On Ranges of Variants of the Divisor Functions that are Dense

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Abstract

For a real number t, let s_t be the multiplicative arithmetic function defined by $s_t(p^\alpha) = \sum_{j=0}^{\alpha}(-p^t)^j$ for all primes p and positive integers α. We show that the range of a function s_{-r} is dense in the interval $(0,1]$ whenever $r \in (0,1]$. We then find a constant $\eta_A \approx 1.9011618$ and show that if $r > 1$, then the range of the function s_{-r} is a dense subset of the interval $\left(\frac{1}{\zeta(r)}, 1\right]$ if and only if $r \leq \eta_A$. We end with an open problem.

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1 Introduction

Let $\mathbb{N}$ denote the set of positive integers. We will let p_i be the i^{th} prime number, and we will use ζ to denote the Riemann zeta function.

Consider a multiplicative arithmetic function s_1 defined by

$$s_1(p^\alpha) = \sum_{j=0}^{\alpha}(-p)^j \tag{1.1}$$

for all primes p and positive integers α. This function, which appears as sequence A061020 in Sloane's Online Encyclopedia of Integer Sequences [2], serves as an interesting variant of the well-known sum-of-divisors function σ. We may generalize the function s_1 to a class of functions s_t in the following very natural fashion.

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Definition 1.1. For any real number t, let s_t be the multiplicative arithmetic function defined by

$$s_t(p^\alpha) = \sum_{j=0}^{\alpha} (-p^t)^j. \tag{1.2}$$

for all primes p and positive integers α.

In this paper, we will concentrate on functions s_{-r} for $r > 0$, so we will always use r to denote a positive real number. Notice that, for any prime p and nonnegative integer α, we have $1 - p^{-r} \leq s_{-r}(p^\alpha) \leq 1$ because $\sum_{j=0}^{\alpha} (-p^{-r})^j$ is an alternating series whose terms have strictly decreasing absolute values. Therefore, if $r > 1$ and N is a positive integer with canonical prime factorization $N = \prod_{j=1}^{v} q_j^{\beta_j}$, then we have

$$s_{-r}(N) = \prod_{j=1}^{v} s_{-r}(q_j^{\beta_j}) \geq \prod_{j=1}^{v} (1 - q_j^{-r}) > \prod_{j=1}^{\infty} (1 - p_j^{-r}) = \frac{1}{\zeta(r)}. \tag{1.3}$$

Hence, for $r > 1$, the range of s_{-r} is a subset of the interval $((\zeta(r))^{-1}, 1]$. We will soon show that, for $r \in (0, 1]$, the range of s_{-r} is a dense subset of $(0, 1]$. However, we will find that the range of s_{-2} is not dense in $((\zeta(2))^{-1}, 1]$. Our goal is to find a constant, which we will call η_A, such that if $r > 1$, then the range of s_{-r} is dense in $((\zeta(r))^{-1}, 1]$ if and only if $r \leq \eta_A$.

2 Finding η_A

For the sake of convenience, we introduce a class of functions L_{-r}, which we define, for each $r > 0$, by $L_{-r}(n) = -\log(s_{-r}(n))$ for all $n \in \mathbb{N}$. Note that the functions L_{-r} take nonnegative values. Furthermore, for any prime p, we see that $(L_{-r}(p^{2\alpha+1}))_{\alpha=0}^{\infty}$ forms a decreasing sequence, $(L_{-r}(p^{2\alpha}))_{\alpha=0}^{\infty}$ forms an increasing sequence, and $\lim_{\alpha\to\infty} L_{-r}(p^\alpha)$ exists (because $\lim_{\alpha\to\infty} s_{-r}(p^\alpha)$ exists by the Alternating Series test). This motivates us to define an ordering $\succ$ on the nonnegative integers as follows. If k_1 and k_2 are odd positive integers with $k_1 < k_2$, then $k_1 \succ k_2$. If k_1 and k_2 are even nonnegative integers with $k_1 < k_2$, then $k_2 \succ k_1$. If k_1 is an odd positive integer and k_2 is an even nonnegative integer, then $k_1 \succ k_2$. This ordering has the property that if $r > 0$ and p is a prime, then, for any distinct nonnegative integers k_1 and k_2, $L_{-r}(p^{k_1}) > L_{-r}(p^{k_2})$ if and only if $k_1 \succ k_2$. We are now equipped to prove the following theorem.

Theorem 2.1. *If $r \in (0, 1]$, then the range of s_{-r} is a dense subset of $(0, 1]$.*

Proof. We first observe that the range of s_{-r} is dense in $(0,1]$ if and only if the range of L_{-r} is dense in $[0,\infty)$. To show that the range of L_{-r} is dense in $[0,\infty)$, we consider the subsums of the series $\sum_{i=1}^{\infty} L_{-r}(p_i)$. We see that any finite subsum of this series, say $\sum_{j=1}^{v} L_{-r}(q_j)$, is within the range of L_{-r} because

$$\sum_{j=1}^{v} L_{-r}(q_j) = -\log\left(\prod_{j=1}^{v} s_{-r}(q_j)\right) = L_{-r}\left(\prod_{j=1}^{v} q_j\right). \tag{2.1}$$

Hence, it suffices to show that $\sum_{i=1}^{\infty} L_{-r}(p_i)$ is a divergent series whose terms tend to 0. First, $\lim_{i\to\infty} L_{-r}(p_i) = \lim_{i\to\infty}(-\log(1-p_i^{-r})) = 0$. Second, we know that $\sum_{i=1}^{\infty} L_{-r}(p_i)$ diverges because, for $r \in (0,1]$, we have $\prod_{i=1}^{\infty}(1-p_i^{-r}) = 0$. ☐

Henceforth, we will focus on values of r that are greater than 1. We seek to establish a necessary and sufficient condition for the range of a function s_{-r} to be dense in $((\zeta(r))^{-1}, 1]$. First, however, we need two lemmata.

Lemma 2.1. *If $r > 1$, $m \in \mathbb{N}$, and $w \in \{1, 2, \ldots, m\}$, then*

$$1 - p_w^{-r} + p_w^{-2r} \le 1 - p_m^{-r} + p_m^{-2r}. \tag{2.2}$$

Proof. Fix some $r > 1$. Define $h\colon \mathbb{R} \to \mathbb{R}$ by $h(x) = 1 - x^{-r} + x^{-2r}$. Then $h'(x) = rx^{-r-1}(1-2x^{-r})$. If $x \ge 2$, then $h'(x) > 0$. As $2 \le p_w \le p_m$, the result follows. ☐

Lemma 2.2. *Let $r > 1$ be a real number, and let p be a prime. For any positive integer k, we have $|L_{-r}(p^{k+2}) - L_{-r}(p^k)| < L_{-r}(p^2)$.*

Proof. For simplicity, we will write $y = p^{-r}$. First, suppose k is odd. Then, because $(L_{-r}(p^{2\alpha+1}))_{\alpha=0}^{\infty}$ is a decreasing sequence, we have

$$|L_{-r}(p^{k+2}) - L_{-r}(p^k)| = L_{-r}(p^k) - L_{-r}(p^{k+2})$$

$$= \log\left(\frac{1}{\sum_{j=0}^{k}(-y)^j}\right) - \log\left(\frac{1}{\sum_{j=0}^{k+2}(-y)^j}\right)$$

$$= \log\left(\frac{1+y}{1-y^{k+1}}\right) - \log\left(\frac{1+y}{1-y^{k+3}}\right) = \log\left(\frac{1-y^{k+3}}{1-y^{k+1}}\right). \tag{2.3}$$

Because $L_{-r}(p^2) = \log\left(\frac{1}{1-y+y^2}\right)$, we see that we simply need to show that $\frac{1-y^{k+3}}{1-y^{k+1}} < \frac{1}{1-y+y^2}$.

Noting that $0 < y < \frac{1}{2}$, we have $y^k < y$ and $y^{k+3} < y^{k+2}$. Therefore, $y + y^k + y^{k+3} < 2y + y^{k+2} + y^{k+4} < 1 + y^{k+2} + y^{k+4}$, so we have $y^2 + y^{k+1} + y^{k+4} < y + y^{k+3} + y^{k+5}$. After adding 1 to each side and rearranging terms, we get $1 - y + y^2 - y^{k+3} + y^{k+4} - y^{k+5} < 1 - y^{k+1}$, which we may write as $(1-y+y^2)(1-y^{k+3}) < 1 - y^{k+1}$. Hence, $\frac{1-y^{k+3}}{1-y^{k+1}} < \frac{1}{1-y+y^2}$, so we have completed the proof for the case in which k is odd.

Now, suppose that k is even. Then, because $(L_{-r}(p^{2\alpha}))_{\alpha=0}^{\infty}$ is an increasing sequence, we have

$$|L_{-r}(p^{k+2}) - L_{-r}(p^k)| = L_{-r}(p^{k+2}) - L_{-r}(p^k)$$

$$= \log\left(\frac{1}{\sum_{j=0}^{k+2}(-y)^j}\right) - \log\left(\frac{1}{\sum_{j=0}^{k}(-y)^j}\right)$$

$$= \log\left(\frac{1+y}{1+y^{k+3}}\right) - \log\left(\frac{1+y}{1+y^{k+1}}\right) = \log\left(\frac{1+y^{k+1}}{1+y^{k+3}}\right). \tag{2.4}$$

Again, we have $L_{-r}(p^2) = \log\left(\frac{1}{1-y+y^2}\right)$, so it suffices to show that $\frac{1+y^{k+1}}{1+y^{k+3}} < \frac{1}{1-y+y^2}$. Because $0 < y < \frac{1}{2}$, we have $1 - y^{2(k+1)} < 1 - y^{2(k+3)}$. Therefore, $\frac{1+y^{k+1}}{1+y^{k+3}} < \frac{1-y^{k+3}}{1-y^{k+1}}$, and we have already shown that $\frac{1-y^{k+3}}{1-y^{k+1}} < \frac{1}{1-y+y^2}$. □

Theorem 2.2. *If $r > 1$, then the range of s_{-r} is dense in the interval $((\zeta(r))^{-1}, 1]$ if and only if $s_{-r}(p_m^2) \geq \prod_{i=m+1}^{\infty} s_{-r}(p_i)$ for all positive integers m.*

Proof. First, suppose there exists some positive integer m such that $s_{-r}(p_m^2) < \prod_{i=m+1}^{\infty} s_{-r}(p_i)$. Let N be an arbitrary positive integer with canonical prime factorization $N = \prod_{j=1}^{v} q_j^{\beta_j}$. If $p_w | N$ for some $w \in \{1, 2, \ldots, m\}$, then $s_{-r}(N) \leq 1 - p_w^{-r} + p_w^{-2r}$. By Lemma 2.1, we see that $s_{-r}(N) \leq 1 - p_m^{-r} + p_m^{-2r} =$

$s_{-r}(p_m^2)$. On the other hand, if $p_w \nmid N$ for all $w \in \{1, 2, \ldots, m\}$, then

$$s_{-r}(N) = s_{-r}\left(\prod_{j=1}^{v} q_j^{\beta_j}\right) = \prod_{j=1}^{v} s_{-r}(q_j^{\beta_j}) \geq \prod_{j=1}^{v} s_{-r}(q_j) > \prod_{i=m+1}^{\infty} s_{-r}(p_i). \quad (2.5)$$

This shows that there is no element of the range of s_{-r} in the interval $\left(s_{-r}(p_m^2), \prod_{i=m+1}^{\infty} s_{-r}(p_i)\right)$, so the range of s_{-r} is not dense in $((\zeta(r))^{-1}, 1]$.

To prove the converse, let us suppose that $s_{-r}(p_m^2) \geq \prod_{i=m+1}^{\infty} s_{-r}(p_i)$ for all positive integers m. We will show that the range of L_{-r} is dense in $[0, \log(\zeta(r)))$, which will prove that the range of s_{-r} is dense in $((\zeta(r))^{-1}, 1]$. Choose some arbitrary $x \in (0, \log(\zeta(r)))$. We will construct a sequence $(C_n)_{n=1}^{\infty}$ of elements of the range of L_{-r} such that $\lim_{n\to\infty} C_n = x$. First, define $C_0 = 0$. Now, recall the ordering $\succ$ that we defined at the beginning of this section. We will say that a nonnegative integer k_1 is larger than a nonnegative integer k_2 with respect to the ordering $\succ$ if and only if $k_1 \succ k_2$. Let n be a positive integer. We will ensure by construction that $C_{n-1} \leq x$. If $C_{n-1} + \lim_{k\to\infty} L_{-r}(p_n^k) = x$, then we will define $\alpha_n = -1$. If $C_{n-1} + \lim_{k\to\infty} L_{-r}(p_n^k) \neq x$, then we will define α_n to be the nonnegative integer satisfying $C_{n-1} + L_{-r}(p_n^{\alpha_n}) \leq x$ that is largest with respect to the ordering $\succ$. In this case, we define $C_n = C_{n-1} + L_{-r}(p_n^{\alpha_n})$. For now, let us assume that x is such that $C_{n-1} + \lim_{k\to\infty} L_{-r}(p_n^k) \neq x$ for all positive integers n. In other words, $\alpha_n \geq 0$ and C_n is defined for all positive integers n.

We first show that C_n is in the range of L_{-r} for all positive integers n. Indeed, we have

$$C_n = \sum_{i=1}^{n} L_{-r}(p_i^{\alpha_i}) = L_{-r}\left(\prod_{i=1}^{n} p_i^{\alpha_i}\right). \quad (2.6)$$

Now, we defined $(C_n)_{n=1}^{\infty}$ to be a monotonic sequence with the property that $C_n \leq x$ for all $n \in \mathbb{N}$, so we may write $\lim_{n\to\infty} C_n = \gamma \leq x$. Suppose, for the sake of finding a contradiction, that $\gamma < x$. For each $n \in \mathbb{N}$, we will let $D_n = L_{-r}(p_n) - L_{-r}(p_n^{\alpha_n})$ and $E_n = \sum_{i=1}^{n} D_i$. Then $C_n + E_n = \sum_{i=1}^{n} L_{-r}(p_n)$, so $\lim_{n\to\infty}(C_n + E_n) = \lim_{n\to\infty}\left(-\log\left(\prod_{i=1}^{n} s_{-r}(p_i)\right)\right) = \log(\zeta(r))$. Therefore, $\lim_{n\to\infty} E_n = \log(\zeta(r)) - \gamma > \log(\zeta(r)) - x$, so we may let m be the smallest positive integer such that $E_m > \log(\zeta(r)) - x$. If $\alpha_m = 1$ and $m > 1$, then $D_m = 0$, implying that $E_{m-1} = E_m > \log(\zeta(r)) - x$, which contradicts the minimality of m. On the other hand, if $\alpha_m = 1$ and $m = 1$, then $E_m = 0 > \log(\zeta(r)) - x$, which is also a contradiction. Hence, $\alpha_m \neq 1$. If α_m is odd, then we will let

$A_m = L_{-r}(p_m^{\alpha_m-2}) - L_{-r}(p_m^{\alpha_m})$. In this case, we see, by the definitions of C_m and α_m and the fact that $\alpha_{m-2} \succ \alpha_m$, that $A_m + C_m > x$. If, on the other hand, α_m is even, then we may write $A_m = L_{-r}(p_m^{\alpha_m+2}) - L_{-r}(p_m^{\alpha_m})$. Again, by the definitions of C_m and α_m and the fact that $\alpha_{m+2} \succ \alpha_m$, we have $A_m + C_m > x$. No matter the parity of α_m, we have $x - C_m < A_m$. Using Lemma 2.2, we see that $A_m \le L_{-r}(p_m^2)$, so $x - C_m < L_{-r}(p_m^2) = -\log(s_{-r}(p_m^2))$. As we originally assumed that $s_{-r}(p_m^2) \ge \prod_{i=m+1}^{\infty} s_{-r}(p_i)$, we have

$$x - C_m < -\log(s_{-r}(p_m^2)) \le -\log\left(\prod_{i=m+1}^{\infty} s_{-r}(p_i)\right)$$
$$= \log(\zeta(r)) - (C_m + E_m). \tag{2.7}$$

This implies that $E_m < \log(\zeta(r)) - x$, which is our desired contradiction. This completes the proof of the case in which $\alpha_n \ge 0$ for all $n \in \mathbb{N}$.

Finally, let us assume that there is some positive integer n such that $C_{n-1} + \lim_{k\to\infty} L_{-r}(p_n^k) = x$. In this case, simply let $C_{n-1+j} = C_{n-1} + L_{-r}(p_n^j)$ for all positive integers j. Then, as before, we see that C_{n-1+j} is always in the range of L_{-r}. Furthermore, $\lim_{j\to\infty} C_{n-1+j} = C_{n-1} + \lim_{j\to\infty} L_{-r}(p_n^j) = x$. This completes the proof. □

We now have a way to test whether or not the range of s_{-r} is dense in $((\zeta(r))^{-1}, 1]$ for a given $r > 1$. However, after a short lemma, we will be able to simplify the problem even further.

Lemma 2.3. *If $j \in \mathbb{N}\backslash\{1, 2, 4\}$, then $\frac{p_{j+1}}{p_j} < \sqrt{2}$.*

Proof. A simple manipulation of the corollary to Theorem 3 in [1] shows that $\frac{p_{j+1}}{p_j} < \frac{(j+1)(\log(j+1) + \log\log(j+1))}{j\log j}$ for all integers $j \ge 6$. It is easy to verify that $\frac{(j+1)(\log(j+1) + \log\log(j+1))}{j\log j} < \sqrt{2}$ for all $j \ge 32$. Therefore, the desired result holds for $j \ge 32$. A quick search through the values of $\frac{p_{j+1}}{p_j}$ for $j < 32$ yields the desired result. □

Theorem 2.3. *If $1 < r \le 2$, then the range of s_{-r} is dense in the interval $((\zeta(r))^{-1}, 1]$ if and only if $s_{-r}(p_m^2) \ge \prod_{i=m+1}^{\infty} s_{-r}(p_i)$ for all $m \in \{1, 2, 4\}$.*

Proof. Let us define a function F by $F(m,r) = s_{-r}(p_m^2)\prod_{i=1}^{m} s_{-r}(p_i)$ so that the inequality $s_{-r}(p_m^2) \geq \prod_{i=m+1}^{\infty} s_{-r}(p_i)$ is equivalent to $F(m,r) \geq (\zeta(r))^{-1}$. Due to the validity of Theorem 2.2, we see that, in order to prove the result, it suffices to show that if $F(m,r) \geq (\zeta(r))^{-1}$ for all $m \in \{1,2,4\}$, then $F(m,r) \geq (\zeta(r))^{-1}$ for all $m \in \mathbb{N}$. Therefore, let us assume that $r \in (1,2]$ is such that $F(m,r) \geq (\zeta(r))^{-1}$ for all $m \in \{1,2,4\}$.

If $m \in \mathbb{N}\backslash\{1,2,4\}$, then Lemma 2.3 tells us that $p_{m+1} < \sqrt{2}p_m \leq \sqrt[r]{2}p_m$, which means that we may write $2p_{m+1}^{-r} > p_m^{-r}$. As $p_m^{-r} - 1$ is negative, we have $2p_{m+1}^{-r}(p_m^{-r} - 1) < p_m^{-r}(p_m^{-r} - 1)$, so we may write $-2p_{m+1}^{-r} + 2p_{m+1}^{-2r} = 2p_{m+1}^{-r}(p_{m+1}^{-r} - 1) < 2p_{m+1}^{-r}(p_m^{-r} - 1) < p_m^{-r}(p_m^{-r} - 1) = -p_m^{-r} + p_m^{-2r}$. Therefore,

$$F(m+1,r) = s_{-r}(p_{m+1}^2)s_{-r}(p_{m+1})\prod_{i=1}^{m} s_{-r}(p_i)$$

$$= (1 - p_{m+1}^{-r} + p_{m+1}^{-2r})(1 - p_{m+1}^{-r})\prod_{i=1}^{m} s_{-r}(p_i)$$

$$= (1 - 2p_{m+1}^{-r} + 2p_{m+1}^{-2r} - p_{m+1}^{-3r})\prod_{i=1}^{m} s_{-r}(p_i) < (1 - 2p_{m+1}^{-r} + 2p_{m+1}^{-2r})\prod_{i=1}^{m} s_{-r}(p_i)$$

$$< (1 - p_m^{-r} + p_m^{-2r})\prod_{i=1}^{m} s_{-r}(p_i) = s_{-r}(p_m^2)\prod_{i=1}^{m} s_{-r}(p_i) = F(m,r). \tag{2.8}$$

Thus, if $m \in \mathbb{N}\backslash\{1,2,4\}$, then $F(m+1,r) < F(m,r)$. This means that $F(3,r) > F(4,r) \geq (\zeta(r))^{-1}$. Furthermore, $F(m,r) > (\zeta(r))^{-1}$ for all integers $m \geq 5$ because $(F(m,r))_{m=5}^{\infty}$ is a decreasing sequence and $\lim_{m\to\infty} F(m,r) = (\zeta(r))^{-1}$. □

Using Mathematica 9.0, we may plot the graphs of $(\zeta(r))^{-1}$, $F(1,r)$, $F(2,r)$, and $F(4,r)$. Doing so, we find that the graphs of $F(2,r)$ and $(\zeta(r))^{-1}$ intersect at a point $r_0 \approx 1.9011618$. Furthermore, we see that if $r \in (1,r_0]$, then $F(m,r) \geq (\zeta(r))^{-1}$ for all $m \in \{1,2,4\}$. Therefore, if $r \in (1,r_0]$, then Theorem 2.3 tells us that the range of s_{-r} is dense in the interval $((\zeta(r))^{-1},1]$. One may also verify that $F(2,r) < (\zeta(r))^{-1}$ for all $r \in (r_0,3.2)$, so the range of s_{-r} is not dense in $((\zeta(r))^{-1},1]$ whenever $r \in (r_0,3.2)$. This leads us to our final theorem.

Theorem 2.4. *Let η_A be the unique number in the interval $(1,2)$ that satisfies the equation*

$$(1 - 2^{-\eta_A})(1 - 3^{-\eta_A})(1 - 3^{-\eta_A} + 3^{-2\eta_A}) = \frac{1}{\zeta(\eta_A)}. \tag{2.9}$$

If $r > 1$, then the range of the function s_{-r} is dense in the interval $\left(\frac{1}{\zeta(r)}, 1\right]$ *if and only if $r \leq \eta_A$.*

Proof. It is easy to see that the number η_A is simply the number r_0 discussed in the preceding paragraph. Therefore, in order to prove the theorem, it suffices (in virtue of the preceding paragraph) to show that $F(1,r) < \frac{1}{\zeta(r)}$ for all $r \geq 3.2$. For $r \geq 3.2$, we have $2^{1-2r} + \frac{2}{r-1} < 1$, so

$$2^{1-2r} + \frac{2}{r-1} - \frac{2^{2-r}}{r-1} + \frac{2^{2-2r}}{r-1} - 1 + 2^r < 2^r. \tag{2.10}$$

We may rearrange the left-hand-side of (2.10) to get

$$\left(1 + 2^r + \frac{2}{r-1}\right)\left(1 - 2^{1-r} + 2^{1-2r}\right) < 2^r, \tag{2.11}$$

from which we obtain

$$\left(1 + 2^r + \frac{2}{r-1}\right)\left(1 - 2^{1-r} + 2^{1-2r} - 2^{-3r}\right) < 2^r. \tag{2.12}$$

Therefore, we have

$$F(1,r) = (1-2^{-r})(1-2^{-r}+2^{-2r}) = 1 - 2^{1-r} + 2^{1-2r} - 2^{-3r} < \frac{2^r}{1 + 2^r + \frac{2}{r-1}}$$

$$= \left(1 + \frac{1}{2^r} + \frac{1}{2^{r-1}(r-1)}\right)^{-1} = \left(1 + \frac{1}{2^r} + \int_2^\infty \frac{1}{x^r} dx\right)^{-1} < \frac{1}{\zeta(r)}. \tag{2.13}$$

□

3 An Open Problem

In this paper, we have found necessary and sufficient conditions for the range of a function s_{-r} to be dense in $((\zeta(r))^{-1}, 1]$ (for $r > 1$). In other words, we know exactly when the closure of the range of a function s_{-r} will be the interval $[(\zeta(r))^{-1}, 1]$. This point of view prompts the following more general question. If we are given a positive integer L, then what are the values of $r > 1$ such that the closure of the range of the function s_{-r} is a disjoint union of exactly L subintervals of $[(\zeta(r))^{-1}, 1]$?

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