1, \dots, i_r)$ and $J = (j_1, \dots, j_s)$,

$$(20) \quad IJ = (i_1, \dots, i_r, j_1, \dots, j_s) \text{ and } I \triangleright J = (i_1, \dots, i_r + j_1, \dots, j_s).$$

The graded dual of **Sym** is $QSym$ (quasi-symmetric functions). The dual basis of (S^I) is (M_I) (monomial), and that of (R_I) is (F_I) .

The *evaluation* $\text{Ev}(w)$ of a word w over a totally ordered alphabet A is the sequence $(|w|_a)_{a \in A}$ where $|w|_a$ is the number of occurrences of a in w . The *packed evaluation* $I = \text{pEv}(w)$ is the composition obtained by removing the zeros in $\text{Ev}(w)$.

Two permutations $\sigma, \tau \in \mathfrak{S}_n$ are said to be sylvester-equivalent if the decreasing binary trees¹ of σ^{-1} and τ^{-1} have the same shape. The generating function of the number of inversions on a sylvester class is given by the q -hook-length formula [1, 25].

3. RECURSIVE SOLUTION OF POINCARÉ'S EQUATION

Equation (11) can be written as a q -difference equation

$$(21) \quad g(qA) - g(A) = \sum_{n \geq 1} S_n(A)g(A)^{n+1}.$$

Introducing a homogeneity parameter t , we have

$$(22) \quad g(qtA) - g(tA) = \sum_{n \geq 1} t^n S_n(A)g(tA)^{n+1}.$$

Let g_n be the term of degree n in g , so that

$$(23) \quad g(tA) = \sum_{n \geq 1} t^n g_n(A).$$

Comparing the homogeneous components in both sides of (22), one gets a triangular system allowing to compute the g_n recursively. For $n \leq 3$:

$$(24) \quad \begin{aligned} g_0 &= 1, \\ (q-1)g_1 &= S_1, \\ (q^2-1)g_2 &= 2S_1g_1 + S_2, \\ (q^3-1)g_3 &= 2S_1g_2 + S_1g_1^2 + 3S_2g_1 + S_3. \end{aligned}$$

Define

$$(25) \quad q_n = q^n - 1, \quad (q)_n = q_n q_{n-1} \cdots q_1 \quad \text{and} \quad \tilde{g}_n = (q)_n g_n$$

The first \tilde{g}_n are then

$$(26) \quad \begin{aligned} \tilde{g}_1 &= S_1, \\ \tilde{g}_2 &= (q-1)S_2 + 2S_1^2, \\ \tilde{g}_3 &= (q)_2 S_3 + 3(q^2-1)S_2^2 + 2(q-1)S_1^3 + (5+q)S_1^4. \end{aligned}$$

On the ribbon basis of **Sym**, the expression is quite remarkable:

$$(27) \quad \begin{aligned} \tilde{g}_1 &= S_1, \\ \tilde{g}_2 &= (q+1)R_2 + 2R_{11}, \\ \tilde{g}_3 &= (1+q)(1+q+q^2)R_3 + (2+q+3q^2)R_{21} + 3(1+q)R_{12} + (5+q)R_{111}. \end{aligned}$$

Indeed, \tilde{g}_n is a linear combination of ribbons with positive coefficients which are all q -analogues of $n!$. Note that it is immediate, by induction on n , that $\tilde{g}_n|_{q=1} = n!S_1^n$, but it is not clear that the coefficients are in $\mathbb{N}[q]$. A combinatorial interpretation of these coefficients will be given below (Theorem 6.4).

¹The decreasing tree $T(w)$ of a word without repeated letters $w = unv$ and maximal letter n is the binary tree with root n and left and right subtrees $T(u)$ and $T(v)$.

4. A TREE-EXPANDED SOLUTION

In order to solve (22), define a q -integral by²

$$(28) \quad \int_a^b t^{n-1} d_q t = \left[\frac{t^n}{q^n - 1} \right]_a^b$$

and a q -difference operator

$$(29) \quad \Delta_q f(t) = \frac{f(qt) - f(t)}{t}.$$

Then,

$$(30) \quad \Delta_q \int_0^t f(s) d_q s = f(t)$$

so that g is the unique solution of the fixed point equation

$$(31) \quad g(tA) = g_0 + \sum_{n \geq 1} S_n(A) \int_0^t s^{n-1} g(sA)^{n+1} d_q s.$$

This equation is of the form

$$(32) \quad g = g_0 + \sum_{n \geq 2} F_n(g, \dots, g)$$

where

$$(33) \quad F_n(x_1, \dots, x_n) = S_{n-1} \int_0^1 s^{n-2} x_1(s) \cdots x_n(s) d_q s$$

is an n -linear operator. The solution can therefore be expanded as a sum over reduced plane trees (plane trees in which all internal vertices have at least two descendants), which will be called *Schröder trees* in the sequel.

Proceeding as in [39], we introduce another indeterminate S_0 (noncommuting with the other S_n) and set $g_0 = S_0$. The solution is then a linear combinations of monomials S^I where I is a vector of nonnegative integers, with $i_1 > 0$.

The first \tilde{g}_n are then

$$(34) \quad \begin{aligned} \tilde{g}_1 &= S^{100} \\ \tilde{g}_2 &= (q-1)S^{2000} + S^{11000} + S^{10100} \\ \tilde{g}_3 &= (q)_2 S^{30000} + (q^2-1)(S^{210000} + S^{201000} + S^{200100}) \\ &\quad + (q-1)(S^{120000} + S^{102000}) \\ &\quad + S^{1110000} + S^{1101000} + S^{1011000} + S^{1010100} + (q+1)S^{1100100}. \end{aligned}$$

We can interpret each S_i as the symbol of an $(i+1)$ -ary operation in Polish notation. Then, \tilde{g}_n is a sum over Polish codes of Schröder trees as in [39, Fig. 4]:

²This is just the ordinary q -integral up to conjugation by the transformation $t \mapsto (q-1)t$.

$$\begin{aligned}
 (35) \quad \tilde{g}_2 &= (q-1) \begin{array}{c} 2 \\ / \quad | \quad \backslash \\ 0 \quad 0 \quad 0 \end{array} + \begin{array}{c} 1 \\ / \quad \backslash \\ 0 \quad 0 \end{array} + \begin{array}{c} 1 \\ / \quad \backslash \\ 0 \quad 1 \\ \quad / \quad \backslash \\ \quad 0 \quad 0 \end{array} \\
 (36) \quad &= (q-1)S^{2000} + S^{11000} + S^{10100}.
 \end{aligned}$$

The exponent vectors I encoding Schröder trees as above will be referred to as *Schröder pseudocompositions*.

From (33), we have:

Theorem 4.1. *Let I be a Schröder pseudocomposition, and $T(I)$ be the tree encoded by I . The coefficient of S^I in g is*

$$(37) \quad m_I(q) = \prod_{v \in T(I)} \frac{1}{q^{\phi(v)-1}}$$

where v runs over the internal vertices of $T(I)$ and $\phi(v)$ is the number of leaves of the subtree of v . ■

Note 4.2. These coefficients are precisely those obtained by Ecalle's arborification method [14, 15, 16]. This coincidence will be explained in Section 11.

Example 4.3. For

$$(38) \quad T = \begin{array}{c} 2 \\ / \quad | \quad \backslash \\ 1 \quad 1 \quad 0 \\ / \quad \backslash \quad / \quad \backslash \\ 0 \quad 0 \quad 0 \quad 3 \\ \quad \quad \quad / \quad | \quad \backslash \\ \quad \quad \quad 0 \quad 0 \quad 0 \end{array}$$

decorating each internal vertex v with the number $\phi(v) - 1$, we obtain

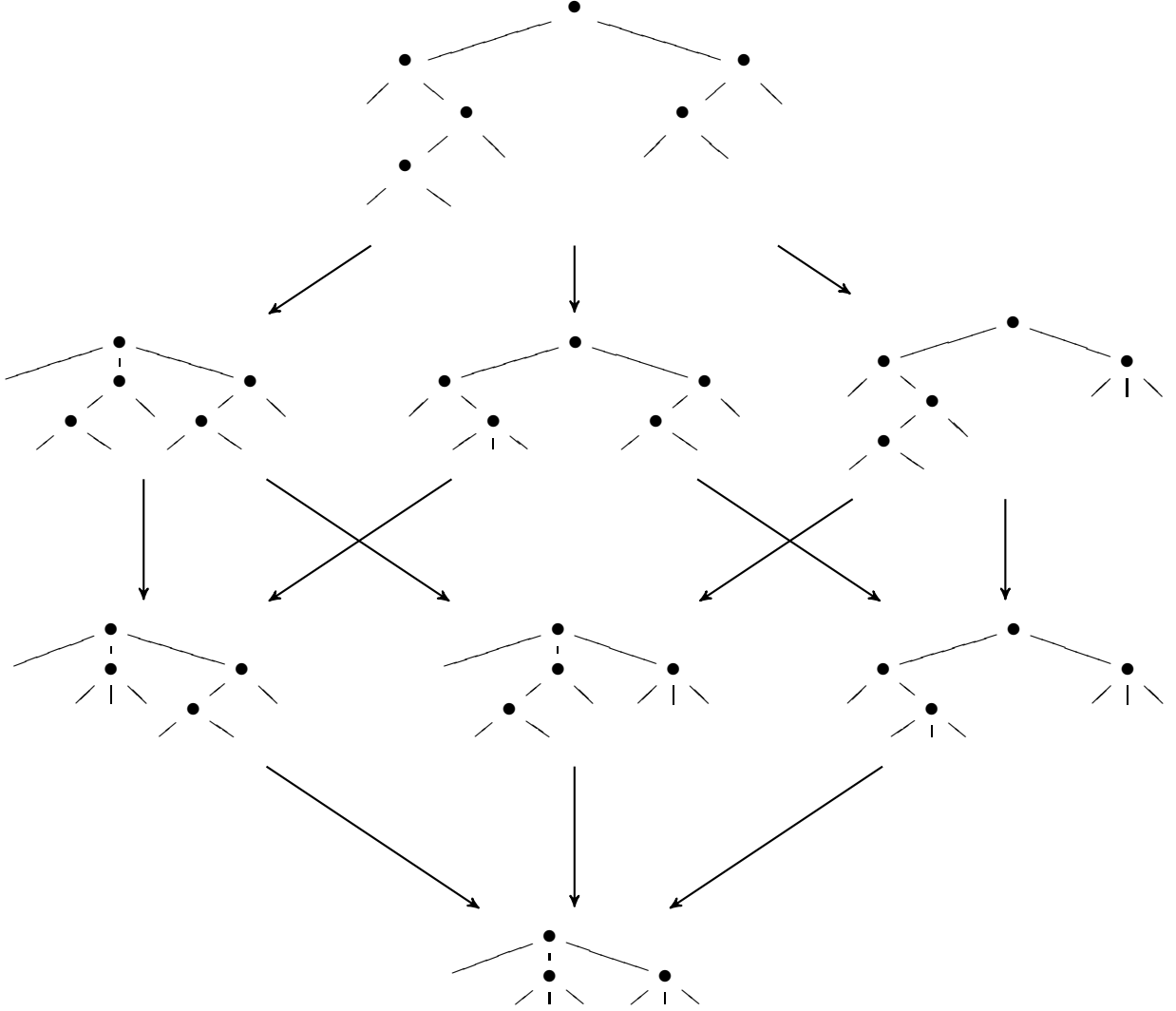
$$(39) \quad \begin{array}{c} 7 \\ / \quad | \quad \backslash \\ 1 \quad 4 \quad \quad \\ / \quad \backslash \quad / \quad \backslash \\ \quad \quad \quad 3 \quad \quad \quad \\ \quad \quad \quad / \quad | \quad \backslash \\ \quad \quad \quad \quad \quad \quad \end{array}$$

so that

$$(40) \quad m_{210010300000} = \frac{1}{q_7 q_4 q_3 q_1}.$$

Note 4.4. The I whose nonzero entries are all equal to 1 correspond to binary trees. The anti-refinements $J \preceq I$ of such an I , obtained by summing consecutive nonzero entries in all possible ways, correspond to the trees $T(J)$ obtained by contracting (internal) left edges in all possible ways in $T(I)$. This procedure provides a way to group Schröder trees into classes labelled by binary trees. An algebraic interpretation of these groups will be provided below.

Example 4.5. The contractions of the binary tree $T(1101100011000)$ are



which are respectively, reading the diagram by rows, $T(201100011000)$, $T(110200011000)$, $T(110110002000)$, $T(20200011000)$, $T(20110002000)$, $T(11020002000)$, and $T(2020002000)$.

5. A BINARY TREE EXPANSION

5.1. A bilinear map on Sym. The preceding remark (Note 4.4) suggests that, as in the case of Lagrange inversion, the conjugacy equation can be cast as a quadratic

fixed point problem. This is easily done at the level of noncommutative symmetric functions.

Let Ω be the linear operator on \mathbf{Sym} introduced in [38, 39], and defined by

$$(41) \quad \Omega S^{(i_1, \dots, i_r)} = S^{(i_1+1, i_2, \dots, i_r)}.$$

Writing

$$(42) \quad g(qtA) - g(tA) = \sum_{n \geq 1} t^n S_n(A) g(tA)^{n+1}$$

$$(43) \quad = \left(\sum_{n \geq 1} t^n S_n(A) g(tA)^n \right) g(tA)$$

$$(44) \quad = tS_1(A)g(tA)^2 + t \left(\sum_{n \geq 2} t^{n-1} S_n(A) g(tA)^n \right) g(tA)$$

$$(45) \quad = tS_1(A)g(tA)^2 + t\Omega \left[\sum_{n \geq 1} t^n S_n(A) g(tA)^{n+1} \right] g(tA)$$

$$(46) \quad = tS_1(A)g(tA)^2 + t\Omega[g(qtA) - g(tA)]g(tA)$$

we see that g is the unique solution of the quadratic functional equation

$$(47) \quad g(tA) = 1 + \int_0^t (S_1(A)g(sA) + t\Omega\Delta_q g(sA)) g(sA) d_q s$$

of the form

$$(48) \quad g = 1 + \mathbf{B}_q(g, g).$$

The bilinear map \mathbf{B}_q has a simple expression in the complete basis. For two compositions $I \vDash i$ and $J \vDash j$,

$$(49) \quad \mathbf{B}_q(S^I, S^J) = \int_0^1 (S_1 S^I S^J s^{i+j} + \Omega S^I (q^i - 1) s^i S^J s^j) d_q s$$

$$(50) \quad = \frac{S^{1IJ} + q_i \Omega S^{IJ}}{q_{i+j+1}}.$$

It follows that in the ribbon basis

$$(51) \quad \mathbf{B}_q(R_I, R_J) = \frac{(R_{1I} + q^i R_{1 \triangleright I}) R_J}{q_{i+j+1}}.$$

As a consequence, the coefficients of \tilde{g}_n on the ribbon basis are in $\mathbb{N}[q]$. A combinatorial interpretation will be provided below.

For example, with $g_0 = 1$, we have,

$$(52) \quad \begin{aligned} g_1 &= \mathbf{B}_q(g_0, g_0) \\ g_2 &= \mathbf{B}_q(g_0, g_1) + \mathbf{B}_q(g_1, g_0) \\ g_3 &= \mathbf{B}_q(g_0, g_2) + \mathbf{B}_q(g_1, g_1) + \mathbf{B}_q(g_2, g_0) \end{aligned}$$

so that one gets

$$\begin{aligned}
(53) \quad \tilde{g}_1 &= q_1 \mathbf{B}_q(\tilde{g}_0, \tilde{g}_0) \\
\tilde{g}_2 &= q_2 (\mathbf{B}_q(\tilde{g}_0, \tilde{g}_1) + \mathbf{B}_q(\tilde{g}_1, \tilde{g}_0)) \\
\tilde{g}_3 &= q_3 (\mathbf{B}_q(\tilde{g}_0, \tilde{g}_2) + \frac{q_2}{q_1} \mathbf{B}_q(\tilde{g}_1, \tilde{g}_1) + \mathbf{B}_q(\tilde{g}_2, \tilde{g}_0))
\end{aligned}$$

and

$$\begin{aligned}
(54) \quad \tilde{g}_1 &= S_1 \\
\tilde{g}_2 &= (S^{11}) + (S^{11} + q_1 S_2) \\
\tilde{g}_3 &= (2S^{111} + q_1 S^{12}) + \frac{q_2}{q_1} (S^{111} + q_1 S^{21}) + (2S^{111} + q_1 S^{12} + 2q_2 S^{21} + q_1 q_2 S_3) \\
&= q_1 q_2 S_3 + 3q_2 S^{21} + 2q_1 S^{12} + \left(\frac{q_2}{q_1} + 4 \right) S^{111},
\end{aligned}$$

which coincides with (26).

5.2. Triduplicial expansion. These equations can be lifted to Schröder trees, by setting as above $g_0 = S_0$ and using (49) without modification. We recover then the same expressions for g_n as in the previous section.

Indeed, start again with (53) and $\tilde{g}_0 = g_0 = S_0$. We then have

$$\begin{aligned}
(55) \quad \tilde{g}_1 &= S^{100}, \\
\tilde{g}_2 &= (S^{10100}) + (S^{11000} + q_1 S^{2000}), \\
\tilde{g}_3 &= (S^{1010100} + S^{1011000} + q_1 S^{102000}) + \frac{q_2}{q_1} (S^{1100100} + q_1 S^{200100}) \\
&\quad + (S^{1101000} + S^{1110000} + q_1 S^{120000} + q_2 S^{201000} + q_2 S^{210000} + q_1 q_2 S^{30000}),
\end{aligned}$$

which again gives back (34).

Now, (51) can be lifted in another way. Indeed, in the same way as Lagrange inversion is directly related to Catalan numbers (in the guise of nondecreasing parking functions) and to the free duplicial algebra on one generator **CQSym** [40], we find here the little Schröder numbers which are related to the free triduplicial algebra on one generator (defined in [40]), into which **CQSym** is naturally embedded.

More precisely, the co-hypoplactic subalgebra of **PQSym**, denoted by **SQSym** in [38], spanned by hypoplactic classes of parking functions (parking quasi-ribbons), has been identified in [40] as the free triduplicial algebra on one generator. Its graded dimension is given by the number of Schröder trees, but no natural bijection between these trees and parking quasi-ribbons had been known up to now. However, this algebra has a basis \mathbf{P}_α which is mapped to the ribbon basis R_I by a Hopf algebra morphism χ . This suggests that the operations on ribbons involved in (51) might be the image under χ of triduplicial operations on parking quasi-ribbons and that an analogue of the S -basis should exist in **SQSym**. We shall see that this is indeed the case, by means of bijections between three families of Schröder objects: parking quasi-ribbons, Schröder trees, and Schröder pseudocompositions. These bijections will allow to transport the triduplicial structure on the latter objects.

Recall from [38] that hypoplactic classes of parking functions are represented as parking quasi-ribbons, or segmented nondecreasing parking functions, *i.e.*, nondecreasing parking functions with bars allowed between different values, for example

$$(56) \quad \{1\}, \quad \{11, 12, 1|2\},$$

$$(57) \quad \{111, 112, 11|2, 113, 11|3, 122, 1|22, 123, 1|23, 12|3, 1|2|3\}.$$

With a parking quasi-ribbon α , we associate the elements

$$(58) \quad \mathbf{P}_\alpha := \sum_{\bar{\mathbf{a}}=\alpha} \mathbf{F}_{\mathbf{a}},$$

where $\bar{\mathbf{a}}$ denotes the hypoplactic class of \mathbf{a} . For example,

$$(59) \quad \mathbf{P}_{11|3} = \mathbf{F}_{131} + \mathbf{F}_{311}, \quad \mathbf{P}_{113} = \mathbf{F}_{113}.$$

The product formula in this basis is

$$(60) \quad \mathbf{P}_\alpha \mathbf{P}_\beta = \mathbf{P}_{\alpha|\beta'} + \mathbf{P}_{\alpha \cdot \beta'}$$

where $\beta' = \beta[|\alpha|]$ (*i.e.*, the word formed by the entries of β shifted by the length of α), and the dot denotes concatenation. The triduplicial operations on parking quasi-ribbons are defined by [40]

$$(61) \quad \alpha \prec \beta = \alpha \cdot \beta[\max(\alpha) - 1],$$

$$(62) \quad \alpha \circ \beta = \alpha | \beta[|\alpha|],$$

$$(63) \quad \alpha \succ \beta = \alpha \cdot \beta[|\alpha|].$$

One easily checks that they satisfy the seven triduplicial relations

$$(64) \quad \begin{aligned} (x \prec y) \prec z &= x \prec (y \prec z) \\ (x \circ y) \circ z &= x \circ (y \circ z) \\ (x \succ y) \succ z &= x \succ (y \succ z) \\ (x \succ y) \prec z &= x \succ (y \prec z) \\ (x \circ y) \prec z &= x \circ (y \prec z) \\ (x \succ y) \circ z &= x \succ (y \circ z) \\ (x \circ y) \succ z &= x \circ (y \succ z). \end{aligned}$$

In order to define the triduplicial operations on Schröder pseudocompositions, we first need a bijection, which will be described below.

5.3. A bijection between parking quasiribbons and Schröder trees. The bijection between Schröder pseudocompositions and Schröder trees is trivial, as it is essentially the Polish notation for the tree. The difficult point is the correspondence between trees and parking quasi-ribbons.

Among all Schröder trees, we have binary trees, and among parking quasi-ribbons, we have parking quasi-ribbons without bars, that are nondecreasing parking functions. Both are counted by Catalan numbers.

We shall first describe the bijection from binary trees to parking quasi-ribbons without bars. Its extension to all Schröder trees will then be straightforward. Let ϕ be this bijection. It is recursively defined as follows. Set $\phi(\emptyset) = \emptyset$ and $\phi(\bullet) = 1$.

Given a tree T with left and right subtrees respectively T_1 and T_2 , we have

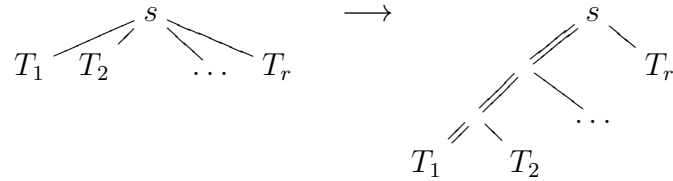
$$(65) \quad \phi(T) = \phi(T_2) \cdot (\phi(T_1)[\max(\phi(T_2)) - 1]) \cdot (|T_1| + \max(\phi(T_2))),$$

with the convention that, if T_2 is empty, $\max(\phi(T_2)) = 1$, and the dot denotes concatenation. This operation can also be described as collecting the vertices of T recursively by visiting first its right subtree, then its left subtree and finally its root, with the rules that a leaf takes the value of the last visited vertex (1 if there were none) and an internal vertex gets as value the size of its left subtree added to the value of its right son (added to 1 if there is no right son).

Example 5.1. We have

$$(66) \quad \begin{array}{c} 8 \\ / \quad \backslash \\ 6 \quad 5 \\ / \quad \backslash \quad / \quad \backslash \\ 6 \quad 4 \quad 4 \\ / \quad \backslash \quad / \quad \backslash \\ 5 \quad 2 \\ / \quad \backslash \\ 1 \quad 1 \end{array} \longrightarrow 1124455668.$$

Let us now extend ϕ to all Schröder trees. First, Schröder trees are in bijection with binary trees with two-colored left edges: if an internal node s has more than two children with corresponding subtrees T_1, T_2, \dots, T_r , draw $r - 1$ left edges (of the second color) from s and attach to the new r nodes the r subtrees in order, as in a binary tree:



This amounts to reverting the contraction process described in Note 4.4.

Having computed this tree, send it with the previous bijection to a nondecreasing parking function, and insert a bar between two letters if they are separated by a left branch of the second color.

Example 5.2. The continuation of (66) is

$$(67) \quad \begin{array}{c} 8 \\ / \quad \backslash \\ 6 \quad 5 \\ / \quad \backslash \quad / \quad \backslash \\ 6 \quad 4 \quad 4 \\ / \quad \backslash \quad / \quad \backslash \\ 5 \quad 2 \\ / \quad \backslash \\ 1 \quad 1 \end{array} \longrightarrow \begin{array}{c} 8 \\ // \quad \backslash \\ 6 \quad 5 \\ / \quad \backslash \quad // \quad \backslash \\ 6 \quad 4 \quad 4 \\ / \quad \backslash \quad // \quad \backslash \\ 5 \quad 2 \\ / \quad \backslash \\ 1 \quad 1 \end{array} \longrightarrow 11|244|5566|8.$$

Theorem 5.3. *The previous algorithm provides a bijection between Schröder trees and parking quasi-ribbons.*

Before proving the theorem, let us describe the inverse bijection $\psi = \phi^{-1}$ from nondecreasing parking functions to binary trees. Set $\psi(1) = \emptyset$. Let $p = a_1 \dots a_r$ be a nondecreasing parking function. Let $w = w_1 \dots w_{r-1}$ be the word such that $w_k = a_k + r - 1 - k$. Let ℓ be greatest index of w such that $a_\ell = a_{\ell+1}$ and $w_\ell = a_r$ (as $a_0 = 1$, if ℓ does not exist, set $\ell = 0$). Then compute recursively the images of $a_1 \dots a_\ell$ as the right subtree of the root, and of $(a_{\ell+1} \dots a_{r-1})[a_\ell - 1]$ as the left subtree of the root.

Example 5.4. Consider all nondecreasing parking functions $112445566X$, where X is bound by the constraint of being a nondecreasing parking function, so that $X \in \{6, 7, 8, 9, 10\}$. The word w is the same for all those parking functions, namely **988988776**, where we write in boldface the w_k such that $a_k = a_{k+1}$. Note that the 6 can be bold or not depending on the value of X : if X is 6, it is indeed in bold. Now, the index ℓ is well-defined in all the examples, so that we can separate the word and apply it recursively:

$$\begin{aligned}
 & 1124455666 \longrightarrow_{\ell=9} (\emptyset, 112445566) \\
 & 1124455667 \longrightarrow_{\ell=8} (1, 11244556) \\
 (68) \quad & 1124455668 \longrightarrow_{\ell=6} (122, 112445) \\
 & 1124455669 \longrightarrow_{\ell=4} (12233, 1124) \\
 & 11244556610 \longrightarrow_{\ell=0} (112445566, \emptyset)
 \end{aligned}$$

Proof – Let us now prove that ϕ is indeed a bijection and that its inverse is ψ as claimed.

First, the values of ϕ are clearly nondecreasing parking functions.

For ψ , the crucial point is to prove that it is well-defined (see (68) for an illustration).

Given a nondecreasing parking function $p = a_1 \dots a_r$, the allowed values for a_r are in the interval $[a_{r-1}, r]$. And this interval corresponds precisely to the values taken by the subword of w obtained by selecting the indices i such that $a_i = a_{i+1}$. Indeed, any of these values belong to this interval, since all values of w do. Conversely, a direct induction on the length of p implies the result, since the only question concerns the index $r - 1$ which is considered in the subword of w iff $a_{r-1} = a_r$. Finally, if one splits p after the rightmost occurrence w_ℓ of such a value, both the prefix of p and its suffix $a_{\ell+1} \dots a_{r-1}$ are parking functions: it is obvious for the prefix and is easy for the suffix, since we considered the *rightmost occurrence*. This occurrence has only strictly smaller values to its left ($w_i - w_{i+1}$ can be at most 1), so that the shifted suffix is a parking function. Now, by definition, the values of ψ are binary trees, so that at this point, we have maps going from each set to the other. Let us now see why they are inverses of each other.

Both maps are recursive, so we just need to prove that they are inverse of each other on the first step. Let $p = a_1 \dots a_r$ be a nondecreasing parking function which is the image under ϕ of a binary tree T having T_1 and T_2 as left and right subtrees.

Since a_r is the sum of the size of T_1 and of the maximal value of $\phi(T_2)$, a_r corresponds to the value w_ℓ in the word w where ℓ is the size of T_2 and $r - \ell - 1$ is the size of T_1 . Now, this value ℓ necessarily satisfies $a_\ell = a_{\ell+1}$, since a_ℓ is the maximal value M of T_2 and $a_{\ell+1} = 1 + M - 1$. Finally, among all indices k satisfying $a_k = a_{k+1}$ and $w_k = w_\ell$, ℓ is the only one such that the suffix $a_{\ell+1} \dots a_{r-1}$ is a parking function, since any other occurrence in w has one equal value to its right, which contradicts the fact of being a nondecreasing parking function.

Since ϕ and ψ both have the right image sets and $\psi \circ \phi$ is the identity map on binary trees, they both are bijections, inverses of each other.

Finally, let us see why the extension of ϕ to Schröder trees is a bijection. First, all left branches relate numbers that cannot be equal, so that separations on nondecreasing parking functions are made between non equal letters, which is the required condition about parking quasi-ribbons. The converse is also true: the number of left branches in a binary tree T is equal to the number of different letters plus one in $\phi(T)$. So the map from binary trees with two-colored left branches to parking quasi-ribbons is a bijection and the composition of both bijections through the middle object of binary trees with two-colored left branches is still a bijection. ■

5.4. Triduplicial operations on Schröder pseudocompositions. Now that we have a bijection between parking quasiribbons and Schröder pseudocompositions, we can translate the triduplicial operations initially defined on parking quasiribbons to Schröder pseudocompositions.

Definition 5.5. *Let I and J be two Schröder pseudocompositions. Define J' such that $J = J'0^m$ and J' does not end by a 0.*

Then

$$(69) \quad I \prec J = J \triangleright I = J'0^{m-1}.I$$

$$(70) \quad I \circ J = J' \triangleright I \cdot 0^{m-1}$$

$$(71) \quad I \succ J = J' \cdot I \cdot 0^{m-1}.$$

Example 5.6. Denoting by a the parking quasi-ribbon 1 and by x the pseudocomposition 100,

$$(72) \quad a \prec a = 11 \quad x \prec x = 10100$$

$$(73) \quad a \circ a = 1|2 \quad x \circ x = 2000$$

$$(74) \quad a \succ a = 12 \quad x \succ x = 11000,$$

which coincide with the bijection described in the previous section.

Theorem 5.7. *The operations above endow the set of Schröder pseudocompositions with the structure of a triduplicial algebra, freely generated by $x = 100$.*

Proof – This is a direct consequence of the translation of the triduplicial operations on Schröder pseudocompositions.

We shall prove it for each rule. Operation \prec on parking quasi-ribbons is defined by $\alpha \prec \beta = \alpha \cdot \beta[\max(\alpha) - 1]$ and via the bijection ψ extended to parking quasi-ribbons, it corresponds to glueing the image of α to the rightmost leaf of β so that, on Schröder pseudocompositions, one obtains $J' \cdot 0^{m-1} \cdot I$.

Operations \circ and \succ on parking quasi-ribbons are defined by $\alpha \circ \beta = \alpha | \beta[|\alpha|]$ and $\alpha \succ \beta = \alpha \cdot \beta[|\alpha|]$ and via the bijection ψ extended to parking quasi-ribbons, it corresponds to putting the image of α as the left child of the rightmost internal node labelled 1 of the image of β (which is also the last visited internal node of the tree in Polish notation) with an edge of the natural color (for \succ) or of the second color (for \circ). The translation on Schröder pseudocompositions is straightforward. \blacksquare

Define now an order \leq on parking quasiribbons by the cover relation

$$(75) \quad \beta \succ \alpha \quad \text{if } \alpha = uv, \beta = u|v'$$

with $v' = v$ if the last letter of u is smaller than the first letter of v , and $v' = v[1]$ otherwise.

For example, the predecessors of $11|23$ are $11|2|3$ and $1|2|34$.

With this order, we can define a basis \mathbf{S}^α by

$$(76) \quad \mathbf{S}^\alpha = \sum_{\alpha \leq \beta} \mathbf{P}_\beta.$$

For example,

$$(77) \quad \mathbf{S}^1 = \mathbf{P}_1, \mathbf{S}^{11} = \mathbf{P}_{11} + \mathbf{P}_{1|2}, \mathbf{S}^{12} = \mathbf{P}_{12} + \mathbf{P}_{1|2}, \mathbf{S}^{1|2} = \mathbf{P}_{1|2}$$

and

$$(78) \quad \mathbf{S}^{11|23} = \mathbf{P}_{11|23} + \mathbf{P}_{1|2|34} + \mathbf{P}_{11|2|3} + \mathbf{P}_{1|2|3|4}.$$

The Hopf epimorphism $\chi : \mathbf{SQSym} \rightarrow \mathbf{Sym}$ is defined by

$$(79) \quad \chi(\mathbf{P}_\alpha) = R_{I \sim}$$

where I is the bar composition of α whose parts are the lengths of the factors between the bars, *e.g.*, for $\alpha = 111|24|5$, $I = 321$.

It is then clear that

$$(80) \quad \chi(\mathbf{S}^\alpha) = S^{I \sim}.$$

For $U_m \in \mathbf{SQSym}_m$ and $V_n \in \mathbf{SQSym}_n$, define

$$(81) \quad B_q(U_m, V_n) = q_{m+n+1} \mathbf{B}_q(U_m, V_n),$$

where \mathbf{B}_q is defined after Eq. (48). Then, in the bases \mathbf{P} and \mathbf{S} with both indexations:

$$(82) \quad B_q(\mathbf{S}^\alpha, \mathbf{S}^\beta) = q_{|\alpha|} \mathbf{S}^{\beta \prec (\alpha \circ 1)} + \mathbf{S}^{\beta \prec (\alpha \succ 1)}$$

$$(83) \quad B_q(\mathbf{S}^I, \mathbf{S}^J) = q_{|I|} \Omega \mathbf{S}^{IJ} + \mathbf{S}^{1IJ}$$

$$(84) \quad B_q(\mathbf{P}_\alpha, \mathbf{P}_\beta) = q^{|\alpha|} \mathbf{P}_{\beta \prec (\alpha \circ 1)} + \mathbf{P}_{\beta \prec (\alpha \succ 1)} + q^{|\alpha|} \mathbf{P}_{\beta \circ \alpha \circ 1} + \mathbf{P}_{\beta \circ \alpha \succ 1}$$

$$(85) \quad B_q(\mathbf{P}_I, \mathbf{P}_J) = q^{|I|} \mathbf{P}_{\Omega IJ} + \mathbf{P}_{1IJ} + q^{|I|} P_{1 \triangleright J' \triangleright I 0^m} + \mathbf{P}_{1J' \triangleright I 0^m}$$

where, as above, $J = J' 0^m$ and J' does not end by a 0.

For example, one can recover the computation of \tilde{g}_3 in (55): start from the expression of \tilde{g}_1 and \tilde{g}_2 in this same equation and then compute \tilde{g}_3 according to Eq. (53):

The first term is $B_q(\tilde{g}_0, \tilde{g}_2)$:

$$(86) \quad \begin{aligned} B_q(S_0, S^{10100}) &= q_0 S^{110100} + S^{1010100} = S^{1010100}, \\ B_q(S_0, S^{11000}) &= q_0 S^{111000} + S^{1011000} = S^{1011000}, \\ B_q(S_0, S^{2000}) &= q_0 S^{12000} + S^{102000} = S^{102000}. \end{aligned}$$

Note that some S^I here are not indexed by Schröder pseudocompositions, but these terms eventually disappear as their coefficient is $q_0 = q^0 - 1 = 0$. The second term is $B_q(\tilde{g}_1, \tilde{g}_1)$:

$$(87) \quad B_q(S^{100}, S^{100}) = q_1 S^{200100} + S^{1100100}.$$

The third term is $\mathbf{B}_q(\tilde{g}_2, \tilde{g}_0)$:

$$(88) \quad \begin{aligned} B_q(S^{10100}, S_0) &= q_2 S^{201000} + S^{1101000}, \\ B_q(S^{11000}, S_0) &= q_2 S^{210000} + S^{1110000}, \\ B_q(S^{2000}, S_0) &= q_2 S^{30000} + S^{120000}, \end{aligned}$$

so that we recover (55).

6. EXPANSION ON THE RIBBON BASIS

The expression of g in **Sym** is recovered by setting $S_0 = 1$. As in the case of the Lagrange series, it is interesting to expand g on the ribbon basis. As we have already seen before, the first terms are

$$(89) \quad \tilde{g}_1 = R_1,$$

$$(90) \quad \tilde{g}_2 = (1 + q)R_2 + 2R_{11},$$

$$(91) \quad \tilde{g}_3 = (1 + q)(1 + q + q^2)R_3 + (2 + q + 3q^2)R_{21} + 3(1 + q)R_{12} + (5 + q)R_{111}.$$

We can observe that each coefficient is a q -analogue of $n!$. We shall now prove this fact, and describe the relevant statistics on permutations.

For a pseudo-composition I , let \hat{I} be the ordinary composition obtained by removing the zero entries.

For a binary tree $t = T(I)$, set

$$(92) \quad P_t = \sum_{J \preceq I} m_J(q) S^J$$

For a pseudo-composition J encoding a tree $T(J)$, let

$$(93) \quad d_J = \sum_{v \in T(J)} (\phi(v) - 1)$$

Then, the coefficient of R_K in $(q)_n P_t$ is equal to

$$(94) \quad (q)_n m_I(q) q^{d_I - d_J}$$

where J is the coarsest anti-refinement of I such that $K \leq \hat{J}$. Indeed, R_K will then occur in all the refinements of J , and if I' is such a refinement, then,

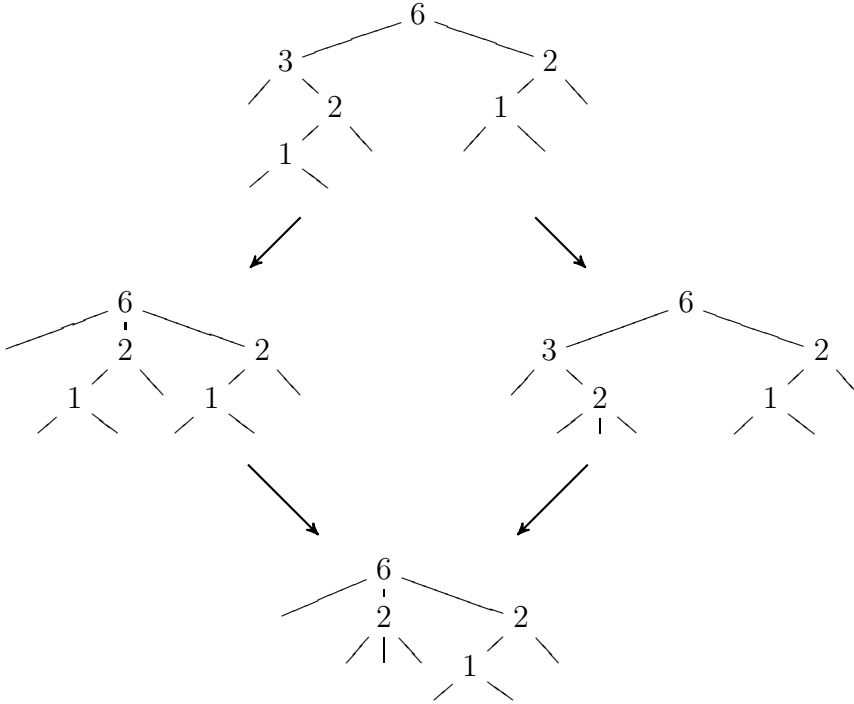
$$(95) \quad m_{I'}(q) = m_I(q) \prod_{v \in C(I, I')} q_{\phi(v)-1}$$

where $C(I, I')$ is the set of vertices of $T(I)$ which have been contracted in $T(I')$. Thus, factoring the coefficient $m_I(q)$, we see that R_K picks up a factor

$$(96) \quad \prod_i (q_i + 1)^{n_i} = q^{d_I - d_J} \quad \text{if} \quad \frac{m_J(q)}{m_I(q)} = \prod_i q_i^{n_i}$$

when summing over the Boolean lattice of refinements of J .

Example 6.1. For $I = (1101100011000)$ and $K = (51)$, we have $J = (20200011000)$, and on the picture (extracted from 4.5)



we can read that the coefficient of R_{51} in the projection of $(q)_6 P_{T(1101100011000)}$ on **Sym** is

$$(97) \quad \frac{(q)_6}{q_6 q_3 q_2^2 q_1^2} q^4 = \frac{q_5 q_4}{q_2 q_1} q^4 = q^{10} + 2q^8 + q^9 + 2q^7 + 2q^6 + q^5 + q^4.$$

For a pseudo-composition I , let $I^\#$ be its coarsest anti-refinement. The polynomial

$$(98) \quad (q)_n m_I(q) q^{d_I - d_{I^\#}}$$

is, up to left-right symmetry, the q -hook-length formula for the binary tree $T(I)$ [1, 25] (with its leaves removed). Indeed, for a vertex v of a binary tree $t = T(I)$, $d_I - d_{I^\#}$ coincides with the number of internal nodes of its left subtree.

Example 6.2. Continuing with $I = (110110011000)$, $I^\# = (2020002000)$,

$$(99) \quad T(I) = \begin{array}{c} 6 \\ / \quad \backslash \\ 3 \quad 2 \\ / \backslash \quad / \backslash \\ 1 \quad 2 \quad 1 \quad \backslash \\ / \quad \backslash \quad / \quad \backslash \\ \quad \quad \quad \end{array} \quad T(I^\#) = \begin{array}{c} 6 \\ / \quad \backslash \\ 2 \quad 2 \\ / \backslash \quad / \backslash \\ 1 \quad \quad 1 \quad \end{array}$$

and (98) yields

$$(100) \quad \frac{(q)_6}{q_6 q_3 q_2^2 q_1^2} q^{6+3+2+1+2+1-(6+2+2)} = \frac{q_5 q_4}{q_2 q_1} q^{3+1+1} = \frac{q_5 q_4}{q_2 q_1} q^5$$

which is the q -hook-length formula for the left-right flip

$$(101) \quad \begin{array}{c} 6 \\ / \quad \backslash \\ 2 \quad 3 \\ / \backslash \quad / \backslash \\ 1 \quad 2 \quad 1 \quad \backslash \\ \quad \quad \quad \end{array} \quad \begin{array}{c} 3 \\ / \quad \backslash \\ 1 \quad 0 \\ / \backslash \quad / \backslash \\ 0 \quad 1 \quad 0 \quad \backslash \\ \quad \quad \quad \end{array}$$

of the skeleton of $T(I)$. Indeed, the power of q is $3 + 1 + 1 = 5$ (given by the sum of the sizes of the right subtrees), and the denominator is $q_6 q_2 q_1 q_3 q_2 q_1$ (recording the sizes of all the subtrees).

Thus, up to a fixed power of q , this expression enumerates by number of non-inversions the permutations in the sylvester class labelled by the binary tree $T(I)$.

Summing over all binary trees, we see that each coefficient $c_K(q)$ of R_K in \tilde{g}_n is indeed a q -analogue of $n!$.

Translating these results at the level of permutations yields the following description of the expansion of \tilde{g}_n on the ribbon basis of **Sym**:

Definition 6.3. Let σ be a permutation and let α be the top of its sylvester class, that is, the permutation with the smallest number of inversions in its sylvester class. Let $I = (i_1, \dots, i_r)$ be a composition and let D be the descent set of the conjugate \tilde{I} . Define $C_I(\sigma)$ as

$$(102) \quad q^{\text{inv}(\sigma) - \text{inv}(\alpha)} q^{\text{inv}(\alpha, D)},$$

where $\text{inv}(\alpha, D)$ is the number of pairs $(i < j)$ such that $\alpha_i > \alpha_j$ and $j \in D$.

Theorem 6.4. The coefficient of R_I in the expansion of \tilde{g}_n is equal to

$$(103) \quad \sum_{\sigma \in \mathfrak{S}_n} C_I(\sigma).$$

■

For example, here are the tables for $n = 3$ and $n = 4$ of all coefficients C_I , where permutations are grouped by sylvester classes.

(104)

σ/I	3	21	12	111
123	1	1	1	1
132	q	q	1	1
312	q^2	q^2	q	q
213	q	1	q	1
231	q^2	q^2	1	1
321	q^3	q^2	q	1

(105)

σ/I	4	31	22	211	13	121	112	1111
1234	1	1	1	1	1	1	1	1
1243	q	q	q	q	1	1	1	1
1423	q^2	q^2	q^2	q^2	q	q	q	q
4123	q^3	q^3	q^3	q^3	q^2	q^2	q^2	q^2
1324	q	q	1	1	q	q	1	1
3124	q^2	q^2	q	q	q^2	q^2	q	q
1342	q^2	q^2	q^2	q^2	1	1	1	1
3142	q^3	q^3	q^3	q^3	q	q	q	q
3412	q^4	q^4	q^4	q^4	q^2	q^2	q^2	q^2
1432	q^3	q^3	q^2	q^2	q	q	1	1
4132	q^4	q^4	q^3	q^3	q^2	q^2	q	q
4312	q^5	q^5	q^4	q^4	q^3	q^3	q^2	q^2
2134	q	1	q	1	q	1	q	1
2143	q^2	q	q^2	q	q	1	q	1
2413	q^3	q^2	q^3	q^2	q^2	q	q^2	q
4213	q^4	q^3	q^4	q^3	q^3	q^2	q^3	q^2
2314	q^2	q^2	1	1	q^2	q^2	1	1
2341	q^3	q^3	q^3	q^3	1	1	1	1
2431	q^4	q^4	q^3	q^3	q	q	1	1
4231	q^5	q^5	q^4	q^4	q^2	q^2	q	q
3214	q^3	q^2	q	1	q^3	q^2	q	1
3241	q^4	q^3	q^4	q^3	q	1	q	1
3421	q^5	q^5	q^3	q^3	q^2	q^2	1	1
4321	q^6	q^5	q^4	q^3	q^3	q^2	q	1

7. SCHRÖDER'S EQUATION FOR THE INVERSE OF h

Let now $f = h^{-1}$ where $h^{-1} \circ \phi \circ h(z) = qz$ or equivalently

$$(106) \quad f \circ \phi(w) = qf(w) \quad (w = h(z)).$$

This is Schröder's equation. In the noncommutative setting, with again $\phi(z) = qz\sigma_z(A)$, it becomes

$$(107) \quad \sum_{k \geq 0} f_k q^{k+1} w^{k+1} \sigma_w((k+1)A) = q \sum_{n \geq 0} f_n w^{n+1}$$

which translates into the recurrence relation

$$(108) \quad f_n = \sum_{k+l=n} q^k f_k S_l((k+1)A).$$

Theorem 7.1. *Let L be the linear endomorphism of \mathbf{Sym} defined by*

$$(109) \quad L(S^I) = S_{i_1}(A) S_{i_2}((i_1+1)A) S_{i_3}((i_1+i_2+1)A) \cdots S_{i_r}((i_1+\cdots+i_{r-1}+1)A).$$

Then,

$$(110) \quad f_n = L \left(S_n \left(\frac{A}{1-q} \right) \right) = \sum_{I \models n} \frac{q^{\text{maj}(I)}}{(1-q^{i_1})(1-q^{i_1+i_2}) \cdots (1-q^{i_1+\cdots+i_r})} L(S^I),$$

where $A/(1-q)$ and maj is defined as in [29, 6.1]³.

For example,

$$(111) \quad f_1 = \frac{1}{1-q} S_1(A)$$

$$(112) \quad f_2 = \frac{1}{1-q^2} S^2(A) + \frac{q}{(1-q)(1-q^2)} S_1(A) S_1(2A)$$

$$(113) \quad f_3 = \frac{1}{1-q^3} S_3(A) + \frac{q^2}{(1-q^2)(1-q^3)} S_2(A) S_1(3A) \\ + \frac{q}{(1-q)(1-q^3)} S_1(A) S_2(2A) + \frac{q^3}{(1-q)(1-q^2)(1-q^3)} S_1(A) S_1(2A) S_1(3A)$$

Proof – Replacing $S_l((k+1)A)$ by $S_l(A)$ in (108), we obtain a recurrence relation satisfied by the expansion on the basis S^I of the $S_n(A/(1-q))$. To recover f_n from this expression, we just have to replace each factor S_{i_k} of S^I by $S_{i_k}((i_1+\cdots+i_{k-1}+1)A)$. ■

³ $\text{maj}(I)$ is the sum of the descents of I , i.e., $\text{maj}(I) = (r-1)i_1 + (r-2)i_2 + \cdots + i_{r-1}$ if $I = (i_1, \dots, i_r)$.

Note 7.2. The noncommutative symmetric function

$$(114) \quad K_n(q) = (q)_n S_n \left(\frac{A}{1-q} \right) = \sum_{I \models n} q^{\text{maj}(I)} R_I$$

is the q -Klyachko function. For q a primitive n th root of 1, it is mapped to Klyachko's Lie idempotent in the descent algebra of the symmetric group [19, 29]. It is also a noncommutative analogue of the Hall-Littlewood function Q'_{1^n} [24].

Naturally, we may also choose to define f by

$$(115) \quad f_n = \sum_{k+l=n} q^k S_l((k+1)A) f_k.$$

With this choice

$$(116) \quad f = \overline{\left(L \left(\sigma_1 \left(\frac{A}{1-q} \right) \right) \right)}$$

and it is what we obtain by mould calculus in the next section (the bar involution is defined by $\overline{S^{\bar{I}}} = S^{\bar{I}}$, where \bar{I} is the mirror composition).

8. A NONCOMMUTATIVE MOULD EXPANSION

Ecalle's approach to the linearization equation (7) is to find a closed expression of the substitution automorphism

$$(117) \quad H : \psi \longmapsto \psi \circ h$$

as a differential operator. The idea is to look for an expansion of the form

$$(118) \quad H = \sum_I \mathcal{M}_I U^I$$

where $I = (i_1, \dots, i_r)$ runs over all compositions, $U^I = U_{i_1} \cdots U_{i_r}$ as usual, and the U_n are the homogeneous component of the differential operator

$$(119) \quad U : \psi \longmapsto \psi \circ u, \quad \text{where } \phi(z) = qu(z),$$

given by the Taylor expansion at z of $\psi(z + (u(z) - z))$,

$$(120) \quad U_n = \sum_{I \models n} \frac{u^I}{\ell(I)} z^{n+\ell(I)} \partial_z^{\ell(I)}.$$

In this setting, the functional equation (7) reads

$$(121) \quad HUM_q = M_q H, \quad \text{where } (M_q \psi)(z) = \psi(qz).$$

We have already identified our generic power series $\phi(z)$ with $qz\sigma_z(X)$, so that $u_n = h_n(X)$. A natural noncommutative analogue is to set $u_n = S_n(A)$. There is another way to introduce noncommutative symmetric functions in this problem. The substitution maps U and H , being automorphisms, are grouplike elements in the Hopf algebra of differential operators. So, it is natural to introduce a second alphabet B (commuting with A) and to identify U_n with $S_n(B)$. The problem amounts to looking for H as an element of (the completion of) $\mathbf{Sym}(A) \otimes \mathbf{Sym}(B)$. Let us

write it as $H(B)$, regarded as a symmetric function of B with coefficients in $\mathbf{Sym}(A)$. Then (121) reads now

$$(122) \quad H(B)\sigma_1(B) = H(qB).$$

This is solved by

$$(123) \quad H(B) = \prod_{i \geq 0}^{\leftarrow} \lambda_{-q^i}(B) = \cdots \lambda_{-q^2}(B)\lambda_{-q}(B)\lambda_{-1}(B)$$

which may be denoted by

$$(124) \quad H(B) = \sigma_1 \left(\frac{B}{q-1} \right)$$

Setting $F = H^{-1}$, we have that

$$(125) \quad F(B) = \sigma_1(B)\sigma_q(B)\sigma_{q^2}(B)\cdots$$

is the image of $\sigma_1(A/(1-q))$ (in the sense of [29]) by the bar involution $S^I \mapsto S^{\bar{I}}$, so that

$$(126) \quad F_n = \sum_{I \models n} \frac{q^{\text{maj}(I)}}{(1-q^{i_1})(1-q^{i_1+i_2})\cdots(1-q^n)} S^{\bar{I}},$$

The function $f(z)$ is obtained by acting on the identity: $f(z) = Fz$. This is obtained from

$$(127) \quad U^I z = \overline{L(S^{\bar{I}}(A))} z^{|I|+1}.$$

Indeed,

$$(128) \quad U_n z^m = \sum_{I \models n} \frac{S^I}{\ell(I)!} \frac{m!}{(m-\ell(I))!} z^{m+n} = \sum_{I \models n} M_I(m) S^I z^{m+n} = S_n(mA) z^{m+n}.$$

9. THE OPERAD OF REDUCED PLANE TREES

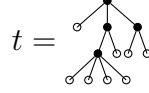
9.1. A free operad. We shall now investigate the relation between our Schröder tree expansion (Section 4) and Ecalle's arborification. So far, the S^I with I a Schröder pseudocomposition have been interpreted as elements of the free triduplicial algebra \mathbf{SQSym} . They can also be interpreted as elements of a free operad (see [3, 5]). We shall see that this operad, which is based on reduced plane trees, is also related to the noncommutative version of the Hopf algebra of formal diffeomorphisms tangent to identity [2].

The set of reduced plane trees with n leaves will be denoted denoted by PT_n , and PT denotes the union $\bigcup_{n \geq 1} \text{PT}_n$.

The number of leaves of a tree t will be called its degree $d(t)$, and we define the grading $\text{gr}(t)$ of a tree as its degree minus 1. In low degrees we have

$$(129) \quad \text{PT}_1 = \{\circ\}, \quad \text{PT}_2 = \{\circ \circ\}, \quad \text{PT}_3 = \left\{ \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \circ \quad \circ \end{array} \right\}, \dots$$

The leaves (in white in the pictures) are also called external vertices whilst the other vertices (in black) are said to be internal (note that \circ has no internal vertex). For instance, the tree



has degree $d(t) = 8$, grading $\text{gr}(t) = 7$ and $i(t) = 4$ internal vertices.

Definition 9.1. *The free non- Σ operad \mathcal{S} in the category of vector spaces is the vector space*

$$(130) \quad \mathcal{S} = \bigoplus_{n \geq 1} \mathcal{S}_n \text{ where } \mathcal{S}_n = \mathbb{C} \text{PT}_n.$$

The composition operations

$$(131) \quad \mathcal{S}_n \otimes \mathcal{S}_{k_1} \otimes \dots \otimes \mathcal{S}_{k_n} \longrightarrow \mathcal{S}_{k_1 + \dots + k_n} \quad (n \geq 1, k_i \geq 1)$$

map the tensor product of trees $t_0 \otimes t_1 \otimes \dots \otimes t_n$ to the tree $t_0 \circ (t_1, \dots, t_n)$ obtained by replacing the leaves of t_0 , from left to right, by the trees t_1, \dots, t_n .

For instance,

$$(132) \quad \begin{array}{c} \bullet \\ / \quad \backslash \\ \circ \quad \bullet \\ | \quad / \quad \backslash \\ \bullet \quad \bullet \quad \bullet \\ / \quad \backslash \\ \circ \quad \circ \end{array} \circ (\circ, \begin{array}{c} \bullet \\ / \quad \backslash \\ \circ \quad \bullet \\ | \quad / \quad \backslash \\ \bullet \quad \bullet \quad \bullet \\ / \quad \backslash \\ \circ \quad \circ \end{array}, \circ, \begin{array}{c} \bullet \\ / \quad \backslash \\ \circ \quad \bullet \\ | \quad / \quad \backslash \\ \bullet \quad \bullet \quad \bullet \\ / \quad \backslash \\ \circ \quad \circ \end{array}) = \begin{array}{c} \bullet \\ / \quad \backslash \\ \circ \quad \bullet \\ | \quad / \quad \backslash \\ \bullet \quad \bullet \quad \bullet \\ / \quad \backslash \quad / \quad \backslash \\ \circ \quad \circ \quad \circ \quad \circ \end{array}$$

The tree \circ of PT_1 is the unit of this composition. A proof of its associativity can be found in [3] or [5], where this operad is called a free S -magmatic operad. Note that \mathcal{S} is also called the operad of Stasheff polytopes (see [26, 32]) so that the letter \mathcal{S} can stand for Stasheff or Schröder as well.

9.2. The group of the operad. Let $\hat{\mathcal{S}}$ be the completion of the vector space \mathcal{S} with respect to the grading $\text{gr}(t) = d(t) - 1$. The group of the operad \mathcal{S} is defined as:

Definition 9.2. *Let*

$$(133) \quad G_{\text{ncdiff}} = \left\{ \circ + \sum_{n \geq 2} p_{(n)}, \quad p_{(n)} \in \mathcal{S}_n \right\} \subset \hat{\mathcal{S}}$$

endowed with the composition product

$$(134) \quad p \circ q = q + \sum_{n \geq 2} p_{(n)} \circ \left(\underbrace{q, \dots, q}_n \right) \in G_{\text{ncdiff}}$$

for $p = \circ + \sum_{n \geq 2} p_{(n)}$ and $q \in G_{\text{ncdiff}}$.

This is indeed a group (see *e.g.*, [5]). Elements of G_{ncdiff} can be described by their coordinates

$$(135) \quad p = \sum_{t \in \text{PT}} p_t t \quad \text{and} \quad q = \sum_{t \in \text{PT}} q_t t.$$

(with $q_\circ = p_\circ = 1$) so that the coordinates of $r = p \circ q$ are given by

$$(136) \quad r_t = \sum_{t=t_0 \circ (t_1, \dots, t_n)} p_{t_0} q_{t_1} \cdots q_{t_n}$$

This expression involves the so-called admissible cuts defining the coproduct in Hopf algebras of the Connes-Kreimer family. It suggests that the elements of G_{ncdiff} can be interpreted as characters of the bialgebra defined as follows.

Let

$$(137) \quad \bar{T}(\mathcal{S}) = \bigoplus_{p \geq 1} \mathcal{S}^{\otimes p}$$

be the reduced tensor algebra over \mathcal{S} (whose basis is given by plane forests $f = t_1 \cdots t_k$) equipped with the coalgebra structure defined on trees by

$$(138) \quad \tilde{\Delta}(t) = \sum_{t=t_0 \circ (t_1, \dots, t_n)} t_0 \otimes (t_1 \cdots t_n)$$

where \cdot means concatenation, and then extended as an algebra morphism on $\bar{T}(\mathcal{S})$. For example,

$$(139) \quad \tilde{\Delta} \left(\begin{array}{c} \bullet \\ / \quad \backslash \\ \circ \quad \circ \\ / \quad \backslash \\ \circ \quad \circ \end{array} \right) = \circ \otimes \begin{array}{c} \bullet \\ / \quad \backslash \\ \circ \quad \circ \\ / \quad \backslash \\ \circ \quad \circ \end{array} + \begin{array}{c} \bullet \\ / \quad \backslash \\ \circ \quad \circ \\ / \quad \backslash \\ \circ \quad \circ \end{array} \otimes \begin{array}{c} \bullet \\ / \quad \backslash \\ \circ \quad \circ \\ / \quad \backslash \\ \circ \quad \circ \end{array} \cdot \circ + \begin{array}{c} \bullet \\ / \quad \backslash \\ \circ \quad \circ \\ / \quad \backslash \\ \circ \quad \circ \end{array} \otimes \circ \cdots \circ$$

It is then clear that any $p \in G_{\text{ncdiff}}$ can be identified as the algebra morphism φ_p defined by

$$(140) \quad \varphi_p(t) = p_t$$

so that, if $r = p \circ q$, then

$$(141) \quad \varphi_{p \circ q} = \varphi_r = \varphi_p * \varphi_q = \mu \circ (\varphi_p \otimes \varphi_q) \circ \tilde{\Delta}$$

where $*$ is the usual convolution product for a bialgebra, and μ is the multiplication of \mathbb{C} . Note that if $\bigwedge(t_1 \cdot t_2 \cdots t_n)$ (with $n \geq 2$) is the tree obtained by grafting the trees t_1, \dots, t_n to a common root (in other words, $\bigwedge(t_1 \cdot t_2 \cdots t_n) = c_n \circ (t_1, \dots, t_n)$, where c_n is the corolla with n leaves) the map $\tilde{\Delta}$ is the unique algebra map such that

$$(142) \quad \tilde{\Delta}(\circ) = \circ \otimes \circ,$$

$$(143) \quad \tilde{\Delta} \left(\bigwedge(t_1, \dots, t_n) \right) = \circ \otimes \bigwedge(t_1, \dots, t_n) + \left(\bigwedge \otimes \text{Id} \right) \circ \tilde{\Delta}(t_1 \cdot t_2 \cdots t_n)$$

9.3. The Hopf algebra of reduced plane trees and its characters. The quotient of the bialgebra $\bar{T}(\mathcal{S})$ by the relations $t \cdot \circ = \circ \cdot t = t$ is a graded unital algebra \mathcal{H}_{PT} , spanned by \circ and the forests $t_1 \cdot t_2 \cdots t_n$ with $t_i \in \cup_{n \geq 2} \text{PT}_n$, with \circ as unit.

It is a Hopf algebra for the coproduct defined on trees by

$$(144) \quad \Delta(t) = (p \otimes p) \circ \tilde{\Delta}(t)$$

where p is the projection from $\bar{T}(\mathcal{S})$ to \mathcal{H}_{PT} .

In the former example:

$$(145) \quad \Delta \left(\begin{array}{c} \bullet \\ / \quad \backslash \\ \circ \quad \circ \\ / \quad \backslash \\ \circ \quad \circ \end{array} \right) = \circ \otimes \begin{array}{c} \bullet \\ / \quad \backslash \\ \circ \quad \circ \\ / \quad \backslash \\ \circ \quad \circ \end{array} + \begin{array}{c} \bullet \\ / \quad \backslash \\ \circ \quad \circ \\ / \quad \backslash \\ \circ \quad \circ \end{array} \otimes \begin{array}{c} \bullet \\ / \quad \backslash \\ \circ \quad \circ \\ / \quad \backslash \\ \circ \quad \circ \end{array} + \begin{array}{c} \bullet \\ / \quad \backslash \\ \circ \quad \circ \\ / \quad \backslash \\ \circ \quad \circ \end{array} \otimes \circ$$

It is now clear that the group G_{ncdiff} is precisely the group of characters of this Hopf algebra (for $p \in G_{\text{ncdiff}}$, $\varphi_p(\circ) = 1$).

This Hopf algebra was first considered in [12] (with the opposite coproduct $P \circ \Delta$, where $P(u \otimes v = v \otimes u)$), where it is called the Hopf algebra of reduced plane trees $\mathcal{H}_{\text{pl}}^{\text{red}}$. The coproduct can be described in terms of admissible cuts of a tree $t \in \text{PT}$, *i.e.*, (possibly empty) subsets c of edges *not connected to a leaf* with the rule that along any path from the root of t to any of its leaves, there is at most one edge in c . The edges in c are naturally ordered from left to right. To any admissible cut c corresponds a unique subforest $P^c(t)$, the *pruning*, concatenation of the subtrees obtained by cutting the edges in c , in the order defined above. The coproduct can then be defined by:

$$(146) \quad \Delta(t) = \sum_{c \in \text{Adm } t} R^c(t) \otimes P^c(t),$$

where $R^c(t)$ is the trunk, obtained by replacing each subtree of $P^c(t)$ with a single leaf.

So far, we have defined the group of the operad of Stasheff polytopes (or Schröder trees), and shown that it coincides with the group of characters of the Hopf algebra of reduced plane trees. We shall see that it is also a group of *formal noncommutative diffeomorphisms* related to the noncommutative Lagrange inversion (see [39]) and to the noncommutative version of Poincaré's equation.

10. NONCOMMUTATIVE FORMAL DIFFEOMORPHISMS

10.1. A group of noncommutative diffeomorphisms. As pointed out in [2], it is possible to consider formal diffeomorphisms in one variable with coefficients in an associative algebra, but if this algebra is not commutative, the set of such diffeomorphisms is not anymore more a group because associativity is broken. Nevertheless, there is still a noncommutative version of the Faà di Bruno Hopf algebra.

We can recover a group by regarding the coefficients as well as the variable as formal noncommutative variables. Heuristically, let us start with a fixed diffeomorphism u of

$$(147) \quad G_{\text{diff}} = \{u(z) = z + \sum_{n \geq 1} u_n z^{n+1} \in \mathbb{C}[[z]]\}$$

in the variable z with coefficients u_n . Consider now that z is replaced by S_0 and that each u_n is replaced by a variable S_n . We get a series

$$(148) \quad g_c = S_0 + \sum_{n \geq 1} S_n S_0^{n+1}$$

in an infinity of noncommuting variables. Nothing prevents us from “iterating” g_c as we would do with an ordinary power series

$$(149) \quad g_c \circ g_c = g_c + \sum_{n \geq 1} S_n g_c^{n+1} = S_0 + S_1 S_0^2 + S_2 S_0^3 + S_1 (S_1 S_0^2) S_0 + S_1 S_0 (S_1 S_0^2) + \dots$$

Further iterations lead to words in the variables S_0, S_1, \dots indexed by Schröder pseudocompositions, which will eventually represent all reduced plane trees.

Let $S^\circ = S_0$ and, if $t = \bigwedge(t_1, \dots, t_n)$, $S^t = S_{n-1}S^{t_1} \dots S^{t_n}$. We recover the correspondence with Polish codes. For example,

$$(150) \quad \begin{array}{c} \circ \\ \swarrow \quad \searrow \\ \circ \quad \circ \\ \swarrow \quad \searrow \quad \swarrow \quad \searrow \\ \circ \quad \circ \quad \circ \quad \circ \\ \swarrow \quad \searrow \quad \swarrow \quad \searrow \\ \circ \quad \circ \quad \circ \quad \circ \end{array} = S_2 S_0 S_1 S_3 S_0 S_0 S_0 S_0 S_0 S_1 S_0 S_0 = S^{201300000100}$$

Identifying trees with their Polish codes, the group G_{ncdiff} can be described as

$$(151) \quad G_{\text{ncdiff}} = \left\{ g = \sum_{t \in \text{PT}} g_t S^t, g_t \in \mathbb{C}, g_\circ = 1 \right\} \subset \mathbb{C}\langle\langle S_0, S_1, \dots \rangle\rangle$$

where $\mathbb{C}\langle\langle S_0, S_1, \dots \rangle\rangle$ is the completion of the algebra of polynomials, with respect to the degree in S_0 .

If we set $g = g(S_0, S_1, \dots) = g(S_0; \mathbf{S})$ ($\mathbf{S} = S_1, \dots$), then,

Theorem 10.1. *The composition $f \circ g = h$ in G_{ncdiff} is given by*

$$(152) \quad h(S_0; \mathbf{S}) = f(g; \mathbf{S}).$$

In other words we substitute g to the variable S_0 in f . Graphically we substitute trees to leaves. It is easy to check that this group coincides with the previous one. If

$$(153) \quad g = \sum_{t \in \text{PT}} g_t S^t, \quad f = \sum_{t \in \text{PT}} f_t S^t, \quad h = f \circ g = \sum_{t \in \text{PT}} h_t S^t \in G_{\text{ncdiff}},$$

then h_t is a sum of contributions from f and g . The contributions to h_t can be

- $f_t S^t$ if we substitute the S_0 part of g to any S_0 (a leaf) of the term $f_t S^t$ of f .
- $g_t S^t$ if we substitute the term $g_t S^t$ of g to the S_0 part of f
- $f_{t_0} g_{t_1} \dots g_{t_n} S^t$ if when substituting (from left to right) the terms $g_{t_1} S^{t_1}, \dots, g_{t_n} S^{t_n}$ to n S_0 variables (leaves) in $f_{t_0} S^{t_0}$, we get the monomial S^t .

This means that

$$(154) \quad h_t = \sum_{(t_0; t_1 \dots t_n) = (R^c(t); P^c(t))} f_{t_0} g_{t_1} \dots g_{t_n}$$

which is precisely the convolution of characters in \mathcal{H}_{PT} .

In the sequel, we shall denote by the same letter (for instance g) an element of G_{ncdiff} regarded as a series of trees

$$(155) \quad g = \sum_t g_t t,$$

as a series of noncommutative monomials

$$(156) \quad g = \sum_t g_t S^t,$$

or as the character g sending t on g_t .

10.2. Inversion in G_{ncdiff} and Lagrange inversion. One can compute the compositional inverse of f_c (defined by $f_t = 1$ if t is a corolla and 0 otherwise). This yields a signed series involving all trees.

Consider the series

$$(157) \quad f_c = S^\circ + \sum_{n \geq 1} S^{\wedge(\circ \cdot n+1)}.$$

We shall work here in $\bar{T}(\mathcal{S})$ (where \circ is not the unit). Let $i(t)$ be the number of internal vertices of a tree t . The inverse of f_c is then given by

$$(158) \quad g_c = \sum_{t \in \text{PT}} (-1)^{i(t)} S^t$$

since

$$(159) \quad g_c = S_0 + \sum_{n \geq 1} \sum_{\substack{t = \wedge(t_1 \dots t_{n+1}) \\ t_i \in \text{PT}}} (-1)^{i(t)} S^{\wedge(t_1 \dots t_{n+1})}$$

$$(160) \quad = S_0 + \sum_{n \geq 1} \sum_{\substack{t = \wedge(t_1 \dots t_{n+1}) \\ t_i \in \text{PT}}} (-1)^{1+i(t_1)+\dots+i(t_{n+1})} S_n S^{t_1} \dots S^{t_{n+1}}$$

$$(161) \quad = S_0 - \sum_{n \geq 1} S_n g_c^{n+1}$$

so that $S_0 = g_c + \sum_{n \geq 1} S_n g_c^{n+1} = f_c \circ g_c$.

In order to establish a link with the noncommutative analogue of Lagrange inversion (see [39]), we can look for the compositional inverse of

$$(162) \quad f_L = \left(1 + \sum_{n \geq 1} S_n S_0^n \right)^{-1} \cdot S_0 \in G_{\text{ncdiff}},$$

where the exponent -1 means here the multiplicative inverse as a formal power series. Working with trees, consider the series of trees L such that

$$(163) \quad L = \circ + \sum_{k \geq 1} \wedge(L \cdot^k \cdot \circ)$$

then $L = \sum L^t t$ with $L^t = 1$ or 0.

The inverse g_L of f_L is $\sum L^t S^t$. Indeed,

$$(164) \quad g_L = S^\circ + \sum_{k \geq 1} S^{\wedge(L \cdot^k \cdot \circ)}$$

$$(165) \quad = S_0 + \sum_{k \geq 1} S_k g_L^k S_0$$

$$(166) \quad = \left(1 + \sum_{n \geq 1} S_n g_L^n \right) S_0$$

and obviously $f_L \circ g_L = S_0$.

Apart from \circ , all the trees of PT occurring in g_L are such that the rightmost subtree of each internal vertex is a leaf (S_0). Let PT_L be the set of such trees, and let α be the map sending \circ to itself and $t \in \text{PT}_L$ to the tree (with possible unary internal vertices) obtained by removing all the rightmost leaves of its internal vertices. We obtain in this way the tree expansion of Section 5.3 in [39].

One can also define S^t for such trees. Now

$$(167) \quad \alpha(L) = \circ + \sum_{n \geq 1} \bigwedge (\alpha(L)^{\cdot n})$$

and if we replace trees by their Polish codes, the resulting series g satisfies

$$(168) \quad g = S_0 + \sum_{n \geq 1} S_n g^n,$$

the functional equation considered in [39]. This correspondence will be explained in details in Section 12, using a group morphism from G_{ncdiff} to the group $G_{\mathcal{C}}$ of the Catalan operad.

10.3. The conjugacy equation. Let Y be the grading operator on trees ($Y(t) = (d(t) - 1)t$), and let $q^Y(t) = q^{d(t)-1}t$. The noncommutative analogue of the conjugacy equation can be written as

$$(169) \quad g(qS_0; \mathbf{S}) = qg_c(g; \mathbf{S})$$

where the initial diffeomorphism is the corolla series. This equation also reads

$$(170) \quad q^{-1}g(qS_0; \mathbf{S}) = q^Y g = g_c \circ g.$$

It is not difficult to compute the coefficients of the solution $g = \sum_t c_t(q)t$, noticing that $c_{\circ}(q) = 1$ and, if $t = \bigwedge(t_1, \dots, t_n)$ then,

$$(171) \quad q^{d(t)-1}c_t(q) = c_t(q) + c_{t_1}(q) \dots c_{t_n}(q).$$

As we have already seen, the coefficients have the closed form

$$(172) \quad c_t(q) = \prod_{v \in t} \frac{1}{q^{\phi(v)-1} - 1}$$

where v runs over the internal vertices of t and $\phi(v)$ is the number of leaves of the subtree of v .

For example, for

$$(173) \quad t = \begin{array}{c} \circ \\ \swarrow \quad \searrow \\ \circ \quad \circ \\ \swarrow \quad \searrow \quad \swarrow \quad \searrow \\ \circ \quad \circ \quad \circ \quad \circ \end{array}, \quad c_t(q) = \frac{1}{(q^7 - 1)(q - 1)(q^4 - 1)(q^3 - 1)}.$$

Surprisingly, the same coefficients appear in the commutative case, in Ecalle's arborified solution of the conjugacy equation, which has been interpreted in [15] in terms of characters on the Connes-Kreimer Hopf algebra \mathcal{H}_{CK} of (non plane) rooted trees decorated by positive integers. We shall see that there is indeed a kind of noncommutative arborification, which will be eventually explained by a morphism of Hopf algebras.

11. COMMUTATIVE VERSUS NONCOMMUTATIVE

11.1. **Commutative diffeomorphisms and the Connes-Kreimer algebra.** In this section we recall briefly how certain (commutative) formal diffeomorphisms can be obtained as characters of a Hopf algebra (see [15] and [16]), in particular the solution of the conjugacy equation

$$(174) \quad qu(h(z)) = h(qz)$$

where u and h are formal diffeomorphisms of G_{diff} (tangent to the identity). The use of trees to encode diffeomorphisms appears in [14] and is related to differential operators indexed by trees, an idea originally due to Cayley.

We shall rely upon the references [6], [17] and [18], except that we use the opposite coproduct, in order to avoid antimorphisms.

A rooted tree⁴ T is a connected and simply connected set of oriented edges and vertices such that there is precisely one distinguished vertex (the root) with no incoming edge. A forest F is a (commutative) monomial in rooted trees.

Let $l(F)$ be the number of vertices in F . One can decorate a forest by \mathbb{N}^* , that is, with each vertex v of F , we associate an element n_v of \mathbb{N}^* . We denote by $\mathcal{T}_{\mathbb{N}}$ (resp. $\mathcal{F}_{\mathbb{N}}$) the set of decorated trees (resp. forests). It includes the empty tree, denoted by \emptyset . As for sequences, if a forest F is decorated by n_1, \dots, n_s ($l(F) = s$), we write

$$(175) \quad |F| = n_1 + \dots + n_s.$$

For n in \mathbb{N}^* , the operator B_n^+ associates with a forest of decorated trees the tree with root decorated by n connected to the roots of the forest : $B_n^+(\emptyset)$ is the tree with one vertex decorated by n . For example,

$$(176) \quad B_n^+ \left(\begin{array}{c} n_4 \\ \bullet \\ n_5 \end{array} \quad n_3 \quad \begin{array}{c} n_1 \\ \bullet \\ \bullet \\ n_2 \end{array} \right) = \begin{array}{c} n \\ \bullet \\ \begin{array}{c} n_4 \quad \bullet \quad n_1 \\ \bullet \quad \bullet \quad \bullet \\ n_5 \quad \bullet \quad n_3 \quad n_2 \end{array} \end{array}$$

The linear span \mathcal{H}_{CK} of $\mathcal{F}_{\mathbb{N}}$ is the graded Connes-Kreimer Hopf algebra of trees decorated by \mathbb{N}^* for the product

$$(177) \quad \pi(F_1 \otimes F_2) = F_1 F_2$$

and the unit \emptyset .

The coproduct Δ can be defined by induction

$$(178) \quad \Delta(\emptyset) = \emptyset \otimes \emptyset,$$

$$(179) \quad \Delta(T_1 \dots T_k) = \Delta(T_1) \dots \Delta(T_k),$$

$$(180) \quad \Delta(B_n^+(F)) = \emptyset \otimes B_n^+(F) + (B_n^+ \otimes \text{Id}) \circ \Delta(F).$$

There exists a combinatorial description of this coproduct (see [17]). For a given tree $T \in \mathcal{T}_{\mathbb{N}}$, an admissible cut c is a subset of its vertices such that, on the path from the root to an element of c , no other vertex of c is encountered. For such an admissible cut, $P^c(T)$ is the product of the subtrees of T whose roots are in c and

⁴Not to be confused with rooted plane trees, of which Schröder trees are a special case.

$R^c(T)$ is the remaining tree, once these subtrees have been removed. With these definitions, for any tree T , we have

$$(181) \quad \Delta(T) = \sum_{c \text{ adm.}} R^c(T) \otimes P^c(T).$$

For example,

$$\Delta \left(\begin{array}{c} n_1 \\ \swarrow \quad \searrow \\ n_3 \quad n_2 \end{array} \right) = \begin{array}{c} n_1 \\ \swarrow \quad \searrow \\ n_3 \quad n_2 \end{array} \otimes \emptyset + \begin{array}{c} n_1 \\ \bullet \\ \downarrow \\ n_3 \end{array} \otimes \begin{array}{c} n_1 \\ \bullet \\ \downarrow \\ n_2 \end{array} + \begin{array}{c} n_1 \\ \bullet \\ \downarrow \\ n_2 \end{array} \otimes \begin{array}{c} n_1 \\ \bullet \\ \downarrow \\ n_3 \end{array} + \begin{array}{c} n_1 \\ \bullet \\ \downarrow \\ n_3 \end{array} \otimes \begin{array}{c} n_1 \\ \bullet \\ \downarrow \\ n_2 \end{array} + \begin{array}{c} n_1 \\ \bullet \\ \downarrow \\ n_2 \end{array} \otimes \begin{array}{c} n_1 \\ \bullet \\ \downarrow \\ n_3 \end{array} + \begin{array}{c} n_1 \\ \bullet \\ \downarrow \\ n_3 \end{array} \otimes \begin{array}{c} n_1 \\ \bullet \\ \downarrow \\ n_2 \end{array} + \begin{array}{c} n_1 \\ \bullet \\ \downarrow \\ n_2 \end{array} \otimes \begin{array}{c} n_1 \\ \bullet \\ \downarrow \\ n_3 \end{array} + \begin{array}{c} n_1 \\ \bullet \\ \downarrow \\ n_3 \end{array} \otimes \begin{array}{c} n_1 \\ \bullet \\ \downarrow \\ n_2 \end{array}$$

Definition 11.1. *Given a formal diffeomorphism*

$$(182) \quad u(z) = z + \sum_{n \geq 1} u_n z^{n+1},$$

we associate with any tree T (see [15, 16, 36, 37]) a monomial $A_T(z)$ recursively defined as follows:

- For the empty tree $A_\emptyset(z) = z$,
- If $T = B_+^n(\emptyset)$ then $A_T(z) = u_n z^{n+1}$,
- If $T = B_+^n(F)$ where $F = T_1^{a_1} \dots T_k^{a_k}$ is a non empty product of k distinct decorated trees, with multiplicities a_1, \dots, a_k ($a_1 + \dots + a_k = s$), then

$$(183) \quad A_T(z) = \frac{1}{a_1! \dots a_k!} A_{T_1}^{a_1}(z) \dots A_{T_k}^{a_k}(z) (\partial_z^{a_1 + \dots + a_k} u_n z^{n+1})$$

The main result is then:

Theorem 11.2. *The map ρ_u sending a character φ of \mathcal{H}_{CK} to the formal diffeomorphism of G_{diff}*

$$(184) \quad \rho_u(\varphi)(z) = \sum_{T \in \mathcal{T}_{\mathbb{N}}} \varphi(T) A_T(z)$$

is a group homomorphism from the group of characters G_{CK} of \mathcal{H}_{CK} to G_{diff} .

See [15, 16, 36, 37] for proofs.

Going back to the equation $u(h(z)) = q^{-1}h(qz)$, one can observe that

- $u = \rho_u(\varphi_0)$, where φ_0 is the character given by $\varphi_0(T) = 1$ (resp. 0) if $T = \emptyset$ or $T = B_+^n(\emptyset)$ (resp. otherwise).
- If the conjugating h is given by a character θ ($h = \rho_u(\theta)$) then $q^{-1}h(qz)$ is given by the character $\theta \circ q^Y$ where $q^Y(F) = q^{|F|}F$.

Therefore, the conjugacy equation can be lifted to the character equation

$$(185) \quad \varphi_0 * \theta = \theta \circ q^Y.$$

This equation is easily solved. For a tree $T = B_+^n(T_1 \dots T_s)$, we get

$$(186) \quad (q^{|T|} - 1)\theta(T) = \theta(T_1) \dots \theta(T_s)$$

so that

$$(187) \quad \theta(T) = \prod_{v \in T} \frac{1}{q^{|T_v|} - 1}$$


where T_v is the subtree of T whose root is the vertex v .

A more explicit expression can be found in [16]. Such “arborified” expressions are useful for analysis since they allow to prove the analyticity of the conjugating map under some diophantine conditions on q (ensuring a geometric growth of the numbers $\theta(T)$, see *e.g.*, [16]).

11.2. Relating \mathcal{H}_{CK} and \mathcal{H}_{PT} . The similarity of the coefficients $c_t(q)$ and $\theta(T)$ suggests a link between both versions of the conjugacy equation, which turns out to be understandable at the level of Hopf algebras.

Theorem 11.3. *Let sk (for “skeleton”) be the map defined from PT to $\mathcal{T}_{\mathbb{N}}$ by $sk(\circ) = \emptyset$ and, if $t = \wedge(t_1, \dots, t_n)$ ($n \geq 2$), then $sk(t) = B_+^{n-1}(sk(t_1) \dots sk(t_n))$.*

This map extends naturally to an algebra morphism from \mathcal{H}_{PT} to \mathcal{H}_{CK} which is actually a Hopf algebra morphism.

For example if $t =$ 

$$(188) \quad T = sk(t) = \begin{array}{c} 2 \\ \swarrow \downarrow \searrow \\ 1 \quad \bullet \quad 1 \\ \downarrow \\ 3 \end{array}$$

The proof follows immediately by comparing the recursive (or combinatorial) definition of the coproducts.

This Hopf morphism induces a group morphism sk^* sending a character $\varphi \in G_{CK}$ to the character $sk^*(\varphi) = \varphi \circ sk$. It sends the character θ to the character defined by

$$(189) \quad g = \sum_t c_t(q)t.$$

In other words, for any reduced plane tree t , $c_t(q) = \theta(sk(t))$.

11.3. The final diagram. Let us summarize the situation:

- We have a noncommutative analogue of the conjugacy equation whose solution is a character on \mathcal{H}_{PT} defined by the coefficients $c_t(q)$,
- In the commutative case, the solution can be computed as a character on \mathcal{H}_{CK} ,
- Both characters are related by the group morphism sk^* .

There is also a morphism from G_{ncdiff} to G_{diff} , so that we also get the solution in the commutative case. For $u \in G_{diff}$, the algebra map α_u sending S_0 to z and S_n to u_n defines a group morphism and, if we denote by $g_c \in G_{ncdiff}$ the sum of all corollas, then $\alpha_u(g_c) = u$. This morphism also sends the solution of the noncommutative conjugacy equation to the solution of the commutative one.

The final picture is then:

Theorem 11.4. *The diagram*

$$(190) \quad \begin{array}{ccc} G_{CK} & \xrightarrow{\text{sk}^*} & G_{ncdiff} \\ & \searrow \rho_u & \downarrow \alpha_u \\ & & G_{diff} \end{array}$$

is commutative.

Proof – Consider a character $\varphi \in G_{CK}$. Then, on the one hand

$$(191) \quad \rho_u(\varphi) = \sum_T \varphi(T) A_T(z)$$

and, on the other hand, as $\phi = \text{sk}^*(\varphi)$ ($\phi(t) = \varphi(\text{sk}(t))$),

$$(192) \quad \alpha_u(\phi) = \alpha_u \left(\sum_t \phi(t) S^t \right) = \sum_T \varphi(T) \alpha_u \left(\sum_{\text{sk}(t)=T} S^t \right).$$

The above diagram commutes if, for any tree T ,

$$(193) \quad A_T(z) = \alpha_u \left(\sum_{\text{sk}(t)=T} S^t \right).$$

The result is obvious for $T = \emptyset$, and for any tree $C_n = B_n^+(\emptyset)$, since $A_{C_n}(z) = u_n z^{n+1}$ and the only reduced tree with such a skeleton is the corolla c_{n+1} :

$$(194) \quad \alpha_u(S^{c_{n+1}}) = \alpha_u(S_n S_0^{n+1}) = u_n z^{n+1} = A_{C_n}(z).$$

Now, from Definition 11.1, if $T = B_n^+(T_1^{a_1} \dots T_k^{a_k})$ where T_1, \dots, T_k are distinct rooted trees, then

$$(195) \quad A_T(z) = \binom{n+1}{a_1, a_2, \dots, a_k} A_{T_1}^{a_1}(z) \dots A_{T_k}^{a_k}(z) u_n z^{n+1-a_1-\dots-a_k}.$$

Let

$$(196) \quad X_0 = S_0 \text{ and } X_i = \sum_{\text{sk}(t)=T_i} S^t.$$

If $W(a_0, a_1, \dots, a_k)$ is the set of all possible words obtained by concatenating a_i copies of X_i , then

$$(197) \quad \sum_{\text{sk}(t)=T} S^t = \sum_{w \in W(n+1-a_1-\dots-a_k, a_1, \dots, a_k)} S_n w$$

since a tree has skeleton T if and only if it can be written $c_n \circ (t_1, \dots, t_n)$ for the operadic composition, where the n -tuple (t_1, \dots, t_n) contains exactly a_i trees with skeleton T_i and $n+1-a_1-\dots-a_k$ trees \circ . There are exactly $\binom{n+1}{a_1, a_2, \dots, a_k}$ such n -tuples, and, by induction,

$$(198) \quad \alpha_u \left(\sum_{\text{sk}(t)=T} S^t \right) = A_{T_1}^{a_1}(z) \dots A_{T_k}^{a_k}(z) u_n z^{n+1-a_1 \dots - a_k} \left(\sum_{w \in W(n+1-a_1 \dots - a_k, a_1, \dots, a_k)} 1 \right) = A_T(z)$$

■

12. THE CATALAN OPERAD

The preceding considerations can be repeated almost word for word for the free operad on one generator in each degree

$$(199) \quad \mathcal{C} = \bigoplus_{n \geq 1} \mathbb{C} \mathcal{T}_n$$

where \mathcal{T}_n denotes the set of all plane trees on n vertices:

$$(200) \quad \mathcal{T}_1 = \{\bullet\}, \mathcal{T}_2 = \{\bullet\!\!\!\! \uparrow \bullet\}, \mathcal{T}_3 = \{\bullet\!\!\!\! \uparrow \bullet\!\!\!\! \uparrow \bullet, \bullet\!\!\!\! \uparrow \bullet\!\!\!\! \uparrow \bullet\!\!\!\! \uparrow \bullet\}, \mathcal{T}_4 = \{\bullet\!\!\!\! \uparrow \bullet\!\!\!\! \uparrow \bullet\!\!\!\! \uparrow \bullet, \bullet\!\!\!\! \uparrow \bullet\!\!\!\! \uparrow \bullet\!\!\!\! \uparrow \bullet\!\!\!\! \uparrow \bullet, \bullet\!\!\!\! \uparrow \bullet\!\!\!\! \uparrow \bullet\!\!\!\! \uparrow \bullet\!\!\!\! \uparrow \bullet\!\!\!\! \uparrow \bullet, \bullet\!\!\!\! \uparrow \bullet\!\!\!\! \uparrow \bullet\!\!\!\! \uparrow \bullet\!\!\!\! \uparrow \bullet\!\!\!\! \uparrow \bullet\!\!\!\! \uparrow \bullet\}, \dots$$

endowed with the same composition as in \mathcal{S} . We shall call it the *Catalan operad*, although it probably has other names [26, 5].

General plane trees can be represented by their Polish codes, as monomials S^I , where S_n is now the symbol of an n -ary operation, that is $S^\bullet = S_0$ and $S^{B+(t_1, \dots, t_k)} = S_k S^{t_1} \dots S^{t_k}$. For instance,

$$(201) \quad \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \diagup \quad \diagdown \quad \diagup \quad \diagdown \\ \bullet \quad \bullet \quad \bullet \quad \bullet \end{array} = S^{3100200}.$$

As observed in [39], these I are also the evaluation vectors on nondecreasing parking functions.

The discussion of Lagrange inversion and related problems given in [39] can be interpreted as calculations in the group $G_{\mathcal{C}}$ of this operad. The functional equation

$$(202) \quad g = \sum_{n \geq 0} S_n g^n$$

can be rewritten as

$$(203) \quad (S_0 - S^{10} - S^{200} - \dots) \circ g = S_0$$

so that it amounts to inverting the series

$$(204) \quad f = S_0 - \sum_{n \geq 1} S^{n0^n}$$

in the group $G_{\mathcal{C}}$. The relation with the computations in Section 10.2 can be elucidated with the help of a surjective group morphism from G_{ncdiff} to $G_{\mathcal{C}}$. Recall that PT_L is the set of Schröder trees such that the rightmost subtree of each internal vertex is a leaf. Let α be the map which sends \circ to \bullet and any other tree t of PT :

- to the the plane tree obtained by removing all the rightmost leaves of its internal vertices if t is in PT_L ,
- to 0 otherwise.

This map is surjective on plane trees and induces a linear map on monomials in S_0, S_1, \dots which happens to be a group morphism from G_{ncdiff} to $G_{\mathcal{C}}$. If we still denote by α this morphism, then, since

$$(205) \quad \alpha(g_c) = \alpha \left(S^\circ + \sum_{n \geq 1} S^{\wedge(\circ^{n+1})} \right) = S_0 + \sum_{n \geq 1} S^{n0^n} \in G_{\mathcal{C}},$$

$$(206) \quad \alpha(g_L) = \alpha \left(\left(1 + \sum_{n \geq 1} S_n S_0^n \right)^{-1} S_0 \right) = S_0 - \sum_{n \geq 1} S^{n0^n} \in G_{\mathcal{C}}.$$

we can obtain the composition inverse of these series of $G_{\mathcal{C}}$ as $\alpha(f_c)$ and $\alpha(f_L)$, which gives back the formulas of [39].

The functional equation for g can also be written

$$(207) \quad g = S_0 + \Omega g \cdot g =: S_0 + B(g, g)$$

Each plane tree in the solution of (202) corresponds to a unique binary tree $B_T(S_0)$ in the solution of (207). This induces a bijection between plane trees and binary trees: writing (see (41))

$$(208) \quad B(S^I, S^J) = \Omega S^I S^J,$$

there is a unique way to decompose a plane tree S^I on n vertices as

$$(209) \quad S^I = B(S^{I_1}, S^{I_2})$$

so that recursively

$$(210) \quad S^I = B_T(S_0, \dots, S_0)$$

for a unique binary tree T with $n - 1$ internal vertices. For example,

$$(211) \quad S^{\overset{\bullet}{\downarrow}} = S^{10} = B(S_0, S_0),$$

$$(212) \quad S^{\bullet \wedge \bullet} = S^{200} = B(S^{10}, S_0) = B(B(S_0, S_0), S_0),$$

$$(213) \quad S^{\overset{\bullet}{\downarrow} \bullet} = S^{110} = B(S_0, S^{10}) = B(S_0, B(S_0, S_0))$$

We recover in this way the classical rotation correspondence.

In fact, if, in the one to one correspondence with plane tree, $S^{I_1} = S^{t_1}$ and $S^{I_2} = S^{t_2}$, the tree corresponding to $S^I = B(S^{I_1}, S^{I_2}) = \Omega S^I S^J$ is $t = B_+(B_-(t_1), t_2)$. Using this trick we get for instance

$$(214) \quad S^{3100200} = S^{\bullet \wedge \bullet \wedge \bullet} = B(S^{\bullet \wedge \bullet}, S^{\bullet \wedge \bullet}) = B(B(S^{\overset{\bullet}{\downarrow}}, S^\bullet), B(S^{\bullet \wedge \bullet}, S^\bullet))$$

that corresponds finally to the binary tree $B(B(B(S_0, B(S_0, S_0)), S_0), B(B(S_0, S_0), S_0))$:

(215) 

Another question which can be investigated in this context is the formal solution of the generic differential equation

(216)
$$\frac{dx}{ds} = f(x(s)), \quad x(0) = x_0.$$

Rather than stating Cayley's formula for $x^{(k)}$ in terms of rooted trees and derivatives, we shall write down a noncommutative version involving plane trees and the coefficients of the generic power series f . Assuming without loss of generality that $x_0 = 0$, we can look for a series $X(s) \in G_{\mathcal{C}}$ satisfying

(217)
$$\frac{dX}{ds} = \sum_{n \geq 0} S_n X(s)^n.$$

Thus,

(218)
$$X(s) = S_0 s + \sum_{n \geq 1} S_n \int_0^s X(u)^n du =: \sum_{n \geq 1} X_n \frac{s^n}{n!}$$

and solving iteratively as usual, we get successively

(219)
$$X_1 = S_0,$$

(220)
$$X_2 = S^{10},$$

(221)
$$X_3 = 2S^{200} + S^{110},$$

(222)
$$X_4 = 6S^{3000} + 3S^{2100} + 3S^{2010} + 2S^{1200} + S^{1110}.$$

Identifying as before trees and their Polish codes,

(223)
$$X_n = \sum_{t \in \mathcal{T}_n} c_t S^t$$

and setting

(224)
$$F_n(Y_1, \dots, Y_n) = S_n \int_0^s Y_1(u) \cdots Y_n(u) du$$

we have

(225)
$$X_{n+1} = \sum_{k=1}^n \sum_{i_1 + \dots + i_k = n} F_k \left(X_{i_1} \frac{s^{i_1}}{i_1!}, \dots, X_{i_k} \frac{s^{i_k}}{i_k!} \right),$$

which gives for the coefficient c_t of $t = B_+(t_1, \dots, t_k) \in \mathcal{T}_{n+1}$


(226)
$$c_t \frac{s^{n+1}}{(n+1)!} = \int_0^s c_{t_1} \cdots c_{t_k} \frac{u^{|t_1| + \dots + |t_k|}}{|t_1|! \cdots |t_k|!} du$$

so that

$$(227) \quad c_t = \binom{n}{|t_1|, \dots, |t_k|} c_{t_1} \cdots c_{t_k}$$

which is clearly the recursion for the number of decreasing (or increasing) labellings of t , also given by the hook-length formula

$$(228) \quad c_t = (n+1)! \prod_{v \in t} \frac{1}{h_v}$$

where h_v is the number of nodes of the subtree with root v . For instance, for the tree  that corresponds to the monomial S^{2100} , there are 3 decreasing labellings:

$$(229) \quad \begin{array}{c} 4 \\ \swarrow \quad \searrow \\ 2 \quad 3 \\ \swarrow \\ 1 \end{array}, \quad \begin{array}{c} 4 \\ \swarrow \quad \searrow \\ 3 \quad 2 \\ \swarrow \\ 1 \end{array}, \quad \begin{array}{c} 4 \\ \swarrow \quad \searrow \\ 3 \quad 1 \\ \swarrow \\ 2 \end{array} .$$

Replacing each S_k by $\frac{f^{(k)}}{k!}$, we recover Cayley's formula for $x^{(n)}$,

$$(230) \quad \frac{d^n x}{ds^n} = \sum_{|t|=n} a(t) \delta_t,$$

where

$$(231) \quad a(t) = \frac{|t|!}{t! |S_t|},$$

S_t being the symmetry group of t ,

$$(232) \quad B_+(t_1, \dots, t_n)! = |B_+(t_1, \dots, t_n)| \cdot t_1! \cdots t_n!, \quad \bullet! = 1.$$

and the elementary differentials are defined by [4]

$$(233) \quad \delta_{\bullet}^i = f^i, \quad \delta_{B_+(t_1, \dots, t_n)}^i = \sum_{j_1, \dots, j_n=1}^N (\delta_{t_1}^{j_1} \cdots \delta_{t_n}^{j_n}) \partial_{j_1} \cdots \partial_{j_n} f^i$$

In particular, the solution is given by

$$(234) \quad x(s) = x_0 + \sum_t \frac{s^{|t|}}{|t|!} a(t) \delta_t(0)$$

For example,

$$(235) \quad x^{(4)} = f'''(f, f, f) + 3f''(f, f'(f)) + f'(f''(f, f)) + f'(f'(f'(f))).$$

Note that with the interpretation of S_n as an n -linear operation, our formal calculations are valid for $x \in \mathbb{R}^N$: we can write the Taylor expansion of f as

$$(236) \quad f(x) = F_0 + F_1(x) + F_2(x, x) + F_3(x, x, x) + \cdots$$

without expliciting the expression of F_n in terms of partial derivatives.

Once again, the functional equation (217) can be recast as a quadratic fixed point problem:

$$(237) \quad \frac{dX}{ds} = S_0 + (S_1 + S_2X(s) + S_3X^2(s) + \cdots)X(s) = S_0 + (\Omega X'(s))X(s)$$

so that

$$(238) \quad X(s) = S_0s + \int_0^s \Omega X'(u) \cdot X(u) du = S_0s + \mathbf{B}(X(s), X(s))$$

The bilinear map \mathbf{B} acts on trees by

$$(239) \quad \mathbf{B} \left(S^I \frac{s^i}{i!}, S^J \frac{s^j}{j!} \right) = \binom{i+j}{i, j} \Omega S^{IJ} \frac{s^{i+j+1}}{(i+j+1)!}$$

13. APPENDIX: NUMERICAL EXAMPLES

In the case of Lagrange inversion, comparison between the formal noncommutative solution and numerical examples (specializations of the alphabet, or characters) leads to interesting insights. We shall give here a (short) list of known workable examples.

13.1. Warmup: $A = 1$. The alphabet $A = 1$ is defined by $S_n(1) = 1$ for all n . In this case,

$$(240) \quad \phi(z) = \frac{qz}{1-z} = qz\sigma_z(1)$$

is a Möbius transformation, and it is trivial to conjugate it to its linear part when $q \neq 1$. However, it is a good exercise to work out the series solution. We have

$$(241) \quad S_n(m) = \binom{n+m-1}{n} \text{ so that } L(S^I)(1) = \frac{n!}{i_1! \cdots i_r!} = n!S^I(\mathbb{E})$$

where \mathbb{E} is defined by $S_n(\mathbb{E}) = 1/n!$. Hence,

$$(242) \quad f(z) = \int_0^\infty e^{-t} L \left(z\sigma_{zt} \left(\frac{\mathbb{E}}{1-q} \right) \right) dt = \frac{z}{1 - \frac{z}{1-q}}.$$

13.2. The logistic map: $A = -1$. The logistic map is defined by

$$(243) \quad \phi(z) = qz(1-z) = qz\sigma_z(-1).$$

Indeed, by definition, $\sigma_z(-1)$ is the inverse of $\sigma_z(1)$, so that $S_1(-1) = -1$ and $S_n(-1) = 0$ for $n > 1$.

The recurrence for $g_n(-1)$ is here

$$(244) \quad (1-q^n)g_n = \sum_{k=0}^{n-1} g_k g_{n-1-k}$$

yielding

$$(245) \quad g_1 = \frac{1}{1-q}, \quad g_2 = \frac{2}{(1-q)(1-q^2)}, \quad g_3 = \frac{5+q}{(1-q)(1-q^2)(1-q^3)}, \cdots$$

the numerator being a q -analogue of $n!$.

For $q = -2, 2, 4$, these series have explicit forms in terms of elementary functions:

$$\begin{aligned} q = -2 : f(z) &= \frac{\sqrt{3}}{6} \left(2\pi - 3 \arccos \left(z - \frac{1}{2} \right) \right), & h(z) &= \frac{1}{2} - \cos \left(\frac{2z}{\sqrt{3}} + \frac{\pi}{3} \right), \\ q = 2 : f(z) &= -\frac{1}{2} \ln(1 - 2z), & h(z) &= \frac{1}{2} (1 - e^{-2z}), \\ (246) \quad q = 4 : f(z) &= (\arcsin \sqrt{z})^2, & h(z) &= (\sin \sqrt{z})^2. \end{aligned}$$

Numerical investigations, including a conjecture for the radius of convergence in the general case, can be found in [7, 8].

13.3. The Ricker map: $A = \mathbb{E}$. This case corresponds to

$$(247) \quad \phi(z) = qze^z.$$

No closed expression is known for f or g , but a numerical study can be found in [7].

We have

$$(248) \quad S_n(m\mathbb{E}) = \frac{m^n}{n!}, \text{ so that } L(S^I) = \frac{1^{i_1}(i_1+1)^{i_2}(i_1+i_2+1)^{i_3} \cdots (i_1+\cdots+i_{r-1}+1)^{i_r}}{i_1!i_2! \cdots i_r!}$$

and we can compute

$$(249) \quad f_1 = \frac{1}{1-q}, \quad f_2 = \frac{3q+1}{2!(1-q)(1-q^2)}, \quad f_3 = \frac{16q^3+11q^2+8q+1}{3!(1-q)(1-q^2)(1-q^3)}, \cdots$$

The numerators are q -analogues of $(n!)^2$, whose combinatorial interpretation requires further investigations.

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