TAUTOLOGICAL INTEGRALS ON SYMMETRIC PRODUCTS OF CURVES

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Abstract. We propose a conjecture on the generating series of Chern numbers of tautological bundles on symmetric products of curves and establish the rank 1 and rank -1 case of this conjecture. Thus we compute explicitly the generating series of integrals of Segre classes of tautological bundles of line bundles on curves, which has a similar structure as Lehn's conjecture for surfaces.

1. Introduction

Let X be a smooth quasi-projective connected complex variety of dimension d, and denote by $X^{[n]}$ the Hilbert scheme of n points on X. Let $\mathcal{Z}_n \subset X \times X^{[n]}$ be the universal family with natural projections $p_1 : \mathcal{Z}_n \to X$ and $\pi : \mathcal{Z}_n \to X^{[n]}$ onto the X and $X^{[n]}$ respectively. For any locally free sheaf F on X, let $F^{[n]} = \pi_*(\mathcal{O}_{\mathcal{Z}_n} \otimes p_1^* F)$, which is called the tautological sheaf of F.

When $d = 2$, many invariants of the Hilbert schemes of points on a projective surface can be determined explicitly by the corresponding invariants of the surface, including the Betti numbers [10], Hodge numbers [11], cobordism classes [8], and elliptic genus [5], etc.. G. Ellingsrud, L. Gottsche and M. Lehn showed in [8] that for a polynomial in Chern classes of tautological sheaves and the tangent bundle of $X^{[n]}$, there exists a universal polynomials in Chern classes of the corresponding sheaves and the tangent bundle on X such that the integrals of these two polynomials over $X^{[n]}$ and X are equal. A direct consequence is that the generating series of certain tautological integrals can be written in universal forms of infinite products; though it is not easy to find explicit expressions. For example, various authors have considered the computation of the integrals of top Segre classes of tautological sheaves of a line bundle on a surface [9, 14, 16, 22, 23, 24]. M. Lehn made a conjecture on the generating series as follows:

Conjecture 1. (M. Lehn [16]) For a smooth projective surface S and a line bundle L on it, define

$$N_n = \int_{S^{[n]}} s_{2n}(L^{[n]}),$$

then

$$\sum_{n\geq 0} N_n z^n = \frac{(1-k)^a(1-2k)^b}{(1-6k+6k^2)^c}. \tag{1}$$

Here $a = HK - 2K^2$, $b = (H-K)^2 + 3\chi(\mathcal{O}_S)$ and $c = \frac{1}{2}H(H-K) + \chi(\mathcal{O}_S)$, where H is the corresponding divisor of L and K is the canonical divisor.

And
$$k = z - 9z^2 + 94z^3 - \cdots \in \mathbb{Q}[[z]]$$
is the inverse of the function
$$z = \frac{k(1-k)(1-2k)^4}{(1-6k+6k^2)^3}.$$

This conjecture is still open up to now. Recently M. Marian, D. Oprea and R. Pandharipande showed that this conjecture holds for K3 surfaces [18] by considering integrals over Quot schemes and the recursive localization relations.

J. V. Rennemo [20] gave a generaliztion of the theorem of G. Ellingsrud, L. Gottsche and M. Lehn: when $d = 1$ and $d = 2$ the universal property of polynomials in Chern classes of tautological sheaves and tangent bundles holds; when $d > 2$, one should consider the universal property of integrals of polynomials only in Chern classes of tautological sheaves over geometric subsets of $X^{[n]}$.

In this article, we focus on the case of $d = 1$. For C a smooth projective curve, the Hilbert scheme of n points on C is isomorphic to the n-th symmetric product, so it is a smooth projective variety of dimension n. We have the following conjecture:

Conjecture 2. For C a smooth projective curve and E_r a vector bundle of rank r on C, one has

$$\sum_{n=0}^{\infty} z^n \int_{C^{[n]}} c(E_r^{[n]}) = \exp(\sum_{n=1}^{\infty} \frac{z^n}{n}(A_n^r d_r + B_n^r e)), \tag{2}$$

$$\sum_{n=0}^{\infty} \frac{z^n}{n} \int_{C^{[n]}} c(-E_r^{[n]}) = \exp(\sum_{n=1}^{\infty} \frac{z^n}{n}(C_n^r(-d_r) + D_n^r e)), \tag{3}$$

Here $c(E_r^{[n]})$ is the total Chern class, $d_r = \displaystyle\int_C c(E_r)$ and e is the Euler number of C. A_n^r, B_n^r, C_n^r and D_n^r are integers depending only on r and n, which satisfy $A_n^r = (-1)^{n+1}\binom{rn-1}{n-1}$, $C_n^r = (-1)^n\binom{-rn-1}{n-1} = (-1)^{n-1}A_n^{r+1}$, $D_n^r = (-1)^n B_n^{r+1}$.

Conjecture 2 shares some similarities as in the surface case, which has been established in an unpublished work by Jian Zhou and the author and will appear in a subsequent work.

We will see the existence of the universal coefficients A_n^r, B_n^r, C_n^r and D_n^r is a direct consequence of Theorem 2.3 in Section 2 and it will be explained in Section 3.2. The mysterious part of Conjecture 2 to the author is that A_n^r, B_n^r, C_n^r and D_n^r are all integers and there are relationships between them. More precisely, if two vector bundles E_r and E_{r+1} of ranks r and $r+1$ respectively satisfy $\int_C c(E_r) = \int_C c(E_{1+r})$, then it is implied by Conjecture 2 that

$$\sum_{n=0}^{\infty} z^n \int_{C^{[n]}} c(-E_r^{[n]}) = \sum_{n=0}^{\infty} (-z)^n \int_{C^{[n]}} c(E_{1+r}^{[n]})$$

For some special r, we can determine B_n^r explicitly and prove the conjecture.

When $r = 1$, $B_n^1 = 0$. We have the following theorem:

Theorem 1.1. *For C a smooth projective curve and L a line bundle on C, one has*

$$\sum_{n=0}^{\infty} z^n \int_{C^{[n]}} c(L^{[n]}) = \exp\Big(\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} z^n \int_C c(L)\Big). \tag{4}$$

For the rank -1 case, it is the generating series of the integrals of top Segre classes of tautological sheaves of a line bundle on C. We can also prove the conjecture in this case. Analogous to Conjecture 1, we have the following theorem:

Theorem 1.2. *For C a smooth projective curve and L a line bundle on C, one has*

$$\begin{aligned} &\sum_{n=0}^{\infty} z^n \int_{C^{[n]}} s(L^{[n]}) \\ =&\exp\Big(\sum_{n=1}^{\infty} \frac{z^n}{n}\Big(-\binom{2n-1}{n-1} d + \Big(4^{n-1} - \binom{2n-1}{n-1}\Big)e\Big)\Big) \\ =&\frac{(1-k)^{e+d}}{(1-2k)^{\frac{e}{2}}}. \end{aligned} \tag{5}$$

Here $s(L^{[n]})$ is the total Segre class, d is the degree of the line bundle L, e is the Euler number of S and $z = k(1-k)$.

Theorem 1.2 is related to an enumerative problem. A. S. Tikhomirov in [22] has interpreted N_n in Conjecture 1 as the number of $(n-2)$-dimensional n-secant planes in the image to the surface in $\mathbb{P}^{3n-1}$. In a similar fashion, $(-1)^n \int_{C^{[n]}} s(L^{[n]})$ counts the number of n-secant $(n-2)$-planes to C in $\mathbb{P}^{2n-2}$. For example, it is easy to compute from Theorem 1.2 that $\int_{C^{[2]}} s(L^{[2]}) = \frac{1}{2}(d^2 - 3d + 2 - 2g)$, which coincides the classical formula of the number of nodes of a curve in $\mathbb{P}^2$.

In 2007 Le Barz [15] and E. Cotterill [7] have already independently derived the generating formula of such numbers. Le Barz's approach is via the multisecant loci, and E. Cotterill uses a formula by Macdonld (cf. [1] Chapter VIII, Prop. 4.2) and the graph theory. However, our method is different from Le Barz's and E. Cotterill's.

We use the similar strategy used in [25] to prove the above theorems. Firstly we establish a universal formula theorem for curves as Theorem 4.2 in [8] for surfaces, and hence we only need to prove the cases of certain line bundles on $\mathbb{P}^1$. Using the natural torus action on $\mathbb{P}^1$ and the induced action on the Hilbert scheme, we can consider the equivariant case of $\mathbb{P}^1$, or we can reduce it further to the equivariant case of $\mathbb{C}$, which becomes a combinatoric problem.

Remark 1.1. After the first version of this article, Professor Oprea tells the author in an email that Conjecture 2 has been solved by M. Marian and himself, and the universal coefficients are also explicitly determined by them in [17]. To be more precise, the relationships between the universal coefficients conjectured in Conjecture 2 do hold; if we write

$$C(z) = \sum_{n=1}^{\infty} \frac{z^n}{n} C_n^r, D(z) = \sum_{n=1}^{\infty} \frac{z^n}{n} D_n^r,$$

then

$$C(-t(1-t)^r) = -\log(1+t)$$

and

$$D(-t(1+t)^r) = \frac{r+1}{2}\log(1+t) - \frac{1}{2}\log(1+t(r+1))$$

.

A direct consequence of the above is that (3) can be written in a form which is similar to Theorem 1.2:

$$\sum_{n=0}^{\infty} z^n \int_{C^{[n]}} c(-E_r^{[n]})$$
$$= \frac{(1-k)^{\frac{r+1}{2}e+d}}{(1-(r+1)k)^{\frac{e}{2}}},$$

where $z = k(1-k)^r$.

2. Universal properties of tautological integrals over Hilbert schemes of points on curves

Let C be a smooth projective connected curve. It is well-known that $C^{[n]}$ is smooth of dimension n and in particular isomorphic to the n-th symmetric product. In this section, we will see that the theorems on the cobordism rings of Hilbert schemes of points of surfaces established by Ellingsrud, Göttsche and Lehn in [8] can be generalized to curves.

We will follow what has been done in [8]. Let $\Omega = \Omega^U \otimes \mathbb{Q}$ be the complex cobordism ring with rational coefficients. For a smooth projective curve C we denote its cobodism class by $[C]$, and define an invertible element in the formal power series ring $\Omega[[z]]$:

$$H(C) := \sum_{n=0}^{\infty} [C^{[n]}] z^n.$$

We have the following theorem:

Theorem 2.1. *$H(S)$ depends only on the cobordism class $[C] \in \Omega$.*

Two stably complex manifolds have the same cobordism class if and only if their collection of Chern numbers are identical. Theorem 2.1 is proved in [20] and there is a generalized version of the this theorem:

For a smooth projective variety X, let $K(X)$ be the Grothendieck group generated by locally free sheaves. Let $E_1, \cdots, E_m \in K(C)$ and $r_1, \cdots, r_m$ are the ranks respectively.

Theorem 2.2. *(J. V. Rennemo [20]) Let P be a polynomial in the Chern classes of $C^{[n]}$ and the Chern classes of $E_1^{[n]}, \cdots, E_m^{[n]}$. Then there is a universal polynomial $\tilde{P}$, depending only on P, in the Chern classes of the tangent bundles of $C^{[n]}$, the ranks $r_1, \cdots, r_m$ and the Chern classes of $E_1, \cdots, E_m$, such that*

$$\int_{C^{[n]}} P = \int_S \tilde{P}.$$

These theorems can be used in the computations of generating series of tautological integrals. Let $\Psi : K(X) \to H^\times$ be a group homomorphism from the additive group $K(X)$ to the multiplicative group $H^\times$ of units of $H(X;\mathbb{Q})$. We require Ψ is functorial with respect to pull-backs and is a polynomial in Chern classes of its argument. Also let $\phi(x) \in \mathbb{Q}[[x]]$ be a formal power series and put $\Phi(X) := \phi(x_1)\cdots\phi(x_n) \in H^*(X;\mathbb{Q})$ with $x_1, \cdots, x_n$ the Chern roots of T_X. For $x \in K(X)$, define a power series in $\mathbb{Q}[[z]]$ as follows:

$$H_{\Psi,\Phi}(X,x) := \sum_{n=0}^{\infty} \int_{X^{[n]}} \Psi(x^{[n]})\Phi(X^{[n]})z^n.$$

Theorem 2.3. *For each integer r there are universal power series $A_i \in \mathbb{Q}[[z]]$, $i = 1, 2$, depending only on Ψ, Φ and r, such that for each $x \in K(C)$ of rank r we have*

$$H_{\Psi,\Phi}(C,x) = \exp(\int_C (c_1(x)A_1 + c_1(C)A_2)).$$

The proof of the above theorem is similar as the proof of Theorem 4.2 in [8] so we omit the details here. The main idea is that $H_{\Psi,\Phi}$ factors through $\mathbb{Q}^2$ to $\mathbb{Q}[[z]]$ and we can choose $(\mathbb{P}^1, r\mathcal{O})$ and $(\mathbb{P}^1, (r-1)\mathcal{O} \oplus \mathcal{O}(-1))$ as the "basis".

3. Proof of theorems

3.1. **Localizations on Hilbert schemes of points.** The linear coordinates on $\mathbb{C}^{[n]}$ are given by $p_i(z_1, \ldots, z_n) = z_1^i + \cdots + z_n^i$. The induced torus action on $\mathbb{C}^{[n]}$ is given by

$$q \cdot p_i = q^i p_i, \quad q = \exp(t) \in \mathbb{C}^*.$$

This action has only one fixed point at $p_1 = \cdots = p_n = 0$, and the tangent bundle and the tautological bundle $\mathcal{O}_{\mathbb{C}}^{[n]}$ have the following weight decompositions at this point:

$$T_{\mathbb{C}}^{[n]} = q^{-1} + \cdots + q^{-n},$$
$$\mathcal{O}_{\mathbb{C}}^{[n]} = 1 + q + \cdots + q^{n-1}.$$

For $A = (a_1, \cdots, a_r)$, where $a_1, \cdots, a_r \in \mathbb{Z}$, denote by $\mathcal{E}_r^A = \mathcal{O}_{\mathbb{C}}^{a_1} \oplus \cdots \oplus \mathcal{O}_{\mathbb{C}}^{a_r}$ the rank r T-equivariant vector bundle of weight $(a_1, \cdots, a_r)$. The tautological bundle $(\mathcal{E}_r^A)^{[n]}$ has the following weight decomposition at the fixed point:

$$(\mathcal{E}_r^A)^{[n]} = \sum_{i=1}^{r} q^{a_i}(1 + q + \cdots + q^{n-1}).$$

For a smooth (quasi-)projective curve C which admits a torus action with isolated fixed points $P_1, \cdots, P_l$ and $u_i = q^{c_i}$ the weights of $T_{P_i}^* C$, this torus action induces a T-action on $S^{[n]}$. The fixed points on $S^{[n]}$ are parameterized by nonnegative integers $(n_1, \cdots, n_l)$ such that

$$n_1 + \cdots + n_l = n.$$

The weight decomposition of the tangent space at the fixed point is given by:

$$\sum_{i=1}^{l}(u_i^{-1}+\cdots+u_i^{-n_i}).$$

Suppose E is a rank r equivariant vector bundle on C such that

$$E|_{P_i}=q^{a_1}+\cdots+q^{a_r}.$$

Then the weight of $E^{[n]}$ at the fixed point $(n_1,\cdots,n_l)$ is given by

$$\sum_{i=1}^{l}\sum_{j=1}^{r}(t^{a_j}(1+u_i+\cdots+u_i^{n_i-1})).$$

Let $\psi(x)\in\mathbb{Q}[[x]]$ be a formal power series. For $x\in K(C)$ of rank r, Let $\Psi(x^{[n]})=\psi(e_1(x^{[n]}))\cdots\psi(e_{rn}(x^{[n]}))$, where $e_1(x^{[n]}),\cdots,e_{rn}(x^{[n]})$ are Chern roots of $x^{[n]}$. It is obvious to see that such Ψ satisfies the conditions in Theorem 2.3.

Let $v=(v_1=a_1t,\cdots,v_r=a_rt)$ and $w_i=c_it$. Using the localization formula, the equivariant version of $H_{\Psi,\Phi}$ for $\mathbb{C}$ and $\mathcal{E}_r^A$ which we denote by $H_{\Psi,\Phi}(\mathbb{C},\mathcal{E}_r^A)(t)$ is as follows:

$$H_{\Psi,\Phi}(\mathbb{C},\mathcal{E}_r^A)(t)=\sum_{n=0}^{\infty}z^n\frac{\phi(-st)}{-st}\prod_{j=1}^{r}\psi(v_j+(s-1)t).$$

Assume that

$$\begin{aligned}H_{\Psi,\Phi}(\mathbb{C},\mathcal{E}_r^A)(t)&=\exp(\sum_{n=1}^{\infty}z^n\int_{\mathbb{C}}^{t}(\sum_{j=0}^{r}(A_{0,j}^n c_0^t(\mathbb{C})c_j^t(\mathcal{E}_r^A)+A_{1,j}^n c_1^t(\mathbb{C})c_j^t(\mathcal{E}_r^A)))\\&=\exp(\sum_{n=1}^{\infty}z^n\int_{\mathbb{C}}^{t}(\sum_{j=0}^{r}(A_{0,j}^n\frac{\sigma_j(v_1,\cdots,v_r)}{t}+A_{1,j}^n\sigma_j(v_1,\cdots,v_r))),\end{aligned}$$

where we denote by $\int^t$ the equivariant integral, c_i^t is the equivariant Chern classes and σ_j is the j-th elementary symmetric polynomial.

Denote by $H_{\Psi,\Phi}(C,E)(t)$ the equivariant version of $H_{\Psi,\Phi}(C,E)$. It can be computed by localization as follows:

$$
\begin{aligned}
H_{\Psi,\Phi}(C,E)(t) &= \sum_{n=0}^{\infty} z^n \prod_{i=1}^{l}\prod_{s=1}^{n_i} \frac{\phi(-sw_i)}{-sw_i} \prod_{j=1}^{r} \psi(v_j+(s-1)w_i) \\
&= \prod_{i=1}^{l} \sum_{n_i=0}^{\infty} z^{n_i} \prod_{s=1}^{n_i} \frac{\phi(-sw_i)}{-sw_i} \prod_{j=1}^{r} \psi(v_j+(s-1)w_i) \\
&= \prod_{i=1}^{l} H_{\Psi,\Phi}(\mathbb{C},\mathcal{E}_r^A)(w_i) \\
&= \prod_{i=1}^{l} \exp(\sum_{n=1}^{\infty} z^n \int_{\mathbb{C}}^{t} (\sum_{j=0}^{r} (A_{0,j}^n \frac{\sigma_j(v_1,\cdots,v_r)}{w_i} + A_{1,j}^n \sigma_j(v_1,\cdots,v_r))) \\
&= \exp(\sum_{n=1}^{\infty} z^n (\sum_{j=0}^{r} (A_{0,j}^n \sum_{i=1}^{l} \frac{\sigma_j(v_1,\cdots,v_r)}{w_i} + A_{1,j}^n \sum_{i=1}^{l} \sigma_j(v_1,\cdots,v_r))) \\
&= \exp(\sum_{n=1}^{\infty} z^n \int_{C}^{t} (\sum_{j=0}^{r} (A_{0,j}^n c_0^t(C) c_j^t(E^A) + A_{1,j}^n c_1^t(\mathbb{C}) c_j^t(E))).
\end{aligned}
$$

If C is projective, by taking nonequivariant limit one has

$$H_{\Psi,\Phi}(C,E) = \exp(\sum_{n=1}^{\infty} z^n \int_{C} (\sum_{j=0}^{r} (A_{0,j}^n c_0(C) c_j(E^A) + A_{1,j}^n c_1(\mathbb{C}) c_j(E)))$$

3.2. **Proof of Theorem 1.1.** Let us see the general case of Conjecture 2 first. Taking $\Psi : K(X) \to H^\times$ to be the total Chern class and $\Phi = 1$, we see that such $H_{\Psi,\Phi}$ satisfies the conditions in Theorem 2.3 and hence can be written in the desired form. So $\frac{1}{n}A_n^r$, $\frac{1}{n}B_n^r$, $\frac{1}{n}C_n^r$ and $\frac{1}{n}D_n^r$ exist as the n-th coefficients of the corresponding universal power series. Now clearly A_n^r, B_n^r, C_n^r and D_n^r are rational numbers depending only on r and n, and we hope to determine them explicitly as integers. As we have discussed in Section 3.1, in order to prove Conjecture 2, it suffices to prove the equivariant version of $(\mathbb{C},\mathcal{E}_r^A)$. Using localization, we have checked Conjecture 2 for $r = 2,3,4,5$ and $n < 10$. We also conjecture that

$$B_n^2 = (-1)^n (4^n - \binom{2n-1}{n-1}),$$

$$B_n^3 = (-1)^n (\sum_{i=0}^{n-1} (\frac{2^{n-2-i}}{n}(n-i)(3n-3i-1)\binom{3n}{i} - \binom{3n-1}{n-1})$$

and have checked them up to $n = 10$. However effort fails to find the explicit expressions for higher r.

Before proving Theorem 1.1, recall that in Section 2.5 of [25] the following identity is established:

$$\begin{aligned}
&\sum_{n=0}^{\infty} z^n \chi(\mathbb{C}^{[n]}, \Lambda_{-y}(\mathcal{E}_1^A)^{[n]})(q) \\
&= \sum_{n=0}^{\infty} z^n \prod_{i=1}^{n} \frac{1 - yq^a q^{i-1}}{1 - q^i} \\
&= \exp(\sum_{n=1}^{\infty} \frac{z^n}{n} \frac{1 - q^{na} y^n}{1 - q^n}) \\
&= \exp(\sum_{n=1}^{\infty} \frac{z^n}{n} \chi(\mathbb{C}, \Lambda_{-y^n} \mathcal{E}_1^A)(q^n)).
\end{aligned} \tag{6}$$

Here $\Lambda_u E = \sum_{i=0}^{n} u^i \Lambda^i E$ and $\chi(\mathbb{C}^{[n]}, \Lambda_{-y}(\mathcal{O}_{\mathbb{C}})^{[n]})(q)$ the equivariant Euler characteristic of $\Lambda_{-y}(\mathcal{O}_{\mathbb{C}})^{[n]}$ on $\mathbb{C}^{[n]}$.

Moreover, the discussion in the last subsection implies that (6) can be generalized as the following:

Proposition 3.1. *For a smooth projective curve C and a line bundle L on C,*

$$\sum_{n=0}^{\infty} z^n \chi(C^{[n]}, \Lambda_{-y} L^{[n]}) = \sum_{n=0}^{\infty} z^n \chi(C, \Lambda_{-y^n} L).$$

To prove Theorem 1.1, we only need to prove the following lemma by using (6):

Lemma 3.2.

$$\sum_{n=0}^{\infty} z^n \int_{C^{[n]}}^{t} c^t((\mathcal{E}_1^A)^{[n]}) = \exp(\sum_{n=1}^{\infty} (-1)^{n+1} z^n \int_{\mathbb{C}}^{t} c^t(\mathcal{E}_1^A)),$$

where $A = (a)$, $a \in \mathbb{Z}$ is the equivariant weight of $\mathcal{O}_{\mathbb{C}}$.

Proof. Similarly as in [12], let $q = \exp(\beta t)$, $y = \exp(\beta)$, and take $\beta \to 0$ in (6), one has

$$\begin{aligned}
&\sum_{n=0}^{\infty} z^n \prod_{i=1}^{n} \frac{1 + at + (i-1)t}{it} \\
&= \exp(\sum_{n=1}^{\infty} \frac{z^n}{n} \frac{n + ant}{nt}) \\
&= \exp(\sum_{n=1}^{\infty} \frac{z^n}{n} \frac{1 + at}{t}).
\end{aligned}$$

Hence

$$\begin{aligned}
&\sum_{n=0}^{\infty} z^n \int_{\mathbb{C}^{[n]}}^t c^t((\mathcal{E}_1^A)^{[n]}) \\
=&\sum_{n=0}^{\infty} z^n \prod_{i=1}^{n} \frac{1+at+(i-1)t}{-it} \\
=&\sum_{n=0}^{\infty} (-z)^n \prod_{i=1}^{n} \frac{1+at+(i-1)t}{it} \\
=&\exp(\sum_{n=1}^{\infty} \frac{(-z)^n}{n} \frac{1+at}{t}) \\
=&\exp(\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} z^n \frac{1+at}{-t}) \\
=&\exp(\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} z^n \int_{\mathbb{C}}^t c^t(\mathcal{E}_1^A)).
\end{aligned}$$

□

3.3. **Proof of Theorem 1.2.** We also have an equivariant version of (5) of $\mathbb{C}$. However, it is also difficult to compute. We have to find some other way to give a proof.

Denote $\mathcal{O}(d)$ over $\mathbb{P}^1$ by L_d and $\int_{(\mathbb{P}^1)^{[n]}} s(L_d^{[n]})$ by N_n^d. As it has been discussed, Theorem 1.2 is true if the following lemma holds:

Lemma 3.3.

$$\sum_{n=0}^{\infty} z^n N_n^0 = \frac{(1-k)^2}{1-2k}, \tag{7}$$

and

$$\sum_{n=0}^{\infty} z^n N_n^{-1} = \frac{1-k}{1-2k}. \tag{8}$$

Here $z = k(1-k)$.

We will use the localization formula to prove this lemma. Recall that the homogeneous coordinates on $\mathbb{P}^1$ are given by $[\zeta_1 : \zeta_2]$ and there is a torus-action on $\mathbb{P}^1$:

$$q \cdot [\zeta_1 : \zeta_2] = [\zeta_1 : q \cdot \zeta_2]$$

.

There are two fixed points $P_1 = [1 : 0]$ and $P_2 = [0 : 1]$ on $\mathbb{P}^1$. We choose the canonical lifting to the tangent bundle of $\mathbb{P}^1$ and the weight decompositions of the cotangent space at P_1 and P_2 are given by q^{-1} and q respectively. We also choose a lifting to L_d such that the weight decomposition of L_d is given by $L_d|_{P_1} = 1$ and $L_d|_{P_1} = q^{-d}$. Denote the equivariant integral $\int_{(\mathbb{P}^1)^{[n]}} s_x^t(L_d^{[n]})$ by $N_n^d(t)$, where

$$s_x^t(L_d^{[n]}) = \frac{1}{c_x^t(L_d^{[n]})} = \frac{1}{1 + xc_1^t(L_d^{[n]}) + \cdots + x^n c_n^t(L_d^{[n]})}$$

is the equivariant total Segre class. N_n^d is the coefficient of x^n in $N_n^d(t)$. By localization formula one has

$$N_n^d(t) = \sum_{k=0}^{n} \prod_{i=1}^{k} \frac{1}{(1-x(i-1)t)(it)} \prod_{i=1}^{n-k} \frac{1}{(1+x((i-1)t-dt))(-it)}.$$

Here we write $\prod_{i=s}^{s-1}(\cdot) = 1$ for convenient notation.

We have the following lemma:

Lemma 3.4. *One has*

$$N_n^d(t) = \frac{\binom{2n-2-d}{n} x^n}{\prod_{i=0}^{n-1}(1+(d-i)xt)(1-ixt)},$$

and hence it is easy to see by comparing the coefficient that $N_n^d = \binom{2n-2-d}{n}$.

Proof.

$$\begin{aligned}
&N_n^d(t) \\
=&\sum_{k=0}^{n} \prod_{i=1}^{k} \frac{1}{(1-x(i-1)t)(it)} \prod_{i=1}^{n-k} \frac{1}{(1+x((i-1)t-dt))(-it)} \\
\overset{y=\frac{1}{tx}}{=}&\sum_{k=0}^{n} \prod_{i=1}^{k} \frac{1}{(1-\frac{(i-1)}{y})(it)} \prod_{i=1}^{n-k} \frac{1}{(1+\frac{(i-1)-d}{y})(-it)} \\
=&\sum_{k=0}^{n} y^n \prod_{i=1}^{k} \frac{1}{(y-(i-1))(it)} \prod_{i=1}^{n-k} \frac{1}{(y+((i-1)-d))(-it)} \\
=&\frac{y^n}{\prod_{i=0}^{n-1}(y+(d-i))(y-i)t^n} \sum_{k=0}^{n} \frac{\prod_{i=k}^{n-1}(-y+i)\prod_{i=n-k}^{n-1}(y+i-d)}{k!(n-k)!} \\
=&\frac{y^n}{\prod_{i=0}^{n-1}(y+(d-i))(y-i)t^n} \sum_{k=0}^{n} \binom{-y+n-1}{n-k}\binom{y+n-1-d}{k} \\
=&\frac{y^n}{\prod_{i=0}^{n-1}(y+(d-i))(y-i)t^n} \binom{2n-2-d}{n}.
\end{aligned}$$

The last identity is a special case of the Chu-Vandermonde's identity (cf. [13], P. 45 Exercise 3.2 (a)).

Take $x = \frac{1}{ty}$ and one gets

$$N_n^d(t) = \frac{\binom{2n-2-d}{n} x^n}{\prod_{i=0}^{n-1}(1+(d-i)xt)(1-ixt)}.$$

□

Now we are going to prove (7).

(N_n^0) is the integer sequence A001791 in the on-line encyclopedia of integer sequences [21] and the generating series is given by

$$\sum_{n=0}^{\infty} N_n^0 z^n = \frac{1-2z+\sqrt{1-4z}}{2\sqrt{1-4z}}.$$

Take $z = k(1-k)$ and one can easily get (7).

Applying similar arguments we can prove (8), so we omit the proof here.

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