ENUMERATION OF A DUAL SET OF STIRLING PERMUTATIONS BY THEIR ALTERNATING RUNS

SHI-MEI MA AND HAI-NA WANG

ABSTRACT. In this paper, we count a dual set of Stirling permutations by the number of alternating runs. Properties of the generating functions, including recurrence relations, grammatical interpretations and convolution formulas are studied.

Keywords: Stirling permutations; Alternating runs; Eulerian polynomials

1. INTRODUCTION

Let $[n]=\{1,2,\ldots,n\}$. The *Stirling number of the second kind* $\left\{ {n \atop k} \right\}$ is the number of ways to partition $[n]$ into k blocks. Denote by D the differential operator $\frac{d}{dx}$, and let $\vartheta=xD$. It is well known that

$$\vartheta^n=\sum_{k=1}^n \left\{ {n \atop k} \right\} z^k D^k.$$

Let

$$r(x)=\frac{\sqrt{1+x}}{\sqrt{1-x}}.$$

By induction, one can easily verify that there are positive integers $T(n,k)$, $k\in[2n-1]$, such that

$$\vartheta^n(r(x))=\frac{\sum_{k=1}^{2n-1}T(n,k)x^k}{(1-x)^n(1+x)^{n-1}\sqrt{1-x^2}} \quad \text{for } n\geq 1.$$

It is clear that the numbers $T(n,k)$ satisfy the initial conditions $T(1,1)=1$ and $T(1,k)=0$ for $k\neq 1$. Let $T_n(x)=\sum_{k=1}^{2n-1}T(n,k)x^k$. Using $\vartheta^{n+1}(r(x))=\vartheta(\vartheta^n(r(x)))$, we get that the polynomials $T_n(x)$ satisfy the recurrence relation

$$T_{n+1}(x)=(2nx+1)xT_n(x)+x(1-x^2)T_n'(x) \tag{1}$$

for $n\geq 0$, with the initial values $T_0(x)=1$ and $T_1(x)=x$. In particular,

$$T_n(1)=-T_{n+1}(-1)=(2n-1)!! \quad \text{for } n\geq 1.$$

Equating the coefficients of x^k on both sides of (1), we get that the numbers $T(n,k)$ satisfy the recurrence relation

$$T(n+1,k)=kT(n,k)+T(n,k-1)+(2n-k+2)T(n,k-2). \tag{2}$$

The motivating goal of this paper is to find a combinatorial interpretation of the numbers $T(n,k)$.

Let $\mathfrak{S}_n$ denote the symmetric group of all permutations of $[n]$, where $[n]=\{1,2,\ldots,n\}$. Let $\pi=\pi(1)\pi(2)\cdots\pi(n)\in\mathfrak{S}_n$. An *interior peak* in π is an index $i\in\{2,3,\ldots,n-1\}$ such that $\pi(i-1)<\pi(i)>\pi(i+1)$. A *left peak* in π is an index $i\in[n-1]$ such that $\pi(i-1)<\pi(i)>\pi(i+1)$, where we take $\pi(0)=0$. Let $\mathrm{ipk}\,(\pi)$ (resp. $\mathrm{lpk}\,(\pi)$) be the number of interior peaks (resp. left peaks) in π. We say that π changes direction at position i if either $\pi(i-1)<\pi(i)>\pi(i+1)$, or $\pi(i-1)>\pi(i)<\pi(i+1)$, where $i\in\{2,3,\ldots,n-1\}$. We say that π has k *alternating runs* if there are $k-1$ indices i such that π changes direction at these positions. Denote by $\mathrm{altrun}\,(\pi)$ the number of alternating runs in π.

Define

$$W_n(x)=\sum_{\pi\in\mathfrak{S}_n}x^{\mathrm{ipk}\,(\pi)},\ \widehat{W}_n(x)=\sum_{\pi\in\mathfrak{S}_n}x^{\mathrm{lpk}\,(\pi)},\ R_n(x)=\sum_{\pi\in\mathfrak{S}_n}x^{\mathrm{altrun}\,(\pi)}.$$

From [14, Corollary 2, Theorem 3], we get

$$\frac{(1+x)^2}{2x}R_n(x)=xW_n(x^2)+\widehat{W}_n(x^2).$$

Let $R_n(x)=\sum_{k=1}^{n-1}R(n,k)x^k$. The study of alternating runs of permutations was initiated by André [2], and he proved that the numbers $R(n,k)$ satisfy the recurrence relation

$$R(n,k)=kR(n-1,k)+2R(n-1,k-1)+(n-k)R(n-1,k-2) \tag{3}$$

for $n,k\geq 1$, where $R(1,0)=1$ and $R(1,k)=0$ for $k\geq 1$. It follows from (3) that the polynomials $R_n(x)$ satisfy the recurrence relation

$$R_{n+2}(x)=x(nx+2)R_{n+1}(x)+x\left(1-x^2\right)R'_{n+1}(x), \tag{4}$$

with the initial value $R_1(x)=1$. Recall that a *descent* of a permutation $\pi\in\mathfrak{S}_n$ is a position i such that $\pi(i)>\pi(i+1)$. Denote by $\mathrm{des}\,(\pi)$ the number of descents of π. Then the equations

$$A_n(x)=\sum_{\pi\in\mathfrak{S}_n}x^{\mathrm{des}\,(\pi)+1}=\sum_{k=1}^{n}\left\langle {n \atop k}\right\rangle x^k,$$

define the *Eulerian polynomial* $A_n(x)$ and the *Eulerian number* $\left\langle {n \atop k}\right\rangle$. The polynomial $R_n(x)$ is closely related to $A_n(x)$:

$$R_n(x)=\left(\frac{1+x}{2}\right)^{n-1}(1+w)^{n+1}A_n\left(\frac{1-w}{1+w}\right),\quad w=\sqrt{\frac{1-x}{1+x}}, \tag{5}$$

which was first established by David and Barton [7, 157-162] and then stated more concisely by Knuth [11, p. 605]. There is a large literature devoted to the polynomials $R_n(x)$ (see [22, A059427]). The reader is referred to [4, 14] for recent results on this subject.

In [5], Carlitz introduced $C_n(x)$ defined by

$$\sum_{n=0}^{\infty}\left\{ {n+k \atop k}\right\} x^n=\frac{C_n(x)}{(1-x)^{2k+1}},$$

and asked for a combinatorial interpretation of $C_n(x)$. Riordan [19] noted that $C_n(x)$ is the enumerator of trapezoidal words with n elements by number of distinct elements, where trapezoidal words are such that the i-th element takes the values $1,2,\ldots,2i-1$. Gessel and Stanley [8] gave another combinatorial interpretation of $C_n(x)$ in terms of descents of Stirling permutations. A *Stirling permutation* of order n is a permutation $\sigma=\sigma(1)\sigma(2)\cdots\sigma(2n-1)\sigma(2n)$ of the multiset $\{1,1,2,2,\ldots,n,n\}$ such that for each i, $1\leq i\leq n$, all entries between the two occurrencies of i are larger than i. Denote by $\mathcal{Q}_n$ the set of Stirling permutation of order n. For $\sigma\in\mathcal{Q}_n$, let $\sigma(0)=\sigma(2n+1)=0$, and let

$$\begin{aligned}\mathrm{des}\,(\sigma)&=\#\{i\mid\sigma(i)>\sigma(i+1)\},\\ \mathrm{asc}\,(\sigma)&=\#\{i\mid\sigma(i-1)<\sigma(i)\},\\ \mathrm{plat}\,(\sigma)&=\#\{i\mid\sigma(i)=\sigma(i+1)\}\end{aligned}$$

denote the number of descents, ascents and plateaux of σ, respectively. Gessel and Stanley [8] proved that

$$C_n(x)=\sum_{\sigma\in\mathcal{Q}_n}x^{\mathrm{des}\,\sigma}.$$

Bóna [3, Theorem 1] introduced the plateau statistic on $\mathcal{Q}_n$, and proved that descents, ascents and plateaux are equidistributed over $\mathcal{Q}_n$. The reader is referred to [9, 10, 12] for recent progress on the study of statistics on Stirling permutations.

In the next section, we show that $T_n(x)$ is the enumerator of a dual set of Stirling permutations of order n by number of alternating runs.

2. Combinatorial interpretation of $T(n,k)$

Let $\sigma = \sigma(1)\sigma(2)\cdots\sigma(2n) \in \mathcal{Q}_n$. Let Φ be the injection which maps each first occurrence of entry j in σ to $2j$ and the second j to $2j-1$, where $j \in [n]$. For example, $\Phi(221331) = 432651$. The *dual set* $\Phi(\mathcal{Q}_n)$ of $\mathcal{Q}_n$ is defined by

$$\Phi(\mathcal{Q}_n) = \{\pi \mid \sigma \in \mathcal{Q}_n, \Phi(\sigma) = \pi\}.$$

Clearly, $\Phi(\mathcal{Q}_n)$ is a subset of $\mathfrak{S}_{2n}$. For $\pi \in \Phi(\mathcal{Q}_n)$, the entry $2j$ is to the left of $2j-1$, and all entries in π between $2j$ and $2j-1$ are larger than $2j$, where $1 \le j \le n$. Let ab be an ascent in σ, so $a < b$. Using Φ, we see that ab maps into $(2a-1)(2b-1)$, $(2a-1)(2b)$, $(2a)(2b-1)$ or $(2a)(2b)$, and vice versa. Note that $\operatorname{asc}(\sigma) = \operatorname{asc}(\Phi(\sigma)) = \operatorname{asc}(\pi)$. Therefore, we have

$$C_n(x) = \sum_{\pi \in \Phi(\mathcal{Q}_n)} x^{\operatorname{asc}(\pi)}.$$

It should be noted that $\pi \in \Phi(\mathcal{Q}_n)$ always ends with a descending run. We now present the following result.

Theorem 1. *We have*

$$T(n,k) = \#\{\pi \in \Phi(\mathcal{Q}_n) \mid \operatorname{altrun}(\pi) = k\}.$$

Proof. There are three ways in which a permutation $\pi \in \Phi(\mathcal{Q}_{n+1})$ with $\operatorname{altrun}(\pi) = k$ can be obtained from a permutation $\sigma \in \Phi(\mathcal{Q}_n)$ by inserting the pair (2n+2)(2n+1) into consecutive positions.

(a) If $\operatorname{altrun}(\sigma) = k$, then we can insert the pair $(2n+2)(2n+1)$ right before the beginning of each descending run, and right after the end of each ascending run. This accounts for $kT(n,k)$ possibilities.

(b) If $\operatorname{altrun}(\sigma) = k-1$, then we distinguish two cases: when σ starts in an ascending run, we insert the pair $(2n+2)(2n+1)$ to the front of σ; when σ starts in an descending run, we insert the pair $(2n+2)(2n+1)$ right after the first entry of σ. This gives $T(n,k-1)$ possibilities.

(c) If $\operatorname{altrun}(\sigma) = k-2$, then we can insert the pair $(2n+2)(2n+1)$ into the remaining $(2n+1)-(k-2)-1 = 2n-k+2$ positions. This gives $(2n-k+2)T(n,k-2)$ possibilities.

Therefore, the numbers $T(n,k)$ satisfy the recurrence relation (2), and this completes the proof. □

Define

$$M_n(x) = \sum_{\pi \in \Phi(\mathcal{Q}_n)} x^{\operatorname{ipk}(\pi)},\ N_n(x) = \sum_{\pi \in \Phi(\mathcal{Q}_n)} x^{\operatorname{lpk}(\pi)}.$$

It follows from [16, Theorem 4] that $M_n(x) = x^n N_n\left(\frac{1}{x}\right)$. Moreover, from [16, Theorem 5], we have

$$(1+x)T_n(x) = xM_n(x^2) + N_n(x^2).$$

We now recall some properties of $N_n(x)$. Let $N_n(x) = \sum_{k=1}^n N(n,k)x^k$. Apart from counting permutations in the set $\Phi(\mathcal{Q}_n)$ with k left peaks, the number $N(n,k)$ also has the following combinatorial interpretations:

(m_1) Let $e=(e_1,e_2,\ldots,e_n)\in\mathbb{Z}^n$, and let $I_{n,k}=\{e\in\mathbb{Z}^n|0\leq e_i\leq (i-1)k\}$, which known as the set of n-dimensional *k-inversion sequences* (see [20]). The number of *ascents* of e is defined by

$$\operatorname{asc}(e)=\#\left\{i:1\leq i\leq n-1|\frac{e_i}{(i-1)k+1}<\frac{e_{i+1}}{ik+1}\right\}.$$

Savage and Viswanathan [21] discovered that $N(n,k)=\#\{e\in I_{n,2}:\operatorname{asc}(e)=n-k\}$.

(m_2) We say that an index $i\in[2n-1]$ is an *ascent plateau* of $\pi\in\mathcal{Q}_n$ if $\pi(i-1)<\pi(i)=\pi(i+1)$. The number $N(n,k)$ counts Stirling permutations in $\mathcal{Q}_n$ with k ascent plateaux (see [16, Theorem 3]).

(m_3) The number $N(n,k)$ counts perfect matching on $[2n]$ with the restriction that only k matching pairs with odd minimal elements (see [18]).

The polynomials $N_n(x)$ satisfy the recurrence relation

$$N_{n+1}(x)=(2n+1)xN_n(x)+2x(1-x)N_n'(x)$$

with initial value $N_0(x)=1$. The first few of $N_n(x)$ are

$$N_1(x)=x, N_2(x)=2x+x^2, N_3(x)=4x+10x^2+x^3, N_4(x)=8x+60x^2+36x^3+x^4.$$

The exponential generating function for $N_n(x)$ is given as follows (see [13, Section 5]):

$$N(x,z)=\sum_{n\geq 0}N_n(x)\frac{z^n}{n!}=\sqrt{\frac{1-x}{1-xe^{2z(1-x)}}}. \tag{6}$$

A polynomial $f(x)=\sum_{k=0}^n a_kx^k$ is *symmetric* if $a_k=a_{n-k}$ for all $0\leq k\leq n$, while it is *unimodal* if there exists an index $0\leq m\leq n$, such that

$$a_0\leq a_1\leq\cdots\leq a_{m-1}\leq a_m\geq a_{m+1}\geq\cdots\geq a_n.$$

Theorem 2. *The polynomial $T_n(x)$ is symmetric and unimodal.*

Proof. It is immediate from (2) that $T_n(x)$ is a symmetric polynomial. We show the unimodality by induction on n. Note that $T_1(x)=x, T_2(x)=x+x^2+x^3$ and $T_3(x)=x+3x^2+7x^3+3x^4+x^5$ are all unimodal. Thus it suffices to consider the case $n\geq 3$. Assume that $T_n(x)$ is symmetric and unimodal. For $1\leq k\leq n+1$, it follows from (2) that

$$\begin{aligned}T(n+1,k)-T(n+1,k-1)&=(k-1)(T(n,k)-T(n,k-1))+(T(n,k-1)-T(n,k-2))\\&\quad+(2n-k+2)(T(n,k-2)-T(n,k-3))+(T(n,k)-T(n,k-3))\\&\geq 0,\end{aligned}$$

where the inequalities are follow from the induction hypothesis. This completes the proof. $\square$

In the next section, we present a grammatical interpretation of $T_n(x)$.

3. Grammatical interpretations

The grammatical method was introduced by Chen [6] in the study of exponential structures in combinatorics. For an alphabet A, let $\mathbb{Q}[[A]]$ be the rational commutative ring of formal power series in monomials formed from letters in A. A context-free grammar over A is a function $G:A\rightarrow\mathbb{Q}[[A]]$ that replace a letter in A by a formal function over A. The formal derivative D is a linear operator defined with respect to a context-free grammar G. More precisely, the derivative $D=D_G$: $\mathbb{Q}[[A]]\rightarrow\mathbb{Q}[[A]]$ is defined as follows: for $x\in A$, we have $D(x)=G(x)$; for a monomial u in $\mathbb{Q}[[A]]$, $D(u)$ is defined so that D is a derivation, and for a general element $q\in\mathbb{Q}[[A]]$, $D(q)$ is defined by linearity.

The *hyperoctahedral group* B_n is the group of signed permutations of the set $\pm[n]$ such that $\pi(-i) = -\pi(i)$ for all i, where $\pm[n] = \{\pm 1, \pm 2, \ldots, \pm n\}$. For each $\pi \in B_n$, we define

$$\begin{aligned} \operatorname{des}_A(\pi) &:= \#\{i \in \{1,2,\ldots,n-1\} | \pi(i) > \pi(i+1)\}, \\ \operatorname{des}_B(\pi) &:= \#\{i \in \{0,1,2,\ldots,n-1\} | \pi(i) > \pi(i+1)\}, \end{aligned}$$

where $\pi(0) = 0$. Following [1], the *flag descent number* of π is defined by

$$\operatorname{fdes}(\pi) := \begin{cases} 2\operatorname{des}_A(\pi) + 1, & \text{if } \pi(1) < 0; \\ 2\operatorname{des}_A(\pi), & \text{otherwise.} \end{cases}$$

Let

$$\begin{aligned} B_n(x) &= \sum_{\pi \in B_n} x^{\operatorname{des}_B(\pi)} = \sum_{k=0}^{n} B(n,k) x^k, \\ S_n(x) &= \sum_{\pi \in B_n} x^{\operatorname{fdes}(\pi)} = \sum_{k=1}^{2n} S(n,k) x^{k-1}. \end{aligned}$$

The polynomial $B_n(x)$ is called an *Eulerian polynomial of type* B, while $B(n,k)$ is called an *Eulerian number of type* B (see [22, A060187]). It follows from [1, Theorem 4.3] that the numbers $S(n,k)$ satisfy the recurrence relation

$$S(n,k) = kS(n-1,k) + S(n-1,k-1) + (2n-k+1)S(n-1,k-2)$$

for $n, k \geq 1$, where $S(1,1) = S(1,2) = 1$ and $S(1,k) = 0$ for $k \geq 3$. The polynomials $S_n(x)$ is closely related to the Eulerian polynomial $A_n(x)$:

$$S_n(x) = \frac{1}{x}(1+x)^n A_n(x) \quad \text{for } n \geq 1,$$

which was established by Adin, Brenti and Roichman [1]. It should be noted that $S_n(x)$ and $A_n(x)$ are both symmetric.

Consider the context-free grammar

$$A = \{x, y, z\}, \; G = \{x \to p(x,y,z), y \to q(x,y,z), z \to r(x,y,z)\},$$

where $p(x,y,z), q(x,y,z)$ and $r(x,y,z)$ are polynomials in x, y and z. The *diamond product* of z with the grammar G is defined by

$$G \diamond z = \{x \to p(x,y,z)z, y \to q(x,y,z)z, z \to r(x,y,z)z\}.$$

We now recall two results on context-free grammars.

Proposition 3 ([14, Theorem 6])**.** *If*

$$G = \{x \to xy, y \to yz, z \to y^2\}, \tag{7}$$

then

$$D^n(x^2) = x^2 \sum_{k=0}^{n} R(n+1,k) y^k z^{n-k}.$$

Setting $x = z = 1$, *we have* $D^n(x^2)|_{x=z=1} = R_{n+1}(y)$.

Proposition 4 ([15, Theorem 10])**.** *Consider the context-free grammar*

$$G' = \{x \to xyz, y \to yz^2, z \to y^2 z\}, \tag{8}$$

which is the diamond product of z with the grammar G defined by (7)*. For $n \geq 1$, we have*

$$\begin{aligned}
D^n(xy) &= x\sum_{k=1}^{2n} S(n,k)y^{2n-k+1}z^k,\\
D^n(yz) &= \sum_{k=0}^{n} B(n,k)y^{2n-2k+1}z^{2k+1},\\
D^n(y) &= \sum_{k=1}^{n} N(n,k)y^{2n-2k+1}z^{2k},\\
D^n(z) &= \sum_{k=1}^{n} N(n,n-k+1)y^{2n-2k+2}z^{2k-1},\\
D^n(y^2) &= 2^n\sum_{k=1}^{n} \left\langle {n \atop k} \right\rangle y^{2n-2k+2}z^{2k}.
\end{aligned}$$

We can now conclude the following result.

Theorem 5. *Let G' be the context-free grammar given by* (8)*. Then for $n \geq 1$, we have*

$$\begin{aligned}
D^n(x) &= x\sum_{k=1}^{2n-1} T(n,k)y^k z^{2n-k},\\
D^n(x^2) &= 2x^2(y+z)^{n-1}\sum_{k=1}^{n} \left\langle {n \atop k} \right\rangle y^k z^{n-k+1}.
\end{aligned}$$

Setting $x=z=1$, we have $D^n(x)|_{x=z=1} = T_n(y)$ and $D^n(x^2)|_{x=z=1} = 2(1+y)^{n-1}A_n(y)$.

Proof. Note that $D(x)=xyz$ and $D^2(x)=xyz^3+xy^2z^2+xy^3z$. For $n\geq 1$, we define $t(n,k)$ by

$$D^n(x) = x\sum_{k\geq 1} t(n,k)y^k z^{2n-k}.$$

Then

$$\begin{aligned}
D^{n+1}(x) &= D(D^n(x))\\
&= x\sum_{k\geq 1} t(n,k)y^{k+1}z^{2n-k+1} + x\sum_{k\geq 1} kt(n,k)y^k z^{2n-k+2} + x\sum_{k\geq 1}(2n-k)t(n,k)y^{k+2}z^{2n-k}.
\end{aligned}$$

Hence

$$t(n+1,k) = kt(n,k) + t(n,k-1) + (2n-k+2)t(n,k-2). \tag{9}$$

By comparing (9) with (2), we see that the numbers $t(n,k)$ satisfy the same recurrence relation and initial conditions as $T(n,k)$, so they agree. The assertion for $D^n(x^2)$ can be proved in a similar way. □

It follows from *Leibniz's formula* that

$$D^n(uv) = \sum_{k=0}^{n} \binom{n}{k} D^k(u)D^{n-k}(v).$$

Hence

$$D^n(x^2) = \sum_{k=0}^{n} \binom{n}{k} D^k(x)D^{n-k}(x),$$

$$D^{n+1}(x) = D^n(xyz) = \sum_{k=0}^{n} \binom{n}{k} D^k(x)D^{n-k}(yz) = \sum_{k=0}^{n} \binom{n}{k} D^k(xy)D^{n-k}(z).$$

Therefore, we can use Proposition 4 and Theorem 5 to get several convolution identities.

Corollary 6. *For $n \geq 1$, we have*

$$2(1+x)^{n-1}A_n(x) = \sum_{k=0}^{n}\binom{n}{k}T_k(x)T_{n-k}(x), \tag{10}$$

$$T_{n+1}(x) = x\sum_{k=0}^{n}\binom{n}{k}T_k(x)B_{n-k}(x^2),$$

$$T_{n+1}(x) = x\sum_{k=0}^{n}\binom{n}{k}S_k(x)N_{n-k}(x^2).$$

Let $T(x,z) = \sum_{n=0}^{\infty} T_n(x)\frac{z^n}{n!}$. Recall that the exponential generating function for $A_n(x)$ is given as follows (see [22, A008292]):

$$A(x,t) = \sum_{n\geq 0} A_n(x)\frac{t^n}{n!} = \frac{1-x}{1-xe^{t(1-x)}}. \tag{11}$$

Combining (10) and (11), we get

$$T(x,z) = \frac{e^{z(x-1)(x+1)}+x}{1+x}\sqrt{\frac{1-x^2}{e^{2\,z(x-1)(x+1)}-x^2}}. \tag{12}$$

From (6), we have

$$\sum_{n\geq 0} M_n(x^2)\frac{z^n}{n!} = \sum_{n\geq 0} x^{2n}N_n\left(\frac{1}{x^2}\right)\frac{z^n}{n!} = \sqrt{\frac{1-x^2}{e^{2\,z(x-1)(x+1)}-x^2}}.$$

Note that

$$\frac{e^{z(x-1)(x+1)}+x}{1+x} = 1 + \sum_{n\geq 1}(x-1)^n(x+1)^{n-1}\frac{z^n}{n!}.$$

Therefore, from (12), we obtain

$$T_n(x) = M_n(x^2) + \sum_{k=0}^{n-1}\binom{n}{k}M_k(x^2)(x-1)^{n-k}(x+1)^{n-k-1} \quad \text{for } n \geq 1.$$

4. Concluding remarks

In this paper, we show that the polynomial $T_n(x)$ has many similar properties to $R_n(x)$. In fact, there are more similar properties deserve to be studied. From the relation (5) and the fact that $A_n(x)$ have only real zeros, Wilf [23] proved that $R_n(x)$ have only real zeros for $n \geq 2$. Moreover, it follows from (4) that all zeros of $R_n(x)$ belong to $[-1,0]$, and the zeros of $R_n(x)$ separate that of $R_{n+1}(x)$ (see [17, Corollary 8]).

Let $f(x)$ and $F(x)$ be two polynomials with only real coefficients. Suppose that $f(x)$ and $F(x)$ both have only imaginary zeros. We say that $f(x)$ *separates* $F(x)$ if $\deg F = \deg f + 2$ and the sequences of real and imaginary parts of the zeros of $f(x)$ respectively separate that of $F(x)$. In other words, let $f(x) = a\prod_{j=1}^{n-1}(x+p_j+q_j\mathrm{i})(x+p_j-q_j\mathrm{i})$, and let $F(x) = b\prod_{j=1}^{n}(x+s_j+t_j\mathrm{i})(x+s_j-t_j\mathrm{i})$, where a,b are respectively leading coefficients of $f(x)$ and $F(x)$, $p_1 \geq p_2 \geq \cdots \geq p_{n-1},\ q_1 \geq q_2 \geq \cdots \geq q_{n-1}, s_1 \geq s_2 \geq \cdots \geq s_n$ and $t_1 \geq t_2 \geq \cdots \geq t_n$. Then we have

$$s_1 \geq p_1 \geq s_2 \geq p_2 \geq \cdots \geq s_{n-1} \geq p_{n-1} \geq s_n,$$

$$t_1 \geq q_1 \geq t_2 \geq q_2 \geq \cdots \geq t_{n-1} \geq q_{n-1} \geq t_n.$$

Based on empirical evidence, we propose the following conjecture.

Conjecture 7. *For $n \geq 2$, all zeros of $T_n(x)/x$ are imaginary and $T_n(x)/x$ separates $T_{n+1}(x)/x$.*

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School of Mathematics and Statistics, Northeastern University at Qinhuangdao, Hebei 066004, P.R. China
E-mail address: `shimeimapapers@163.com (S.-M. Ma)`

Department of Mathematics, Northeastern University, Shenyang, 110004, China
E-mail address: `hainawangpapers@163.com (H.-N. Wang)`