A conjecture on integer arithmetic which implies that there are infinitely many twin primes

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Abstract

Let $f(1)=2$, $f(2)=3$, and $f(n+1)=f(n)!$ for every integer $n \geqslant 2$. We write $x \nmid y$ to denote that x does not divide y. Let Φ_n denote the statement: if a system $\mathcal{S} \subseteq \{x_i+1=x_k,\ x_i!=x_k,\ x_i \nmid x_k:\ i,j,k \in \{1,\ldots,n\}\}$ has only finitely many solutions in positive integers $x_1,\ldots,x_n$, then each such solution $(x_1,\ldots,x_n)$ satisfies $x_1,\ldots,x_n \leqslant f(n)$. We prove: (1) the statement Φ_4 implies that there are infinitely many primes of the form $m!+1$, (2) the statement Φ_5 implies that there are infinitely many primes of the form $m!-1$, (3) the statement Φ_6 implies that there are infinitely many primes p such that $p!+1$ is also prime, (4) the statement Φ_5 implies that there are infinitely many twin primes. We conjecture that the statements $\Phi_1,\ldots,\Phi_6$ are true.

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Let $\mathcal{P}$ denote the set of prime numbers. Let $f(1)=2$, $f(2)=3$, and $f(n+1)=f(n)!$ for every integer $n \geqslant 2$. We write $x \nmid y$ to denote that x does not divide y. Let Φ_n denote the statement: if a system $\mathcal{S} \subseteq \{x_i+1=x_k,\ x_i!=x_k,\ x_i \nmid x_k:\ i,j,k \in \{1,\ldots,n\}\}$ has only finitely many solutions in positive integers $x_1,\ldots,x_n$, then each such solution $(x_1,\ldots,x_n)$ satisfies $x_1,\ldots,x_n \leqslant f(n)$.

Observation. *The equation $x_1!=x_1$ has exactly two solutions in positive integers, namely $x_1=1$ and $x_1=2$. The system* $\begin{cases} x_1! = x_1 \\ x_1+1 = x_2 \end{cases}$ *has exactly two solutions in positive integers, namely $(1,2)$ and $(2,3)$. For every integer $n \geqslant 3$, the system*

$$\begin{cases} x_1! &= x_1 \\ x_1+1 &= x_2 \\ \forall i \in \{2,\ldots,n-1\}\ x_i! &= x_{i+1} \end{cases}$$

has exactly two solutions in positive integers, namely $(1,2,\ldots,2)$ and $(f(1),\ldots,f(n))$.

The Observation leads to the following conjecture.

Conjecture. *The statements $\Phi_1,\ldots,\Phi_6$ are true.*

Lemma 1. *([4, pp. 214–215]) For every positive integer x, $x \in \mathcal{P} \cup \{4\}$ if and only if $x \nmid (x-1)!$*

Lemma 2. *For every integer $n \geqslant 4$, $n \in \mathcal{P} \cup \{4\}$ if and only if $n \nmid (n-3)!$*

Proof. If n is prime and $n \geqslant 4$, then $n \nmid (n-3)!$ If $n = 4$, then $n \nmid (n-3)!$ Assume that an integer $n > 4$ is composite. Hence, $n \geqslant 6$.

Case 1: n is a perfect square.

In this case, $n \geqslant 9$ and $\sqrt{n} \in \mathbb{N} \setminus \{0, 1, 2\}$. Hence, $\sqrt{n} < 2 \cdot \sqrt{n} \leqslant n - 3$. Therefore, $\sqrt{n} \cdot \left(2 \cdot \sqrt{n}\right)$ divides $(n-3)!$ In particular, $n = \sqrt{n} \cdot \sqrt{n}$ divides $(n-3)!$

Case 2: n is not a perfect square.

Let m denote the smallest prime factor of n. We have:

$$\left(m \cdot \frac{n}{m} = n\right) \wedge (m > 1) \wedge \left(\frac{n}{m} > 1\right) \wedge \left(m \neq \frac{n}{m}\right)$$

Hence, $m \leqslant \frac{n}{2} \leqslant n - 3$ and $\frac{n}{m} \leqslant \frac{n}{2} \leqslant n - 3$. Since $m \neq \frac{n}{m}$, we conclude that $m \cdot \frac{n}{m} = n$ divides $(n-3)!$ □

Let $\mathcal{A}$ denote the system

$$\begin{cases} x! = x_1 \\ x_1 + 1 = x_2 \\ x_1! = x_3 \\ x_2 \nmid x_3 \end{cases}$$

which is illustrated in Figure 1.

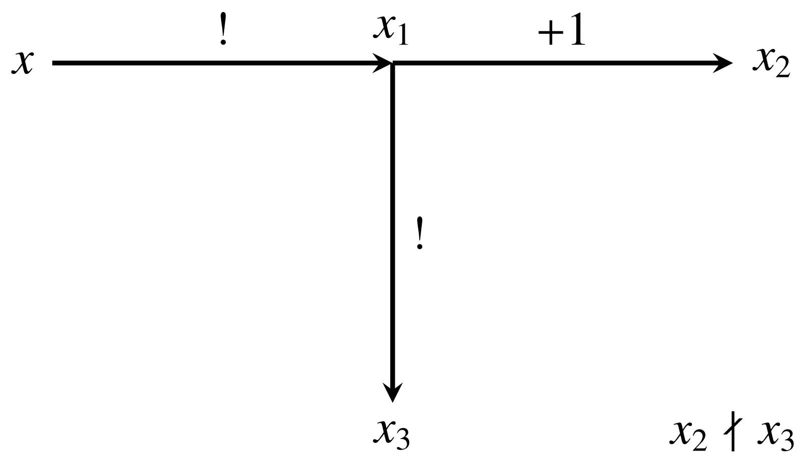

Fig. 1 The diagram of the system $\mathcal{A}$

Lemma 3. *For every positive integer x, the system $\mathcal{A}$ is solvable in positive integers x_1, x_2, x_3 if and only if $x! + 1$ is prime. In this case, the numbers x_1, x_2, x_3 are uniquely determined by the following equalities:*

$$\begin{aligned} x_1 &= x! \\ x_2 &= x! + 1 \\ x_3 &= (x!)! \end{aligned}$$

Proof. The system $\mathcal{A}$ is solvable in positive integers x_1, x_2, x_3 if and only if $x! + 1 \nmid (x!)!$ By Lemma 1, this condition means that $x! + 1 \in \mathcal{P} \cup \{4\}$. For every positive integer x, $x! + 1 \neq 4$. Hence, $x! + 1 \in \mathcal{P} \cup \{4\}$ if and only if $x! + 1$ is prime. □

Theorem 1. *The statement Φ_4 implies that there are infinitely many primes of the form $x! + 1$.*

Proof. We take $x = 11$ and remark that $x! + 1$ is prime, see [1, p. 441], [4, p. 215], and [5]. By Lemma 3, there exists a unique tuple (x_1, x_2, x_3) of positive integers such that the tuple (x, x_1, x_2, x_3) solves the system $\mathcal{A}$. Hence, $x_1 = 11! > 6! = f(4)$. The statement Φ_4 and the inequality $x_1 > f(4)$ imply that the system $\mathcal{A}$ has infinitely many solutions in positive integers x, x_1, x_2, x_3. According to Lemma 3, there are infinitely many positive integers x such that $x! + 1$ is prime. □

Let $\mathcal{B}$ denote the system

$$\begin{cases} x! &= x_1 \\ x_2 + 1 &= x_1 \\ x_3 + 1 &= x_2 \\ x_3! &= x_4 \\ x_2 &\nmid x_4 \end{cases}$$

which is illustrated in Figure 2.

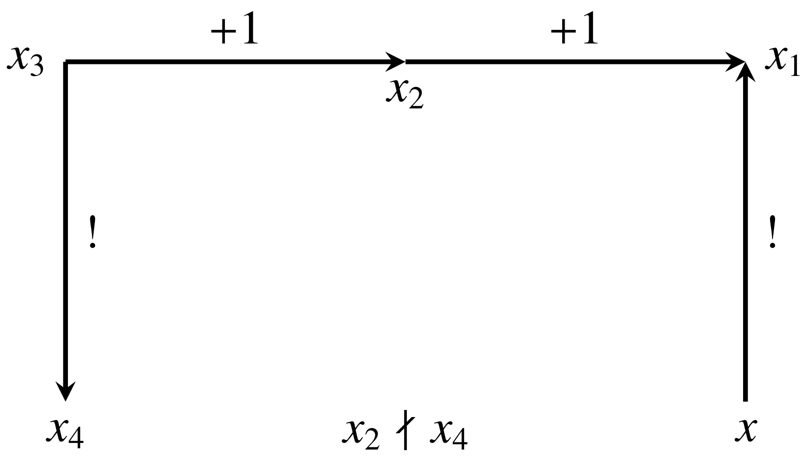

Fig. 2 The diagram of the system $\mathcal{B}$

Lemma 4. *For every positive integer x, the system $\mathcal{B}$ is solvable in positive integers x_1, x_2, x_3, x_4 if and only if $x! - 1$ is prime. In this case, the numbers x_1, x_2, x_3, x_4 are uniquely determined by the following equalities:*

$$\begin{aligned} x_1 &= x! \\ x_2 &= x! - 1 \\ x_3 &= x! - 2 \\ x_4 &= (x! - 2)! \end{aligned}$$

Proof. The system $\mathcal{B}$ is solvable in positive integers x_1, x_2, x_3, x_4 if and only if

$$\Big(x \geqslant 3\Big) \wedge \Big(x! - 1 \nmid (x! - 2)!\Big)$$

By Lemma 1, the above conjunction means that

$$\Big(x \geqslant 3\Big) \wedge \Big(x! - 1 \in \mathcal{P} \cup \{4\}\Big) \tag{1}$$

The condition $x! - 1 \in \mathcal{P} \cup \{4\}$ implies that $x \geqslant 3$. For every positive integer x, $x! - 1 \neq 4$. The last two sentences prove that formula (1) equivalently expresses that $x! - 1$ is prime. □

Theorem 2. *The statement Φ_5 implies that there are infinitely many primes of the form $x! - 1$.*

Proof. We take $x = 7$ and remark that $x! - 1$ is prime, see [1, p. 441], [4, p. 215], and [6]. By Lemma 4, there exists a unique tuple (x_1, x_2, x_3, x_4) of positive integers such that the tuple (x, x_1, x_2, x_3, x_4) solves the system $\mathcal{B}$. Hence, $x_4 = (x! - 2)! = (7! - 2)! > 720! = f(5)$. The statement Φ_5 and the inequality $x_4 > f(5)$ imply that the system $\mathcal{B}$ has infinitely many solutions in positive integers x, x_1, x_2, x_3, x_4. According to Lemma 4, there are infinitely many positive integers x such that $x! - 1$ is prime. □

Let $\mathcal{C}$ denote the system

$$\begin{cases} x_1 + 1 &= x \\ x! &= x_2 \\ x_2 + 1 &= x_3 \\ x_1! &= x_4 \\ x_2! &= x_5 \\ x &\nmid x_4 \\ x_3 &\nmid x_5 \end{cases}$$

which is illustrated in Figure 3.

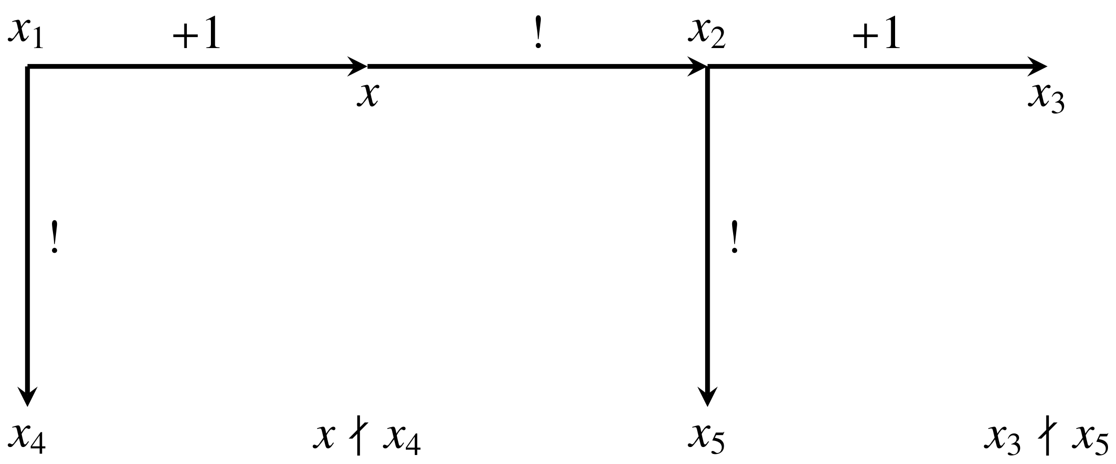

Fig. 3 The diagram of the system $\mathcal{C}$

Lemma 5. *For every positive integer x, the system $\mathcal{C}$ is solvable in positive integers x_1, x_2, x_3, x_4, x_5 if and only if x and $x! + 1$ are prime. In this case, the numbers x_1, x_2, x_3, x_4, x_5 are uniquely determined by the following equalities:*

$$\begin{aligned} x_1 &= x - 1 \\ x_2 &= x! \\ x_3 &= x! + 1 \\ x_4 &= (x-1)! \\ x_5 &= (x!)! \end{aligned}$$

Proof. The system $\mathcal{C}$ is solvable in positive integers x_1, x_2, x_3, x_4, x_5 if and only if

$$\Big(x \geqslant 2\Big) \wedge \Big(x \nmid (x-1)!\Big) \wedge \Big(x! + 1 \nmid (x!)!\Big) \tag{2}$$

By Lemma 1, formula (2) is equivalent to

$$\Big(x \geqslant 2\Big) \wedge \Big(x \in \mathcal{P} \cup \{4\}\Big) \wedge \Big(x! + 1 \in \mathcal{P} \cup \{4\}\Big) \tag{3}$$

The condition $x \in \mathcal{P} \cup \{4\}$ implies that $x \geqslant 2$. If $x = 4$, then $x! + 1 \notin \mathcal{P} \cup \{4\}$. For every positive integer x, $x! + 1 \neq 4$. The last three sentences imply that formula (3) is equivalent to

$$\left(x \in \mathcal{P}\right) \wedge \left(x! + 1 \in \mathcal{P}\right)$$

□

Theorem 3. *The statement Φ_6 implies that there are infinitely many primes p such that $p! + 1$ is also prime.*

Proof. We take $x = 26951$ and remark that the numbers x and $x! + 1$ are prime ([7]). By Lemma 5, there exists a unique tuple $(x_1, x_2, x_3, x_4, x_5)$ of positive integers such that the tuple $(x, x_1, x_2, x_3, x_4, x_5)$ solves the system C. Hence, $x_5 = (x!)! > ((720)!)! = f(6)$. The statement Φ_6 and the inequality $x_5 > f(6)$ imply that the system C has infinitely many solutions in positive integers $x, x_1, x_2, x_3, x_4, x_5$. According to Lemma 5, there are infinitely many primes p such that $p! + 1$ is also prime. □

A twin prime is a prime number that is either 2 less or 2 more than another prime number. The twin prime conjecture states that there are infinitely many twin primes, see [2], [3, p. 39], [4, p. 120], and [8, p. 3091]. Let $\mathcal{D}$ denote the system

$$\begin{cases} x_1 + 1 &= x \\ x + 1 &= x_2 \\ x_2 + 1 &= y \\ x_1! &= x_3 \\ x &\nmid x_3 \\ y &\nmid x_3 \\ x &\nmid y \end{cases}$$

which is illustrated in Figure 4.

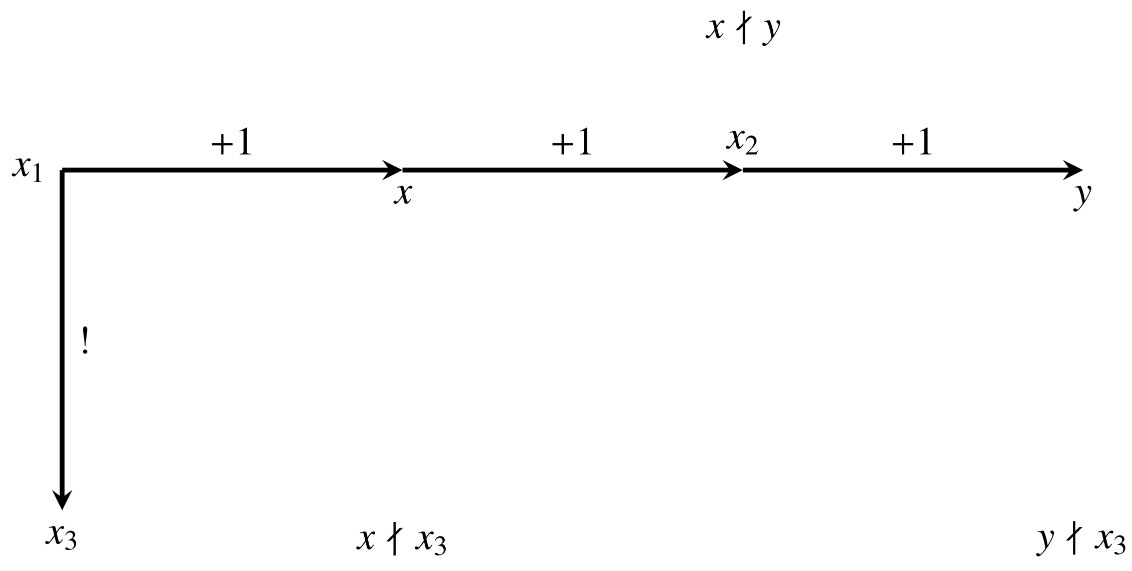

Fig. 4 The diagram of the system $\mathcal{D}$

Lemma 6. *For every positive integers x and y, the system $\mathcal{D}$ is solvable in positive integers x_1, x_2, x_3 if and only if x and y are prime and $x + 2 = y$. In this case, the numbers x_1, x_2, x_3 are uniquely determined by the following equalities:*

$$\begin{aligned} x_1 &= x - 1 \\ x_2 &= y - 1 \\ x_3 &= (x-1)! \end{aligned}$$

Proof. The system $\mathcal{D}$ is solvable in positive integers x_1, x_2, x_3 if and only if

$$\Big(x \geqslant 2\Big) \wedge \Big(x \nmid x + 2\Big) \wedge \Big(x + 2 = y\Big) \wedge \Big(x \nmid (x-1)!\Big) \wedge \Big(y \nmid (y-3)!\Big) \tag{4}$$

The conjunction $(x \geqslant 2) \wedge (x \nmid x + 2)$ is equivalent to $x \geqslant 3$. By this and Lemmas 1 and 2, formula (4) is equivalent to

$$\Big(x \geqslant 3\Big) \wedge \Big(x + 2 = y\Big) \wedge \Big(x \in \mathcal{P} \cup \{4\}\Big) \wedge \Big(y \in \mathcal{P} \cup \{4\}\Big) \tag{5}$$

The conditions $x + 2 = y$ and $y \in \mathcal{P} \cup \{4\}$ imply that $x \neq 4$. The conditions $x \geqslant 3$ and $x + 2 = y$ imply that $y \neq 4$. The last two sentences prove that formula (5) is equivalent to

$$\Big(x \geqslant 3\Big) \wedge \Big(x + 2 = y\Big) \wedge \Big(x \in \mathcal{P}\Big) \wedge \Big(y \in \mathcal{P}\Big)$$

The condition $x \in \mathcal{P}$ implies that $x \geqslant 2$. The conditions $x + 2 = y$ and $y \in \mathcal{P}$ imply that $x \neq 2$. Therefore, we can omit the inequality $x \geqslant 3$. □

Theorem 4. *The statement Φ_5 implies that there are infinitely many twin primes.*

Proof. Let $x = 809$ and $y = 811$. The numbers x and y are prime and $x + 2 = y$. By Lemma 6, there exists a unique tuple (x_1, x_2, x_3) of positive integers such that the tuple (x, y, x_1, x_2, x_3) solves the system $\mathcal{D}$. Hence, $x - 1 > 720$. Therefore, $x_3 = (x-1)! > 720! = f(5)$. The statement Φ_5 and the inequality $x_3 > f(5)$ imply that the system $\mathcal{D}$ has infinitely many solutions in positive integers x, y, x_1, x_2, x_3. According to Lemma 6, there are infinitely many twin primes. □

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