Some Problems Arising from Partition Poset Homology

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Dedicated to Richard Stanley on the occasion of his 70th birthday.

Abstract. We describe some open problems related to homology representations of subposets of the partition lattice, beginning with questions first raised in Stanley's work on group actions on posets.

1. Introduction

This paper is about a part of Richard Stanley's work that has exerted a tremendous influence on my mathematical career. I am privileged to have had Richard as my thesis adviser. I recall that during the first few months of our interaction I was in a somewhat catatonic state of awe; this was not exactly conducive to producing new mathematics. All that changed when Richard taught 18.318 (M.I.T.'s yearly "Topics in Combinatorics" course for graduate students) in the Spring of 1984. I forget the exact subtitle of the course, but the emphasis was on symmetric functions and representation theory.

Richard's lectures were always a model of clarity and exposition. He had a knack for making very difficult results seem obvious and effortless (until I tried to reconstruct the arguments myself). From Richard's 18.318 I learnt about Schur-Weyl duality, and what the mysterious plethysm operation of symmetric functions meant in concrete representation-theoretic terms, for both the symmetric group and the general linear group. I still remember the sense of excitement I felt upon finally gaining a useful understanding of these deep ideas. I had already taken an earlier 18.318 offered by Phil Hanlon, on the character theory of the symmetric group, and I had read the first chapter of Ian Macdonald's *Symmetric Functions and Hall Polynomials*. Richard's course, with his inimitable style, dry wit and understated humour, tied everything together for me, and gave me an appreciation of the endless possibilities, and the elegance, of the symmetric functions technique.

Much of the material in that course has now been handed down to future generations in *Enumerative Combinatorics, Volume 2* (EC 2). In addition to the lectures, the other feature that made Richard's courses invaluable was his problem sets (also handed down to posterity in both EC1 and EC2). I remember working on them for

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hours on end, late into the night. They were truly addictive, although I was not successful in solving many of the problems. Some are still open. Today, after many years of my own teaching, I marvel at Richard's patience in reading all those papers (we were about ten in the class, I think). Many of my attempts resulted in only partial solutions, and yet he clearly read them all, and they were returned interspersed with comments here and there indicating a better direction to pursue. It was always very exciting and tremendously encouraging to see a "good" at the end of a solution, a couple of times "very elegant proof," and once even "I couldn't do this one myself." (For this one I have always suspected he was being a little too generous.) In addition to Richard's remarkable patience and meticulousness in grading these problem sets, something I did not know at the time was that he also recorded what he deemed to be the best solutions, and generously and unfailingly credited his students for them in both EC 1 and EC 2.

I believe it was here that I first understood how to dissect virtual modules via the Frobenius characteristic (something which turned out to be very useful in predicting *topological* properties of posets). Some years later, when I realised how inextricably the plethysm operation is linked to the partition lattice, I would be very happy to be able to use what I had learnt in Richard's class, in my own research. That particular 18.318 of Richard's is, hands down, for me the most influential course I have ever taken.

Likewise, the most influential research paper that I read as a graduate student is Richard's *Some aspects of group actions on posets* [**23**]. It was at about the same time as Richard's course, but curiously, I would realise its profound impact only when I came back to it in a few years. That paper introduced me to poset homology. The order complex of a partially ordered set provides a beautifully concrete way to illustrate many of the big theorems in algebraic topology, and Richard's paper highlights the fascinating interplay between combinatorics, algebraic topology and representation theory. Although my thesis was on a different subject, the large majority of my later work was on topics related to the partition lattice, rank-selected and other subposets, and homology representations.

In this paper I will focus on questions raised in one section of Richard's *Some aspects of group actions on posets*, Section 7. This part alone immediately attracted a considerable amount of attention ([**10**], [**12**], [**13**]) and eventually spawned a vast body of research connecting the partition lattice to the free Lie algebra, subspace arrangements, configuration spaces, Lie operads, complexes of graphs and various generalisations. The comprehensive survey paper of Wachs [**36**] is an excellent reference for the literature and for more recent developments. Although extensively studied, partition posets continue to be a source of interesting open problems, often with tantalising ramifications. The partition lattice and its subposets also serve as a guiding first example of many curious phenomena, both topological and representation-theoretic, that have since been generalised and rediscovered in many other contexts in the last decade. Richard foreshadows many of these developments in his paper with his characteristic uncanny insight.

2. Partition lattice homology and the trivial representation

Recall that Π_n is the lattice of set partitions of an n-element set, ordered by refinement. We say a block of a partition is nontrivial if it consists of more than one element (for this and other basic definitions see [**24**], EC1). For a bounded poset P (i.e., one with a unique minimal element $\hat{0}$ and a unique maximal element $\hat{1}$) we denote by $\hat{P}$ the *proper part* of P, i.e., the poset P with the greatest element $\hat{1}$ and the least element $\hat{0}$ removed. We write $\Delta(P)$ for the order complex of P; the simplices of $\Delta(P)$ are the chains of $\hat{P}$. By the ith (reduced) homology $\tilde{H}_i(P)$ of P we mean the ith (reduced) simplicial homology of its order complex $\Delta(P)$. All homology in this paper is reduced homology taken with integer coefficients except for representation-theoretic discussions, in which case we take coefficients over the complex field. All posets are bounded unless explicitly stated otherwise, although we will always consider the proper part of the poset so as to avoid topologically trivial situations.

Stanley's paper [**23**] considers a finite group G acting in an order-preserving fashion on a poset P, and the resulting action of G on the unique novanishing homology of P in the special case when P is Cohen-Macaulay, that is, if the reduced homology of the order complex of the proper part of every interval $[x, y]$ of P, $\hat{0} \le x \le y \le \hat{1}$, vanishes below the top dimension. (See [**24**] for definitions.) If P is a ranked poset, then the rank-selected subposet P_S of P consists of all elements with rank belonging to the subset S. Stanley [**22**], and independently Baclawski [**1**], had shown that rank-selection preserves the Cohen-Macaulay property.

We will confine our discussion to the case of rank-selection in the partition lattice Π_n. Here the nontrivial ranks are $1, 2, \ldots, n-2$. It will be convenient to write $[a, b]$ for the interval of consecutive ranks $\{a, a+1, \ldots, b\}$. The automorphism group of the lattice is the symmetric group S_n, and Stanley observed that the rank-selected homology modules refine the permutation representation of S_n on the maximal chains of Π_n according to the subsets S of the ranks, i.e., of $\{1, 2, \ldots, n-2\}$. That is, if α_n denotes the permutation representation of S_n on the maximal chains of Π_n and $\beta_S(n)$ denotes the representation on the homology of the rank-selected subposet $\Pi_n(S)$, then

(1) $$\alpha_n = \sum_{S \subseteq [1, n-2]} \beta_S(n).$$

More generally, (see [**23**], Theorem 1.1), if $\alpha_S(n)$ denotes the permutation representation of S_n on the maximal chains of $\Pi_n(S)$, one has

(2) $$\alpha_S(n) = \sum_{T \subseteq S} \beta_T(n),$$

and hence, by an application of the Hopf-Lefschetz formula,

(3) $$\beta_S(n) = \sum_{T \subseteq S} (-1)^{|S-T|} \alpha_T(n)$$

Stanley asked for a characterisation of the homology representations $\beta_S(n)$. He also gave a complete description of $\beta_S(n)$ in the case when S is the full set of ranks $[1, n-2]$, i.e, of the top homology module of Π_n. Let sgn denote the sign representation of S_n, and let ψ_n be the S_n-representation obtained by inducing any faithful irreducible representation of a cyclic subgroup of order n. Write β_n for $\beta_{[1,n-2]}(n)$, i.e., the top homology representation of Π_n. Then, using a crucial Möbius function computation of Hanlon ([**12**]), Stanley showed that

Theorem 2.1. *([**23**], Theorem 7.3 and Corollary 7.5)*

$$\beta_n = (\text{sgn})\ \psi_n.$$

The restriction of β_n to S_{n-1} is the regular representation.

As mentioned in the introduction, this observation would eventually generate a large literature on the partition lattice (see bibliography).

We note that the rank-selection question was settled completely by Stanley in [**23**] for the Boolean lattice of subsets of a set of size n, the subspace lattice $Gl_n(q)$ and the lattice of faces of a cross-polytope.

Stanley's paper and the questions raised in it were the primary motivation for the paper [**28**]. Here rank-selection in Π_n was studied more extensively, and recursive plethystic descriptions were given for the rank-selected homology representations. Although a considerable amount of representation-theoretic and enumerative information can be extracted from these formulas (see [**28**], [**30**]), the problem of giving a nice characterisation of these representations appears to be a difficult one. There is, however, a fundamental three-term plethystic recurrence which can be used to compute the representations efficiently. Let h_n denote the homogeneous symmetric function of degree n, and let $\beta_S(n)$ denote both the rank-selected representation of S_n and its Frobenius characteristic (which is thus a symmetric function of degree n). The square brackets denote the plethysm operation. (See [**19**] and [**25**] for definitions.) Likewise, let $\alpha_S(n)$ denote both the S_n-representation and its Frobenius characteristic on the maximal chains of $\Pi_n(S)$. Finally, let $S(n, k)$ denote the Stirling number of the second kind, that is, the number of set partitions with k nonempty blocks.

Theorem 2.2. *([**28**], Theorem 2.13, Proposition 2.16, Proposition 3.1) Let $S = \{s_1 < s_2 < \ldots < s_r\}$, $r \geq 2$ be a subset of the nontrivial ranks $[1, n-2]$ of Π_n, and let $S - s_1$ denote the subset $\{s_2 - s_1, s_3 - s_1, \ldots, s_r - s_1\}$ of ranks in Π_{n-s_1}. Then*

$$\beta_S(n) + \beta_{S\setminus\{s_1\}}(n) = \beta_{S-s_1}(n - s_1)\big[\sum_{i\geq 1} h_i\big]|_{\deg n}.$$

Hence, if $d_S(n)$ denotes the unique nonvanishing Betti number for the order complex of $\Pi_n(S)$, one has the recurrence

$$d_S(n) + d_{S\setminus\{s_1\}}(n) = d_{S-s_1}(n - s_1)\ S(n, n - s_1).$$

For the permutation representation of S_n on the rank-selected maximal chains, one has

$$\alpha_S(n) = \alpha_{S-s_1}(n - s_1)[\sum_{i\geq 1} h_i]|_{\deg n}.$$

The second question raised by Stanley concerns the multiplicity $b_S(n)$ of the trivial representation in the rank-selected homology $\beta_S(n)$ of Π_S. Hanlon had shown in [**13**] that the multiplicity $b_S(n)$ is zero for $S = [1, r], r \geq 1$. This result is recovered, and more conditions on S for vanishing multiplicity are given in [**28**], where the action restricted to S_{n-1} is also studied. The primary tool was Theorem 2.2 above. Hanlon and Hersh [**15**] strengthened and proved some of the results and conjectures of [**28**] using completely different techniques, spectral sequences of filtered complexes and an intricate partitioning of the quotient complex $\Delta(\Pi_n)/S_n$ [**14**].

Let E_n be the nth Euler number defined by the generating function

$$\sum_{n\geq 0} E_n \frac{x^n}{n!} = \tan x + \sec x.$$

(E_{2n-1} is the *tangent* number, and E_{2n} is the *secant* number.) It is well-known ([**24**]) that E_n counts the alternating permutations in S_n, i.e., those permutations $\sigma \in S_n$ such that $\sigma(1) > \sigma(2) < \sigma(3) > \dots$. There is a large literature on the Euler numbers. Stanley's paper also contains the following result (for which he gives a second proof in ([**26**], Theorem 3.4)):

THEOREM 2.3. *([**23**], Theorem 7.7) The multiplicity of the trivial representation in the S_n-action α_n on the maximal chains of Π_n (which is also the total number of S_n-orbits) is the Euler number E_{n-1}.*

As observed by Stanley, the multiplicity $b_S(n)$ of the trivial representation in the rank-selected homology module $\beta_S(n)$ gives rise to a refinement of E_{n-1} into nonnegative integers indexed by subsets of $[1, n-2]$.

THEOREM 2.4. *([**23**])*

$$E_{n-1} = \sum_{S \subseteq [1,n-2]} b_S(n).$$

Interestingly, there is a second such refinement that occurs naturally when considering the homology representation: the subsets of $[1, n-2]$ also index a refinement of the succeeding Euler number E_n into nonnegative integers. Let $b'_S(n)$ denote the mutiplicity of the trivial representation of $S_{n-1} \times S_1$ in $\beta_S(n)$. Then

THEOREM 2.5. *([**28**], Proposition 3.4 (1) and p.269) The multiplicity of the trivial representation of $S_{n-1} \times S_1$ on the maximal chains of Π_n is E_n, and hence*

$$E_n = \sum_{S \subseteq [1,n-2]} b'_S(n).$$

Tables of values of $b_S(n)$ and $b'_S(n)$ for $4 \le n \le 9$ are given in ([**28**], pp. 286-288). (The entries for $n = 7$ and the subset $S = \{2, 4, 5\}$ appear to have been omitted: The missing entries are $b_7(\{2, 4, 5\}) = 5$ and $b'_7(\{2, 4, 5\}) = 23$.)

QUESTION 2.6. As far as we know, it is an open problem to describe combinatorially the refinements of Theorems 2.4 and 2.5. Is there a way to use more sophisticated topological techniques, by embedding quotient complexes, or using partitionings as in [**15**], that would shed some light on them? It seems curious that there should be two such combinatorial refinements of the Euler numbers.

An elegant unified generalisation of Stanley's Theorem 2.3, for each finite root system R and corresponding Coxeter group W, appears in a recent paper [**18**] of Josuat-Vergès, who computes the number $K(W)$ of W-orbits on the maximal chains of the intersection lattice I of R. In the particular case $R = A_{n-1}$, one has $I = \Pi_n$ and $W = S_n$, and thus by Theorem 2.3, Josuat-Vergès' number is

$$K(S_n) = K(\Pi_n) = E_{n-1} = \#\{S_n\text{–orbits on maximal chains of } \Pi_n\}. \tag{4}$$

However, decades earlier, Springer [**21**], in the same general setting, had also defined and computed an integer $T(W)$ as follows: if S is a set of simple roots for R, and $J \subset S$, let $\sigma(J, S)$ denote the number of elements $w \in W$ such that $w\alpha > 0$ for $\alpha \in J$ and $w\alpha < 0$ for $\alpha \in S \backslash J$. Then $T(W)$ is the maximum value of $\sigma(J, S)$ as J ranges over all subsets of S. For $R = A_{n-1}$, Springer showed that $T(W) = E_n$, and thus it transpires that his number "matches" Theorem 2.5:

$$T(S_n) = T(\Pi_n) = E_n = \#\{(S_{n-1} \times S_1) - \text{orbits on maximal chains of } \Pi_n\}. \tag{5}$$

Thus, the phenomenon of having *two* successive Euler numbers associated to Π_n manifests itself in this context; it is the only context known to us other than [**28**]. It seems natural to ask:

QUESTION 2.7. Does this generalise to the rank n root systems B_n and D_n? What is the result of considering orbits of parabolic subgroups (e.g., the Weyl group of B_{n-1}) on the maximal chains of the intersection lattices ?

A useful recurrence relating the two families of numbers $b_S(n)$ and $b'_S(n)$ is as follows:

PROPOSITION 2.8. *([**28**], Proposition 4.9, p. 284) Let S be a subset of $[1, n-2]$ and suppose $1 \notin S$. Let $S-1$ denote the subset of $\{1, \ldots, n-3\}$ obtained by subtracting 1 from each element of S. Then*

$$b_{S\cup\{1\}}(n) + b_S(n) = b'_{S-1}(n-1).$$

Proposition 2.8 is derived from the three-term recurrence of Theorem 2.2, which completely determines all the rank-selected homology representations. This recurrence is intrinsic to the recursive nature of the partition lattice; it arises from the fact that upper intervals are isomorphic to smaller partition lattices. See [**28**] for details.

QUESTION 2.9. Is there a natural topological explanation, again in terms of quotient complexes, for the recurrence of Proposition 2.8?

Recall that $[a, b]$ denotes the subset of consecutive ranks $\{a, a+1, \ldots, b\}$. Partitionings of the quotient complex of $\Delta(\Pi_n)$ by S_n and $S_{n-1}\times S_1$ were used successfully by Hanlon and Hersh to prove two conjectures of ([**28**], p. 289):

THEOREM 2.10. [**15**] *(Theorems 2.1 and 2.2) $b_S(n) \neq 0$ if $1 \notin S$ or if $S = [1, r] \cup T$ where T is a subset such that $\min T \geq r + 2$ and $|T| \geq r$.*

When $S = [1, r]$, it was shown in ([**28**], Proposition 4.10 (2)) that $b'_S(n) = 1$, and it was conjectured that this uniquely characterises the first r consecutive ranks ([**28**], p. 289), and that $b'_S(n)$ is always nonzero. This conjecture was also proved by Hanlon and Hersh.

THEOREM 2.11. [**15**] *$b'_S(n) > 1$ unless $S = [1, r]$ for some r with $2 \leq r \leq n-1$.*

We summarise the other known results about $b_n(S)$ and $b'_n(S)$ below. Some partial results concerning the case of an arbitrary consecutive set of ranks are recorded in ([**28**], Theorem 4.8), but even this case seems to be difficult in general.

THEOREM 2.12. *$b_n(S) = 0$ whenever any of the following conditions are met:*

(1) *([**12**];[**23**], Proposition 7.8) $S = [1, i]$, $1 \leq i \leq n-2$.*
(2) *([**28**], Theorem 4.2 and Remark 4.10.1) $S \supseteq [1, \lfloor \frac{n+1}{2} \rfloor]$*
(3) *([**28**], Theorem 4.3 and Remark 4.10.1) $S = [1, r] \cup \{a\}$ for $a \notin [\binom{r+2}{2}, n-r-1]$.*
(4) *([**15**], Theorem 2.4); [**28**], Theorem 4.7(2) and Remark 4.10.1) $S = [1, r]\backslash\{k\}$ for $k > r/2$.*

3. Permutation modules in homology

Stanley's paper focuses as much on the maximal chains in the partition lattice as on the homology. In fact, the homology representation on the maximal chains has a very interesting structure, which we now describe.

As before, let h_n be the homogeneous symmetric function of degree n. Thus h_n is the Frobenius characteristic of the trivial representation of S_n. See [**19**] and [**25**] for definitions. Define positive integers $a_i(n)$ by the recurrence

$$a_i(n+1) = ia_i(n) + (n-2i+2)a_{i-1}(n) \tag{6}$$

with initial conditions $a_0(1) = 1 = a_1(2), a_0(n) = 0, n > 1$, and $a_i(n) = 0$ if $2i > n$. The $a_i(n)$ count the *simsun* permutations in S_{n-2} with $(i-1)$ descents; they were defined combinatorially in [**28**]. The recurrence (4) arose naturally from the recursive plethystic generating function for the action of S_n on the maximal chains of Π_n, which also led to the following more elegant description of the S_n-action on the maximal chains:

Theorem 3.1. *([**28**], Theorem 3.2) The S_n-action on the maximal chains of Π_n decomposes into orbits as follows:*

$$\alpha_n = \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} a_i(n) W_i,$$

where W_i is the transitive permutation representation of S_n acting on the cosets of the Young subgroup $S_2^i \times S_1^{n-2i}$. In symmetric function terms, the Frobenius characteristic of α_n is

$$\sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} a_i(n) h_2^i h_1^{n-2i}.$$

By extracting the multiplicity of the trivial representation, one obtains a refinement of the Euler number E_{n-1} via the simsun permutations. The simsun permutations appear extensively in the literature. See [**28**] for the original definition, and also [**26**] and [**27**]. (Note that the recurrence given in ([**26**], Theorem 3.3) for the number $f_k(n)$ of simsun permutations in S_n with k descents is obtained from equation (4) above by means of the substitution $a_i(n) = f_{i-1}(n-2)$.)

Now let $m = 2n$ be even, and consider the subposet Π_{2n}^e of Π_{2n} consisting of partitions with an even number of blocks. This is the rank-selected poset corresponding to selecting alternate ranks in Π_{2n}, beginning with rank 2. Let R_{2n} denote the Frobenius characteristic (see [**19**], [**25**]]). Define integers $\{b_i(n),\ 2 \le i \le n\}$, by means of the recurrence $b_2(n) = 1,\ n \ge 2$, ($b_i(n) = 0$ unless $2 \le i \le n$),

$$b_i(n) = \sum_{k \ge 0} \binom{2n-2i+k}{k} \sum_{r \ge 1} (-1)^{r-1} \binom{i-k}{i-2r} b_{i-k}(n-r).$$

The main result of [**30**] states that:

Theorem 3.2. *([**30**], Theorem 2.5)*

$$R_{2n} = \sum_{i=2}^{n} b_i(n) h_2^i h_1^{2n-2i},$$

and thus the character values are nonzero only on the involutions of S_{2n}.

It was conjectured in [**30**] that the integers $b_i(n)$ are nonnegative; this result was proved by Benjamin Joseph (a student of Stanley) in his thesis. By analysing the recurrence for the $b_i(n)$ within the framework of a sign-reversing involution, Joseph devises an intricate and ingenious set of objects enumerated by the $b_i(n)$. The objects are cleverly constructed to be the fixed points of the sign-reversing involution.

THEOREM 3.3. *([**17**], Chapter 4, Section 3, Theorem 8) $b_i(n)$ is a positive integer for all $n \geq i \geq 2$.*

Hence, putting together these two results, since $h_2^i h_1^{2n-2i}$ is the Frobenius characteristic of the transitive permutation module induced from the Young subgroup $S_2^i \times S_1^{2n-2i}$, another conjecture ([**30**], Conjecture 2.7) is established:

COROLLARY 3.4. *The top homology of Π_{2n}^e is a permutation module for S_{2n} whose character values are supported on the set of involutions.*

Note that by Theorem 3.1, the identical statement is true (mutatis mutandis) for the action of S_n on the maximal chains of Π_n.

QUESTION 3.5. Corollary 3.4 is still a somewhat mysterious fact; is there a more natural topological explanation?

Define integers $E_k(n)$ by

$$E_k(n) = \sum_{i=2}^{n} b_i(n) \binom{n-i}{k-i}, \quad 2 \leq k \leq n.$$

It follows from Joseph's work that the $E_k(n)$ are also nonnegative integers (see [**30**], Conjecture 3.1). Their representation-theoretic significance lies in the fact that the Frobenius characteristic of the homology of Π_{2n}^e can be rewritten as (see [**30**], Corollary 2.8)

$$R_{2n} = \sum_{i=2}^{n} E_i(n) h_2^i e_2^{n-i}.$$

In this formulation, $E_n(n)$ is the multiplicity of the trivial representation in R_{2n}. But R_{2n} is a submodule of α_{2n}, and the number of S_{2n}-orbits in the latter module is E_{2n-1} by Stanley's result. Thus $E_n(n)$ must count a subset of the alternating permutations enumerated by E_{2n-1}.

QUESTION 3.6. Is there a combinatorial description of the integers $E_n(n)$ as a subset of alternating permutations in S_{2n-1}? Is there one that would lead to a similar description for the integers $b_i(n)$?

Finally, let $\Pi_{2n}^e(\bar{k})$ denote the subposet of Π_{2n}^e obtained by selecting the top k nontrivial ranks. We have one more conjecture which generalises Theorem 3.3:

CONJECTURE 1. *([**30**], Conjecture 2.10) For $1 \leq k \leq n-1$, S_{2n} acts on the homology of $\Pi_{2n}^e(\bar{k})$ as a permutation module which can be written as a sum of induced modules, in which the stabilisers are all Young subgroups of S_{2n}. In terms of symmetric functions, the Frobenius characteristic of the homology is h-positive.*

In [**15**], Hanlon and Hersh describe a partitioning for the quotient complex of $\Delta(\Pi_n)/S_\lambda$ for an arbitrary Young subgroup S_λ.

QUESTION 3.7. Can such a partitioning be adapted to prove the above conjecture, using the sequence of poset inclusions

$$\Pi_{2n}^e(\bar{1}) \subset \Pi_{2n}^e(\bar{2}) \ldots \subset \Pi_{2n}^e(\overline{n-1}) = \Pi_{2n}^e?$$

Recall from Theorem 3.1 that the Frobenius characteristic of α_{2n} is also a nonnegative integer combination of the homogeneous symmetric functions $h_2^i h_1^{2n-2i}$. By Stanley's observation (2.1), R_{2n} is (the Frobenius characteristic of) a submodule of α_{2n}. The data supports a stronger statement:

CONJECTURE 2. *$\alpha_{2n} - R_{2n}$ is a nonnegative integer combination of homogeneous symmetric functions h_λ indexed by integer partitions λ of n with parts equal to 1 or 2. This is equivalent to the enumerative statement that*

$$b_i(n) \leq a_i(2n) \text{ for all } 2 \leq i \leq n. \tag{7}$$

The data for $n \leq 7$ overwhelmingly verifies the above inequality. In view of this, one may ask:

QUESTION 3.8. Is there a way to explain Theorem 3.2 by isolating specific chains in Π_{2n}?

QUESTION 3.9. Is there an injection from Joseph's set of objects counted by the $b_i(n)$ to the simsun permutations in S_{2n-2} with $(i-1)$ descents, or a different set of objects counted by $a_i(2n)$?

QUESTION 3.10. Does the topology of the quotient complex $\Delta(\Pi_{2n}^e)/S_{2n}$ offer any insight in this special case?

For many other enumerative and completely elementary conjectures (still open, as far as we know) regarding refinements of the simsun numbers $a_i(2n)$ and the Genocchi numbers, see [**30**]. Tables of values for the $b_i(n)$ appear in [**31**]. For values of the $a_i(m),\ m \leq 14$, see ([**39**], sequence No. A113897).

4. Partitions with forbidden blocks: pure and non-shellable

A central theme of Stanley's paper [**23**] is to take a representation-theoretic result and extract interesting enumerative identities from it. In [**32**] and [**33**], representation-theoretic results were used to predict the topology of the order complex.

Our study of the posets described in this section begins with a Lefschetz module calculation (see [**23**] for definitions), and the discovery that the resulting Frobenius characteristic has interesting properties. The results of the preceding sections were obtained by using two tools introduced in [**28**] which have proved to be very powerful. The first exploits the acyclicity of Whitney homology ([**28**], Lemma 1.1). Coupled with the innate plethystic nature of the partition lattice, this technique allows one to write down, with relative ease, generating functions for the Lefschetz module of any subposet of partitions with arbitrary block sizes ([**29**]). The considerable machinery of symmetric functions can then be used to analyse the representations, and thereby predict topological properties. (As a simple example, if the Lefschetz module is *not* plus or minus a true module, one concludes immediately that there is homology in more than one degree.) These ideas were applied successfully in [**34**] and [**35**] to the k-equal lattice and the k-equal subspace arrangement [**8**]. The k-equal lattice served as the motivating example for the theory of nonpure shellability developed by

Björner and Wachs [**6**]; their shelling was a crucial ingredient in isolating the homology representations by degree. The Whitney homology technique was subsequently generalised by Wachs to obtain far-reaching and beautiful results (see [**36**] for many applications), and combined with shellings and other constructive methods, to refine the resulting plethystic identities for homology by degree.

A second technique in [**28**] proved particularly useful in computing homology in the case when an antichain is deleted from a poset (rank-selection falls into this category by repeated deletion of ranks). It may be viewed as a group-equivariant homology version of ([**2**], Lemma 4.6). The general group-equivariant result appears in ([**28**], Theorem 1.10 and Remark 1.10.1), while the result for antichains appears explicitly in ([**32**], Theorem 4.2). In [**32**] the antichain result was also derived in a more general context using the long exact homology sequence of a pair. The key step is to determine the relative homology of the pair, and can also be obtained from the Homotopy Complementation Formula of Björner and Walker [**7**]. The homological antichain result is crucial to the Lefschetz module computations described here, as well as the three-term recurrence (Theorem 2.2) of the preceding section, and once again can be effectively combined with the recursive and plethystic nature of the partition lattice.

Recall that the modular elements of the partition lattice ([**24**], Example 3.13.4) are precisely the partitions with a unique non-singleton block. For $k \geq 2$ let Q_n^k be the subposet of Π_n consisting of all partitions except those with $(n-k)$ blocks of size 1 and one block of size k, and let $P_n^k = \cap_{i=2}^k Q_n^i$. Thus Q_n^k is obtained by deleting from Π_n all the modular elements in which the singleton block has size k, while P_n^k is obtained by removing all modular elements where the singleton block has size i, for $2 \leq i \leq k$. These posets were studied in [**32**]. For $k \geq 3$, Q_n^k is pure of full rank $n-1$, but P_n^k is pure of rank $n-2$, since all the atoms have been deleted.

Let π_n denote the Frobenius characteristic of the S_n-module afforded by the homology of Π_n, and define the symmetric function $\pi_{n,k}$ by

$$\pi_{n,k} = \pi_k h_1^{n-k} - \pi_n.$$

Using the preceding methods, it was shown ([**32**], Theorem 4.2 and Theorem 4.4) that the Lefschetz module of the order complex of Q_n^k has Frobenius characteristic $(-1)^{n-4}\pi_{n,k}$. It is easy to see that $\pi_{n,k}$ is in fact a true representation of S_n. Even more interestingly, the restriction of $\pi_{n,n-1}$ to S_{n-1} is π_{n-1}. Hence (with unabashedly wishful thinking)[1] the representation theory might be said to predict that Q_n^k has homology only in degree $(n-4)$, and that Q_n^{n-1} and Π_{n-1} have the same homology, or even the same homotopy type. Miraculously, this is indeed the case.

THEOREM 4.1. *(See ([**32**], Theorem 2.1, Theorem 3.5, Theorems 4.5 - 4.6.)*

(1) *The inclusion map between the $(n-4)$-dimensional order complex of P_n^k and the $(n-3)$-dimensional order complex of Q_n^k is an S_n-equivariant homotopy equivalence.*
(2) *There is a map from the poset of nonmodular partitions P_n^{n-1} to Π_n whose image is Π_{n-1}, and which induces an $(S_{n-1} \times S_1)$-equivariant homotopy equivalence.*
(3) *P_n^k is Cohen-Macaulay of rank $(n-2)$, Q_n^k (of rank $n-1$) is not. The pure poset Q_n^k has unique nonvanishing homology in degree $(n-4)$, one less than the top degree.*

[1] cf. Richard Stanley's talk at his 70th birthday conference.

(4) *The action of the symmetric group S_n on the unique nonvanishing homology of each of the posets P_n^k and Q_n^k has Frobenius characteristic*

$$\pi_k h_1^{n-k} - \pi_n.$$

The representation described by $\pi_{n,k}$ is what was called the generalised Whitehouse module in [**32**]; in the special case $k = n-1$, it is the Whitehouse module (also named thus in [**32**]) or tree representation of [**20**], whose restriction to S_{n-1} is π_{n-1}, the representation of S_{n-1} on the top homology of the partition lattice Π_{n-1}. The generalised Whitehouse module shows up again in [**33**].

It was also shown that

THEOREM 4.2. *([**32**], Theorem 2.12) For $k \geq 2$, the order complex of Q_n^k has the homotopy type of a wedge of spheres of dimension* $(n-4)$.

The proof proceeds somewhat indirectly, by establishing that the order complex is simply-connected using a technical lemma of Bouc ([**9**] (9, Section 2.2.2, Lemme 6). Bouc's lemma is misstated in his original paper. The proof in [**32**] therefore requires a slight modification which appears in ([**33**], p. 276).

QUESTION 4.3. Is there a more direct way to prove Theorem 4.2, i.e., to establish the homotopy type of the posets Q_n^k or P_n^k?

The restriction of $\pi_{n,k}$ to $S_{n-1} \times S_1$ has Frobenius characteristic equal to $(n-k)\pi_k h_1^{n-k-1}$. This leads to

QUESTION 4.4. (More wishful thinking?) There is an $S_k \times (S_{n-k-1} \times S_1)$-module isomorphism between the homology of the poset Q_n^k (or P_n^k), and $(n-k)$ copies of the homology of $\Pi_k \times \Pi_{n-k}$. (Note that dimensions agree for the order complexes if we consider $P_n^k : (k-3)+(n-k-3)+2 = n-4$.) Can this be explained topologically via a simplicial (poset) map, as it was in Theorem 4.1 for the case $k = n-1$?

It follows from Theorem 4.1 that the relative homology group of the pair (Π_n, P_n^k) vanishes in degrees different from $n-3$, and, in degree $(n-3)$, equals the direct sum of $\tilde{H}_{n-3}(\Pi_n)$ and $\tilde{H}_{n-4}(P_n^k)$. Hence the relative homology group $\tilde{H}_{n-3}(\Pi_n, P_n^k)$ affords the representation $\pi_k h_1^{n-k}$. Note that the relative chain complex of the pair consists of chains passing through a modular element with at least $(n-k)$ singletons.

QUESTION 4.5. Does the quotient complex $\Delta(\Pi_n)/\Delta(P_n^k)$ have a nice topological description that is consistent with the representation it affords? (When P_n^k is replaced by Q_n^k, the answer is provided by the Homotopy Complementation Formula of [**7**].)

The posets Q_n^k appear to be related to two other classes of subposets, which we now describe. First let $\Pi_{n,\leq k}$ denote the subposet of Π_n consisting of partitions with block size at most k. Note that one has natural inclusions

$$\Pi_{n,\leq 2} \subset \ldots \subset \Pi_{n,\leq k} \subset \Pi_{n,\leq k+1} \ldots \subset \Pi_{n,\leq n-2}, \tag{8}$$

and that $\Pi_{n,\leq n-2}$ is precisely the poset Q_n^{n-1}. By (1) and (2) of Theorem 4.1, this last order complex has the same S_{n-1}-homotopy type as that of Π_{n-1}, and its S_n homology representation is given by the Whitehouse module.

For $k = 2$, this is the matching complex studied by many authors in different guises (see [**36**] for an extensive list). Homology and homotopy type were determined by Bouc [**9**], who also showed that there is torsion. For arbitrary k these posets were first considered in [**5**], where their Möbius function was computed; they also arise in certain examples of relative arrangements, a concept introduced in [**38**]. Welker conjectured that the integral homology of these posets is free. This was established for certain values of n :

Theorem 4.6. *([**33**], Theorem 2.8, Theorem 2.10, Corollary 2.15) Let $k \geq 3$.*

(1) *For fixed n, when $n < 2k+2$, all the posets $\Pi_{n,\leq k}$ have the same S_n-homotopy type as the posets Q_n^{n-1}; hence the S_n-representation on the homology is the Whitehouse module (Frobenius characteristic $\pi_{n-1}h_1 - \pi_n$).*
(2) *When $n = 2k+2$, the $(2k-2)$-dimensional order complex of $\Pi_{2k+2,\leq k}$ is homotopy equivalent to a wedge of $(2k-3)$-dimensional spheres.*
(3) *When $n = 3k+2$, the $(3k-3)$-dimensional order complex of $\Pi_{3k+2,\leq k}$ is homotopy equivalent to a wedge of $(3k-4)$-dimensional spheres.*

The case $n = 2k+2$ deserves special mention. The homotopy equivalence was again proved indirectly. Interestingly, there is an S_{2k+1}-isomorphism in homology with the order complex of Q_{2k+1}^{k+1}.

Question 4.7. Is there a natural S_{2k+1}-homotopy equivalence between (the order complexes of) $\Pi_{2k+2,\leq k}$ and Q_{2k+1}^{k+1}? What, if anything, does restricting block sizes have to do with removing modular elements?

Now let $\Pi_{n,\neq k}$ denote the poset of partitions with no block of size k. (Again the Möbius function for these posets was computed in [**5**].) Observe that $\Pi_{n,\neq 2}$ is the 3-equal lattice of [**8**]. Björner and Welker determined the homotopy type of the k-equal lattice to be a wedge of spheres of varying dimensions, while the homology representations were determined in [**34**]. Our original motivation for the study of both $\Pi_{n,\neq k}$ and $\Pi_{n,\leq k}$ arose from Lefschetz module calculations appearing in [**29**].

Theorem 4.8. *([**33**], Theorem 4.3, Theorem 4.8) Assume that $k \geq 3$. For $n \leq 2k$ the posets $\Pi_{n,\neq k}$ and the poset Q_n^k have homotopy equivalent order complexes; when $n < 2k$ they have isomorphic S_n-homology modules, namely, the generalised Whitehouse module described in Theorem 4.1 (4) above.*

Conjecture 3. *The $(2k-2)$-dimensional order complex of $\Pi_{2k+1,\neq k}$ has the homotopy type of a wedge of spheres of dimension $2k-3$.*

It is shown in [**33**] that the integral homology can only occur in degrees $2k-3$ and $2k-4$. The Möbius number of the poset is computed in ([**33**], Theorem 4.5).

The posets described in this section have the property that they are pure with non-vanishing homology only in one less than the top dimension, and also have the homotopy type of a wedge of spheres. These posets are clearly not shellable, so the methods of [**3**] and [**6**] do not apply.

Question 4.9. Is there a general edge labelling that would detect this phenomenon of collapse in homology and homotopy type from the top dimension to a lower one, and corroborate the homotopy type of these posets?

5. Stability in Homology

This section is prompted by another observation in Stanley's paper [**23**]: For a fixed subset S of ranks, the numbers $b_S(n)$ stabilise at $n = 2\max(S)$. That is, $b_S(m) = b_S(2(\max S))$ for all $m \geq 2\max S$. In particular, if $b_S(2\max(S)) = 0$, then $b_S(n) = 0$ for all $n \geq 2(\max S)$.

In view of equations (2) and (3) of Section 2, Stanley's statement is equivalent to the analogous statement for the multiplicities $a_S(n)$ of the trivial representation in the action of S_n on the maximal chains of the rank-selected subposet $\Pi_n(S)$. Stability in the special case of the trivial representation is "not difficult to see" ([**23**]); this will be evident in the proof of Theorem 5.3 below, where we establish a somewhat more general result. Stanley's stability result for $b_S(n)$ was also proved using Hersh's partitioning of $\Delta(\Pi_m)/S_m$ ([**15**], Theorem 2.5).

Coincidentally, there has been a recent surge of interest in such stability questions, in the work of Church and Farb as well as others. See [**11**] and the references therein.

To avoid confusion and trivialities, we point out that $a_\emptyset(n) = 1 = b_\emptyset(n)$ and $a_{\{1\}}(n) = 1$ for all $n \geq 2$.

Proposition 5.1. *([**23**], p. 152) Let S be any (nonempty) subset of the nontrivial ranks* $[1, n-2]$.

1. *(implicit in [**23**]) $a_S(n) = a_S(2\max S) = \overline{a}_S,\ n \geq 2(\max S)$, and hence*
2. *$b_S(n) = b_S(2(\max S) = \overline{b}_S$ for all $n \geq 2(\max S)$.*

It turns out that this stability is transferred to the multiplicities $b'_S(n)$ as well. Although this is also a special case of Theorem 5.3 below, it is worth mentioning separately because in this case the formulas derived in [**28**] allow us to write down relatively simple and interesting relationships between the stable values. The stability also transfers to the numbers $a'_S(n)$ which denote the number of $(S_{n-1} \times S_1)$-orbits in the S_n-action on the rank-selected subposet corresponding to the set S. For a fixed set S, we write $\overline{a}_S$ for the limiting value of $a_S(n)$ as n approaches ∞, and similarly $\overline{b}_S = \lim_{n\to\infty} b_S(n)$, $\overline{b'}_S = \lim_{n\to\infty} b'_S(n)$ and $\overline{a'}_S = \lim_{n\to\infty} a'_S(n)$. We record the actual expressions relating the limiting values in:

Proposition 5.2. *Fix n and a nonempty subset $S \subseteq [1, n-2]$ such that $n \geq 2\max(S) + 1$. Then*

1. *$a'_S(n) = a'_S(2\max S + 1) = \overline{a'}_S$ for all $n \geq 2(\max S) + 1$, and $\overline{a'}_S = \overline{a}_{(S+1)\cup\{1\}}$.*
2. *$b'_T(n) = b'_T(2\max T + 1) = \overline{b'}_T$ for all $n \geq 2(\max T) + 1$, and $\overline{b'}_T = \overline{b}_{\{1\}\cup(T+1)} + \overline{b}_{(T+1)}$.*
3. *The multiplicity of the irreducible indexed by $(n-1, 1)$ (the reflection representation of S_n) also stabilises in both $\alpha_S(n)$ and $\beta_S(n)$ when $n \geq 2\max(S)+1$. The stable values are $\overline{a}_{\{1\}\cup(S+1)} - \overline{a}_S$ and $\overline{b}_{\{1\}\cup(S+1)} + \overline{b}_{(S+1)} - \overline{b}_S$ respectively.*

Proof. Part (1) can be proved using the following special case of the plethystic recurrence ([**28**], Proposition 3.1; see also Theorem 2.2 of this paper) which was crucial to the construction of the simsun permutations of Rodica Simion and this author (see Theorem 3.1). Let $S = \{s_1 < \ldots < s_k\}$ be a nonempty subset not containing 1, and let $S - 1$ denote the subset obtained by subtracting 1 from each element of S. Then, in terms of symmetric functions,

$$\alpha_{\{1\}\cup S}(n) = h_2 \frac{\partial}{\partial p_1} \alpha_{S-1}(n-1). \tag{9}$$

The multiplicity of the trivial representation on the left-hand side is $a_{\{1\}\cup S}(n)$. On the right-hand side, it is

$$\langle \frac{\partial}{\partial p_1}\alpha_{S-1}(n-1), h_{n-2}\rangle,$$

which equals $a'_{S-1}(n-1)$. Thus we have

$$a_{\{1\}\cup S}(n) = a'_{S-1}(n-1)$$

Now replace $S-1$ by S, and $n-1$ by n. Then

$$a_{\{1\}\cup(S+1)}(n+1) = a'_S(n) \tag{10}$$

The left-hand side equals its limiting value $\overline{a}_{\{1\}\cup(S+1)}$ when $n+1 \geq 2\max(S+1)$, i.e., when $n \geq 2\max S + 1$ as stated in Part (1).

Note that while this also establishes the stability of $b'_S(n)$ for n sufficiently large, in order to compute the actual stable value, it is more efficient to invoke the recurrence of Proposition 2.8. Writing T for the set $S-1$ and m for $n-1$ in that recurrence, we have

$$b'_T(m) = b_{(T+1)\cup\{1\})}(m+1) + b_{(T+1)}(m+1) \tag{11}$$

which in turn equals $\overline{b}_{(T+1)\cup\{1\}} + \overline{b}_{(T+1)}$, provided $m+1 \geq 2\max(T+1) = 2(\max T)+2$, i.e., provided $m \geq 2(\max T)+1$ as desired. (Here $T+1$ is the subset obtained from T by adding 1 to each element of T.)

For Part (3) it suffices to note that the multiplicity of the reflection representation is the difference $a'_S(n) - a_S(n)$ (respectively $b'_S(n) - b_S(n)$). $\square$

By analysing the S_n-orbits of maximal chains in $\Pi_n(S)$, we are able to derive a slightly stronger stability result. The reader is referred to [**19**] for the following definitions. Recall that s_λ denotes the Schur function indexed by the integer partition λ; it is the Frobenius characteristic of the irreducible indexed by λ. Also recall the operation D_μ of *skewing* a (possibly virtual) representation f of S_n by the irreducible indexed by a partition μ of m : its image is the representation $D_\mu f$ of S_{n-m} defined by $\langle D_\mu f, s_\nu\rangle = \langle f, s_\mu s_\nu\rangle$ for all partitions ν of $n-m$. Here $\langle , \rangle$ denotes the inner product of S_n-characters (or Frobenius characteristics) which makes the Schur function an orthonormal basis in the ring of symmetric functions.

THEOREM 5.3. *Fix a nonempty subset S of the nontrivial ranks $[1, n-2]$ and an integer $k \geq 0$. Consider the Schur functions appearing in the linear span of the set $U_k = \{h_{n-\ell}s_\mu : 0 \leq \ell \leq k, \mu \vdash \ell\}$. Then for $n \geq 2\max(S)+k$, the multiplicities of the irreducible indexed by λ in $\alpha_S(n)$ and $\beta_S(n)$ stabilise for every partition λ of n such that s_λ is in U_k. Equivalently, for $n \geq 2\max(S)+k$, the operation of skewing the representations $\alpha_S(n)$ and $\beta_S(n)$ by the irreducible indexed by $(n-\ell)$ results in stable modules for all $0 \leq \ell \leq k$.*

PROOF. Let ν be any integer partition of k, and denote by S_ν the Young subgroup determined by ν. The argument we give here will show that the number of $(S_{n-k}\times S_k)$-orbits, and more generally that the number of $S_{n-k}\times S_\nu$-orbits, in $\alpha_S(n)$ stabilises; in the case $k=0$ it must reduce to Stanley's original argument (unstated in [**23**]). Let $S = \{s_1 < s_2 < \ldots < s_r\}$, and assume $n \geq 2s_r + k$. The partitions at rank s_r in Π_n have $n - s_r$ blocks, and their *type*, (defined as the integer partition of n whose parts encode the block sizes) must be an integer partition of n of length $n - s_r$, of the form

$$\mu_1 + 1, \mu_2 + 1, \ldots, \mu_\ell + 1, 1, 1, \ldots, 1$$

where $\mu = (\mu_1 \geq \mu_2 \geq \ldots \geq \mu_\ell)$ is an integer partition of s_r. The fact that $n \geq 2s_r + k$ implies that $s_r \leq n - s_r - k \leq n - s_r$, ensuring that the set partitions $\tau \in \Pi_n$ at rank s_r are in one-to-one correspondence with the integer partitions μ of s_r. Furthermore, it is clear that there must be at least $n - s_r - |\mu| = n - 2s_r$ parts of size 1, i.e., each set partition has at least $n - 2s_r$ singleton blocks. But $n - 2s_r \geq k$, so this means every set partition at rank s_r has at least k singleton blocks. Each such partition determines a distinct $(S_{n-k} \times S_\nu)$-orbit of the chains supported by the rank set S. Since the remaining elements of the chain lie below the elements at rank s_r, they depend only on the choice of the integer partition μ of $s_r = \max S$, and not on n. Hence the number of $(S_{n-k} \times S_\nu)$-orbits in $\alpha_S(n)$ is constant for such values of n. Since the homogeneous symmetric function h_ν is the Frobenius characteristic of the trivial representation of S_ν, we conclude that $\langle \alpha_S(n), h_{n-k} h_\nu \rangle$ is constant for $n \geq 2\max(S) + k$.

Now use the fact [**19**] that the functions h_ν form a basis for the symmetric functions of degree k, and in fact each Schur function s_ν is an integer combination of the $\{h_\lambda\}$, as λ runs over all integer partitions of k. We conclude that the inner product $\langle \alpha_S(n), h_{n-k} s_\nu \rangle$ is stable for all partitions ν of k and $n \geq 2\max(S) + k$. The result follows with the final observation that if $n \geq 2\max(S) + k$, then necessarily $n \geq 2\max(S) + \ell$ for all ℓ between 0 and k. □

Remark 5.4. Restated in the language of [**11**], this theorem shows that, for fixed k and nonempty S, the sequences of (skew) representations $\{D_{(n-k)}\alpha_S(n)\}_n$ and $\{D_{(n-k)}\beta_S(n)\}_n$ are *representation stable*, whereas the sequences $\{\alpha_S(n)\}_n$ and $\{\beta_S(n)\}_n$ are λ-*representation stable* for some choices (see Corollaries 5.6 and 5.8 below) of λ, e.g., $(n-k, k)$ and $(n-k, 1^k)$.

Question 5.5. Can this stability of the action on the chains (simplices) be explained via the topology of the quotient complex $\Delta(\Pi_n(S))/(S_{n-k} \times S_\nu)$ for an integer partition ν of k? Note that this would involve looking only at flag f-vectors, rather than h-vectors as in [**15**].

Theorem 5.3 allows us to establish stability for more irreducibles; we single out the cases of two-part partitions and hooks:

Corollary 5.6. *Let $V(k)$ denote the S_n-irreducible indexed by the integer partition $(n-k, k)$, where $n - k \geq k \geq 0$. Then the multiplicity of $V(k)$ in $\alpha_S(n)$ and hence in $\beta_S(n)$ stabilises for $n \geq 2\max(S) + k$.*

Proof. Applying Theorem 5.3 with $\nu = (k)$ yields the stability of the inner products $\langle \alpha_S(n), h_{n-\ell} h_\ell \rangle$ for $0 \leq \ell \leq k$. Since the Frobenius characteristic of $V(k)$ is $h_{n-k} h_k - h_{n-k+1} h_{k-1}$, the result follows. □

Remark 5.7. The stable values of the preceding multiplicities can be determined as before. For instance, we have that the stable multiplicity of $(n-2, 2)$ in $\alpha_S(n)$ is (for $n \geq 2\max(S) + 2$) $a_{(S+2)\cup 2} - 2a'_S$.

Corollary 5.8. *Let $V(1^k)$ denote the irreducible indexed by the hook shape $(n-k, 1^k)$ (with k parts equal to 1). Then the multiplicity of $V(1^k)$ in $\alpha_S(n)$ and hence in $\beta_S(n)$ is stable for $n \geq 2\max(S) + k$.*

Proof. Take $\nu = (1^k)$ in Theorem 5.5. Then $s_\nu = e_k$, the kth elementary symmetric function. The set U_k in the statement of Theorem 5.5 then contains the set $\{(-1)^{k-i}h_{n-i}e_i\}_{i=o}^{k}$. But a well-known symmetric function identity [**19**] says that the Schur function corresponding to the hook $(n-k, 1^k)$ equals the alternating sum

$$\sum_{i=0}^{k}(-1)^{k-i}h_{n-i}e_i.$$

The result now follows as before. □

For definitive and far-reaching results on the stability question in the rank-selected homology of Π_n, see the forthcoming paper [**16**].

Finally, we have:

Remark 5.9. Stability of another sort is exhibited by the homology of the order complexes (discussed in Section 4) of the sequences of posets $\Pi_{n,\leq k}$ and $\Pi_{n,\neq k}$, for $k > \frac{n-2}{2}$ and $k > \frac{n}{2}$ respectively (see the chain of inclusions in equation (8)). The significance (if any) of the half-way mark is unclear.

The families of subposets described in this paper often have natural filtrations associated with them. The topological results in [**32**] and [**33**] were obtained by first establishing that the order complexes are simply connected, and then showing that homology is concentrated in a single degree. There has been an explosion in the development of tools in geometric and topological combinatorics in recent years: see [**36**]. Perhaps these techniques can be used successfully on the problems discussed here.

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