On the number of open knight's tours

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July 15, 2015

Abstract

We review the state of the art in the problem of counting the number *open knight tours*, since the publication in internet of a computation of this quantity [1].

1 Problem description

"A knight's tour is a sequence of moves of a knight on a chessboard such that the knight visits every square only once. If the knight ends on a square that is one knight's move from the beginning square (so that it could tour the board again immediately, following the same path), the tour is closed, otherwise it is open. The knight's tour problem is the mathematical problem of finding a knight's tour"[5]. In this note we are interested in the determination of the *number of solutions* of the knight's tour problem, i.e. in the computation of the number of open knight tours.

The history of the problem dates back to the Hindu culture, in the 9th century BC [5]. Many mathematicians made contributions to this problem finding solutions with special properties, and providing methods to find them. The list includes De Moivre, Euler, Legendre and Vandermonde [3, 6, 9, 11].

To define precisely the quantities we are interested in, we borrow the notation from George Jelliss's web page [4], denoting by

G the number of **G**eometrically distinct open tours.

T the number of open **T**our diagrams, by rotation and symmetry $\mathbf{T} = 8\,\mathbf{G}$.

N the number of open tour **N**umberings: $\mathbf{N} = 2\,\mathbf{T} = 16\,\mathbf{G}$ since each can tour diagram be numbered from either end.

D is the number of closed tour **D**iagrams.

These quantities refer to the standard 8×8 chessboard.

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2 Number of closed tours

The number of closed tours was independently computed by McKay [8] and Wegener [7], obtaining that the number of classes of re-entrant solutions (not taking into account the order and the initial square) is $\mathbf{D} = 13,267,364,410,532$, that is approximately 10^{13}.

3 An estimation of the number of open tours

The number of open tours in an a classical 8×8 chessboard remained an open question, as reported in a comprehensive web page about Knight tours [4], where some bounds and estimation are discussed. The method of importance sampling combined with the Warnsdorff's Rule [3] was used by Cancela and Mordecki in 2006 [2] to conjecture that $G \simeq 1.22 \times 10^{15}$, providing a 99%-confidence interval of the form $[1.220, 1.225] \times 10^{15}$.

4 A computation of the number of open tours

A comprehensive computation of the total number of open tours was carried out by Alexander Chernov, and published on the Internet [1]. The final amount is $\mathbf{T} = 9,795,914,085,489,952$ open tours diagrams, that gives $\mathbf{G} = 1,224,489,260,686,244$. This number belongs to the confidence interval in [2]. This number is also published as the 8-th term of the sequence A165134 in [10], by A. Chernov. Although we believe that Chernov's result is correct, we expect to have an independent verification, that we hope will be carried out soon.

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