Dalian notes on rational Pontryagin classes

Michael S. Weiss

1. Overview

1.1. Introduction. The ring $H^*(BO;\mathbb{Q})$ is a polynomial ring $\mathbb{Q}[p_1, p_2, p_3, \dots]$ where $p_i \in H^{4i}(BO;\mathbb{Q})$ is the Pontryagin class. The restrictions of the classes p_i to the classifying spaces for finite-dimensional vector bundles satisfy well-known relations:

(1.1.1) $p_n = e^2 \in H^{4n}(BSO(2n);\mathbb{Q}), \quad \forall k > 0 : \; p_{n+k} = 0 \in H^{4n+4k}(BO(2n);\mathbb{Q})$

where e denotes the Euler class.

Let $BTOP(m)$ be the classifying space for *fiber bundles* with fiber $\cong \mathbb{R}^m$ and let $BTOP = \bigcup_{m\geq 0} BTOP(m)$ be the colimit of the spaces $BTOP(m)$. According to Novikov, the rational Pontryagin classes come from the cohomology of BTOP. Indeed with the work of Sullivan and Kirby-Siebenmann [**22**] in the late 1960s it became clear that the inclusion $BO \to BTOP$ induces an isomorphism in rational cohomology,

$$H^*(BTOP;\mathbb{Q}) \xrightarrow{\cong} H^*(BO;\mathbb{Q}).$$

Therefore we can write unambiguously $p_i \in H^{4i}(BTOP;\mathbb{Q})$. We can also write unambiguously $p_i \in H^{4i}(BTOP(m);\mathbb{Q})$ using the restriction map from $H^*(BTOP;\mathbb{Q})$ to $H^*(BTOP(m);\mathbb{Q})$. In other words, the rational Pontryagin classes can be viewed as characteristic classes for fiber bundles with fiber $\mathbb{R}^m$ for some m, and as such they are stable by construction; they do not change under fiberwise product of such fiber bundles with trivial line bundles.

From now on homology and cohomology are taken with rational coeffficients unless otherwise stated. — The main result of these notes is that the analogues of relations (1.1.1) fail to hold in the cohomology or $BTOP(2n)$ or $BSTOP(2n)$. For large enough n we have $p_n \neq e^2$ in $H^{4n}(BSTOP(2n))$ and even more surprisingly, $p_{n+k} \neq 0$ in $H^{4n+4k}(BTOP(2n))$ where $4n + 4k$ can be nearly as big as $9n$. Here is a more precise statement.

Theorem 1.1.2. *There exist positive constants c_1 and c_2 such that, for all positive integers n and k where $n \geq c_1$ and $k < 5n/4 - c_2$, the class p_{n+k} is nonzero in $H^{4n+4k}(BTOP(2n))$.*

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Note in passing that $p_{n+k} \neq 0$ in $H^{4n+4k}(\mathrm{BTOP}(2n))$ (for some $n, k > 0$) implies $p_{n+k} \neq 0$ in $H^{4n+4k}(\mathrm{BSTOP}(2n))$, and that in turn implies $p_{n+k} \neq e^2$ in $H^{4n+4k}(\mathrm{BSTOP}(2n+2k))$.

There is a geometric formulation of theorem 1.1.2 which is in fact slightly stronger.

THEOREM 1.1.3. *For n and k satisfying the conditions in theorem 1.1.2, there exists a fiber bundle $E \to M$ where*

- *M is a closed smooth stably parallelized manifold of dimension $2n + 4k$,*
- *the fibers are closed oriented topological manifolds of dimension $2n$,*
- *the signature of the total space E is nonzero, but*
- *all decomposable Pontryagin numbers of E are zero.*

(The *decomposable* Pontryagin numbers of E are those corresponding to monomials in the Pontryagin classes, of cohomological degree $4n + 4k$ but distinct from p_{n+k}.) It is an exercise to show that theorem 1.1.3 implies theorem 1.1.2, along the following lines. Let $T_{\mathrm{vert}}E$ be the vertical tangent (micro-)bundle of E, classified by a map $E \to \mathrm{BSO}(2n)$. The scalar product of $p_{n+k}(T_{\mathrm{vert}}E)$ with the fundamental class of E must be equal to a nonzero scalar multiple of the signature of E by the Hirzebruch signature theorem.

The specific construction of the fiber bundle $E \to M$ above also implies that the class p_{n+k} detects nonzero elements in $\pi_{4n+4k}(\mathrm{BTOP}(2n)) \otimes \mathbb{Q}$. This is explained in appendix B, which I added to this article at a late stage in reaction to a question asked by Diarmuid Crowley.

REMARK 1.1.4. It is a well-known theorem of smoothing theory, Morlet style [**22**], that $\mathrm{diff}_\partial(D^m) \simeq \Omega^{m+1}(\mathrm{TOP}(m)/\mathrm{O}(m))$ for $m \neq 4$, where $\mathrm{TOP}(m)/\mathrm{O}(m)$ is short for the homotopy fiber of $\mathrm{BO}(m) \to \mathrm{BTOP}(m)$. This fact makes the spaces $\mathrm{BTOP}(m)$ important in differential topology. See also remark 4.5.5.

This article grew out of notes intended to clarify an obscure story about Pontryagin classes and manifold calculus, told by me to Soren Galatius and Oscar Randal-Williams at the ICM satellite conference in Dalian, China, in August 2014. — For many years, from around 1994 to 2012, I believed that p_n *is* equal to e^2 in $H^{4n}(\mathrm{BSTOP}(2n))$ for all n, and consequently that p_{n+k} is zero in $H^{4n+4k}(\mathrm{BTOP}(2n))$ for all $n, k > 0$. An elaborate strategy for proving this was developed, based on remark 1.1.4 and using ideas from singularity theory. I offer my apologies to those who sat through talks or series of talks on this unfortunate project, and to Rui Reis who, more misguided than guided by me, participated in the project.

Acknowledgments. I am grateful to both Soren Galatius and Oscar Randal-Williams for their encouragement and interest, and for their work on parametrized surgery and homological stability [**16**], [**17**] which is used here. I am also indebted to Soren Galatius for reading an older version of this article and making many suggestions for improvement. Two of these are incorporated in the formulation of theorem 1.1.2. In the older formulation, $n + k$ was subject to a curious divisibility condition, satisfied in most cases but not all. This became unnecessary with lemma A.1, which I learned from Galatius. In the older formulation of theorem 1.1.2, there was a bound of the form $k \leq n/2-$const.; he had some ideas on how that could be improved and they were implemented. Galatius also pointed out that my proof of theorem 1.1.2 proved the stronger statement theorem 1.1.3.

An earlier version of section A contained arguments based on the multiplicativity of the sequence of the Hirzebruch $\mathcal{L}$-polynomials which were incomplete or careless. This was kindly pointed out to me by Martin Olbermann. He drew my attention to [**15**].

1.2. Two useful manifolds and their automorphisms. Let W be the smooth manifold equal to a connected sum of "many" copies of $S^n \times S^n$ minus the interior of a codimension zero disk. Then $\partial W \cong S^{2n-1}$. Select a point z in ∂W once and for all. Put $W_z = W \smallsetminus \{z\}$. There is a forgetful map

$$\mathrm{Bdiff}_\partial(W) \longrightarrow \mathrm{Bdiff}_\partial(W_z).$$

Here $\mathrm{diff}_\partial(-)$ generally refers to (topological or simplicial groups of) diffeomorphisms from a manifold to itself which extend the identity on the boundary. Therefore $\mathrm{diff}_\partial(W_z)$ is already defined, no less than $\mathrm{diff}_\partial(W)$. But it is often useful to think of W as the one-point compactification of W_z , and so to think of elements of $\mathrm{diff}_\partial(W_z)$ as *homeomorphisms* $W \to W$ which extend the identity on ∂W and restrict to diffeomorphisms $W_z \to W_z$.

As will be explained in section 1.4, both $\mathrm{Bdiff}_\partial(W)$ and $\mathrm{Bdiff}_\partial(W_z)$ are homologically rather accessible, but for very different reasons.

1.3. Homotopical description of some kappa classes. Let $\mathcal{P}$ be a polynomial with rational coefficients in the Pontryagin classes $p_1, p_2, p_3, \ldots$, homogeneous of degree $4n + 4k$ in the cohomological sense. (For example, if $n + k = 10$, then $p_{10} + 5p_2p_3p_5 - 7p_2^5$ qualifies.) By $\kappa_t(\mathcal{P}) \in H^{2n+4k}(\mathrm{Bhomeo}_\partial(W))$ we mean the class obtained as follows. Let $(E, \partial E) \to \mathrm{Bhomeo}_\partial(W)$ be the tautological fiber bundle pair with fiber pair $(W, \partial W)$. Evaluate $\mathcal{P}$ on the vertical tangent bundle of E to obtain a class in $H^{4n+4k}(E, \partial E)$. Apply integration along the fibers of $(E, \partial E) \to \mathrm{Bhomeo}_\partial(W)$ to get a class in $H^{2n+4k}(\mathrm{Bhomeo}_\partial(W))$. This is $\kappa_t(\mathcal{P})$.

A practical description is as follows. Think of $H_{2n+4k}(\mathrm{Bhomeo}_\partial(W))$ as rationalized stable homotopy $\pi^s_{2n+4k}((\mathrm{Bhomeo}_\partial(W)) \otimes \mathbb{Q}$ and represent an element x of the stable homotopy group $\pi^s_{2n+4k}(\mathrm{Bhomeo}_\partial(W))$ by a map $M \to \mathrm{Bhomeo}_\partial(W)$, where M is a stably framed closed manifold of dimension $2n+4k$. This determines a fiber bundle $E_M \to M$ with fiber $W/\partial W$ (by pulling back $E \to \mathrm{Bhomeo}_\partial(W)$ and collapsing boundary spheres to points). Note that $W/\partial W$ is a closed manifold which could also be described as $W \cup D^{2n}$. Therefore E_M is a closed oriented manifold of dimension $4n + 4k$. Then we have

$$\langle \kappa_t(\mathcal{P}), x \rangle = \langle \mathcal{P}(TE_M), [E_M] \rangle,$$

i.e., $\langle \kappa_t(\mathcal{P}), x \rangle$ is the Pontryagin number determined by E_M and $\mathcal{P}$. Furthermore, we can also write $\mathcal{P}(T_{\mathrm{vert}}E_M)$ instead of $\mathcal{P}(TE_M)$, since the tangent bundle of M is stably trivialized. This point of view will be important later.

In the following we write $\kappa_t(\mathcal{P}) \in H^{2n+4k}(\mathrm{Bdiff}_\partial(W_z))$ for the image of $\kappa_t(\mathcal{P})$, as just defined, under the homomorphism in cohomology induced by the inclusion of $\mathrm{Bdiff}_\partial(W_z)$ in $\mathrm{Bhomeo}_\partial(W)$. Here we obtain a description of the classes $\kappa_t(\mathcal{P})$ in some special cases. Let $\mathrm{haut}_\partial(W)$ consist of the homotopy automorphisms of W relative to the boundary. View W as a based space by taking the antipode of z in

$\partial W = S^{2n-1}$ as the base point. There are forgetful maps

$$\begin{tikzcd} \mathrm{Bdiff}_\partial(W_z) \arrow[d, "\nu"] \\ \mathrm{B}(\mathrm{haut}_\partial(W) \ltimes \mathrm{map}_*(W, \mathrm{SO}(2n))) \arrow[d, "u"] \\ \mathrm{Bhaut}_\partial(W). \end{tikzcd}$$

In particular the map ν is obtained by choosing a trivialization of TW once and for all, and viewing an element of $\mathrm{diff}_\partial(W_z)$ first as a homeomorphism $W \to W$ which restricts to a diffeomorphism $W_z \to W_z$, then forgetfully as a homotopy automorphism of W rel ∂W covered by a vector bundle map from TW to TW (the derivative of the diffeomorphism). It does not matter that the derivative is undefined at z because the inclusion $W_z \to W$ is a homotopy equivalence of based spaces.

Let $\mathcal{L}_{n+k}$ be the Hirzebruch polynomial of cohomological degree $4n + 4k$ in the Pontryagin classes, so that the Pontryagin number associated with $\mathcal{L}_{n+k}$ and a closed oriented manifold of dimension $4n + 4k$ is the signature of that manifold. (This works for topological manifolds just as it does for smooth manifolds, again because the inclusion BO $\to$ BTOP is a rational equivalence and the inclusion of Thom spectra MSO $\to$ MSTOP is a rational equivalence.)

Proposition 1.3.1. (i) *The class* $\kappa_t(\mathcal{L}_{n+k}) \in H^{2n+4k}(\mathrm{Bdiff}_\partial(W_z))$ *is in the image of* $(u\nu)^*$*, i.e., it comes from* $H^{2n+4k}(\mathrm{Bhaut}_\partial(W))$.

(ii) *For any decomposable polynomial* $\mathcal{P}$ *in the Pontryagin classes, of cohomological degree* $4n+4k$*, the class* $\kappa_t(\mathcal{P}) \in H^{2n+4k}(\mathrm{Bdiff}_\partial(W_z))$ *is* $\nu^*(\kappa_h(\mathcal{P}))$ *for some class* $\kappa_h(\mathcal{P}) \in H^{2n+4k}(\mathrm{B}(\mathrm{haut}_\partial(W) \ltimes \mathrm{map}_*(W, \mathrm{SO}(2n))))$.

Proof. Statement (i) is an easy consequence of the Hirzebruch signature theorem. For $x \in H_{2n+4k}(\mathrm{Bdiff}_\partial(W_z))$ represented by a map $M \to \mathrm{Bdiff}_\partial(W_z)$, where M is a closed smooth stably framed manifold of dimension $2n + 4k$, we have

$$\langle \kappa_t(\mathcal{L}_{n+k}), x\rangle = \text{signature of } E_M$$

as explained earlier. Here $E_M \to M$ is the fiber bundle with fiber $W/\partial W$ determined by $M \to \mathrm{Bdiff}_\partial(W_z) \subset \mathrm{Bhomeo}_\partial(W_z)$. This description of $\kappa_t(\mathcal{L}_{n+k})$ makes it clear that $\kappa_t(\mathcal{L}_{n+k})$ comes from $\mathrm{Bhaut}_\partial(W)$, because a bordism class of maps $M \to \mathrm{Bhaut}_\partial(W)$ with M as above still determines a fibration $E_M \to M$ where the fibers are oriented Poincaré duality spaces of formal dimension $2n$, and E_M is therefore a Poincaré duality space of formal dimension $4n + 4k$. Poincaré duality spaces also have signatures. (But see also remark 4.5.3.)

For the proof of statement (ii), let us abbreviate

$$\begin{aligned} B_0 &:= \mathrm{B}(\mathrm{haut}_\partial(W) \ltimes \mathrm{map}_*(W, \mathrm{SO}(2n))), \\ B_1 &:= \mathrm{B}(\mathrm{haut}_\partial(W) \ltimes \mathrm{map}_*(W/\partial W, \mathrm{SO}(2n))) \end{aligned}$$

so that $B_1 \subset B_0$. By the definition of B there is a tautological fibration pair

$$(E_0, \partial E_0) \longrightarrow B_0$$

with fiber pair $\simeq (W, \partial W)$. Also by the definition of B_0, there is a distinguished oriented vector bundle V_0 of fiber dimension $2n$ on E_0. The situation can be summarized in a commutative diagram

$$\begin{tikzcd} V_1 \arrow[r, "\subset"] \arrow[d] & V_0 \arrow[d] \\ (E_1, \partial E_1) \arrow[r, "\subset"] \arrow[d] & (E_0, \partial E_0) \arrow[d] \\ B_1 \arrow[r, "\subset"] & B_0 \end{tikzcd}$$

where E_1 for example is the restriction of E_0 to B_1 . Since V_1 is trivialized over ∂E_1, we have a well defined class $\mathcal{P}(V_1) \in H^{4n+4k}(E_1, \partial E_1)$. The essence of statement (ii) is that this class comes from a class in $H^{4n+4k}(E_0, \partial E_0)$. To show this, we introduce $\zeta(B_0) \subset \partial E_0$, the image of the zero section $B_0 \to \partial E_0$. Since V_0 is trivialized over $\partial E_1 \cup \zeta(B_0)$, there is still a well defined class

$$\mathcal{P}(V_0) \in H^{4n+4k}(E_0, \partial E_1 \cup \zeta(B_0)).$$

This class $\mathcal{P}(V_0)$ lifts to an element $\bar{\mathcal{P}}(V_0)$ of $H^{4n+4k}(E_0, \partial E_0)$ because it maps to zero in $H^{4n+4k}(\partial E_0, \partial E_1 \cup \zeta(B_0))$. Reason: the quotient $\partial E_0/(\partial E_1 \cup \zeta(B_0))$ is a suspension and $\mathcal{P}$ is decomposable. Finally let $\kappa_h(\mathcal{P})$ be the class obtained from $\bar{\mathcal{P}}(V_0)$ in $H^{4n+4k}(E_0, \partial E_0)$ by integration along the fibers of $(E_0, \partial E_0) \to B_0$. □

REMARK 1.3.2. The class $\kappa_h(\mathcal{P})$ in proposition 1.3.1 is not unique. In the notation of the proof, and with the reasoning of that proof, it it is determined only modulo a subgroup J of $H^{2n+4k}(B_0)$ which is contained in the kernel of the homomorphism ν^* from $H^{2n+4k}(B_0)$ to $H^{2n+4k}(\mathrm{Bdiff}_\partial(W_z))$. The subgroup J is the image of the composition

$$\begin{tikzcd} H^{4n+4k-1}(\partial E_0, \partial E_1 \cup \zeta(B_0)) \arrow[d, "\text{boundary operator}"] \\ H^{4n+4k}(E_0, \partial E_0) \arrow[d, "\text{integration along fibers}"] \\ H^{2n+4k}(B_0). \end{tikzcd}$$

Perhaps $\kappa_h(\mathcal{P})$ can be made (more) unique with some more work, but the definition of $\kappa_h(\mathcal{P})$ as given, with indeterminacy, has the following advantage. Let $q\colon \mathrm{BSO}(2n) \to \mathrm{BSO}(2n)$ be any based map; a rational map is also enough. This induces a (rational) map $\bar{q}\colon B_0 \to B_0$, where $B_0 = B(\mathrm{haut}_\partial(W) \ltimes \mathrm{map}_*(W, \mathrm{SO}(2n)))$ as before. Then $q^\star\mathcal{P}$ is again decomposable, and we clearly have

$$\bar{q}^\star(\kappa_h(\mathcal{P})) = \kappa_h(q^\star\mathcal{P}) \in H^{2n+4k}(B_0)/J\,.$$

1.4. Reduction to a technical lemma. Manifold calculus gives us tools to construct self-maps of $\mathrm{Bdiff}_\partial(W_z)$ with curious properties. This is based on the well-known applications of manifold calculus to the homotopy theory of spaces of smooth embeddings, and the following (non-technical) observation.

LEMMA 1.4.1. *For* $n \geq 3$, *the inclusion* $\mathrm{diff}_\partial(W_z) \to \mathrm{emb}_\partial(W_z, W_z)$ *is a homotopy equivalence.*

Proof. Let U be a standard open neighborhood of z in W, so that $W_U := W \smallsetminus U$ is a compact smooth manifold with corners. The boundary ∂W_U is a union $\partial_0 W_U \cup \partial_1 W_U$ where $\partial_0 W_U = W_U \cap \partial W$ and $\partial_1 W_U$ is the closure of the complement of $\partial_0 W_U$ in ∂W_U. It is easy to see that in the diagram

$$\mathrm{diff}_\partial(W_z) \hookrightarrow \mathrm{emb}_\partial(W_z, W_z) \longrightarrow \mathrm{emb}_{\partial_0}(W_U, W_z),$$

both the right-hand arrow and the composite arrow are homotopy equivalences. (Indeed, a smooth embedding $g\colon W_U \to W_z$ which is the identity on $\partial_0 W_U$ must preserve the intersection form on the n-dimensional integral homology. Since that intersection form is nondegenerate, g has to be a homotopy equivalence. Since $n \geq 3$, this implies that the closure of the complement of $\mathrm{im}(g)$ is a collar.) □

In manifold calculus, applied to spaces of smooth embeddings $\mathrm{emb}(M, N)$, it is customary to ask for a sufficiently high codimension. The codimension should be at least three for the machine to function. But it is important to have the correct interpretation of codimension. The geometric dimension of the target matters, and the homotopical dimension (e.g., handle dimension, maximal index of handles in a handle decomposition) of the source. In our situation, $M = W_z = N$, the geometric dimension of the target is $2n$ but the homotopical dimension of the source is n. Therefore the codimension count is $2n - n = n$.

To formulate the key lemma, we start with the choice of a positive integer b, less than n. The space $\mathrm{BSO}(2n)$ is rationally a product of Eilenberg-MacLane spaces. The Pontryagin classes $p_1, \dots p_{n-1}$ in degrees $4, 8, \dots, 4n-4$ together with the Euler class e in degree $2n$ define such a splitting. It follows that there is a unique homotopy class of rational maps $q_b\colon \mathrm{BSO}(2n) \to \mathrm{BSO}(2n)$ such that $q_b^*(p_j) = p_j$ for $j \neq b$ and $q_b^*(e) = e$, whereas $q_b^*(p_b) = 0$. Let

$$\bar{q}_b\colon \mathrm{B}(\mathrm{haut}_\partial(W) \ltimes \mathrm{map}_*(W, \mathrm{SO}(2n)_{\mathbb{Q}}) \longrightarrow \mathrm{B}(\mathrm{haut}_\partial(W) \ltimes \mathrm{map}_*(W, \mathrm{SO}(2n)_{\mathbb{Q}})$$

be the (rational) map induced by q_b.

Lemma 1.4.2. *There exist positive integers c_1 and c_2 such that, if $n \geq c_1$ and $b = n - c_2$, there is a map*

$$f\colon \mathrm{Bdiff}_\partial(W_z) \longrightarrow \mathrm{Bdiff}_\partial(W_z)$$

(rational, defined in the range $< 4b + 3n$ only) *which satisfies $\nu f \simeq \bar{q}_b \nu$ for the map ν of proposition* 1.3.1. *Such a map can also be defined in the range $< 4b + n$ for all $b \leq n - c_2$.*

$$\begin{tikzcd}
\mathrm{Bdiff}_\partial(W_z) \arrow[r, dotted, "f"] \arrow[d, "\nu"] & \mathrm{Bdiff}_\partial(W_z) \arrow[d, "\nu"] \\
\mathrm{B}(\mathrm{haut}_\partial(W) \ltimes \mathrm{map}_*(W, \mathrm{SO}(2n)_{\mathbb{Q}}) \arrow[r, "\bar{q}_b"] & \mathrm{B}(\mathrm{haut}_\partial(W) \ltimes \mathrm{map}_*(W, \mathrm{SO}(2n)_{\mathbb{Q}})
\end{tikzcd}$$

Here the phrase *defined in the range* $< 4b + 3n$ *only* indicates that in order to construct f we make the sacrifice of killing the homotopy groups of $\mathrm{Bdiff}_\partial(W_z)$ in dimensions $\geq 4b + 3n$. — Lemma 1.4.2 will be proved in section 4. Specifically, the ultimate reason for f being defined in the range $7n$ minus constant only is given at the end of section 4.4. Sections 2 and 3 provide background for the proof of lemma 1.4.2.

The property $\nu f \simeq \bar{q}_b \nu$ in lemma 1.4.2 implies $u\nu f \simeq u\bar{q}_b\nu = u\nu$, and it follows that $f^*(\kappa_t(\mathcal{L}_{n+k})) = \kappa_t(\mathcal{L}_{n+k})$ by proposition 1.3.1 part (i). On the other hand, for a decomposable polynomial $\mathcal{P}$ in the Pontryagin classes, of cohomological degree $4n+4k$, the property $\nu f \simeq \bar{q}_b \nu$ implies $f^*(\kappa_t(\mathcal{P})) = \kappa_t(q_b^*\mathcal{P})$ by proposition 1.3.1 part (ii) and remark 1.3.2. If $b = n - c_2$ as in the first part of the lemma, then we need to ensure that $2n+4k < 3n+4b$ because of the sacrifice of homotopy groups in dimensions $\geq 3n+4b$. This means $k < 5n/4 - c_2$, in agreement with the condition on k in theorems 1.1.2 and 1.1.3. If b is chosen as in the second part of the lemma, then we need $2n+4k < n+4b$.

Proof of theorem 1.1.3 modulo lemma 1.4.2. Take n and k as given. We distinguish three cases.

Case 1. Suppose that $n/4 < k + c_2 \leq n$. Set $b := n - c_2$ as in the first part of lemma 1.4.2, and $a := n + k - b = k + c_2$. By the Galatius-RandalWilliams theorems [**16**], [**17**], there exists $x \in H_{2n+4k}(\mathrm{Bdiff}_\partial(W))$ such that $\langle \kappa(p_a p_b), x\rangle \neq 0$ while $\langle \kappa(\mathcal{P}), x\rangle = 0$ for all other *monomials* $\mathcal{P}$ of cohomological degree $4n+4k$ in the Pontryagin classes. Without loss of generality, x is represented by a map

$$\alpha\colon M \to \mathrm{Bdiff}_\partial(W)$$

where M is a closed smooth stably framed manifold of dimension $2n+4k$. By lemma A.1, the coefficient of $p_a p_b$ in the Hirzebruch polynomial $\mathcal{L}_{n+k}$ is nonzero. Therefore $\langle \kappa(\mathcal{L}_{n+k}), x\rangle \neq 0$. Let y be the image of x in $H_{2n+4k}(\mathrm{Bdiff}_\partial(W_z))$. By the above calculations, f_*y has a nonzero scalar product with $\kappa_t(\mathcal{L}_{n+k})$, but a trivial scalar product with $\kappa_t(\mathcal{P})$ for any decomposable polynomial of cohomological degree $4n+4k$ in the Pontryagin classes. The class f_*y is represented by

$$\beta = f\iota\alpha\colon M \longrightarrow \mathrm{Bdiff}_\partial(W_z)$$

where M is a smooth closed stably framed manifold and $\iota\colon \mathrm{Bdiff}_\partial(W) \to \mathrm{Bdiff}_\partial(W_z)$ is the inclusion. Let $E_M \to M$ be the bundle of $2n$-dimensional closed oriented topological manifolds with fiber $W/\partial W$ obtained by composing

$$M \xrightarrow{\beta} \mathrm{Bdiff}_\partial(W_z) \hookrightarrow \mathrm{Bhomeo}(W) \to \mathrm{Bhomeo}(W/\partial W).$$

The bundle $E_M \to M$ has the properties that we require.

Case 2. Suppose that $k + c_2 \leq n/4$. Then we have $2n+4k < 3n$. Therefore we go for the second part of lemma 1.4.2, choosing b in such a way that $3n \leq 4b+n$, that is to say, $b \geq n/2$. We require $b \leq 3n/4$ because we want $a := n+k-b > n/4$, and we require $b \leq n - c_2$. (The inequalities $n/2 \leq b \leq n - c_2$ force $n \geq 2c_2$, so we ought to ensure that $c_1 \geq 2c_2$.) Then we proceed as in Case 1.

Case 3. Suppose that $5n/4 > k + c_2 > n$. Set $b = n - c_2$ as in the first part of lemma 1.4.2. Now $a = n + k - b = k + c_2 > n$, so that p_a is zero in $H^{4a}(BSO(2n))$. Therefore, instead of working with a directly, we choose integers a_1 and a_2 such that $a = a_1 + a_2$ and $n/4 < a_1$, $n/4 < a_2$. Then $a_1, a_2 < n$. So by the Galatius-RandalWilliams theorem there exists $x \in H_{2n+4k}(\mathrm{Bdiff}_\partial(W))$ such that $\langle \kappa(p_{a_1}p_{a_2}p_b), x\rangle \neq 0$ while $\langle \kappa(\mathcal{P}), x\rangle = 0$ for all other monomials $\mathcal{P}$ of cohomological degree $4n+4k$ in the Pontryagin classes. Lemma A.6 shows that the coefficient of $p_{a_1}p_{a_2}p_b$ in the Hirzebruch polynomial $\mathcal{L}_{n+k}$ is nonzero. Therefore $\langle \kappa(\mathcal{L}_{n+k}), x\rangle \neq 0$. Continue as in Case 1. □

2. Manifold calculus, embeddings and configuration categories

2.1. The manifold calculus view on wild derivatives. It is clear from the foregoing, especially lemma 1.4.1, that we must look for embeddings or families of embeddings from W_z to W_z rel ∂ with violent or otherwise unexpected derivatives. If we were interested in smooth immersions $W_z \to W_z$ rel ∂, then (by the Smale-Hirsch h-principle for immersions) making families with any derivatives whatsoever would not be a problem. Therefore we can also reformulate the task as follows: we must look for families of smooth immersions from W_z to W_z rel ∂ with violent or unusual derivatives which are regularly homotopic rel ∂ to families of smooth embeddings.
In [**4**], Pedro Boavida and I have reformulated the standard theorems of manifold calculus applied to spaces of smooth embeddings in such a way that the obstructions to deforming families of immersions to families of embeddings are particularly visible. Here is a sample which is close to being the most useful variant for us. It is for manifolds without boundary M and N; later a version for manifolds with boundary will be stated. It uses certain categories $\mathsf{con}(M;r)$ and $\mathsf{con}(N;r)$ of ordered configurations in M and N, of cardinality bounded above by a positive integer r. Taken by itself it is neither useful nor difficult. To make it useful, combine it with the difficult theorem that the standard comparison map $\mathrm{emb}(M,N) \to T_r\mathrm{emb}(N,M)$ is highly-connected. (In fact it is $((r+1)(c-2)+3-\dim(N))$-connected where c is the codimension, wisely determined as the difference between geometric dimension of N and homotopy dimension or handle dimension of M.)

THEOREM 2.1.1. *For any integer $r \geq 1$, the commutative square*

$$\begin{tikzcd} T_r\mathrm{emb}(M,N) \arrow[r] \arrow[d] & \mathbb{R}\mathrm{map}_{\mathsf{Fin}}(\mathsf{con}(M;r),\mathsf{con}(N;r)) \arrow[d, "\text{specialization}"] \\ T_1\mathrm{emb}(M,N) \arrow[r] & \mathbb{R}\mathrm{map}_{\mathsf{Fin}}(\mathsf{con}^{\mathrm{loc}}(M;r),\mathsf{con}^{\mathrm{loc}}(N;r)) \end{tikzcd}$$

is homotopy cartesian.

Now I need to explain what these configuration categories are. Let $k \in \mathbb{N}$. The space of maps from $\underline{k}$ to M comes with an obvious stratification. There is one stratum for each equivalence relation η on $\underline{k}$. The points of that stratum are precisely the maps $\underline{k} \to M$ which can be factorized as projection from $\underline{k}$ to $\underline{k}/\eta$ followed by an injection of $\underline{k}/\eta$ into M.
We construct a topological category $\mathsf{con}(M)$ (category object in the category of topological spaces) whose object space is

$$\coprod_{k\geq 0} \mathrm{emb}(\underline{k}, M),$$

that is, the topological disjoint union of the ordered configuration spaces of M for each cardinality $k \geq 0$. By a morphism from $f \in \mathrm{emb}(\underline{k}, M)$ to $g \in \mathrm{emb}(\underline{\ell}, M)$ we mean a pair consisting of a map $v\colon \underline{k} \to \underline{\ell}$ and a Moore homotopy $\gamma = (\gamma_t)_{t\in[0,a]}$ from f to gv such that γ in reverse is an *exit path* in the stratified space of *all* maps from $\underline{k}$ to M. Equivalently, if $\gamma_s(x) = \gamma_s(y)$ for some $s \in [0,a]$ and $x, y \in \underline{k}$, then $\gamma_t(x) = \gamma_t(y)$ for all $t \in [s,a]$. (Andrade [**2**] uses the expression *sticky homotopy.*)

The space of all morphisms is therefore a coproduct

$$\coprod_{\substack{k,\ell\ge 0\\ \nu:\underline{k}\to\underline{\ell}}} \mathsf{P}(\nu)$$

where $\mathsf{P}(\nu)$ consists of triples (f,g,γ) as above: $f\in\mathrm{emb}(\underline{k},M)$, $g\in\mathrm{emb}(\underline{\ell},M)$ and γ is the reverse of an exit path in $\mathrm{map}(\underline{k},M)$ from $g\nu$ to f. Composition of morphisms is obvious. It is obvious that in $\mathsf{con}(M)$, the maps *source* and *target* from morphism space to object space are fibrations. This is a very useful property for topological categories to have.

For fixed objects $f\in\mathrm{emb}(\underline{k},M)$ and $g\in\mathrm{emb}(\underline{\ell},M)$, the space of morphisms from f to g is homotopy equivalent, by a result of David Miller [**26**], to the space of pairs (ν,γ) where $\nu:\underline{k}\to\underline{\ell}$ as before and $\gamma=(\gamma_t)_{t\in[0,1]}$ is a path in $\mathrm{map}(\underline{k},M)$ from f to $g\nu$ such that γ_t is *injective* for all t strictly less than 1. It follows that the space of morphisms in $\mathsf{con}(M)$ with fixed target $g\in\mathrm{emb}(\underline{\ell},M)$ is homotopy equivalent (in a fairly obvious way) to

$$\coprod_{j\ge 0}\mathrm{emb}(\underline{j},U)$$

where U is a standard (open tubular) neighborhood of $\mathrm{im}(g)$ in M. In short, we have a good understanding of morphism spaces and object spaces in $\mathsf{con}(M)$, and they all somehow boil down to ordered configuration spaces of M or something very closely related.

Next some variants: $\mathsf{con}(M;r)$ is the full subcategory of $\mathsf{con}(M)$ with object space

$$\coprod_{k=0}^{r}\mathrm{emb}(\underline{k},M)$$

The localized form $\mathsf{con}^{\mathrm{loc}}(M)$ is a type of comma category. Its objects are the morphisms in $\mathsf{con}(M)$ whose target has cardinality 1, in other words has the form $x:\underline{1}\to M$. The morphisms are commutative triangles

$$\begin{tikzcd}
(f:\underline{k}\to M) \arrow[rr, "\gamma"] \arrow[dr, "\tau\circ\gamma"'] & & (g:\underline{\ell}\to M) \arrow[dl, "\tau"] \\
& (x:\underline{1}\to M) &
\end{tikzcd}$$

in $\mathsf{con}(M)$. (In making the step from $\mathsf{con}(M)$ to $\mathsf{con}^{\mathrm{loc}}(M)$ we lose one good property: the map *source* from morphism space to object space is no longer a fibration. But we still have a good degreewise understanding of the nerve as a simplicial set.)

By the David Miller result, the object space of $\mathsf{con}^{\mathrm{loc}}(M)$ is homotopy equivalent to the total space of a fiber bundle on M whose fiber at $x\in M$ is

$$\coprod_{k\ge 0}\mathrm{emb}(\underline{k},T_xM)\ .$$

And $\mathsf{con}^{\mathrm{loc}}(M;r)$ is like $\mathsf{con}^{\mathrm{loc}}(M)$ but with cardinalities of configurations bounded above by r. We write Fin for the category whose objects are the finite sets $\underline{\ell}$, where $\ell\ge 0$, and whose morphisms are all maps between these (not required to be order preserving). There are obvious forgetful functors

$$\mathsf{con}^{\mathrm{loc}}(M)\to\mathsf{con}(M)\to\mathsf{Fin}\,.$$

Next, the meaning of $\mathbb{R}\mathrm{map}_{\mathsf{Fin}}$ in the right-hand column of the square in the theorem ought to be explained. It wants to say *space of simplicial maps over the nerve of* Fin *in the right derived sense*. More details are given in section 2.2.
Last not least, a few words on the horizontal maps in the square of the theorem are in order. Looking at the top row for example, the point is that we have a natural transformation

$$\mathrm{emb}(M,N) \longrightarrow \mathbb{R}\mathrm{map}_{\mathsf{Fin}}(\mathsf{con}(M;r),\mathsf{con}(N;r))$$

of contravariant functors in the variable M. It is easy to verify that the target, as a functor of M, satisfies the conditions for a polynomial functor of degree r, in the sense of manifold calculus. Therefore, by something like a universal property (in the derived sense) of T_r , that natural map factors canonically through $T_r\mathrm{emb}(M,N)$. The reasoning for the lower row is similar. Here one should also be aware that $T_1\mathrm{emb}(M,N)$ is another way to write $\mathrm{imm}(M,N)$. For the construction of the diagram it is actually wiser to write $T_1\mathrm{emb}(M,N)$ as I have done. For an interpretation, maybe writing $\mathrm{imm}(M,N)$ is not bad. How do we think of the lower horizontal arrow in the diagram? The idea is as follows: inspired by the Smale-Hirsch h-principle, we think of a smooth immersion $M \to N$ mainly as a continuous map $f\colon M \to N$ together with linear injections $\psi_x\colon T_xM \to T_{f(x)}N$, one for each $x \in M$ and depending continuously on $x \in M$. The lower right-hand term in the square of the theorem has a very similar description: an element of it can be seen as a continuous map $f\colon M \to N$ together with functors (in the right derived sense)

$$\psi_x\colon \mathsf{con}(T_xM) \to \mathsf{con}(T_{f(x)}N)$$

depending continuously on $x \in M$. So the lower horizontal map in the square is obtained by noting that a linear injection $T_xM \to T_{f(x)}N$ determines a functor between configuration categories, $\mathsf{con}(T_xM) \to \mathsf{con}(T_{f(x)}N)$.

The theorem has a variant for manifolds with boundary. For that we assume that M and N are manifolds with boundary and a smooth embedding $\partial M \to \partial N$ has been fixed in advance. Write $M_- := M \smallsetminus \partial M$. We are interested in the space $\mathrm{emb}_\partial(M,N)$ of smooth neat embeddings which extend the selected embedding $\partial M \to \partial N$. (Our main reason for avoiding the situation where we look at all neat embeddings $M \to N$, taking boundary to boundary, is that it appears to lead to a kind of two-variable manifold calculus.) Here we need a new definition of the configuration category $\mathsf{con}(M)$.
Let Fin_* be the category with objects $[k] = \{0,1,\dots,k\}$ for $k \ge 0$. These objects are viewed as based sets with base point 0, so the morphisms are the based maps. Let $k \in \{0,1,2,\dots\}$. The space $\mathrm{map}(\underline{k},M)$ comes with a stratification. There is one stratum for each pair (S,η) where $S \subset \underline{k}$ and η is an equivalence relation on $\underline{k}$ such that S is a union of equivalence classes. The points of that stratum are the maps $\underline{k} \to M$ taking S to ∂M and the complement of S to M_-, and which can be factored as projection from $\underline{k}$ to $\underline{k}/\eta$ followed by an injection of $\underline{k}/\eta$ into M.
The category $\mathsf{con}(M)$ has object space

$$\coprod_{k\ge 0} \mathrm{emb}(\underline{k},M_-).$$

A morphism from $f \in \mathrm{emb}(\underline{k},M_-)$ to $g \in \mathrm{emb}(\underline{\ell},M_-)$ is a pair consisting of a morphism ν from $[k]$ to $[\ell]$ in Fin_* and a Moore path $\gamma = (\gamma_t)_{t\in[0,a]}$ in $\mathrm{map}(\underline{k},M)$

which is an exit path in reverse. It should satisfy $\gamma_0 = f$ and $\gamma_a(x) = g(\nu(x))$ if $\nu(x) \in \underline{\ell}$, but $\gamma_a(x) \in \partial M$ if $\nu(x) = 0$. Composition of morphisms is (almost) obvious.
The category $\mathsf{con}(M)$, in the case where M has boundary, comes with a forgetful functor to Fin_*. We define $\mathsf{con}^{\mathrm{loc}}(M)$ to be the same as $\mathsf{con}^{\mathrm{loc}}(M_-)$, so this comes with a forgetful functor to Fin as before.

THEOREM 2.1.2. *For any integer* $r \geq 1$*, the commutative square*

$$\begin{tikzcd} T_r\mathrm{emb}_\partial(M,N) \arrow[r] \arrow[d] & \mathbb{R}\mathrm{map}_{\partial,\,\mathsf{Fin}_*}(\mathsf{con}(M;r),\mathsf{con}(N;r)) \arrow[d, "\text{specialization}"] \\ T_1\mathrm{emb}_\partial(M,N) \arrow[r] & \mathbb{R}\mathrm{map}_{\partial,\,\mathsf{Fin}}(\mathsf{con}^{\mathrm{loc}}(M;r),\mathsf{con}^{\mathrm{loc}}(N;r)) \end{tikzcd}$$

is homotopy cartesian. □

Perhaps the right-hand column is self-explanatory. If not, let's agree to assume that M is equipped with a closed collar $\bar{U} \cong \partial M \times [0,1]$, closure of an open collar U, and that we can focus attention on the neat smooth embeddings $M \to N$ which are prescribed on $\bar{U}$ in some way. Then we have also prescribed functors from $\mathsf{con}(U;r)$ to $\mathsf{con}(N;r)$ and from $\mathsf{con}^{\mathrm{loc}}(U;r)$ to $\mathsf{con}^{\mathrm{loc}}(N;r)$. In the right-hand column, the notation $\mathbb{R}\mathrm{map}_\partial$ means that we look for functors in the derived sense which extend these specified functors from $\mathsf{con}(U;r)$ to $\mathsf{con}(N;r)$ and from $\mathsf{con}^{\mathrm{loc}}(U;r)$ to $\mathsf{con}^{\mathrm{loc}}(N;r)$. More precisely, the upper right hand term for example is the homotopy fiber of the restriction map

$$\mathbb{R}\mathrm{map}_{\mathsf{Fin}_*}(\mathsf{con}(M;r),\mathsf{con}(N;r)) \longrightarrow \mathbb{R}\mathrm{map}_{\mathsf{Fin}_*}(\mathsf{con}(U;r),\mathsf{con}(N;r))$$

over the point determined by that selected functor $\mathsf{con}(U;r) \to \mathsf{con}(N;r)$.

REMARK 2.1.3. Configuration categories were not invented in [**4**]. We (Boavida de Brito and Weiss) took the concept as we found it in [**2**]. In our enthusiasm we provided many alternative but weakly equivalent descriptions. Configuration categories and closely related notions have also been very important in the recent development of factorization homology. See for example [**3**]. But in my opinion, the underlying idea that the ordered configuration spaces $\mathrm{emb}(\underline{k}, M)$ of a manifold M, taken together for all k, should be organized into some bigger structure goes back to Fulton-MacPherson [**14**] in a complex algebraic geometry setting. Axelrod and Singer [**1**] exported this idea to differential topology. Sinha used the Axelrod-Singer formulation and emphasized its usefulness in manifold calculus, concentrating on the case of a 1-dimensional source manifold. See for example [**29**].

2.2. Functors in the right derived sense. Generally we like to replace topological categories $\mathcal{A},\mathcal{B},\mathcal{C},\dots$ by their nerves $N\mathcal{A},N\mathcal{B},N\mathcal{C},\dots$ which are simplicial spaces. If these nerves are well behaved, we interpret *functor from* $\mathcal{A}$ *to* $\mathcal{B}$ *in the derived sense* to mean *simplicial map from* $\mathcal{A}$ *to* $\mathcal{B}$ *in the derived sense.* The space of these is denoted

$$\mathbb{R}\mathrm{map}(\mathcal{A},\mathcal{B}).$$

It remains to be said what is meant by *well-behaved* and what is meant by the notation $\mathbb{R}\mathrm{map}(X,Y)$ for two simplicial spaces X and Y.
Rezk has invented the concept of *complete Segal space.* This is a simplicial space which has roughly the properties that we expect from a nerve, but formulated in a

homotopical way. I have copied the following from the Boavida-Weiss paper that I mentioned earlier.

DEFINITION 2.2.1. A *Segal space* is a simplicial space X satisfying condition (σ) below. If condition (κ) below is also satisfied, then X is a *complete Segal space.*

- (σ) For each $n \geq 2$ the map $(u_1^*, u_2^*, \ldots, u_n^*)$ from X_n to the homotopy inverse limit of the diagram

$$X_1 \xrightarrow{d_0} X_0 \xleftarrow{d_1} X_1 \xrightarrow{d_0} \cdots \qquad \cdots \xrightarrow{d_0} X_0 \xleftarrow{d_1} X_1$$

  is a weak homotopy equivalence. (The u_i^* are iterated face operators corresponding to the weakly order-preserving maps $u_i : \{0,1\} \to \{0,1,2,\ldots,n\}$ defined by $u_i(0) = i-1$ and $u_i(1) = i$.)

In order to formulate condition (κ) we introduce some vocabulary based on (σ). We call an element z of $\pi_0 X_1$ *homotopy left invertible* if there is an element x of $\pi_0 X_2$ such that $d_0 x = z$ and $d_1 x$ is in the image of $s_0 : \pi_0 X_0 \to \pi_0 X_1$. (In such a case $d_2 x$ can loosely be thought of as a left inverse for $z = d_0 x$. Indeed $d_1 x$ can loosely be thought of as the composition $d_2 x \circ d_0 x$, and by assuming that this is in the image of s_0 we are saying that it is in the path component of an identity morphism. We have written d_0, d_1, s_0 etc. for maps induced on π_0 by the face and degeneracy operators.) We call z *homotopy right invertible* if there is an element y of $\pi_0 X_2$ such that $d_2 y = z$ and $d_1 x$ is in the image of $s_0 : \pi_0 X_0 \to \pi_0 X_1$. Finally $z \in \pi_0 X_1$ is *homotopy invertible* if it is both homotopy left invertible and homotopy right invertible. Let X_1^w be the union of the homotopy invertible path components of X_1. It is a subspace of X_1.

- (κ) The map d_0 restricts to a weak homotopy equivalence from X_1^w to X_0.

DEFINITION 2.2.2. A *functor* from a complete Segal space X to another complete Segal space Y is just a simplicial map $f : X \to Y$.

- (i) Such a functor is a *weak equivalence* if and only if $f_n : X_n \to Y_n$ is a weak homotopy equivalence for all $n \geq 0$.
- (ii) Suppose that X and Y are complete Segal spaces. By $\mathrm{map}(X, Y)$, the space of all functors from X to Y, we mean a simplicial set whose set of n-simplices is the set of simplicial maps from $\Delta^n \times X$ to Y (where $\Delta^n \times X$ has k-th term equal to $\Delta^n \times X_k$). For a homotopy invariant (= right derived) notion of *space of all functors from* X *to* Y, we require a cofibrant replacement of X^c of X and a fibrant replacement $Y^f \to Y$ of Y. Then we define $\mathbb{R}\mathrm{map}(X, Y)$ as $\mathrm{map}(X^c, Y^f)$, the space of simplicial maps from X^c to Y^f. Here we rely on a standard model category structure (to make sense of cofibrant and fibrant replacements) on the category of simplicial spaces in which a morphism $X \to Y$ is a weak equivalence, respectively fibration, if $X_n \to Y_n$ is a weak equivalence, respectively fibration, for every $n \geq 0$. The cofibrant and fibrant replacements can be constructed in a functorial way.

(*End of quotation from Boavida-Weiss.*)

These ideas are not directly applicable to the nerves of $\mathsf{con}(M)$ and $\mathsf{con}(N)$, which are "Segal" but not complete in the sense of Rezk. In fact the nerve of Fin is "Segal" but not complete. But the reference functor $\mathsf{con}(M) \to \mathsf{Fin}$ induces a map of the nerves (which are Segal spaces) which is *fiberwise complete* as in the following definition (again a quotation from Boavida-Weiss).

Let Y and Z be simplicial spaces which satisfy (σ). Let $\mathsf{f}\colon \mathsf{Y} \to \mathsf{Z}$ be a simplicial map. The following condition is an obvious variation on condition (κ) above.

(κ_{ver}) The square

$$\begin{tikzcd} \mathsf{Y}_1^w \arrow[r] \arrow[d, "d_0"] & \mathsf{Z}_1^w \arrow[d, "d_0"] \\ \mathsf{Y}_0 \arrow[r] & \mathsf{Y}_0 \end{tikzcd}$$

is a homotopy pullback.

Definition 2.2.3. We say that $\mathsf{f}\colon \mathsf{Y} \to \mathsf{Z}$ constitutes a a fiberwise complete Segal space over Z if Y and Z satisfy (σ) and f satisfies (κ_{ver}). A *functor* from a simplicial space X over Z to a fiberwise complete Segal space Y over Z is just a simplicial map $\mathsf{g}\colon \mathsf{X} \to \mathsf{Y}$ over Z.

To make sense of $\mathbb{R}\mathrm{map}_{\mathsf{Z}}(\mathsf{X}, \mathsf{Y})$, space of derived simplicial maps from X to Y over Z, we use a model category structure on the category of simplicial spaces over Z (and at this point we should try forget the fact that Y is Segal and fiberwise complete over Z). The notion of weak equivalence is in any case degreewise. Other details are omitted.

2.3. Local configuration categories and little disk operads. From the definitions and a result [**4**, 4.1] which expresses the locality of local configuration categories, the space

$$\mathbb{R}\mathrm{map}_{\partial,\,\mathsf{Fin}}(\mathsf{con}^{\mathrm{loc}}(\mathsf{M};\mathsf{r}), \mathsf{con}^{\mathrm{loc}}(\mathsf{N};\mathsf{r}))$$

in the diagram of theorem 2.1.2 can be described as the space of sections of a fibration on M whose fiber over $x \in \mathsf{M}$ is the space of pairs (y, g) where $y \in \mathsf{N}$ and

$$g \in \mathbb{R}\mathrm{map}_{\mathsf{Fin}}(\mathsf{con}(\mathsf{T}_x\mathsf{M};\mathsf{r}), \mathsf{con}(\mathsf{T}_y\mathsf{N};\mathsf{r})).$$

More precisely, these are sections defined on all of M and *prescribed* over $\partial\mathsf{M}$. The prescription on $\partial\mathsf{M}$ comes from the fact that a preferred embedding $\partial\mathsf{M} \to \partial\mathsf{N}$ has been selected.

Theorem 2.3.1. [**4**, §7] *Let V and W be finite dimensional real vector spaces. Let $\mathcal{E}_{\mathsf{V}}$ and $\mathcal{E}_{\mathsf{W}}$ be the operads of little disks in V and W, respectively. There is a weak equivalence*

$$\mathbb{R}\mathrm{map}(\mathcal{E}_{\mathsf{V}}, \mathcal{E}_{\mathsf{W}}) \longrightarrow \mathbb{R}\mathrm{map}_{\mathsf{Fin}}(\mathsf{con}(\mathsf{V}), \mathsf{con}(\mathsf{W}).$$

Similarly, for an integer $\mathsf{r} \geq 1$ there is a weak equivalence

$$\mathbb{R}\mathrm{map}(\mathcal{E}_{\mathsf{V},\leq \mathsf{r}}, \mathcal{E}_{\mathsf{W},\leq \mathsf{r}}) \longrightarrow \mathbb{R}\mathrm{map}_{\mathsf{Fin}}(\mathsf{con}(\mathsf{V};\mathsf{r}), \mathsf{con}(\mathsf{W};\mathsf{r})$$

where $\mathcal{E}_{\mathsf{V},\leq \mathsf{r}}$ is the variant of $\mathcal{E}_{\mathsf{V}}$ truncated at cardinality r; see remark 2.3.2 below.

Remark 2.3.2. The operads in this theorem are plain operads in the category of spaces. In order to make sense of spaces of derived maps between such operads, we need at least a notion of weak equivalence for maps between such operads [**12**]. This is *levelwise* equivalence. Cisinski and Moerdijk [**7**], [**8**], [**9**] have developed a more tractable setting for operads in spaces by associating to an operad P as above (for simplicity) its dendroidal nerve $\mathsf{N}_{\mathsf{d}}\mathsf{P}$, a contravariant functor from a certain category of trees to spaces. Briefly, the functor $\mathsf{N_d}$ gives (in the case of plain operads) a faithful translation so that $\mathbb{R}\mathrm{map}(\mathsf{P},\mathsf{Q})$ can be identified with $\mathbb{R}\mathrm{map}(\mathsf{N_d P},\mathsf{N_d Q})$. And here the correct interpretation of $\mathbb{R}\mathrm{map}(\mathsf{N_d P},\mathsf{N_d Q})$ can be obtained by using any of the familiar model structure with *levelwise* weak equivalences on the category of contravariant functors from that tree category to spaces. Also, the correct interpretation of the *truncation at level* r of an operad P is obtained by restricting $\mathsf{N_d P}$ to a certain full subcategory of the tree category (consisting of those trees where every vertex has at most r incoming edges).

As a consequence, the space in the lower right-hand corner of the diagram of theorem 2.1.2 can be described as the space of sections, prescribed over ∂M, of a fibration on M whose fiber over $x \in M$ is the space of pairs (y, f) where $y \in N$ and $f \in \mathbb{R}\mathrm{map}(\mathcal{E}_{T_x M,\leq r}, \mathcal{E}_{T_y N,\leq r})$.

3. Configuration categories and homotopy automorphisms

3.1. The case of unrestricted cardinalities. Let K be a smooth compact manifold with boundary and let $g\colon \mathsf{K} \to \mathsf{K}$ be a homeomorphism. The homeomorphism induces a functor of configuration categories,

$$\mathsf{con}(\mathsf{K} \smallsetminus \partial\mathsf{K}) \to \mathsf{con}(\mathsf{K} \smallsetminus \partial\mathsf{K}).$$

Can we recover the *homotopy automorphism* of the pair $(\mathsf{K}, \partial\mathsf{K})$ determined by g from the above functor of configuration categories? (This problem has a fairly trivial solution when $\partial\mathsf{K}$ is empty, because then $\mathsf{con}(\mathsf{K};1)$ is essentially just a space and as such it is K. But the cases where $\partial\mathsf{K} \neq \emptyset$ appear to be much harder.) To be more demanding, we look for an arrow making the following diagram commutative (up to specified homotopies):

$$\begin{tikzcd}
\mathrm{Bhomeo}(\mathsf{K}) \arrow[r] \arrow[d] & \mathrm{Bhomeo}(\mathsf{K} \smallsetminus \partial\mathsf{K}) \arrow[d] \\
\mathrm{Bhaut}(\mathsf{K}, \partial\mathsf{K}) \arrow[d, "\text{res.}"] & \mathrm{Bhaut}_{\mathsf{NFin}}(\mathsf{con}(\mathsf{K} \smallsetminus \partial\mathsf{K})) \arrow[l, dotted] \arrow[d, "\text{res.}"] \\
\mathrm{Bhaut}(\mathsf{K}) \arrow[r, equal] & \mathrm{Bhaut}_{\mathsf{NFin}}(\mathsf{con}(\mathsf{K} \smallsetminus \partial\mathsf{K};1))
\end{tikzcd}$$

Here $\mathrm{haut}_{\mathsf{NFin}}(\mathsf{con}(\mathsf{K} \smallsetminus \partial\mathsf{K}))$ is shorthand for the union of the weakly invertible path components of $\mathbb{R}\mathrm{map}_{\mathsf{NFin}}(\mathsf{con}(\mathsf{K} \smallsetminus \partial\mathsf{K}))$. — Such an arrow exists under some (severe) conditions on K. See [**34**] and [**35**]. The message of these two papers is that, under some conditions on K, the space $\partial\mathsf{K}$ can be recovered (functorially, up to weak equivalence etc.) from the configuration category of $\mathsf{K} \smallsetminus \partial\mathsf{K}$.

There is a more complicated version for a smooth compact K with a codimension zero smooth compact submanifold $\partial_0\mathsf{K}$ of $\partial\mathsf{K}$. Let $\partial_1\mathsf{K}$ be the closure of $\partial\mathsf{K} \smallsetminus \partial_0\mathsf{K}$ in $\partial\mathsf{K}$. We look for an arrow making the following diagram commutative up to specified homotopies:

$$
\begin{tikzcd}
\mathrm{Bhomeo}_{\partial_0}(\mathsf{K}) \arrow[r] \arrow[d] & \mathrm{Bhomeo}_{\partial}(\mathsf{K} \smallsetminus \partial_1\mathsf{K}) \arrow[d] \\
\mathrm{Bhaut}_{\partial_0}(\mathsf{K}, \partial_1\mathsf{K}) \arrow[d, "\text{res.}"] & \mathrm{Bhaut}_{\partial,\mathsf{NFin}_*}(\mathsf{con}(\mathsf{K} \smallsetminus \partial_1\mathsf{K})) \arrow[l, dotted] \arrow[d, "\text{res.}"] \\
\mathrm{Bhaut}_{\partial_0}(\mathsf{K}) \arrow[r, equal] & \mathrm{Bhaut}_{\partial,\mathsf{NFin}_*}(\mathsf{con}(\mathsf{K} \smallsetminus \partial_1\mathsf{K}; 1))
\end{tikzcd}
$$

(The configuration category $\mathsf{con}(\mathsf{K} \smallsetminus \partial_1\mathsf{K})$ has a subcategory which can be described vaguely as the over category associated with the essentially unique object taken to the object $[0]$ in Fin_* by the reference functor. The notation $\mathrm{Bhaut}_{\partial,\mathsf{NFin}_*}(\dots)$ indicates that we are after homotopy automorphisms which cover the identity of Fin_* and induce the identity automorphism of that subcategory, in a derived sense.) Again, such a factorization exists under severe conditions on K and $\partial_0\mathsf{K}$. See [**34**] and [**35**]. The case that we are interested in is $\mathsf{K} = W$ and $\partial_0\mathsf{K}$ equal to a closed hemisphere of ∂W not containing the selected point z. The severe conditions are then satisfied. In this very special case we have $\mathrm{Bhomeo}_{\partial_0}(\mathsf{K}) \simeq \mathrm{Bhomeo}_{\partial}(\mathsf{K})$ by the Alexander trick applied to the disk $\partial_1\mathsf{K}$, and $\mathrm{Bhaut}_{\partial_0}(\mathsf{K}, \partial_1\mathsf{K}) \simeq \mathrm{Bhaut}_{\partial}(\mathsf{K})$ by a similar but easier argument for homotopy automorphisms of disks. Therefore we can simplify the above diagram slightly and write

$$
\begin{tikzcd}
\mathrm{Bhomeo}_{\partial}(W) \arrow[r] \arrow[d] & \mathrm{Bhomeo}_{\partial}(W_z) \arrow[d] \\
\mathrm{Bhaut}_{\partial}(W) \arrow[d, "\text{res.}"] & \mathrm{Bhaut}_{\partial,\mathsf{NFin}_*}(\mathsf{con}(W_z)) \arrow[l, dotted] \arrow[d, "\text{res.}"] \\
\mathrm{Bhaut}_{\partial}(W_z) \arrow[r, equal] & \mathrm{Bhaut}_{\partial,\mathsf{NFin}_*}(\mathsf{con}(W_z; 1))
\end{tikzcd}
$$

3.2. The case of restricted cardinalities. These statements have truncated versions. We truncate the configuration categories by allowing only configurations of cardinality $\leq \ell$, say. In that case there is an arrow making the following commutative up to specified homotopies:

$$
\begin{tikzcd}
\mathrm{Bhomeo}_{\partial_0}(\mathsf{K}) \arrow[r] \arrow[d] & \mathrm{Bhomeo}_{\partial}(\mathsf{K} \smallsetminus \partial_1\mathsf{K}) \arrow[d] \\
\mathrm{Bhaut}_{\partial_0}(\mathsf{K}, \partial_1\mathsf{K})^{\natural} \arrow[d, "\text{res.}"] & \mathrm{Bhaut}_{\partial,\mathsf{NFin}_*}(\mathsf{con}(\mathsf{K} \smallsetminus \partial_1\mathsf{K}; \ell)) \arrow[l, dotted] \arrow[d, "\text{res.}"] \\
\mathrm{Bhaut}_{\partial_0}(\mathsf{K})^{\natural} \arrow[r, equal] & \mathrm{Bhaut}_{\partial,\mathsf{NFin}_*}(\mathsf{con}(\mathsf{K} \smallsetminus \partial_1\mathsf{K}; 1))
\end{tikzcd}
\tag{3.2.1}
$$

The superscript $(-)^\natural$ means that we have to sacrifice/kill homotopy groups in dimensions $\ge c$, where c depends on ℓ and goes to infinity with ℓ. Again this specializes to a diagram of the shape

$$\begin{tikzcd}
\mathrm{Bhomeo}_\partial(W) \arrow[r] \arrow[d] & \mathrm{Bhomeo}_\partial(W_z) \arrow[d] \\
\mathrm{Bhaut}_\partial(W)^\natural \arrow[d, "\text{res.}"] & \mathrm{Bhaut}_{\partial,\mathsf{NFin}_*}(\mathsf{con}(W_z;\ell)) \arrow[l, dotted] \arrow[d, "\text{res.}"] \\
\mathrm{Bhaut}_\partial(W_z)^\natural \arrow[r, equal] & \mathrm{Bhaut}_{\partial,\mathsf{NFin}_*}(\mathsf{con}(W_z;1))
\end{tikzcd}$$

It turns out that $\ell = 8$ is big enough for our purposes. Indeed, the estimates in [**35**] show that killing the homotopy groups of $\mathrm{Bhaut}_\partial(W)$ in dimensions $\ge 7n - 20$ is the price to pay for restricting cardinalities like that. But for reasons given below in section 4.1, we have the permission to kill anything above dimension $2n + 4k$, and $7n - 20$ is greater than $2n + 4k$ by our assumption on k.

4. Dissonance

This section is mainly about the proof of lemma 1.4.2.

4.1. Homotopical description of some kappa classes revisited. Using convergence in manifold calculus and theorem 2.1.2 we may replace the grouplike topological monoid $\mathrm{emb}_\partial(W_z, W_z)$ by the homotopy pullback of

$$\begin{tikzcd}
& \mathrm{haut}_{\partial,\,\mathsf{NFin}_*}(\mathsf{con}(W_z;8)) \arrow[d, "\text{specialization}"] \\
\mathrm{imm}^\times_\partial(W_z, W_z) \arrow[r] & \mathrm{haut}_{\partial,\,\mathsf{NFin}}(\mathsf{con}^{\mathrm{loc}}(W_z;8))
\end{tikzcd}$$

This should be seen as a diagram of topological grouplike monoids. (We have already discarded the non-invertible components; in one case this is indicated by the superscript $\times$.) The choice of integer 8 is justified by the standard convergence estimates in manifold calculus applied to smooth embedding spaces. (Strictly speaking this is not the same argument that led us to view $\ell = 8$ as a good choice in section 3.2, but loosely speaking it is not very surprising that the same choice of cardinality bound serves us well on two closely related occasions.)

This gives us immediately a better understanding of the map ν in section 1.3. Namely, ν is (up to delooping) the map between homotopy pullbacks induced by a natural transformation from the diagram

$$\begin{tikzcd}
& \mathrm{haut}_{\partial,\,\mathsf{NFin}_*}(\mathsf{con}(W_z;8)) \arrow[d] \\
\mathrm{imm}^\times_\partial(W_z, W_z) \arrow[r] & \mathrm{haut}_{\partial,\mathsf{NFin}}(\mathsf{con}^{\mathrm{loc}}(W_z;8))
\end{tikzcd}$$

to the diagram

$$\begin{tikzcd} & \mathrm{haut}_\partial(W)^\natural \arrow[d] \\ \mathrm{haut}_\partial(W_z) \ltimes \mathrm{map}_*(W_z, \mathrm{SO}(2n)) \arrow[r] & \mathrm{haut}_\partial(W_z) \end{tikzcd}$$

(The decoration $(-)^\natural$ should be read as in section 3.2.) The natural transformation specializes to a weak equivalence on the lower left-hand terms. This is just a special case of the Smale-Hirsch theorem. On the upper right-hand terms, the natural transformation is given by the dotted arrows of section 3. On the lower right-hand terms, it is given by restriction from $\mathsf{con}^{\mathrm{loc}}(W_z; 8)$ to $\mathsf{con}^{\mathrm{loc}}(W_z; 1)$.

Therefore it suffices for the proof of lemma 1.4.2 to show that the rational map

$$\bar{q}_b \colon \mathrm{Bimm}^\times_\partial(W_z, W_z) \longrightarrow \mathrm{Bimm}^\times_\partial(W_z, W_z)$$

induced by the rational map $q_b \colon \mathrm{BSO}(2n) \to \mathrm{BSO}(2n)$ of section 1.4 is a map over $\mathrm{Bhaut}_{\partial,\mathsf{NFin}}(\mathsf{con}^{\mathrm{loc}}(W_z; 8))$. And that reduces immediately to showing that $q_b \colon \mathrm{BSO}(2n) \to \mathrm{BSO}(2n)$ itself is rationally a map over $\mathrm{Bhaut}_{\mathsf{NFin}}(\mathsf{con}(\mathbb{R}^{2n}; 8))$. See remark 4.1.1 below. That is what we will show — neglecting (suppressing, killing) homotopy groups of $\mathrm{Bhaut}_{\mathsf{NFin}}(\mathsf{con}(\mathbb{R}^{2n}; 8))$ in dimensions which are too high or too low to be of interest here. Obviously the homotopy groups in dimensions greater than $4n + 4k$ can be neglected. The homotopy groups in dimensions $\leq n$ can also be neglected because W_z is $(n-1)$-connected.

Remark 4.1.1. Firstly there is a weak equivalence of topological monoids

$$\mathrm{haut}_{\partial,\mathsf{Fin}}(\mathsf{con}^{\mathrm{loc}}(W_z; 8)) \;\simeq\; \mathrm{map}_*(W_z, \mathrm{haut}(\mathsf{con}^{\mathrm{loc}}(\mathbb{R}^{2n}; 8))).$$

Secondly there is no essential difference between the categories $\mathsf{con}(\mathbb{R}^{2n}; 8)$ and $\mathsf{con}^{\mathrm{loc}}(\mathbb{R}^{2n}; 8)$. They are weakly equivalent as complete Segal spaces.

4.2. Stabilization. To confirm that $q_b \colon \mathrm{BSO}(2n) \to \mathrm{BSO}(2n)$ of section 1.4 is indeed a (rational) map over $\mathrm{Bhaut}_{\mathsf{NFin}}(\mathsf{con}(\mathbb{R}^{2n}; 8))$, up to homotopy, and after suppressing some irrelevant homotopy groups, we use a homotopy commutative square

$$\begin{tikzcd} \mathrm{BSO}(2n-s) \arrow[r] \arrow[d, "\text{stab.}"] & \mathrm{Bhaut}_{\mathsf{NFin}}(\mathsf{con}(\mathbb{R}^{2n-s}; 8)) \arrow[d, "\text{stab.}"] \\ \mathrm{BSO}(2n) \arrow[r] & \mathrm{Bhaut}_{\mathsf{NFin}}(\mathsf{con}(\mathbb{R}^{2n}; 8))\,. \end{tikzcd} \tag{4.2.1}$$

This works for any $s \in \{0, 1, \dots, 2n-1\}$. The vertical arrows are stabilization maps, but it turns out that the stabilization map on the right is not a trivial matter; it uses something inspired by the Boardman-Vogt tensor product of operads. More details are given in [**5**]. — The idea is now to select s in such a way that the right-hand arrow in diagram (4.2.1) is rationally nullhomotopic (after killing of homotopy groups of the target in dimensions high enough or low enough to be irrelevant for us). Why should this be possible? As will be explained in sections 4.3 and 4.4, the rational homotopy groups of the top right-hand term (so far as they are relevant for us) are clustered in dimensions close to the integer multiples of $2n - s$, and the rational homotopy groups of the bottom right-hand term (so far as relevant) are clustered in dimensions close to the integer multiples of $2n$. Therefore, if s has the right size, then obstruction theory lets us deduce that the map is rationally nullhomotopic (after killing of homotopy groups of the target in dimensions high enough and low enough to be irrelevant for us). This observation gave rise to the section title: dissonance.

4.3. Spaces of derived maps between certain configuration categories. Fix positive integers s,t and ℓ where $\mathsf{s} \le \mathsf{t}$. To justify what has been said about dissonance, let us look at the homotopy fiber of the restriction map

$$\begin{array}{c} \mathbb{R}\mathrm{map}_{\mathsf{NFin}}(\mathsf{con}(\mathbb{R}^{\mathsf{s}};\ell),\mathsf{con}(\mathbb{R}^{\mathsf{t}};\ell)) \\ \downarrow \\ \mathbb{R}\mathrm{map}_{\mathsf{NFin}}(\mathsf{con}(\mathbb{R}^{\mathsf{s}};\ell-1),\mathsf{con}(\mathbb{R}^{\mathsf{t}};\ell-1)) \end{array}$$

over the base point. (There is a base point.)
We have theorem 2.3.1 as a translation tool. Therefore we are led to a question about operads. Let P and Q be (plain) operads P and Q in the category of spaces. We ask for a practical description of the homotopy fibers $\Phi(\ell)$ of the forgetful map

$$\begin{array}{c} \mathbb{R}\mathrm{map}(\mathsf{P}_{\le \ell},\mathsf{Q}_{\le \ell}) \\ \downarrow \\ \mathbb{R}\mathrm{map}(\mathsf{P}_{\le \ell-1},\mathsf{Q}_{\le \ell-1}) \end{array}$$

where P_ℓ for example means the truncation of P at level ℓ (where operations of arity greater than ℓ are suppressed). Note that there is a plural here: the map has many homotopy fibers, and we ought to specify a point in the target to specify one of them, and we do but the notation does not show it. — To make it easier, we assume that P and Q are both contractible in degrees 0 and 1. (Beware: for an operad, being contractible in degree 0 is quite different from being empty in degree 0. The operad $\mathcal{E}_{\mathsf{s}}$ of little disks in $\mathbb{R}^{\mathsf{s}}$ is a fine example of an operad which is contractible both in degree 0 and 1.)

The following sketch of an answer is a quotation from [**18**]. A practical description of $\Phi(\ell)$ is suggested by the Fulton-MacPherson-Axelrod-Singer description of the little disk operads or the corresponding configuration categories. (This is sketched in [**4**, §3].) The operad P has a space $\mathsf{P}(\underline{\ell})$ of ℓ-ary operations, which is obviously going to be important in the description of $\Phi(\ell)$, but it determines two more spaces: the ℓ-th latching space $\mathsf{LP}(\ell)$ and the ℓ-th matching space[1] $\mathsf{MP}(\ell)$ of P. Of these two, $\mathsf{MP}(\ell)$ is less interesting and less difficult; it is the homotopy inverse limit of the spaces $\mathsf{P}(\mathsf{S})$ where S runs through the proper subsets of $\underline{\ell} = \{1,2,\dots,\ell\}$. To make a diagram out of these $\mathsf{P}(\mathsf{S})$, use the assumption that $\mathsf{P}(\underline{0})$ is contractible. — By analogy $\mathsf{LP}(\ell)$ should be defined as a homotopy direct limit. I skip the description of it for general P. In the case where P is the operad of little s-disks, the latching space $\mathsf{LP}(\ell)$ can be described as the Axelrod-Singer artificial boundary of the manifold of normalized ordered configurations of ℓ points in $\mathbb{R}^{\mathsf{s}}$. (*Normalized* means that two ordered configurations are identified if they agree up to scaling and translation.) According to Axelrod-Singer the space of normalized ordered configurations is the interior of a compact manifold with boundary, and the boundary is

[1]Perhaps this choice of words is unwise. I use these words, latching and matching, because they are reminiscent of constructions in the context of Reedy model category structures which have such names.

$\mathsf{LP}(\ell)$, in the case where P is the operad of little s-disks. For general P again, there are canonical maps

$$\mathsf{LP}(\ell) \longrightarrow \mathsf{P}(\underline{\ell}) \longrightarrow \mathsf{MP}(\ell)$$

which should be viewed as maps of spaces equipped with an action of the symmetric group Σ_ℓ. I emphasize that $\mathsf{P} \mapsto \mathsf{LP}(\ell)$ and $\mathsf{P} \mapsto \mathsf{MP}(\ell)$ are functors and, for the present purposes, they should be understood and can be designed as functors taking levelwise weak equivalences of operads to weak equivalences of spaces with an action of the symmetric group Σ_ℓ. The expected description of $\Phi(\ell)$ is then: space of derived Σ_ℓ-maps, subject to boundary conditions, from diagram

$$\mathsf{LP}(\ell) \longrightarrow \mathsf{P}(\underline{\ell}) \longrightarrow \mathsf{MP}(\ell)$$

to diagram

$$\mathsf{LQ}(\ell) \longrightarrow \mathsf{Q}(\underline{\ell}) \longrightarrow \mathsf{MQ}(\ell).$$

The boundary conditions depend on which homotopy fiber $\Phi(\ell)$ we are looking at. That is to say, we have already selected an element $\mathbb{R}\mathrm{map}(\mathsf{P}_{\le \ell-1}, \mathsf{Q}_{\le \ell-1})$ and it is easy to see that this determines (sufficiently) a derived Σ_ℓ-map from diagram

$$\mathsf{LP}(\ell) \longrightarrow \mathsf{MP}(\ell)$$

to diagram

$$\mathsf{LQ}(\ell) \longrightarrow \mathsf{MQ}(\ell).$$

That map is the boundary condition. The spaces $\mathsf{P}(\underline{\ell})$ and $\mathsf{Q}(\underline{\ell})$ do not appear in the description of it. It is deliberate. Therefore $\Phi(\ell)$ should be thought of as the space of dotted arrows in a diagram of Σ_ℓ-spaces and (derived) Σ_ℓ-maps

$$\begin{tikzcd}
\mathsf{LP}(\ell) \arrow[r] \arrow[d] & \mathsf{P}(\underline{\ell}) \arrow[r] \arrow[d, dotted] & \mathsf{MP}(\ell) \arrow[d] \\
\mathsf{LQ}(\ell) \arrow[r] & \mathsf{Q}(\underline{\ell}) \arrow[r] & \mathsf{MQ}(\ell)
\end{tikzcd}$$

commutative up to specified homotopies. The solid arrows are given in advance. The specified homotopy making the outer rectangle homotopy commutative is also given in advance and must be respected, in a derived sense.

4.4. Calculations. For arithmetic progressions in $\mathbb{Z}$ we use the notation

$$A[u{:}v{:}w] \ := \{u, u+v, u+2v, \dots, w\}$$

where $u, v, w \in \mathbb{Z}$ and $w = u + rv$ for some non-negative integer r. Taking $w = \infty$ is also allowed; in that case $A[u : v : \infty] \ := \{u, u+v, u+2v, u+3w, \dots\}$. In connection with that, we shall also use arithmetic notation for operations involving subsets of $\mathbb{Z}$. For example, $S + T$ means $\{s + t \mid (s,t) \in S \times T\}$ and $S - T$ means $\{s - t \mid (s,t) \in S \times T\}$. Therefore: do not confuse $S - T$ with $S \smallsetminus T$.

(i) For fixed $n \ge 2$, the space $\mathsf{P}(\ell) \simeq \mathrm{emb}(\{1, 2, \dots, \ell\}, \mathbb{R}^{2n})$ is the total space or final stage of a fibration tower

$$\mathrm{emb}(\{1,2,\dots,\ell\}, \mathbb{R}^{2n}) \to \mathrm{emb}(\{2,3,\dots,\ell\}, \mathbb{R}^{2n}) \to \mathrm{emb}(\{3,4,\dots,\ell\}, \mathbb{R}^{2n}) \to \cdots$$

whose layers are wedges of $(2n-1)$-spheres. The rational homotopy groups of a wedge of $(2n-1)$-spheres are concentrated in dimensions $t(2n-2)+1$ where $t = 1, 2, 3, \dots$ by a theorem of Hilton, subsumed in the Hilton-Milnor theorem. Therefore the rational homotopy groups of $\mathrm{emb}(\{1, 2, \dots, \ell\}, \mathbb{R}^{2n})$ are also concentrated in dimensions which belong to $A[2n-1{:}2n-2{:}\infty]$.

(ii) The space $\mathsf{MP}(\ell)$ is the total space or final stage of a tower of fibrations whose layers are as follows: a finite product of copies of $\mathrm{emb}(\{1,2,\dots,\ell-1\},\mathbb{R}^{2n})$, a finite product of copies of $\Omega\mathrm{emb}(\{1,2,\dots,\ell-2\},\mathbb{R}^{2n})$, ..., a finite product of copies of $\Omega^{\ell-3}\mathrm{emb}(\{1,2\},\mathbb{R}^{2n})$. The layers are connected if $\ell-2<2n-1$, which we can assume since we will have to impose a stronger condition later anyway. Therefore, using (i), the rational homotopy groups of $\mathsf{MP}(\ell)$ are concentrated in dimensions which belong to $\mathsf{A}[2n-1{:}2n-2{:}\infty]-\mathsf{A}[0{:}1{:}\ell-3]$.

(iii) By (i) and (iii) the rational homotopy groups of hofiber$[\mathsf{P}(\ell)\to\mathsf{MP}(\ell)]$ are concentrated in dimensions which belong to $\mathsf{A}[2n-1{:}2n-2{:}\infty]-\mathsf{A}[0{:}1{:}\ell-2]$.

(iv) By Poincaré duality, justified by the Axelrod-Singer description of the standard map $\mathsf{LP}(\ell)\to\mathsf{P}(\ell)$, the reduced cohomology group in degree j of the homotopy cofiber of $\mathsf{LP}(\ell)\to\mathsf{P}(\ell)$ for $\ell\geq 2$ is identified with the homology group $\mathsf{H}_{2n(\ell-1)-1-j}$ of $\mathsf{P}(\ell)$. The latter groups are well known [**10**, III.6, III.7] and concentrated in dimensions $t(2n-1)$ where $t\in\{0,1,\dots,\ell-1\}$. So the reduced cohomology groups of the homotopy cofiber of $\mathsf{LP}(\ell)\to\mathsf{P}(\ell)$ are concentrated in dimensions which belong to $\mathsf{A}[\ell-2{:}2n-1{:}2n(\ell-1)-1]$.

(v) By Poincaré duality, the cohomology groups of $\mathsf{LP}(\ell)$ for $\ell\geq 3$ are concentrated in dimensions which belong to $\mathsf{A}[0{:}2n-1{:}(\ell-1)(2n-1)]+\mathsf{A}[0{:}\ell-3{:}\ell-3]$.

(vi) It follows from (i)-(v) that the based space of (derived) solutions of

$$\begin{tikzcd}
\mathsf{LP}(\ell) \arrow[r] \arrow[d, "\mathrm{id}"] & \mathsf{P}(\underline{\ell}) \arrow[r] \arrow[d, dotted] & \mathsf{MP}(\ell) \arrow[d, "\mathrm{id}"] \\
\mathsf{LP}(\ell) \arrow[r] & \mathsf{P}(\underline{\ell}) \arrow[r] & \mathsf{MP}(\ell)
\end{tikzcd}$$

(with specified primary homotopies to make the little squares homotopy commutative, and a specified secondary homotopy to make the connection with a given homotopy for the outer rectangle) has higher rational homotopy groups concentrated in dimensions which belong to

$$\big((\text{set in (iii)})-(\text{set in (iv)})\big)\ \cup\ \big((\text{set in (ii)})-(\text{set in (v)})-2\big)$$

if $\ell\geq 3$. (*Higher* means that no claims are made for π_0 and π_1 .) This is

$$\begin{aligned}
&\big(\mathsf{A}[2n-1{:}2n-2{:}\infty]-\mathsf{A}[0{:}1{:}\ell-2]-\mathsf{A}[\ell-2{:}2n-1{:}2n(\ell-1)-1]\,\big)\\
\cup\ \ &\big(\mathsf{A}[2n-1{:}2n-2{:}\infty]-\mathsf{A}[2{:}1{:}\ell-1]\\
&\qquad\qquad\qquad -\mathsf{A}[0{:}2n-1{:}(\ell-1)(2n-1)]-\mathsf{A}[0{:}\ell-3{:}\ell-3]\,\big)\\
=\ \ &\big(\mathsf{A}[2n-1{:}2n-2{:}\infty]-\mathsf{A}[0{:}1{:}\ell-2]-\mathsf{A}[\ell-2{:}2n-1{:}2n(\ell-1)-1]\big)\\
&\cup\big(\mathsf{A}[2n-1{:}2n-2{:}\infty]-\mathsf{A}[2{:}1{:}2(\ell-2)]-\mathsf{A}[0{:}2n-1{:}(\ell-1)(2n-1)]\big)\\
=\ \ &\big(\mathsf{A}[2n-1{:}2n-2{:}\infty]-\mathsf{A}[\ell-2{:}1{:}2(\ell-2)]-\mathsf{A}[0{:}2n-1{:}(\ell-1)(2n-1)]\big)\\
&\cup\big(\mathsf{A}[2n-1{:}2n-2{:}\infty]-\mathsf{A}[2{:}1{:}2(\ell-2)]-\mathsf{A}[0{:}2n-1{:}(\ell-1)(2n-1)]\big)\\
\subset\ \ &\big(\mathsf{A}[2n-1{:}2n-2{:}\infty]-\mathsf{A}[0{:}1{:}2(\ell-2)]-\mathsf{A}[0{:}2n-1{:}(\ell-1)(2n-1)]\,\big).
\end{aligned}$$

An easy direct calculation shows that this estimate is also correct for $\ell=2$. Note that $\mathsf{LP}(2)$ is empty, $\mathsf{P}(2)\simeq S^{2n-1}$ and $\mathsf{MP}(2)$ is contractible, so that direct calculations are feasible when $\ell=2$.

(vii) Although in (vi) we have not asked for solutions with derived Σ_ℓ-invariance, obstruction theory leads to the same estimates if we do ask for derived Σ_ℓ-invariance. (The rational Borel cohomology groups for a space or pair of spaces with an action of a finite group G, with coefficients in a $\mathbb{Q}[G]$-module K, are direct summands of the ordinary cohomology groups with coefficents K of that space or pair of spaces.)

(viii) Suppose that $\ell \geq 2$. By (vi) and (vii), the identity component of $\mathbb{R}\mathrm{map}(P_{\leq\ell}, P_{\leq\ell})$ has nontrivial higher rational homotopy groups only in dimensions which belong to

$$\begin{aligned} & (\, A[2n-1{:}2n-2{:}\infty] - A[0{:}1{:}2(\ell-2)] - A[0{:}2n-1{:}(\ell-1)(2n-1)]\,)\ \cap\ \mathbb{N} \\ = \ & (\, A[2n-1{:}2n-2{:}\infty] - A[0{:}1{:}3(\ell-2)+1]\,)\ \cap\ \mathbb{N}\,. \end{aligned}$$

(ix) By (viii), the n-connected cover of $B(\mathbb{R}\mathrm{map}(P_{\leq 8}, P_{\leq 8}))$ has nontrivial rational homotopy groups only in dimensions which belong to

$$T_{2n} := (\, A[2n-1{:}2n-2{:}\infty] - A[0{:}1{:}19]\,)\ \cap\ \mathbb{N}\,.$$

(x) By (ix), the n-connected cover of $B(\mathbb{R}\mathrm{map}(P_{\leq 8}, P_{\leq 8}))$ has nontrivial rational cohomology groups only in dimensions which belong to the (additive) submonoid $\langle T_{2n}\rangle_{\mathbb{N}}$ of $\mathbb{N}$ generated by T_{2n}.

(xi) We return to diagram (4.2.1), with a determination to suppress (at least) the homotopy groups in dimensions $\leq n$ in the upper row and homotopy groups in dimensions $\geq 8n$ in the lower row. (This can be done by taking Postnikov covers and Postnikov bases, respectively.) The right-hand vertical arrow then becomes rationally nullhomotopic provided s is even and has been selected in such a way that $\langle T_{2n-s}\rangle_{\mathbb{N}}$ has empty intersection with the part of T_{2n} that matters, i.e., with

$$(\, A[2n-1{:}2n-2{:}8n-7] - A[0{:}1{:}19]\,)\ \cap\ \mathbb{N}\,.$$

This leads to the following conditions on s and n:

$$\langle T_{2n-s}\rangle_{\mathbb{N}} \ni 5(2n-s-19) \quad > \quad (8n-7) \in T_{2n}$$

which means $2n > 5s + 88$, and

$$\langle T_{2n-s}\rangle_{\mathbb{N}} \ni (2n-s-1) \quad < \quad (2n-1-19) \in T_{2n}$$

which means $s > 19$, and other conditions implied by these two. We can take $s = 20$ but then we also need to ensure $n \geq 95$.

COMPLETION OF PROOF OF LEMMA 1.4.2. We take this up from sections 4.1 and 4.2. Let X be the n-connected cover of $BSO(2n)$ and let $Y = B\mathbb{R}\mathrm{map}(P_{\leq 8}, P_{\leq 8})$ as above. We have a map $w{:}\,X \to Y$ and we have a self-map $q = q_b{:}\,X \to X$ and we want to show $w \simeq wq_b$, but we are willing to kill some higher homotopy groups (of the target Y, in dimensions $\geq 8n$ minus constant). We take

$$b = n - 11 = n - s/2 - 1$$

to begin. Write $X = X_\lambda \times X_\mu \times X_\rho \times X_\eta$ where

- X_λ is the product of the factors corresponding to p_r where $r < b$;
- X_μ is the factor corresponding to p_b;
- X_ρ is the product of the factors corresponding to p_r where $r > b$;
- X_η is the factor corresponding to the Euler class e.

The maps u and uq_b agree on $X_\lambda \times X_\mu$ because, as we have just shown, u and uq_b are both zero there. (Think $X_\lambda \times X_\mu \subset BSO(2(n-11))$ and remember diagram (4.2.1).) They agree on $X_\lambda \times X_\rho \times X_\eta$ because q_b is the identity there. So they agree on

$$X_\lambda \times \big(X_\mu \vee X_\rho \times X_\eta\big).$$

So the remaining obstructions to agreement of u and uq_b on all of X (as in obstruction theory) are in the reduced cohomology $\tilde{H}^j$ of

$$\frac{X_\lambda \times (X_\mu \times X_\rho \times X_\eta)}{X_\lambda \times (X_\mu \vee X_\rho \times X_\eta)} = (X_\lambda)_+ \wedge X_\mu \wedge (X_\rho \times X_\eta)$$

with coefficients in the homotopy groups $\pi_j(Y)$. By a straightforward computation, the minimal j making both the reduced cohomology of $(X_\lambda)_+ \wedge X_\mu \wedge (X_\rho \times X_\eta)$ in degree j and the homotopy group $\pi_j(Y)$ rationally nontrivial is at least $8n - 26$. (The first nontrivial reduced cohomology group of $(X_\lambda)_+ \wedge X_\mu \wedge (X_\rho \times X_\eta)$ is in dimension $j = 4b + 2n = 6n - 4c_2 = 6n - 44$, but $\pi_j(Y)$ is zero for this j. The next run of nontrivial cohomology groups appears in dimensions j greater than $7n$ approximately, but this is not matched by nontrivial $\pi_j(Y)$ until we reach $j = 8n - 7 - 19 = 8n - 26$.) To sum up, we can achieve $w \simeq wq_b$ at the price of killing the homotopy groups of Y in dimensions $\geq 8n - 26$. Now, turning to lemma 1.4.2, we see that we can achieve $v \simeq v\bar{q}_b$ by killing the homotopy groups of the target of v in dimensions $\geq 7n - 26$, since the manifold W has the homotopy type of a wedge of n-spheres. This completes the proof of the first part of lemma 1.4.2. (It seemed convenient for one reason and another to replace $7n - 26$ by the smaller number $4b + 3n = 7n - 44$ in the statement.)

For the second part of the lemma, we reason similarly, but here it is enough to note that $(X_\lambda)_+ \wedge X_\mu \wedge (X_\rho \times X_\eta)$ is $(4b + 2n - 1)$-connected. Therefore we can achieve $w \simeq wq_b$ at the price of killing the homotopy groups of Y in dimensions $\geq 4b + 2n$, and it follows that we can achieve $v \simeq v\bar{q}_b$ at the price of killing the homotopy groups of the target of v in dimensions $\geq 4b + n$.

4.5. Looking back. Many of the remarks collected here reflect discussions with Soren Galatius.

Remark 4.5.1. Looking back on section 4.4, it is worth emphasizing that two homotopical properties of W_z were used in two different places. Firstly, W_z is homotopically n-dimensional (although it is geometrically $2n$-dimensional); this was used at the end of the section. But secondly, W_z is $(n - 1)$-connected; this was used at the beginning of the section. It allowed us to replace $BSO(2n)$ by its n-connected cover. Both observations contribute to a sharpening of the estimates in different ways.

Remark 4.5.2. Too little has been said about rationalization as a tool in calculations. Here is an important principle which was taken for granted in section 4.4. Let $f : X \to Y$ be a based map of based connected CW-spaces, where Y is simply connected, has only finitely many nontrivial homotopy groups and X has compact skeletons. We might be interested in the rational homotopy type of the path component of $\mathrm{map}(X, Y)$ containing f. It is not obvious that this agrees with the rational homotopy type of the appropriate component of $\mathrm{map}(X, Y_{\mathbb{Q}})$, and perhaps such a statement does not even make sense. But after passage to 1-connected covers (of the two mapping spaces) there is agreement. This can easily be shown by induction on the number of nontrivial homotopy groups that Y has.

A related use of rational homotopy appears in lemma 1.4.2. Here it is tempting to ask whether the *rational* map f from $\mathrm{Bdiff}_\partial(W_z)$ to itself, as in the lemma, can be replaced by an honest map with rationally similar properties. But we do not have to answer that question here. It is enough to know that the rational map f acts on the rational homology and cohomology of $\mathrm{Bdiff}_\partial(W_z)$.

This is also a good opportunity to admit that the concept of rational map was used without a crystal clear definition. For a based and path-connected space Y, homotopy equivalent to a CW-space, the rationalization $Y_\mathbb{Q}$ of Y is understood (in this article) to be the homotopy orbit space of the standard action of $\pi_1(Y)$ on the rationalization of the universal cover of Y. Consequently there is a canonical map $Y \to Y_\mathbb{Q}$ which induces an isomorphism in π_1 and isomorphisms $\pi_k(Y)\otimes\mathbb{Q} \to \pi_k(Y_\mathbb{Q})$ for $k > 1$. A *rattional map* from X to Y is a map from X to $Y_\mathbb{Q}$.

REMARK 4.5.3. Let $p\colon E \to M$ be a fibration where base M and fiber F are oriented Poincaré duality spaces, and M is connected. It is known that the signature of E depends only on M and $H^n(F;\mathbb{Q})$ with the intersection form and the action of $\pi_1(M)$, where n is half the formal dimension of F. (If the formal dimension of F is not even, the statement remains correct in the sense that the signature of E is zero.) I am told by Andrew Ranicki that this is [**23**, Thm. 2.7], to be used with rational coefficients; and that in the case of a fiber bundle where base and fiber are closed manifolds, the work of W. Meyer [**25**] can be used. The conclusion from this is that part (i) of proposition 1.3.1 has more radical variants: for example $\kappa_t(\mathcal{L}_{n+k})$ comes from the cohomology of a covering space of $\mathrm{Bhaut}_\star(W)$. From this point of view, it might seem that section 3 with its emphasis on ∂W is superfluous. But there is part (ii) of proposition 1.3.1, too. I cannot think of a way to take ∂W out of that. Therefore section 3 could be essential after all.

REMARK 4.5.4. Manifold calculus wants to say that $\mathrm{emb}_\partial(W_z, W_z)$ is computable. If we take that seriously then we might ask whether it was really necessary to use the Galatius-RandalWilliams results on parameterized surgery [**16**], [**17**] in the proof of theorem 1.1.2. Tentative answer: manifold calculus gives a pointillistic picture of $\mathrm{emb}_\partial(W_z, W_z)$, and with that picture it is not easy to understand the composite map

$$\mathrm{emb}_\partial(W_z, W_z) \simeq \mathrm{diff}_\partial(W_z) \hookrightarrow \mathrm{homeo}_\partial(W) \hookrightarrow \mathrm{haut}_\partial(W).$$

REMARK 4.5.5. Farrell-Hsiang [**13**] and Watanabe [**32**], [**33**] show in very different ways that the inclusion $\mathrm{BO}(m) \to \mathrm{BTOP}(m)$ fails to be a rational homotopy equivalence for many $m \gg 0$. The Farrell-Hsiang argument uses the smoothing theory connection between $\mathrm{TOP}(m+1)/\mathrm{TOP}(m)$ and the space of smooth pseudo-isotopies of D^m (similar in spirit to remark 1.1.4) and then, following Waldhausen, the deep relationship between smooth pseudo-isotopy spaces and algebraic K-theory. The force of this approach is limited by the known stability range (currently [**21**], Kiyoshi Igusa's stability theorem) for smooth pseudo-isotopy spaces. That is to say, Farrell and Hsiang get a complete description of $H^*(\mathrm{BTOP}(m))$ for $\star \le 4m/3 - c$ for a constant c, and nothing beyond that range. Watanabe writes about $\mathrm{diff}_\partial(D^m)$ (related to $\mathrm{TOP}(m)$ by remark 1.1.4) and uses Kontsevich integrals (configuration space integrals) to find nontrivial rational cohomology classes in rather high degrees. It is not absurd to assume a close connection between Kontsevich integrals and functor calculus. See [**30**], [**31**] for example. Therefore the articles [**32**], [**33**] by Watanabe could be closely related to this article here.

A. Coefficients in the Hirzebruch $\mathcal{L}$-polynomials

For positive integers $j_1, j_2 \dots, j_r$ let

$$h_{j_1,j_2,\dots,j_r} \in \mathbb{Q}$$

be the coefficient of $p_{j_1}p_{j_2}\cdots p_{j_r}$ in the Hirzebruch polynomial $\mathcal{L}_{j_1+j_2+\cdots+j_r}$. Here we are mostly interested in the cases $r = 1, 2, 3$. — The multiplicativity property of the sequence of the $\mathcal{L}$-polynomials [**28**, §19] implies

$$h_{i,j} + h_{i+j} = h_i h_j \qquad \text{if } i \neq j\ ; \quad 2h_{i,j} + h_{i+j} = h_i h_j \qquad \text{if } i = j.$$

(Here is an uneducated argument which avoids the concept of a multiplicative sequence of polynomials. Apply the Hirzebruch signature formula to M_i , M_j and $M_i \times M_j$ where M_i and M_j are closed smooth oriented manifolds of dimension $4i$ and $4j$, respectively, both having nonzero signature and both almost parallelizable, i.e., having tangent bundles which are trivial over the complement of a point. In the case of $M_i \times M_j$, use the Cartan formula for Pontryagin classes [**28**, Thm.15.3] to make simplifications; the cases $i \neq j$ and $i = j$ require separate treatment. Then use the multiplicativity of the signature.)

Lemma A.1. *For all* $i, j \geq 1$ *we have* $h_{i+j} > h_i h_j$ *, so that* $h_{i,j} < 0$.

Proof. (After Galatius.) We use the formula

$$h_n = 2^{2n}(2^{2n-1} - 1)\frac{B_n}{(2n)!} = \zeta(2n)\frac{2^{2n} - 2}{\pi^{2n}}$$

where $\zeta(s) = \sum_{m=1}^{\infty} m^{-s}$ is Riemann's zeta function. Then we get

$$\pi^{2i+2j}(h_i h_j - h_{i+j}) = \zeta(2i)\zeta(2j)(2^{2i} - 2)(2^{2j} - 2) - \zeta(2i + 2j)(2^{2i+2j} - 2).$$

Setting $\zeta(2i) = 1 + x$ and $\zeta(2j) = 1 + y$ and $\zeta(2i + 2j) = 1 + z$ turns the right-hand side into

$$\begin{array}{rl} & 6 - 2^{2i+1} - 2^{2j+1} \\ + & (2^{2i+2j} - 2^{2i+1} - 2^{2j+1} + 4)(x + y + xy) \\ + & (2 - 2^{2i+2j})z\,. \end{array} \tag{A.2}$$

Since $x, y, z > 0$ this is less than

$$(6 - 2^{2i+1} - 2^{2j+1}) + (2^{2i+2j} + 4)(x + y + xy) + 2z \tag{A.3}$$

and since $x, y, z < 1$ that is less than

$$20 - 2^{2i+1} - 2^{2j+1} + 2^{2i+2j}x + 2^{2i+2j}y + 2^{2i+2j}xy\,. \tag{A.4}$$

Now we use the estimate $\zeta(s) < 1 + 2^{-s} + (5/2)\cdot 3^{-s}$ which is valid for real $s \geq 2$. (Compare $\sum_{n=4}^{\infty} n^{-s}$ with $\int_3^{\infty} t^{-s}dt = 3(3^{-s})/2$.) Therefore

$$2^{2i+2j}x < \big(1 + (5/2)(2/3)^{-2i}\big)2^{2j}, \qquad 2^{2i+2j}y < \big(1 + (5/2)(2/3)^{-2j}\big)2^{2i}$$

and $2^{2i+2j}xy < 7$. It follows that (A.4) is less than

$$27 + \big((5/2)(2/3)^{-2j} - 1\big)2^{2i} + \big((5/2)(2/3)^{-2i} - 1\big)2^{2j}. \tag{A.5}$$

It is easy to see that this quantity is negative when i, j are both > 1 and one of them is ≥ 4. If $i = 1$ but $j \geq 4$, then we return to (A.2) and note that we unwisely added at least $2^{2j+1}x$ in going from there to (A.3), where $x > 1/4$. We are therefore allowed to subtract this from (A.5), which leads to a negative quantity. This takes care of all cases where $i+j \geq 5$. In the remaining cases one can consult tables to see that $h_{i,j}$ is negative. $\square$

The uneducated argument based on the multiplicativity of signatures implies, if $i<j<k$ and $i+j \neq k$, that

$$h_{i,j,k} + h_{i+j,k} + h_{i+k,j} + h_{k+j,i} + h_{i+k+j} = h_i h_j h_k\,.$$

This is also confirmed in [**15**, Expl.A.3]. Therefore using $h_{s,t} = h_s h_t - h_{s+t}$ for $s \neq t$ we obtain $h_{i,j,k} = 2h_{i+k+j} + h_i h_j h_k - h_{i+j}h_k - h_{i+k}h_j - h_{k+j}h_i$. The same formula for $h_{i,j,k}$ can be obtained with the same argument in the cases where $i<j$ and $k=i+j$, but here $2h_{s,s} = h_s h_s - h_{2s}$ should also be used. Using [**15**, Prop.A.1] and [**15**, Expl.A.3] we obtain without any restrictions on i,j,k that

$$h_{i,j,k} = \big(2h_{i+k+j} + h_i h_j h_k - h_{i+j}h_k - h_{i+k}h_j - h_{k+j}h_i\big)/c!$$

where c is 1, 2 or 3 depending on the cardinality of $\{i,j,k\}$.

LEMMA A.6. *For all* $i,j,k \geq 1$ *we have* $h_{i,j,k} \geq 0$.

PROOF. Write $\rho_i := \pi^{2i}h_i = \zeta(2i)(2^{2i}-2)$ etc.; write

$$\begin{array}{ccc} \zeta(2i) = 1+x, & \zeta(2j) = 1+y, & \zeta(2k) = 1+z, \\ \zeta(2j+2k) = 1+yz-u, & \zeta(2i+2k) = 1+xz-v, & \zeta(2i+2j) = 1+xy-w, \\ & \zeta(2i+2j+2k) = 1+xyz-t\,. & \end{array}$$

We assign weight one to x,y,z, weight two to u,v,w and weight three to t. Note that x for example is an infinite sum of terms n^{-2i} where n is an integer greater than 1; similarly u for example is an infinite sum of terms $n_1^{-2j}n_2^{-2k}$ where n_1 and n_2 are *distinct* integers, both greater than 1; and t is an infinite sum of terms $n_1^{-2i}n_2^{-2j}n_3^{-2k}$ where n_1,n_2,n_3 are integers greater than 1, not all identical. Important: $x,y,z,u,v,w,t>0$. We want to show $C>0$ where

$$C := 2\rho_{i+j+k} + \rho_i\rho_j\rho_k - \rho_{i+j}\rho_k - \rho_{i+k}\rho_j - \rho_{j+k}\rho_i\,.$$

With the abbreviations above, C turns into

$$\begin{array}{ll} & 2(1+xyz-t)(2^{2i+2j+2k}-2) \\ + & (1+x)(1+y)(1+z)(2^{2i}-2)(2^{2j}-2)(2^{2k}-2) \\ - & (1+yz-u)(1+x)(2^{2j+2k}-2)(2^{2i}-2) \\ - & (1+xz-v)(1+y)(2^{2i+2k}-2)(2^{2j}-2) \\ - & (1+xy-w)(1+z)(2^{2i+2j}-2)(2^{2k}-2). \end{array}$$

Sorting this by monomials we obtain

$$\begin{array}{l} -24 \\ +(2^{2i}-2)(2^{2j}-2)(2^{2k}-2)x \\ +(2^{2i}-2)(2^{2j}-2)(2^{2k}-2)y \\ +(2^{2i}-2)(2^{2j}-2)(2^{2k}-2)z \\ +\big((2^{2i}-2)(2^{2j}-2)(2^{2k}-2) - (2^{2j+2k}-2)(2^{2i}-2)\big)yz \\ +\big((2^{2i}-2)(2^{2j}-2)(2^{2k}-2) - (2^{2i+2k}-2)(2^{2j}-2)\big)xz \\ +\big((2^{2i}-2)(2^{2j}-2)(2^{2k}-2) - (2^{2i+2j}-2)(2^{2k}-2)\big)xy \\ +(2^{2j+2k}-2)(2^{2i}-2)u \\ +(2^{2i+2k}-2)(2^{2j}-2)v \\ +(2^{2i+2j}-2)(2^{2k}-2)w \\ +\text{contribution from monomials of weight 3.} \end{array} \tag{A.7}$$

The contribution of the monomials of weight 3 can be bounded in absolute value by a constant independent of i, j, k, and so we can (provisionally) neglect it. We can also neglect the -24. The remaining threats are from the coefficients of yz, xz and xy. Expanding the coefficient of yz for example, we get

$$\begin{array}{rl} & \big((2^{2i}-2)(2^{2j}-2)(2^{2k}-2)-(2^{2j+2k}-2)(2^{2i}-2)\big) \\ = & -2(2^{2i+2j}+2^{2i+2k})+6\cdot 2^{2i}+4\cdot 2^{2j}+4\cdot 2^{2k}-12. \end{array} \tag{A.8}$$

Terms not involving i can be neglected, and positive contributions can also be neglected, and so we are left with

$$-\big(2^{2i+2j+1}+2^{2i+2k+1}\big)yz. \tag{A.9}$$

But we have $yz < 7\cdot 2^{-2j-2k}$, so that

$$\big(2^{2i+2j+1}+2^{2i+2k+1}\big)yz \;<\; 7\cdot 2^{2i}.$$

So the combined negative threat is in the worst case $-7(2^{2i}+2^{2j}+2^{2k})$. If at least two of i, j, k are greater than 1, then the contributions in the rows of (A.7) corresponding to monomials x, y and z are greater than $7(2^{2i}+2^{2j}+2^{2k})$. If $j = k = 1$ and $i \geq 8$, say, we argue differently. Then there is a total positive contribution of at least $4\cdot 2^{2i}$ from the rows of (A.7) corresponding to monomials y and z. We reconsider our decision to neglect the term $6\cdot 2^{2i}$ in (A.8). After multiplication with yz this is still worth at least $3/2\cdot 2^{2i}$. Next, u is slightly greater than $1/8$ and so we get a positive contribution of at least $(14/8)\cdot 2^{2i}$ from the row corresponding to monomial u. Then we note that $4+3/2+14/8 > 7$. This seems to take care of all the cases where $i+j+k \geq 10$, but to drown out all the neglected constants, we should assume $i+j+k \geq 14$, say. This means that for the cases where $i+j+k \leq 13$, tables must be consulted to see that $h_{i,j,k} > 0$. □

REMARK A.10. The polynomials $\mathcal{L}_m$ for $m \leq 13$ are written out in [**24**]. Hirzebruch's book [**20**] also contains such a list, but it only goes as far as $m = 5$. Both lists suggest to me that $(-1)^{r-1}h_{j_1,j_2,\dots,j_r} \in \mathbb{Q}$ is always positive. Has this been proved? Carl McTague tells me that a similar sign pattern can be seen in the multiplicative sequence of polynomials associated with the $\widehat{A}$-genus.

B. Detecting elements in rational homotopy groups

Diarmuid Crowley asked me (in March 2016, after a talk which I gave at the Scottish Topology Seminar in Edinburgh) whether the nonzero Pontryagin classes

$$p_{n+k} \in H^{4n+4k}(\mathrm{BTOP}(2n);\mathbb{Q})$$

found in theorem 1.1.2 evaluate nontrivially on $\pi_{4n+4k}(\mathrm{BTOP}(2n))$. The answer is yes. We begin the proof with some general and less general observations. Cohomology is taken with rational coefficients throughout.

(i) Let X be a simply connected based CW-space and $b \in H^s(X)$ where $s \geq 3$. Suppose that $\Omega^2 b \in H^{s-2}(\Omega^2 X)$ is nonzero. (Think of b as a homotopy class of based maps from X to an Eilenberg-MacLane space $K(\mathbb{Q}, s)$. This justifies the notation $\Omega^2 b$ for a homotopy class of based maps from $\Omega^2 X$ to $\Omega^2 K(\mathbb{Q}, s) = K(\mathbb{Q}, s-2)$.) *Then* b *evaluates nontrivially on* $\pi_s(X)$. Proof: if $\Omega^2 b$ is nonzero, then $\Omega b \in H^{s-1}(\Omega X)$ is indecomposable in $H^*(\Omega X)$. Since ΩX is a connected H-space, it is rationally a product of Eilenberg-MacLane spaces (well-known) and it follows that Ωb evaluates nontrivially on $\pi_{s-1}(\Omega X)$.

(ii) Let (X, Y) be a based CW-pair and $b \in H^s(X)$ where $s \geq 4$. Let Z be the homotopy fiber of the inclusion $Y \to X$ and let

$$\varphi(b) \in H^{s-1}(Z) \cong H^s(\mathrm{cone}(Z), Z)$$

be the image of b under the homomorphism in cohomology induced by the canonical based map of pairs from $(\mathrm{cone}(Z), Z)$ to (X, Y). Suppose that Z is simply connected and $\Omega^2\varphi(b) \in H^{s-3}(\Omega^2 Z)$ is nonzero. *Then* b *evaluates nontrivially on* $\pi_s(X, Y)$. Proof: use (i) and $\pi_s(X, Y) \cong \pi_{s-1}(Z)$.

(iii) The class $p_{n+k} \in H^{4n+4k}(\mathrm{BTOP}(2n)) \cong H^{4n+4k}(\mathrm{BSTOP}(2n))$ lifts to

$$H^{4n+4k}(\mathrm{BSTOP}(2n), \mathrm{BSO}(2n)).$$

If a chosen lift evaluates nontrivially on $\pi_{4n+4k}(\mathrm{BSTOP}(2n), \mathrm{BSO}(2n))$, then p_{n+k} itself evaluates nontrivially on $\pi_{4n+4k}(\mathrm{BSTOP}(2n))$. Proof: the inclusion-induced map

$$\pi_{4n+4k}(\mathrm{BSTOP}(2n)) \to \pi_{4n+4k}(\mathrm{BSTOP}(2n), \mathrm{BSO}(2n))$$

is a rational isomorphism.

(iv) Let V be a topological manifold with boundary. Then the space of collars on ∂V is contractible. (This space can be defined as the geometric realization of a simplicial set where a k-simplex is an embedding $\Delta^k \times \partial V \times [0, 1) \to \Delta^k \times V$ over Δ^k, etc.)

Sketch proof of (iv): By Brown [**6**] and Connelly [**11**] the space of collars on ∂V is nonempty. This makes it easy to reduce to the situation where $V \cong \partial V \times [0, 1)$. In that case the space of collars has a preferred base point and it is easy to produce a contraction of the space of collars to that base point.

Next, we use the manifold formulation (theorem 1.1.3) with fiber bundle projection $g: E \to M$. By construction of this fiber bundle,

(v) the map g has a section $s: M \to E$ such that the restricted projection

$$E \smallsetminus s(M) \longrightarrow M$$

has the structure of a smooth fiber bundle (with noncompact fibers);

(vi) there exists an embedding $u: \mathbb{R}^{2n} \times M \to E$ over M such that $s = uj$ where $j: M \to \mathbb{R}^{2n} \times M$ is the standard inclusion.

Sketch proof of (vi): begin with an embedding $v: D^{2n} \times M \to E$ over M such that $s(x) = v(z, x)$ for all $x \in M$, with a fixed $z \in \partial D^{2n}$. Use (iv) to choose a fiberwise collar on $v(\partial D^{2n} \times M)$ in $E \smallsetminus v((D^{2n} \smallsetminus \partial D^{2n}) \times M)$, total space of a fiber bundle on M. Use that fiberwise collar to extend v to an embedding $u_1: \mathbb{R}^{2n} \times M \to E$ over M. Define $u(y, x) = u_1(y + z, x)$.

(vii) There is a commutative square

$$\begin{tikzcd}
H^*(\mathrm{BSTOP}(2n)) \arrow[r] & H^*(E) \\
H^*(\mathrm{BSTOP}(2n), \mathrm{BSO}(2n)) \arrow[r] \arrow[u] \arrow[d, "\Omega^{2n-1}\circ\varphi"] & H^*(E, E \smallsetminus s(M)) \arrow[u] \\
H^{*-2n}(\Omega^{2n-1}(\mathrm{STOP}(2n)/\mathrm{SO}(2n))) \arrow[r] & H^{*-2n}(M) \arrow[u, "\cong"']
\end{tikzcd}$$

The upper and middle horizontal arrows are induced by the classifying map for the vertical tangent bundle of E. The isomorphism in the lower right-hand side uses excision to replace E by the image of $u\colon \mathbb{R}^{2n}\times M \to E$.

Now choose $b \in H^{4n+4k}(\mathrm{BSTOP}(2n), \mathrm{BSO}(2n))$ which lifts the Pontryagin class

$$p_{n+k} \in H^{4n+4k}(\mathrm{BSTOP}(2n)).$$

Then b has a nonzero image in $H^{2n+4k}(\Omega^{2n-1}(\mathrm{STOP}(2n)/\mathrm{SO}(2n)))$ since it has a nonzero image in $H^{4n+4k}(E)$. Using (ii), we deduce that b evaluates nontrivially on the relative homotopy group $\pi_{4n+4k}(\mathrm{BSTOP}(2n), \mathrm{BSO}(2n))$. Using (iii), we deduce that p_{n+k} evaluates nontrivially on $\pi_{4n+4k}(\mathrm{BSTOP}(2n))$. □

Example B.1. This is an example of a simply connected based space X such that the canonical graded homomorphism from the rational cohomology $H^*(X)$ modulo decomposables to $\hom(\pi_*(X), \mathbb{Q})$ is not injective. (I learned this from Diarmuid Crowley.) Choose a non-torsion element in $\pi_4(S^2 \vee S^2)$ and represent it by a based map $f\colon S^4 \to S^2 \vee S^2$. Such an element exists by Hilton's theorem [**19**] or the Hilton-Milnor theorem. Let X be the mapping cone of f. Then $H^5(X) \cong \mathbb{Q}$ and any nonzero element of that group is indecomposable. The inclusion $S^2 \vee S^2 \to X$ induces a homomorphism in H^5 which is zero, and a homomorphism in π_5 which is surjective by the long exact sequence

$$\begin{tikzcd} \cdots \arrow[r] & \pi_5(S^2 \vee S^2) \arrow[r] & \pi_5(X) \arrow[r] & \pi_5(X, S^2 \vee S^2) \arrow[r, "1\mapsto[f]"] \arrow[d, equal] & \pi_4(S^2 \vee S^2) \arrow[r] & \cdots \\ & & & \mathbb{Z} & & \end{tikzcd}$$

Therefore every element of $H^5(X)$ evaluates trivially on $\pi_5(X)$.

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Math. Institut, WWU Münster, 48149 Münster, Einsteinstrasse 62, Germany

E-mail address: `m.weiss@uni-muenster.de`