

FINITE MULTIPLE ZETA VALUES AND FINITE EULER SUMS

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ABSTRACT. The alternating multiple harmonic sums are partial sums of the iterated infinite series defining the Euler sums which are the alternating version of the multiple zeta values. In this paper, we present some systematic structural results of the van Hamme type congruences of these sums, collected as finite Euler sums. Moreover, we relate this to the structure of the Euler sums which generalizes the corresponding result of the multiple zeta values. We also provide a few conjectures with extensive numerical support.

1. INTRODUCTION AND PRELIMINARIES

A very fruitful practice in producing interesting congruences is to consider so-called van Hamme type congruences. One starts with a well-behaved infinite series whose summands are given by rational numbers and then, for suitable primes p , looks at the $(p - 1)$ -st partial sum modulo p , or even modulo higher powers p which leads to super congruences.

A classical result in this spirit is a variant of Wolstenholme's Theorem [17] dating back to the mid nineteenth century: for all prime $p \geq 5$ we have

$$(1) \quad \sum_{k=1}^{p-1} \frac{1}{k^2} \equiv 0 \pmod{p}, \quad \sum_{k=1}^{p-1} \frac{1}{k} \equiv 0 \pmod{p^2}.$$

Congruences in (1) were improved to super congruences later (see [6, 15]) in which one finds that Bernoulli numbers play the key roles by virtue of the following Faulhaber's formula: (see [1, § 23.1.4-7])

$$(2) \quad \sum_{j=1}^{n-1} j^d = \sum_{r=0}^d \binom{d+1}{r} \frac{B_r}{d+1} n^{d+1-r}, \quad \forall n, d \geq 1.$$

Generalizing the congruences in (1) to multiple harmonic sums (MHSs) was the initial motivation for our work in [20]. To define these sums and their alternating version, we start by looking at a sort of double cover of the set \mathbb{N} of positive integers. Let \mathbb{D} be the set of *signed numbers*

$$(3) \quad \mathbb{D} := \mathbb{N} \cup \overline{\mathbb{N}}, \quad \text{where } \overline{\mathbb{N}} = \{\bar{k} : k \in \mathbb{N}\}.$$

Define the absolute value function $|\cdot|$ on \mathbb{D} by $|k| = |\bar{k}| = k$ for all $k \in \mathbb{N}$ and the sign function by $\text{sgn}(k) = 1$ and $\text{sgn}(\bar{k}) = -1$ for all $k \in \mathbb{N}$. On \mathbb{D} we define a commutative and associative binary operation \oplus (called *O-plus*) as follows: for all $a, b \in \mathbb{D}$

$$(4) \quad a \oplus b = \begin{cases} \overline{|a| + |b|}, & \text{if } \text{sgn}(a) \neq \text{sgn}(b); \\ |a| + |b|, & \text{if } \text{sgn}(a) = \text{sgn}(b). \end{cases}$$

For any $d \in \mathbb{N}$ and $\mathbf{s} = (s_1, \dots, s_d) \in \mathbb{D}^d$ we define the *alternating multiple harmonic sums* (AMHSs) by

$$H_n(s_1, \dots, s_d) = \sum_{n \geq k_1 > k_2 > \dots > k_d \geq 1} \prod_{j=1}^d \frac{\text{sgn}(s_j)^{k_j}}{k_j^{|s_j|}},$$

$$H_n^*(s_1, \dots, s_d) = \sum_{n \geq k_1 \geq k_2 \geq \dots \geq k_d \geq 1} \prod_{j=1}^d \frac{\text{sgn}(s_j)^{k_j}}{k_j^{|s_j|}}.$$

When $\mathbf{s} \in \mathbb{N}^d$ both of these have been called *multiple harmonic sums* (MHSs) in the literature. But more precisely, the second should be called *multiple harmonic star sums*. Conventionally, the number d is called the *depth*, denoted by $\text{dp}(\mathbf{s})$, and $|\mathbf{s}| := \sum_{j=1}^d |s_j|$ the *weight*. For convenience we set $H_n(\mathbf{s}) = 0$ if $n < \text{dp}(\mathbf{s})$, $H_n(\emptyset) = H_n^*(\emptyset) = 1$ for all $n \geq 0$. To save space, we put $\{s\}^d = (s, \dots, s)$ with s repeating d times.

For example, we have

Theorem 1.1. ([21, Theorem 2.13]) *Let s and d be two positive integers. Let p be an odd prime such that $p \geq d + 2$ and $p - 1$ divides none of ds and $ks + 1$ for $k = 1, \dots, d$. Then*

$$H_{p-1}(\{s\}^d) \equiv \begin{cases} 0 \pmod{p}, & \text{if the weight } ds \text{ is even;} \\ 0 \pmod{p^2}, & \text{if the weight } ds \text{ is odd.} \end{cases}$$

In particular, the above are always true if $p \geq ds + 3$.

Congruences in (1) have also been generalized to some other non-homogeneous MHSs in [8, 21] and further to the alternating version in [16, 23]. In particular, in [23] we defined an adèle-like structure in which MHSs modulo primes p are collected and form some objects which, after Kaneko and Zagier, we call, in this paper, the *finite multiple zeta values* (FMZVs) and the *finite Euler sums* (FESs). See Definition 2.8 for the precise meaning.

When $\mathbf{s} \in \mathbb{D}^d$ we obtain the Euler (star) sums by taking the limit of AMHSs:

$$\zeta(\mathbf{s}) = \lim_{n \rightarrow \infty} H_n(\mathbf{s}), \quad \zeta^*(\mathbf{s}) = \lim_{n \rightarrow \infty} H_n^*(\mathbf{s}).$$

When $\mathbf{s} \in \mathbb{N}^d$ these become the *multiple zeta values* (MZVs) and *multiple zeta star values* (MZSVs), respectively.

In recent years, MZVs and Euler sums have appeared in many areas of mathematics and mathematical physics. The main theorem obtained by Brown in [3] implies that every period of the mixed Tate motives unramified over \mathbb{Z} is a $\mathbb{Q}[\frac{1}{2\pi i}]$ -linear combination of the MZVs. It also implies that every MZV is a \mathbb{Q} -linear combination of the *Hoffman elements*, i.e., MZVs with arguments equal to 2 or 3. Let MZ_n be the \mathbb{Q} -vector space spanned by the MZVs of weight n . Then Brown's result shows that the \mathbb{Q} -dimension $\dim \text{MZ}_n \leq d_n$ where the Padovan numbers d_n has the generating series

$$\frac{1}{1 - t^2 - t^3} = \sum_{n=0}^{\infty} d_n t^n.$$

Zagier conjectured that in fact $\dim_{\mathbb{Q}} \text{MZ}_n = d_n$.

Similarly, let ES_n be the \mathbb{Q} -vector space spanned by all the weight n Euler sums. A conjecture similar to that of Zagier made by Broadhurst [2] says that

$$(5) \quad \sum_{n=0}^{\infty} (\dim_{\mathbb{Q}} \text{ES}_n) t^n = \frac{1}{1 - t - t^2}.$$

Hence $\dim_{\mathbb{Q}} \text{ES}_n$ should be just Fibonacci numbers. This has been proved by Deligne [5] under the assumption of a variant Grothendieck's period conjecture.

Very recently, a surprising connection between the FMZVs (see Definition 2.8) and MZVs has emerged. A similar connection between AMHSs and Euler sums should also exist. We will present these in § 8 and § 10 after recalling many related results and proving some new ones in §§ 2-7.

The primary goal of this paper is to continue our study of the congruence properties of MHSs and AMHSs initiated in [8, 16, 21, 23]. The new results come from two sources: one is the algebra approach using quasi-symmetric functions with signed powers (see § 5), the other is the shuffle relation first discovered by Kaneko for MHSs and generalized here to AMHSs (see § 3.2).

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2. AN ADELE-LIKE STRUCTURE

We defined an adèle-like structure in [23] as a suitable theoretical framework in which the van Hamme type congruences for MHSs and AMHSs can be organized and further

investigated. Let \mathcal{P} be the set of primes and let ℓ be a positive integer. For any $n \in \mathbb{N}$, put

$$(6) \quad \mathcal{A}_\ell(n) := \prod_{p \in \mathcal{P}, p \geq n} (\mathbb{Z}/p^\ell \mathbb{Z}), \quad \mathcal{A}_\ell := \prod_{p \in \mathcal{P}} (\mathbb{Z}/p^\ell \mathbb{Z}) \Big/ \bigoplus_{p \in \mathcal{P}} (\mathbb{Z}/p^\ell \mathbb{Z}),$$

with componentwise addition and multiplication. We call ℓ the *superbity*. This object seems to appear first in [12].

Remark 2.1. Note that there is a natural projection $\mathcal{A}_\ell(n) \rightarrow \mathcal{A}_\ell$ by inserting 0's in the components corresponding to $p < n$. So every identity that holds in $\mathcal{A}_\ell(n)$ for any particular n is still valid in \mathcal{A}_ℓ , but not vice versa. Hence, results obtained in [23] are more precise than the corresponding ones stated in the framework of this paper.

Suppose the elements of \mathcal{A}_ℓ are represented by $(a_p) := (a_p)_{p \in \mathcal{P}}$. Observe that any two elements (a_p) and (b_p) represent the same one in \mathcal{A}_ℓ if and only if $a_p = b_p$ for all but finitely many primes p . It is straightforward to see that \mathcal{A}_ℓ is a \mathbb{Q} -algebra after embedding \mathbb{Q} in \mathcal{A}_ℓ “diagonally” using the following map:

$$(7) \quad \begin{aligned} \iota_\ell : \mathbb{Q} &\longrightarrow \mathcal{A}_\ell \\ r &\longrightarrow (\iota_{p,\ell}(r))_{p \in \mathcal{P}} \end{aligned}$$

where $\iota_\ell(0) = (0)_{p \in \mathcal{P}}$ and for all nonzero $r \in \mathbb{Q}$

$$\iota_{p,\ell}(r) := \begin{cases} 0, & \text{if } \text{ord}_p(r) < 0; \\ r \pmod{p^\ell}, & \text{if } \text{ord}_p(r) \geq 0. \end{cases}$$

Proposition 2.2. (cf. [23, Lemma 2.1]) *The map ι_ℓ is a monomorphism of algebras. Namely, ι_ℓ gives an embedding of \mathbb{Q} into \mathcal{A}_ℓ as an algebra.*

Proof. Clear. □

Remark 2.3. It is easy to see that the cardinality of \mathcal{A}_ℓ is \aleph_1 , *i.e.*, the same as that of the real numbers. It is not too hard to construct an embedding of \mathbb{R} into \mathcal{A}_ℓ as a set. But we don't know whether there is an embedding of \mathbb{R} as an algebra.

The following conventions are quite convenient for us. First, we abuse our notation by writing a_p for $(a_p)_{p \in \mathcal{P}}$ if no confusion arises. In particular, whenever p appears in an equation in \mathcal{A}_ℓ ($\ell \geq 2$) it means the element $(p)_{p \in \mathcal{P}}$. Second, by writing $(a_p)_{p \geq k} \in \mathcal{A}_\ell$ we mean the element $(a_p)_{p \in \mathcal{P}}$ with $a_p = 0$ for all $p < k$.

Definition 2.4. We define a number $a \in \mathcal{A}_\ell$ to be *algebraic over \mathbb{Q}* if there is a nontrivial polynomial $f(t) \in \mathbb{Q}[t]$ such that $f(a) = 0$. A non-algebraic number of \mathcal{A}_ℓ over \mathbb{Q} is called *transcendental*. Finitely many numbers $a_1, \dots, a_n \in \mathcal{A}_\ell$ are called *algebraically*

independent over \mathbb{Q} if for any nontrivial polynomial $f(t_1, \dots, t_n) \in \mathbb{Q}[t_1, \dots, t_n]$ we have $f(a_1, \dots, a_n) \neq 0$. We call the elements in an infinite subset S of \mathcal{A}_ℓ *algebraically independent over \mathbb{Q}* if any finitely many elements of S are.

Definition 2.5. For any non-negative integer k , we define the \mathcal{A}_1 -Bernoulli numbers

$$\beta_k := \left(\frac{B_{p-k}}{k} \pmod{p} \right)_{p>k}.$$

For all $k \geq 2$, we define the k th \mathcal{A}_1 -Fermat quotient

$$q_k := \left(\frac{k^{p-1} - 1}{p} \pmod{p} \right)_{p>k}.$$

We see that $\beta_{2k} = 0$ for all $k \geq 1$ since all Bernoulli-number $B_{2j+1} = 0$ when $j \geq 1$. As for the \mathcal{A}_1 -Fermat quotient q_2 , according to [11], we know that for all primes less than 1.25×10^{15} the super congruence $2^{p-1} \equiv 1 \pmod{p^2}$ is satisfied by only two primes 1093 and 3511 which are called Wieferich primes. It is also known that $q_2 \neq 0$ in \mathcal{A}_1 under *abc*-conjecture (see [14]). Moreover, note that we have $q_{k\ell} = q_k + q_\ell$ for all k, ℓ and $q_1 = 0$. So q_k is an \mathcal{A}_1 -analog of the logarithm value $\log k$.

Conjecture 2.6. Put $\beta_1 = 1$ by abuse of notation. Suppose $n_1, \dots, n_r \in \mathbb{N}$ such that $\log n_1, \dots, \log n_r$ are \mathbb{Q} -linearly independent. Then the numbers in the set

$$\bigcup_{k=0}^{\infty} \left\{ \beta_{2k+1} q_{n_1}, \dots, \beta_{2k+1} q_{n_r} \right\}$$

are algebraically independent.

One should compare this with the following

Conjecture 2.7. Put $\zeta(1) = 1$ by abuse of notation. Suppose $n_1, \dots, n_r \in \mathbb{N}$ such that $\log n_1, \dots, \log n_r$ are \mathbb{Q} -linearly independent. Then the numbers in the set

$$\bigcup_{k=0}^{\infty} \left\{ \zeta(2k+1) \log n_1, \dots, \zeta(2k+1) \log n_r \right\}$$

are algebraically independent.

Like \mathcal{A}_1 -Bernoulli numbers and \mathcal{A}_1 -Fermat quotients, every object similarly defined for each prime can be put into \mathcal{A}_1 . Here is a very short list for such numbers that we believe are of some interest.

- Wilson quotients $\frac{(p-1)!+1}{p}$
- p -adic Γ values at any rational point $\Gamma_p(a/b)$
- $F_{p-\frac{5}{p}}/p$: the Fibonacci quotient (OEIS A092330)

- $U_{p-\frac{2}{p}}(2, -1)/p$: the Pell quotient (OEIS A000129)
- $U_{p-\frac{3}{p}}(4, 1)/p$: a quotient related to the Lucas sequence 1, 4, 15, 56, 209, ... (OEIS A001353)
- $U_{p-\frac{6}{p}}(10, 1)/p$: a quotient related to the Lucas sequence 1, 10, 99, 980, 9701, ... (OEIS A004189)

Definition 2.8. For $\mathbf{s} = (s_1, \dots, s_d) \in \mathbb{D}^d$ (see (3)), we define

$$\zeta_{\mathcal{A}_\ell}(\mathbf{s}) = \sum_{p > k_1 > \dots > k_d \geq 1} \frac{\text{sgn}(s_1)^{k_1} \dots \text{sgn}(s_d)^{k_d}}{k_1^{|s_1|} \dots k_d^{|s_d|}} \in \mathcal{A}_\ell,$$

$$\zeta_{\mathcal{A}_\ell}^*(\mathbf{s}) = \sum_{p > k_1 \geq \dots \geq k_n \geq 1} \frac{\text{sgn}(s_1)^{k_1} \dots \text{sgn}(s_d)^{k_d}}{k_1^{|s_1|} \dots k_d^{|s_d|}} \in \mathcal{A}_\ell.$$

These elements in \mathcal{A}_ℓ are called *finite Euler sums* (FESs) of superbity ℓ . Further, if all the s_j 's are positive integers then, after Kaneko and Zagier, we call them *finite multiple zeta values* (FMZVs) of superbity ℓ .

Remark 2.9. We have intentionally dropped the word “star” for the $\zeta_{\mathcal{A}_\ell}^*$ -values since we will see that they can be expressed by all the $\zeta_{\mathcal{A}_\ell}$ -values. See (24) in Theorem 6.1.

For example, Theorem 1.1 can be rephrased as

Theorem 2.10. *Let s and d be two positive integers. Then*

$$(8) \quad \zeta_{\mathcal{A}_1}(\{s\}^d) = 0 \quad \text{if the weight } ds \text{ is even;}$$

$$(9) \quad \zeta_{\mathcal{A}_2}(\{s\}^d) = 0 \quad \text{if the weight } ds \text{ is odd.}$$

Theorem 2.10 has been generalized to higher superbities by Zhou and Cai.

Theorem 2.11. ([24, Remark]) *Let $d, s \in \mathbb{N}$. If ds is even then*

$$(10) \quad \zeta_{\mathcal{A}_2}(\{s\}^d) = (-1)^{d-1} s \beta_{ds+1} p.$$

If ds is odd then

$$(11) \quad \zeta_{\mathcal{A}_3}(\{s\}^d) = (-1)^d \frac{s(ds+1)}{2} \beta_{ds+2} p^2.$$

For FESs, by using Euler numbers we obtained the following theorem in [16].

Theorem 2.12. ([16, Corollary 2.3]) *Let $s \in \mathbb{N}$. Then*

$$(12) \quad \zeta_{\mathcal{A}_1}(\bar{s}) = \begin{cases} -2q_2, & \text{if } s = 1; \\ -2(1 - 2^{1-s})\beta_s, & \text{if } s > 2 \text{ is odd;} \end{cases}$$

$$(13) \quad \zeta_{\mathcal{A}_2}(\bar{s}) = s(1 - 2^{-s})p\beta_{s+1}, \quad \text{if } s \text{ is even.}$$

The following results in depth 2 and 3 will be very useful.

Theorem 2.13. ([21, Theorem 3.1], [8, Theorem 6.1]) *For all positive integers s and t*

$$(14) \quad \zeta_{\mathcal{A}_1}(s, t) = \zeta_{\mathcal{A}_1}^*(s, t) = (-1)^s \binom{s+t}{s} \beta_{s+t}.$$

If $s, t \in \mathbb{N}$ and $w = s + t$ is odd, then

$$(15) \quad \zeta_{\mathcal{A}_1}(\bar{s}, t) = \zeta_{\mathcal{A}_1}(s, \bar{t}) = -\zeta_{\mathcal{A}_1}^*(\bar{s}, t) = -\zeta_{\mathcal{A}_1}^*(s, \bar{t}) = (1 - 2^{1-w})\beta_w.$$

Theorem 2.14. ([21, Theorem 3.5], [8, Theorem 6.2]) *Let $(l, m, n) \in \mathbb{N}^3$. If $w = l + m + n$ is odd then*

$$-\zeta_{\mathcal{A}_1}(l, m, n) = \zeta_{\mathcal{A}_1}^*(l, m, n) = \left[(-1)^l \binom{w}{l} - (-1)^n \binom{w}{n} \right] \frac{\beta_w}{2}.$$

3. DOUBLE SHUFFLE RELATIONS OF FINITE EULER SUMS

3.1. Stuffle relations of finite Euler sums. There are many \mathbb{Q} -linear relations among Euler sums. One of the most important tools to study these is so-called (regularized) double shuffle relations. However, in the finite setting, the shuffle structure is not easily seen due to the lack of integral expressions, although the stuffle is obvious. For instance, for any positive integer n , we have

$$H_n(s)H_n(t) = H_n(s, t) + H_n(t, s) + H_n(s \oplus t), \quad \forall s, t \in \mathbb{D},$$

By extending an idea of Hoffman [9] Racinet studied the cyclotomic analogs of MZVs of level N in [13] using algebras of words. At level two, these analogs are exactly the Euler sums considered in [22].

Definition 3.1. Let the level N be a positive integer ($N = 1$ or 2 in this paper). Let Γ_N be the set of N th roots of unity. The set of alphabet $\mathbf{X} = \mathbf{X}_N$ consists of $N + 1$ letters \mathbf{x}_ξ for $\xi \in \{0\} \cup \Gamma_N$. Let X^* be the set of words over \mathbf{X} (*i.e.*, monomials in the letters in \mathbf{X}) including the *empty word* 1. The *weight* of a word \mathbf{w} is the number of letters contained in \mathbf{w} and its *depth* is the number of \mathbf{x}_ξ 's ($\xi \in \Gamma_N$) contained in \mathbf{w} . Define the *Hoffman-Racinet algebra* of level N , denoted by \mathfrak{A}_N , to be the (weight) graded noncommutative polynomial \mathbb{Q} -algebra generated by X^* . Let \mathfrak{A}_N^0 be the subalgebra of \mathfrak{A}_N generated by words not beginning with x_1 and not ending with x_0 . The words in \mathfrak{A}_N^0 are called *admissible words*.

A shuffle product, denoted by \sqcup , is defined on \mathfrak{A}_N as follows: $1 \sqcup \mathbf{w} = \mathbf{w} \sqcup 1 = \mathbf{w}$ for all $\mathbf{w} \in \mathfrak{A}_N$, and

$$x\mathbf{u} \sqcup y\mathbf{v} = x(\mathbf{u} \sqcup y\mathbf{v}) + y(x\mathbf{u} \sqcup \mathbf{v}),$$

for all $x, y \in \mathbf{X}$ and $\mathbf{u}, \mathbf{v} \in X^*$. Then \sqcup is extended \mathbb{Q} -linearly over \mathfrak{A}_N .

Let \mathfrak{A}_N^1 be the subalgebra of \mathfrak{A}_N generated by those words not ending with \mathbf{x}_0 . For every $n \in \mathbb{N}$, we define the weight n element

$$y_{n,\xi} := \mathbf{x}_0^{n-1} \mathbf{x}_\xi, \quad \xi \in \Gamma_N.$$

Then \mathfrak{A}_N^1 is clearly generated over the alphabet $Y_N := \{y_{n,\mu} : n \in \mathbb{N}, \mu \in \Gamma_N\}$. Let Y_N^* be the set of words over Y_N . We now define a stuffle product $*$ on \mathfrak{A}_N^1 as follows: $1 * \mathbf{w} = \mathbf{w} * 1 = \mathbf{w}$ for all $\mathbf{w} \in Y_N^*$, and

$$(16) \quad y_{m,\mu} \mathbf{u} * y_{n,\nu} \mathbf{v} = y_{m,\mu} (\mathbf{u} * y_{n,\nu} \mathbf{v}) + y_{n,\nu} (y_{m,\mu} \mathbf{u} * \mathbf{v}) + y_{m+n,\mu\nu} (\mathbf{u} * \mathbf{v}),$$

for all $m, n \in \mathbb{N}, \mu, \nu \in \Gamma_N$ and $\mathbf{u}, \mathbf{v} \in Y_N^*$. Then $*$ is extended \mathbb{Q} -linearly over \mathfrak{A}_N^1 .

For convenience, we define $W : \bigcup_{d \geq 1} \mathbb{D}^d \rightarrow Y_2^*$, $W(\mathbf{s}) = y_{|s_1|, \text{sgn}(s_1)} \cdots y_{|s_d|, \text{sgn}(s_d)}$ for all $\mathbf{s} = (s_1, \dots, s_d) \in \mathbb{D}^d$. Clearly the map W is a bijection.

Definition 3.2. Define $\zeta_* : \mathfrak{A}_2^0 \rightarrow \mathbb{R}$ as follows: for any admissible word $\mathbf{w} = W(\mathbf{s}) \in \mathfrak{A}_2^0$ where $\mathbf{s} \in \mathbb{D}^d$ we set $\zeta_*(\mathbf{w}) := \zeta(\mathbf{s})$. Then we extend it \mathbb{Q} -linearly to \mathfrak{A}_2^0 .

Definition 3.3. Define $\zeta_{\mathfrak{A}_\ell, *}: \mathfrak{A}_2^1 \rightarrow \mathfrak{A}_\ell$ as follows: for any word $\mathbf{w} = W(\mathbf{s}) \in \mathfrak{A}_2^1$ where $\mathbf{s} \in \mathbb{D}^d$ we set $\zeta_{\mathfrak{A}_\ell, *}(\mathbf{w}) := \zeta_{\mathfrak{A}_\ell}(\mathbf{s})$. Then we extend it \mathbb{Q} -linearly to \mathfrak{A}_2^1 .

Proposition 3.4. *The map $\zeta_* : (\mathfrak{A}_2^0, *) \rightarrow \mathbb{R}$ is an algebra homomorphism. So is the map $\zeta_{\mathfrak{A}_\ell, *} : (\mathfrak{A}_2^1, *) \rightarrow \mathfrak{A}_\ell$.*

Proof. By induction on $|\mathbf{u}| + |\mathbf{v}|$ we can prove easily that

$$\begin{aligned} \zeta_*(\mathbf{u} * \mathbf{v}) &= \zeta_*(\mathbf{u}) \zeta_*(\mathbf{v}) & \forall \mathbf{u}, \mathbf{v} \in \mathfrak{A}_2^0, \\ \zeta_{\mathfrak{A}_\ell, *}(\mathbf{u} * \mathbf{v}) &= \zeta_{\mathfrak{A}_\ell, *}(\mathbf{u}) \zeta_{\mathfrak{A}_\ell, *}(\mathbf{v}) & \forall \mathbf{u}, \mathbf{v} \in \mathfrak{A}_2^1. \end{aligned}$$

We leave the details to the interested reader. \square

Definition 3.5. To find as many \mathbb{Q} -linear relations as possible in weight w we may choose all the known relations in weight $k < w$, multiply them by $\zeta_{\mathfrak{A}_\ell}(\mathbf{s})$ for all \mathbf{s} of weight $w - k$, and then expand all the products using the stuffle relation (16). All the \mathbb{Q} -linear relations among FESs of the same weight produced in this way are called *linear stuffle relations of FESs*.

Example 3.6. By (14) and (15) we have

$$\zeta_{\mathfrak{A}_1}(2, 1) = 4\zeta_{\mathfrak{A}_1}(2, \bar{1}) = 3\beta_3.$$

Multiplying $\zeta_{\mathfrak{A}_1}(\bar{1})$ on both sides we get a linear stuffle relation of FESs of weight 4:

$$(17) \quad \begin{aligned} \zeta_{\mathfrak{A}_1}(\bar{1}, 2, 1) + \zeta_{\mathfrak{A}_1}(\bar{3}, 1) + \zeta_{\mathfrak{A}_1}(2, \bar{1}, 1) + \zeta_{\mathfrak{A}_1}(2, 1, \bar{1}) + \zeta_{\mathfrak{A}_1}(2, \bar{2}) \\ = 8\zeta_{\mathfrak{A}_1}(2, \bar{1}, \bar{1}) + 4\zeta_{\mathfrak{A}_1}(\bar{3}, \bar{1}) + 4\zeta_{\mathfrak{A}_1}(\bar{1}, 2, \bar{1}). \end{aligned}$$

3.2. Shuffle relations of FESs. It is not hard to see that Euler sums can be expressed by iterated integrals. Suppose $s_j \in \mathbb{D}$ and $\text{sgn}(s_j) = \mu_j$ for all $j = 1, \dots, d$. Then

$$(18) \quad \zeta(s_1, \dots, s_d) = \int_0^1 \left(\frac{dt}{t}\right)^{|s_1|-1} \left(\frac{dt}{\xi_1 - t}\right) \cdots \left(\frac{dt}{t}\right)^{|s_d|-1} \left(\frac{dt}{\xi_d - t}\right),$$

where $\xi_i = \prod_{j=1}^i \mu_j$, $i = 1, \dots, d$. We thus define $\mathbf{p}, \mathbf{p}^{-1} : \mathfrak{A}_2^1 \rightarrow \mathfrak{A}_2^1$ by

$$\mathbf{p}(y_{s_1, \xi_1} \cdots y_{s_d, \xi_d}) := y_{s_1, \mu_1} \cdots y_{s_d, \mu_d},$$

where $\mu_i = \prod_{j=1}^i \xi_j$ and

$$\mathbf{p}^{-1}(y_{s_1, \mu_1} \cdots y_{s_d, \mu_d}) := y_{s_1, \xi_1} \cdots y_{s_d, \xi_d},$$

where $\xi_j = \mu_{j-1}^{-1} \mu_j$ (setting $\mu_0 = 1$).

For all $s_1, \dots, s_d \in \mathbb{D}$, we now define the one-variable multiple polylog

$$L_{s_1, \dots, s_d}(z) := \sum_{k_1 > k_2 > \dots > k_d \geq 1} z^{k_1} \prod_{j=1}^d \frac{\text{sgn}(s_j)^{k_j}}{k_j^{|s_j|}}, \quad |z| < 1.$$

Then it is easy to extend (18) to these functions.

Lemma 3.7. ([13, Proposition 2.2.8]) *For all $s_1, \dots, s_d \in \mathbb{D}$, we have*

$$L_{s_1, \dots, s_d}(z) = \int_0^z \left(\frac{dt}{t}\right)^{|s_1|-1} \left(\frac{dt}{\xi_1 - t}\right) \cdots \left(\frac{dt}{t}\right)^{|s_d|-1} \left(\frac{dt}{\xi_d - t}\right),$$

where $\xi_i = \text{sgn}(s_{i-1}) \text{sgn}(s_i)$, $i = 1, \dots, d$. Here we have set $\text{sgn}(s_0) = 1$.

Define the map $L(-; z) : (\mathfrak{A}_2^1, *) \rightarrow \mathbb{R}[[z]]$ by setting $L(\mathbf{w}; z) = L_{\mathbf{s}}(z)$ for all $\mathbf{w} = W(\mathbf{s})$. Then define the map $L_{\sqcup}(-; z) : (\mathfrak{A}_2^1, \sqcup) \rightarrow \mathbb{R}[[z]]$ by setting $L_{\sqcup}(\mathbf{w}; z) = L(\mathbf{p}(\mathbf{w}); z)$.

Proposition 3.8. *The map $L_{\sqcup}(-; z) : (\mathfrak{A}_2^1, \sqcup) \rightarrow \mathbb{R}[[z]]$ is an algebra homomorphism.*

Proof. This follows from Lemma 3.7 and the shuffle relation satisfied by the iterated integrals. \square

When $z \rightarrow 1$ we obtain the algebra homomorphism $\zeta_{\sqcup} : (\mathfrak{A}_2^0, \sqcup) \rightarrow \mathbb{R}$ with $\zeta_{\sqcup}(\mathbf{w}) = \zeta_*(\mathbf{p}(\mathbf{w}))$. One of the main results of [13] is the following theorem.

Theorem 3.9. ([13, Proposition 2.4.14]) *Set $\zeta_{\sqcup}(\mathbf{x}_1) = \zeta_*(\mathbf{x}_1) = T$. Then*

- (i) ζ_* can be extended to an algebra homomorphism $\zeta_* : (\mathfrak{A}_2^1, *) \rightarrow \mathbb{R}[T]$;
- (ii) ζ_{\sqcup} can be extended to an algebra homomorphism $\zeta_{\sqcup} : (\mathfrak{A}_2^1, \sqcup) \rightarrow \mathbb{R}[T]$.

Moreover,

$$\zeta_{\sqcup} = \rho \circ \zeta_* \circ \mathbf{p}$$

where $\rho : \mathbb{R}[t] \rightarrow \mathbb{R}[t]$ is an \mathbb{R} -linear map such that

$$\rho(e^{Tu}) = \exp\left(\sum_{n=2}^{\infty} \frac{(-1)^n}{n} \zeta(n) u^n\right) e^{Tu}, \quad |u| < 1.$$

Example 3.10. Suppose $\mathbf{s} = (\bar{2}, 1)$. Then $W(\mathbf{s}) = \mathbf{y}_{2,-1}\mathbf{y}_{1,1}$ and

$$L(\mathbf{y}_{2,-1}\mathbf{y}_{1,1}; z) = \int_0^1 \frac{dt}{t} \left(\frac{-dt}{1+t}\right)^2 = L_{\bar{2},1}(z).$$

Set $\zeta_{\mathfrak{A}_1, \sqcup} = \zeta_{\mathfrak{A}_1, *} \circ \mathbf{p} : \mathfrak{A}_2^1 \rightarrow \mathfrak{A}_1$. Although $\zeta_{\mathfrak{A}_1, \sqcup}$ is not an algebra homomorphism from $(\mathfrak{A}_2^1, \sqcup)$ we will see in the next theorem that it does provide a kind of shuffle relation. It is the FES analog of [10, Theorem 1.7] for symmetrized Euler sums (see (42) and (43)).

Define $\tau : \mathbf{Y}_1^* \rightarrow \mathbf{Y}_1^*$ by

$$\tau(\mathbf{x}_0^{s_1-1} \mathbf{x}_1 \cdots \mathbf{x}_0^{s_d-1} \mathbf{x}_1) = (-1)^{s_1+\cdots+s_d} \mathbf{x}_0^{s_d-1} \mathbf{x}_1 \cdots \mathbf{x}_0^{s_1-1} \mathbf{x}_1.$$

Theorem 3.11. For all words $\mathbf{w}, \mathbf{u} \in \mathbf{Y}_1^*$ and $\mathbf{v} \in \mathbf{Y}_2^*$, we have

- (i) $\zeta_{\mathfrak{A}_1, \sqcup}(\mathbf{u} \sqcup \mathbf{v}) = \zeta_{\mathfrak{A}_1, \sqcup}(\tau(\mathbf{u})\mathbf{v})$,
- (ii) $\zeta_{\mathfrak{A}_1, \sqcup}((\mathbf{w}\mathbf{u}) \sqcup \mathbf{v}) = \zeta_{\mathfrak{A}_1, \sqcup}(\mathbf{u} \sqcup \tau(\mathbf{w})\mathbf{v})$,
- (iii) For all $s \in \mathbb{N}$, $\zeta_{\mathfrak{A}_1, \sqcup}((\mathbf{x}_0^{s-1} \mathbf{x}_1 \mathbf{u}) \sqcup \mathbf{v}) = (-1)^s \zeta_{\mathfrak{A}_1, \sqcup}(\mathbf{u} \sqcup (\mathbf{x}_0^{s-1} \mathbf{x}_1 \mathbf{v}))$.

Proof. Taking $\mathbf{u} = \emptyset$ and then $\mathbf{w} = \mathbf{u}$ we see that (ii) implies (i). Decomposing \mathbf{w} into strings of the type $\mathbf{x}_0^{s-1} \mathbf{x}_1$ we see that (iii) implies (ii). Now we only need to prove (iii).

For simplicity let $\mathbf{a} = \mathbf{x}_0$ and $\mathbf{b} = \mathbf{x}_1$ in the rest of this proof. Let $s_j \in \mathbb{D}$ with $\text{sgn}(s_j) = \eta_j$ for $j = 1, \dots, d$. Let $\mathbf{u} = W(\mathbf{s})$ and $\mathbf{v} = \mathbf{p}(\mathbf{u}) = \mathbf{y}_{t_1, \xi_1} \cdots \mathbf{y}_{t_d, \xi_d} \in \mathbb{D}^d$. Then clearly $\mathbf{p}(\mathbf{b}\mathbf{u}) = \mathbf{b}\mathbf{v}$.

For any prime $p > 2$, the coefficient of z^p in $L_{\sqcup}(\mathbf{b}\mathbf{u}; z)$ is given by

$$\text{Coeff}_{z^p} [L_{\sqcup}(\mathbf{b}\mathbf{u}; z)] = \frac{1}{p} \sum_{p > k_1 > \cdots > k_d > 0} \frac{\xi_1^{k_1} \cdots \xi_d^{k_d}}{k_1^{t_1} \cdots k_d^{t_d}} = \frac{1}{p} H_{p-1}(\mathbf{p}(\mathbf{u})).$$

Observe that

$$\mathbf{b}\left((\mathbf{a}^{s-1} \mathbf{b}\mathbf{u}) \sqcup \mathbf{v} - (-1)^s \mathbf{u} \sqcup (\mathbf{a}^{s-1} \mathbf{b}\mathbf{v})\right) = \sum_{i=0}^{s-1} (-1)^i (\mathbf{a}^{s-1-i} \mathbf{b}\mathbf{u}) \sqcup (\mathbf{a}^i \mathbf{b}\mathbf{v}).$$

By first applying $L_{\sqcup}(-; z)$ to the above and then extracting the coefficients of z^p from both sides we get

$$\begin{aligned} & \frac{1}{p} \left(H_{p-1} \circ \mathbf{p}((\mathbf{a}^{s-1} \mathbf{b} \mathbf{u}) \sqcup \mathbf{v}) - (-1)^s H_{p-1} \circ \mathbf{p}(\mathbf{u} \sqcup (\mathbf{a}^{s-1} \mathbf{b} \mathbf{v})) \right) \\ &= \sum_{i=0}^{s-1} (-1)^i \text{Coeff}_{z^p} [L_{\sqcup}(\mathbf{a}^{s-1-i} \mathbf{b} \mathbf{u}; z) L_{\sqcup}(\mathbf{a}^i \mathbf{b} \mathbf{v}; z)] \\ &= \sum_{i=0}^{s-1} (-1)^i \sum_{j=1}^{p-1} \text{Coeff}_{z^j} [L_{\sqcup}(\mathbf{a}^{s-1-i} \mathbf{b} \mathbf{u}; z)] \text{Coeff}_{z^{p-j}} [L_{\sqcup}(\mathbf{a}^i \mathbf{b} \mathbf{v}; z)] \end{aligned}$$

by Proposition 3.8. Now the last sum is p -integral since $p-j < p$ and $j < p$. Therefore we get

$$H_{p-1} \circ \mathbf{p}((\mathbf{a}^{s-1} \mathbf{b} \mathbf{u}) \sqcup \mathbf{v}) \equiv (-1)^s H_{p-1} \circ \mathbf{p}(\mathbf{u} \sqcup (\mathbf{a}^{s-1} \mathbf{b} \mathbf{v})) \pmod{p}$$

which completes the proof of (iii). \square

Definition 3.12. A relation produced by Theorem 3.11 is called a *linear shuffle relation* of FES. For each weight $w \geq 2$, by the *double shuffle relations* of FESs of weight w we mean all the linear shuffle relations of weight w and all the linear stuffle relations of w defined in Definition 3.5.

Restricting to FMZVs, we obtain the *linear shuffle relations* and *double shuffle relations* of FMZVs.

4. REVERSAL RELATIONS

For any $\mathbf{s} = (s_1, \dots, s_d) \in \mathbb{D}^d$, denote its *reversal* by $\overleftarrow{\mathbf{s}} = (s_d, \dots, s_1)$ and set $\text{sgn}(\mathbf{s}) = \prod_{j=1}^d \text{sgn}(s_j)$. The following results are called *reversal relations*.

Theorem 4.1. ([23, (6)]) *Let $\mathbf{s} = (s_1, \dots, s_d) \in \mathbb{D}^d$. Then*

$$(19) \quad \zeta_{\mathcal{A}_1}(\overleftarrow{\mathbf{s}}) = (-1)^{|\mathbf{s}|} \text{sgn}(\mathbf{s}) \zeta_{\mathcal{A}_1}(\mathbf{s}), \quad \zeta_{\mathcal{A}_1}^*(\overleftarrow{\mathbf{s}}) = (-1)^{|\mathbf{s}|} \text{sgn}(\mathbf{s}) \zeta_{\mathcal{A}_1}^*(\mathbf{s}).$$

Theorem 4.1 can be lifted to superbity two.

Theorem 4.2. ([21, Theorem 2.1]) *Let $\mathbf{s} = (s_1, \dots, s_d) \in \mathbb{D}^d$. Let $\mathbf{e}_i = (0, \dots, 0, 1, 0, \dots, 0)$ where 1 appears at the i th component. Then*

$$(20) \quad (-1)^{|\mathbf{s}|} \text{sgn}(\mathbf{s}) \zeta_{\mathcal{A}_2}(\overleftarrow{\mathbf{s}}) = \zeta_{\mathcal{A}_2}(\mathbf{s}) + p \sum_{i=1}^d |s_i| \zeta_{\mathcal{A}_2}(\mathbf{s} \oplus \mathbf{e}_i),$$

$$(21) \quad (-1)^{|\mathbf{s}|} \text{sgn}(\mathbf{s}) \zeta_{\mathcal{A}_2}^*(\overleftarrow{\mathbf{s}}) = \zeta_{\mathcal{A}_2}^*(\mathbf{s}) + p \sum_{i=1}^d |s_i| \zeta_{\mathcal{A}_2}^*(\mathbf{s} \oplus \mathbf{e}_i),$$

where the binary operation \oplus is carried out componentwise by (4).

5. QUASI-SYMMETRIC FUNCTIONS WITH SIGNED POWERS

To derive more relations between MHSs and AMHSs we turn to the theory of quasi-symmetric functions. To treat AMHSs with signed indices in \mathbb{D} we have to allow the powers in these quasi-symmetric functions to be signed numbers.

Definition 5.1. We denote by $\mathbb{Z}[x_1, \dots, x_n; \mathbb{D}]$ the set of polynomials in x_1, \dots, x_n with signed powers, namely,

$$\mathbb{Z}[x_1, \dots, x_n; \mathbb{D}] := \left\{ \sum_{e_1=\bar{d}_1}^{d_1} \cdots \sum_{e_n=\bar{d}_n}^{d_n} c_{e_1, \dots, e_n} x_1^{e_1} \cdots x_n^{e_n} \mid d_1, \dots, d_n \in \mathbb{N}_0, c_{e_1, \dots, e_n} \in \mathbb{Z} \right\}.$$

Here, we set $\bar{0} = 0$ and $\sum_{e=\bar{d}}^d$ means e runs through the set $\{\bar{d}, \dots, \bar{1}, 0, 1, \dots, d\}$. Furthermore, $x_j^e x_j^{e'} = x_j^{e \oplus e'}$ for any $j \leq n$ and $e, e' \in \mathbb{D}$. Also we set

$$\deg(x_1^{e_1} \cdots x_n^{e_n}) = |e_1| + \cdots + |e_n|.$$

Definition 5.2. Let $\mathbf{x} = (x_j)_{j \geq 1}$. An element of finite degree $F(\mathbf{x})$ in $\mathbb{Z}[\mathbf{x}; \mathbb{D}]$ is called a *quasi-symmetric function* if for any $i_1 > i_2 > \cdots > i_d$ and $j_1 > j_2 > \cdots > j_d$ and any signed powers $e_1, \dots, e_d \in \mathbb{D}$ the coefficients of the monomials $x_{i_1}^{e_1} \cdots x_{i_d}^{e_d}$ and $x_{j_1}^{e_1} \cdots x_{j_d}^{e_d}$ are the same. The set of all such quasi-symmetric functions is denoted by \mathbf{QSym}_2 .

For positive integer n we define a (weight) graded algebra homomorphism

$$\begin{aligned} \phi_n : (\mathfrak{A}_2^1, *) &\longrightarrow \mathbf{QSym}_2 \\ W(\mathbf{s}) &\longmapsto \sum_{n \geq k_1 > k_2 > \cdots > k_d \geq 1} x_{k_1}^{s_1} \cdots x_{k_d}^{s_d}, \quad \forall d \leq n, \mathbf{s} \in \mathbb{D}^d, \end{aligned}$$

and set $\phi_n(1) = 1$ and $\phi_n(\mathbf{w}) = 0$ if the $|\mathbf{w}| > n$. It is easy to see we can make $(\phi_n)_{n \geq 1}$ into a compatible system to obtain a homomorphism $\phi : \mathfrak{A}_2^1 \rightarrow \mathbf{QSym}_2$.

Example 5.3. We have $\phi(\mathbf{y}_{1,1}\mathbf{y}_{2,-1}) = \sum_{i>j \geq 1} x_i x_j^{\bar{2}} \in \mathbf{QSym}_2$ but it is *not* a symmetric function since the monomial $x_2 x_1^{\bar{2}}$ appears but $x_1 x_2^{\bar{2}}$ does not.

It is not hard to see that an integral basis for \mathbf{QSym}_2 can be chosen as

$$(22) \quad \begin{aligned} E_{\mathbf{s}} = E_{s_1, \dots, s_d} &:= \sum_{k_1 \geq k_2 \geq \cdots \geq k_d} x_{k_1}^{s_1} \cdots x_{k_d}^{s_d}, \quad \text{or} \\ M_{\mathbf{s}} = M_{s_1, \dots, s_d} &:= \sum_{k_1 > k_2 > \cdots > k_d} x_{k_1}^{s_1} \cdots x_{k_d}^{s_d} = \phi(W(s_1, \dots, s_d)). \end{aligned}$$

This yields the following theorem which can be compared to [8, Theorem 2.2].

Theorem 5.4. ϕ provides an isomorphism $(\mathfrak{A}_2^1, *) \cong \mathbf{QSym}_2$.

Proof. Clear. \square

Theorem 5.5. *The antipode S of \mathbf{QSym}_2 is given by the followings: for every $\mathbf{s} \in \mathbb{D}^d$,*

(i) $S(M_{\mathbf{s}}) = (-1)^d E_{\overleftarrow{\mathbf{s}}}$, where $E_{\mathbf{t}} = \sum_{\mathbf{s} \preceq \mathbf{t}} M_{\mathbf{s}}$ and $\mathbf{t} \preceq \mathbf{s}$ means \mathbf{t} can be obtained from \mathbf{s} by combining some of its parts using \oplus .

(ii) $S(M_{\mathbf{s}}) = (-1)^d E_{\overleftarrow{\mathbf{s}}} = \sum_{\sqcup_{j=1}^r \mathbf{s}_j = \mathbf{s}} (-1)^r M_{\mathbf{s}_1} M_{\mathbf{s}_2} \cdots M_{\mathbf{s}_r}$, where $\sqcup_{j=1}^r \mathbf{s}_j$ is the concatenation of \mathbf{s}_1 to \mathbf{s}_r .

(iii) $M_{\overleftarrow{\mathbf{s}}} = (-1)^d \sum_{\sqcup_{j=1}^r \mathbf{s}_j = \mathbf{s}} (-1)^r E_{\mathbf{s}_1} E_{\mathbf{s}_2} \cdots E_{\mathbf{s}_r}$.

Proof. The case for positive compositions \mathbf{s} for part (i) and (ii) is just [9, Theorem 6.2]. A careful reading of their proofs reveals that all the steps are still valid when the components of \mathbf{s} are signed number. Applying antipode S to (ii) and use (i) we can derive (iii) quickly. \square

6. CONCATENATION RELATIONS OF FESS

We now apply the above results concerning quasi-symmetric functions to FESSs. In order to do so, for any $n \in \mathbb{N}$, we define the algebra homomorphism $\rho_n : (\mathfrak{A}_2^1, *) \rightarrow \mathbb{Q}$ such that

$$(23) \quad \rho_n(M_{\mathbf{s}}) = H_n(\mathbf{s}).$$

Theorem 6.1. *For all $\mathbf{s} = (s_1, \dots, s_d) \in \mathbb{D}^d$ and positive integer ℓ*

$$(24) \quad \zeta_{s\ell}^*(\mathbf{s}) = \sum_{\mathbf{t} \preceq \mathbf{s}} \zeta_{s\ell}(\mathbf{t}), \quad \zeta_{s\ell}(\mathbf{s}) = \sum_{\mathbf{t} \preceq \mathbf{s}} (-1)^{\text{dp}(\mathbf{s}) - \text{dp}(\mathbf{t})} \zeta_{s\ell}^*(\mathbf{t}).$$

When $\ell = 1$ we have

$$(25) \quad \zeta_{s1}^*(\overleftarrow{\mathbf{s}}) = (-1)^d \sum_{\sqcup_{j=1}^r \mathbf{s}_j = \mathbf{s}} (-1)^r \prod_{j=1}^r \zeta_{s1}(\mathbf{s}_j),$$

$$(26) \quad \zeta_{s1}(\overleftarrow{\mathbf{s}}) = (-1)^d \sum_{\sqcup_{j=1}^r \mathbf{s}_j = \mathbf{s}} (-1)^r \prod_{j=1}^r \zeta_{s1}^*(\mathbf{s}_j).$$

Proof. These equations follow from the definition of $E_{\mathbf{s}}$, Theorem 5.5(ii) and (iii), respectively, after we apply ρ_{p-1} for all primes p . \square

Definition 6.2. We will call the relations in (25) and (26) the *concatenation relations* between FMZVs and FESSs.

Example 6.3. By (24) and the concatenation relation (25) we have

$$\zeta_{s_1}(1, \bar{2}) + \zeta_{s_1}(\bar{3}) = \zeta_{s_1}^*(1, \bar{2}) = \zeta_{s_1}(\bar{2})\zeta_{s_1}(1) - \zeta_{s_1}(\bar{2}, 1).$$

Thus we get $\zeta_{s_1}(\bar{3}) = -\zeta_{s_1}(\bar{2}, 1)$ since $\zeta_{s_1}(1) = 0$ and $\zeta_{s_1}(1, \bar{2}) = \zeta_{s_1}(\bar{2}, 1)$ by the reversal relation (19).

7. DUALITY OF FMZVS

The FMZVs satisfy a different kind of duality from that of MZVs.

Definition 7.1. For positive integers $r_1, \dots, r_\ell, t_1, \dots, t_\ell$, let

$$\mathbf{s} = (r_1, \{1\}^{t_1-1}, r_2 + 1, \{1\}^{t_2-1}, \dots, r_\ell + 1, \{1\}^{t_\ell-1}).$$

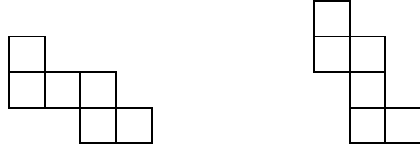
We define the v-dual of \mathbf{s} by

$$(27) \quad \mathbf{s}^\vee = (\{1\}^{r_1-1}, t_1 + 1, \{1\}^{r_2-1}, t_2 + 1, \dots, t_{\ell-1} + 1, \{1\}^{r_\ell-1}, t_\ell).$$

From the definition we clearly have

$$(28) \quad \text{dp}(\mathbf{s}) + \text{dp}(\mathbf{s}^\vee) = |\mathbf{s}| + 1.$$

The v-dual can be easily explained using the conjugation on the ribbons (a kind of skew-Young diagrams) as follows. For a composition $\mathbf{s} = (s_1, \dots, s_d)$ the ribbon $R_{\mathbf{s}}$ is defined to be the skew-Young diagram of d rows whose j th row starts below the last box of $(j-1)$ st row and has exactly s_j boxes. Recall that the conjugate of a (skew-)Young diagram is the mirror image about the diagonal line going from the south-west corner to north-east. For e.g., the following two diagrams give the ribbon $R_{1,3,2}$ and its conjugate:



In general it can be shown without too much difficulty that the ribbon $R_{\mathbf{s}^\vee}$ is exactly the conjugate of the ribbon $R_{\mathbf{s}}$. So $(2, 3, 1)^\vee = (1, 2, 1, 2)$.

Theorem 7.2. ([9, Theorem 6.7]) *Let $\mathbf{s} = (s_1, \dots, s_d) \in \mathbb{N}^d$. Then*

$$(29) \quad \zeta_{s_1}^*(\mathbf{s}) = -\zeta_{s_1}^*(\mathbf{s}^\vee).$$

We now consider duality property in superbity two.

Theorem 7.3. ([21, Theorem 2.11]) *Let \mathbf{s} be any composition of positive integers of weight w . Then*

$$(30) \quad -\zeta_{\mathcal{A}_2}^*(\mathbf{s}^\vee) = \zeta_{\mathcal{A}_2}^*(\mathbf{s}) + p \cdot \sum_{\mathbf{t} \preceq \mathbf{s}} \zeta_{\mathcal{A}_2}(1, \mathbf{t}).$$

Parallel to [10, Corollary 1.12] for the symmetrized MZVs (see (31) and (32)), the following result on another kind of duality provides further evidence for Conjecture 8.2.

Theorem 7.4. ([9, Theorem 6.9]) *For all $\mathbf{w} \in \mathfrak{A}_1^1$ we have*

$$\zeta_{\mathcal{A}_1}(\mathbf{w}) = \zeta_{\mathcal{A}_1}(\varphi(\mathbf{w})),$$

where φ is the involution on \mathfrak{A}_1^1 defined by $\varphi(\mathbf{x}_0) = \mathbf{x}_0 + \mathbf{x}_1$ and $\varphi(\mathbf{x}_1) = -\mathbf{x}_1$.

8. DIMENSION CONJECTURES FOR (FINITE) MZVs

Denote by $\text{FMZ}_{w,\ell}$ the \mathbb{Q} -vector subspace of \mathcal{A}_ℓ generated by all FMZVs of weight w and superbitity ℓ . Further we write $\text{FMZ}_w = \text{FMZ}_{w,1}$. Numerical evidence supports the following conjecture.

Conjecture 8.1. *Let w be any positive integer.*

(i) *Set $d_0 = 1$ and $d_w = \dim \text{FMZ}_w$ for all $w \geq 1$. Then $d_1 = d_2 = 0$ and*

$$d_w = d_{w-2} + d_{w-3} \quad \forall w \geq 3.$$

(ii) *For all $w \geq 3$, FMZ_w has a basis*

$$\{\zeta_{\mathcal{A}_1}(1, 2, a_1, \dots, a_r) : a_1, \dots, a_r = 2 \text{ or } 3, a_1 + \dots + a_r = w - 3\}.$$

(iii) *All the \mathbb{Q} -linear relations among FMZVs can be produced by the double shuffle relations.*

In the spring of 2013, after the author gave an Ober-Seminar talk at the Max Planck Institute for Mathematics at Bonn, Prof. D. Zagier mentioned that he had come to the dimension part of Conjecture 8.1 some time earlier [19]. In fact, Kaneko and Zagier proposed the following more precise relation between FMZVs and MZVs.

Conjecture 8.2. *There is an \mathbb{Q} -algebra isomorphism*

$$\begin{aligned} f_{\text{KZ}} : \text{FMZ}_{w,1} &\longrightarrow \text{MZ}_w / \zeta(2)\text{MZ}_{w-2} \\ \zeta_{\mathcal{A}_1}(\mathbf{s}) &\longmapsto \zeta_{\square}^{\mathcal{S}}(\mathbf{s}) \end{aligned}$$

where for $\mathbf{s} = (s_1, \dots, s_d)$, the symmetrized MZVs

$$(31) \quad \zeta_{\sqcup}^{\mathbf{s}}(\mathbf{s}) = \sum_{i=0}^d (-1)^{s_1 + \dots + s_i} \zeta_{\sqcup}(s_i, \dots, s_1) \zeta_{\sqcup}(s_{i+1}, \dots, s_d),$$

$$(32) \quad \zeta_{*}^{\mathbf{s}}(\mathbf{s}) = \sum_{i=0}^d (-1)^{s_1 + \dots + s_i} \zeta_{*}(s_i, \dots, s_1) \zeta_{*}(s_{i+1}, \dots, s_d),$$

where ζ_{\sqcup} and ζ_{*} on the right-hand side are given by Theorem 3.9.

The following results are contained in [10] (see Remarque 1.3 and Fait 1.8).

Proposition 8.3. *For all composition \mathbf{s} of positive integers,*

- $\zeta_{\sqcup}^{\mathbf{s}}(\mathbf{s})$ and $\zeta_{*}^{\mathbf{s}}(\mathbf{s})$ are all finite, and
- $\zeta_{\sqcup}^{\mathbf{s}}(\mathbf{s}) - \zeta_{*}^{\mathbf{s}}(\mathbf{s}) \in \zeta(2)\text{MZ}_{w-2}$ for all $|\mathbf{s}| = w$.

Therefore, one may also replace ζ_{*} by ζ_{\sqcup} in Conjecture 8.2. Further, the conjectured map is surjective according to the next theorem proved by Yasuda [18].

Theorem 8.4. *Let $\sharp = \sqcup$ or $*$. Then the space MZ_w is generated by symmetrized MZVs $\{\zeta_{\sharp}^{\mathbf{s}}(\mathbf{s}) : |\mathbf{s}| = w\}$.*

However, even in weight 5, it seems impossible to prove the map f_{KZ} is well defined and injective at the moment. Indeed, we have

$$\zeta_{s_1}(4, 1) = 5\beta_5 = (B_{p-5})_p.$$

But we know $\zeta_{\sqcup}^{\mathbf{s}}(4, 1) = 5\zeta(5) - 2\zeta(2)\zeta(3)$ is conjecturally nonzero on the right-hand side so that if f_{KZ} is well defined then $\beta_5 \neq 0$ which would imply that $B_{p-5} \not\equiv 0 \pmod{p}$ for infinitely many primes p , a statement far from proved. On the other hand, even if we could prove $\beta_5 \neq 0$ by some other means, we still don't know whether $\zeta(2)\zeta(3)/\zeta(5) \in \mathbb{Q}$ is true or not, thus we still don't know whether f_{KZ} is injective.

As a further support of Conjecture 8.1,

Theorem 8.5. *For all $3 \leq w \leq 13$, we have*

$$\text{FMZ}_w = \langle \zeta_{s_1}(1, 2, a_1, \dots, a_r) : a_1, \dots, a_r \in \{2, 3\}, a_1 + \dots + a_r = w - 3 \rangle.$$

Moreover, all the relations in these weights can be proved by the double shuffle relations of FMZVs defined as in Definition 3.12.

Proof. This theorem can be proved with the help of Maple. So we leave it to the interested reader. \square

We can also obtain the following detailed results by Maple computation. Note that all depth one or two values are given by Theorems 2.10 and 2.13. All FMZVs of weight 6 and depth at least 4 can be computed from the next theorem by the duality relations in Theorem 7.2.

Theorem 8.6. *The FMZVs of superbity 1 in depth 3 and weight 6 are given by Table 1 (or can be obtained by the reversal relations). Moreover, for all the values in the table, $\zeta_{s_1}^* = -\zeta_{s_1}$ if weight and depth has the same parity while $\zeta_{s_1}^* = \zeta_{s_1}$ otherwise.*

$\zeta_{s_1}(1, 3, 2)$	$-\frac{9}{2}\beta_3^2$	$\zeta_{s_1}(1, 4, 1)$	$3\beta_3^2$	$\zeta_{s_1}(1, 1, 4)$	$-\frac{3}{2}\beta_3^2$
$\zeta_{s_1}(2, 1, 3)$	$\frac{3}{2}\beta_3^2$	$\zeta_{s_1}(2, 3, 1)$	$-\frac{9}{2}\beta_3^2$	$\zeta_{s_1}(3, 1, 2)$	$\frac{3}{2}\beta_3^2$
$\zeta_{s_1}(3, 2, 1)$	$3\beta_3^2$	$\zeta_{s_1}(4, 1, 1)$	$-\frac{3}{2}\beta_3^2$	$\zeta_{s_1}(1, 2, 3)$	$3\beta_3^2$

TABLE 1. FMZVs of superbity 1, depth 3 and weight 6.

Note that all depth 3 odd weight FMZVs are given by Theorems 2.14. All FMZVs of weight 7 and depth at least 5 can be converted to FMZVs of depth at most 3 by the duality relations in Theorem 7.2.

Theorem 8.7. *The FMZVs of superbity 1, depth 4 and weight 7 are given by Table 2 (or can be obtained by the reversal relations). Moreover, $\zeta_{s_1}^* = \zeta_{s_1}$ for all the values in the table.*

$\zeta_{s_1}(1, 1, 1, 4)$	$-27\beta_7'$	$\zeta_{s_1}(1, 1, 2, 3)$	$69\beta_7'$	$\zeta_{s_1}(1, 1, 3, 2)$	$-27\beta_7'$
$\zeta_{s_1}(1, 1, 4, 1)$	$33\beta_7'$	$\zeta_{s_1}(1, 2, 1, 3)$	$-27\beta_7'$	$\zeta_{s_1}(1, 2, 2, 2)$	$-27\beta_7'$
$\zeta_{s_1}(1, 2, 3, 1)$	$-63\beta_7'$	$\zeta_{s_1}(1, 3, 1, 2)$	$-9\beta_7'$	$\zeta_{s_1}(2, 1, 1, 3)$	$33\beta_7'$
$\zeta_{s_1}(2, 1, 2, 2)$	$-63\beta_7'$				

TABLE 2. FMZVs of superbity 1, depth 4 and weight 7 ($\beta_7' = \beta_7/16$).

9. FINITE MULTIPLE ZETA VALUES OF SMALL SUPERBITIES

We now move up to superbity 2 and beyond, namely, we consider congruences modulo p -powers. First, we improve on Theorem 2.13.

Theorem 9.1. ([21, Theorem 3.2]) *Suppose $s, t \in \mathbb{N}$ have the same same parity. Then*

$$\zeta_{s_2}(s, t) = p \left[(-1)^t s \binom{s+t+1}{t} - (-1)^t t \binom{s+t+1}{s} - s - t \right] \frac{\beta_{t+s+1}}{2},$$

$$\zeta_{s_2}^*(t, s) = p \left[(-1)^t s \binom{s+t+1}{t} - (-1)^t t \binom{s+t+1}{s} + s + t \right] \frac{\beta_{t+s+1}}{2}.$$

So up to weight 4 we have the following table (see [21, Proposition 3.7] for detailed computation in weight 4) Note that $\zeta_{\mathcal{A}_2}(1) = \zeta_{\mathcal{A}_2}(3) = \zeta_{\mathcal{A}_2}(5) = \zeta_{\mathcal{A}_2}(\{1\}^3) = \zeta_{\mathcal{A}_2}(\{1\}^5) =$

$\zeta_{\mathcal{A}_2}(2)$	$2p\beta_3$	$\zeta_{\mathcal{A}_2}(1, 1)$	$-p\beta_3$	$\zeta_{\mathcal{A}_2}(1, 2)$	$\zeta_{\mathcal{A}_2}(1, 2)$
$\zeta_{\mathcal{A}_2}(2, 1)$	$-\zeta_{\mathcal{A}_2}(1, 2)$	$\zeta_{\mathcal{A}_2}(4)$	$4p\beta_5$	$\zeta_{\mathcal{A}_2}(1, 3)$	$\frac{1}{2}p\beta_5$
$\zeta_{\mathcal{A}_2}(3, 1)$	$-\frac{9}{2}p\beta_5$	$\zeta_{\mathcal{A}_2}(1, 1, 2)$	$3\beta_5$	$\zeta_{\mathcal{A}_2}(1, 2, 1)$	$-\frac{9}{2}p\beta_5$
$\zeta_{\mathcal{A}_2}(2, 1, 1)$	$\frac{11}{2}\beta_5$	$\zeta_{\mathcal{A}_2}(1, 1, 1, 1)$	$-\beta_5$		

TABLE 3. FMZV of superbity 2 in weight up to 4.

0. For larger weights we have the next two theorems.

Theorem 9.2. *In weight 5 we have*

$$(33) \quad \zeta_{\mathcal{A}_2}^*(1, 3, 1) = \zeta_{\mathcal{A}_2}(1, 3, 1) = 0,$$

$$(34) \quad \zeta_{\mathcal{A}_2}^*(2, 1, 2) = \zeta_{\mathcal{A}_2}(2, 1, 2) = -3p\beta_3^2,$$

$$(35) \quad \zeta_{\mathcal{A}_2}^*(2, 3) = \zeta_{\mathcal{A}_2}(2, 3) = 2\zeta_{\mathcal{A}_2}^*(4, 1) = 2\zeta_{\mathcal{A}_2}(4, 1),$$

$$(36) \quad 2\zeta_{\mathcal{A}_2}^*(2, 2, 1) - 3\zeta_{\mathcal{A}_2}(4, 1) = 9p\beta_3^2,$$

and

$$(37) \quad 2\zeta_{\mathcal{A}_2}^*(1, 1, 3) = -2\zeta_{\mathcal{A}_2}^*(3, 1, 1) = -2\zeta_{\mathcal{A}_2}(1, 1, 3) = 2\zeta_{\mathcal{A}_2}(3, 1, 1) = \zeta_{\mathcal{A}_2}(4, 1).$$

Proof. Throughout this proof all congruences are taken modulo p^2 . First, by applying (30) to $\mathbf{s} = (1, 1, 3)$ we get

$$(38) \quad -\zeta_{\mathcal{A}_2}^*(3, 1, 1) = \zeta_{\mathcal{A}_2}^*(1, 1, 3) + p\left(\zeta_{\mathcal{A}_1}(1, 1, 1, 3) + \zeta_{\mathcal{A}_1}(1, 2, 3) + \zeta_{\mathcal{A}_1}(1, 1, 4) + \zeta_{\mathcal{A}_1}(1, 5)\right) = \zeta_{\mathcal{A}_2}^*(1, 1, 3),$$

which yields the first “=” in (37) by Theorem 8.6. Notice by the reversal relations in Theorem 4.2 $\zeta_{\mathcal{A}_2}^*(1, 3, 1) = \zeta_{\mathcal{A}_2}(1, 3, 1)$ and

$$-\zeta_{\mathcal{A}_2}(1, 1, 3) - \zeta_{\mathcal{A}_2}(3, 1, 1) = p(\zeta_{\mathcal{A}_1}(2, 1, 3) + \zeta_{\mathcal{A}_1}(1, 2, 3) + 3\zeta_{\mathcal{A}_1}(1, 1, 4))$$

by Theorems 8.6 and 8.7. Thus (38) together with stuffle relations yields

$$0 = \zeta_{\mathcal{A}_2}(1, 1)\zeta_{\mathcal{A}_2}(3) = \zeta_{\mathcal{A}_2}(1, 3, 1) + \zeta_{\mathcal{A}_2}(1, 1, 3) + \zeta_{\mathcal{A}_2}(3, 1, 1) + \zeta_{\mathcal{A}_2}(2)\zeta_{\mathcal{A}_2}(3) - \zeta_{\mathcal{A}_2}(5) = \zeta_{\mathcal{A}_2}(1, 3, 1).$$

which gives (33) using Theorem 4.2. Applying (30) to $\mathbf{s} = (2, 1, 2)$ we get

$$\begin{aligned} -\zeta_{\mathcal{A}_2}^*(1, 3, 1) &= p \left(\zeta_{\mathcal{A}_1}(1, 2, 1, 2) + \zeta_{\mathcal{A}_1}(1, 3, 2) + \zeta_{\mathcal{A}_1}(1, 2, 3) + \zeta_{\mathcal{A}_1}(1, 5) \right) \\ &\quad + \zeta_{\mathcal{A}_2}^*(2, 1, 2) = \zeta_{\mathcal{A}_2}^*(2, 1, 2) + 3p\beta_3^2 \end{aligned}$$

using Table 1 on page 17. Thus (34) follows immediately from (33).

Turning to the proof of (35) we note that for any positive integer $k < p$ (setting $h_k = H_{k-1}$)

$$\begin{aligned} (39) \quad & \frac{\binom{p-1}{k}}{\binom{p-1+k}{k}} = \frac{(p-1)(p-2)\cdots(p-k)}{p(p+1)\cdots(p+k-1)} = \frac{(-1)^k k}{p} \prod_{j=1}^k \left(1 - \frac{p}{j}\right) \prod_{j=1}^{k-1} \left(1 + \frac{p}{j}\right)^{-1} \\ & \equiv \frac{(-1)^k k}{p} \left(1 - pH_k(1) + p^2 H_k(1, 1)\right) \left(1 - ph_k(1) + p^2 h_k(2) + p^2 h_k(1, 1)\right) \\ & \equiv \frac{(-1)^k k}{p} \left(1 - 2ph_k(1) - \frac{p}{k} + \frac{2p^2}{k} h_k(1) + 2p^2 h_k(2) + 4p^2 h_k(1, 1)\right). \end{aligned}$$

By [7, Theorem 2.1] we obtain (with abuse of notation $H = H_{p-1}$)

$$\begin{aligned} & H_{p-1}^*(\{2\}^a, 3, \{2\}^b) \\ & \equiv -\frac{2}{p} \sum_{k=1}^{p-1} \frac{1}{k^{2b+2a+2}} \left(1 - 2ph_k(1) - \frac{p}{k} + \frac{2p^2}{k} h_k(1) + 2p^2 h_k(2) + 4p^2 h_k(1, 1)\right) \\ & \quad - \frac{4}{p} \sum_{k=1}^{p-1} \frac{h_k(2b+1)}{k^{2a+1}} \left(1 - 2ph_k(1) - \frac{p}{k} + \frac{2p^2}{k} h_k(1) + 2p^2 h_k(2) + 4p^2 h_k(1, 1)\right) \\ & = -\frac{2}{p} H(2b+2a+2) - \frac{4}{p} H(2a+1, 2b+1) + 4H(2b+2a+2, 1) \\ & \quad + 2H(2b+2a+3) + 8H(2a+1, 1, 2b+1) + 8H(2a+1, 2b+1, 1) \\ & \quad + 8H(2a+1, 2b+2) + 4H(2a+2, 2b+1) \\ & \quad - 4p \left[H(2a+2b+3, 1) + H(2a+2b+2, 2) + 2H(2a+2b+2, 1, 1) \right] \\ & \quad - 8p \left[H(2a+2, 1, 2b+1) + H(2a+2, 2b+1, 1) + H(2a+2, 2b+2) \right. \\ & \quad \quad + H(2a+1, 2, 2b+1) + H(2a+1, 2b+1, 2) + H(2a+1, 2b+3) \\ & \quad \quad + 2H(2a+1, 1, 2b+2) + 2H(2a+1, 2b+2, 1) + 2H(2a+1, 1, 1, 2b+1) \\ & \quad \quad \left. + 2H(2a+1, 1, 2b+1, 1) + 2H(2a+1, 2b+1, 1, 1) \right]. \end{aligned}$$

Taking $a = 1$ and $b = 0$ we get

$$(40) \quad H_{p-1}^*(2, 3) = -\frac{2}{p}H(4) - \frac{4}{p}H(3, 1) + 16H(3, 1, 1) + 8H(3, 2) + 8H(4, 1) \\ - 4p \left[H(5, 1) + 3H(4, 2) + 6H(4, 1, 1) + 6H(3, 2, 1) + 6H(3, 1, 2) + 12H(3, 1, 1, 1) \right].$$

By stuffle relation and using (33) we see that

$$2\zeta_{\mathcal{A}_2}(3, 1, 1) + \zeta_{\mathcal{A}_2}(3, 2) + \zeta_{\mathcal{A}_2}(4, 1) = \zeta_{\mathcal{A}_2}(3, 1)\zeta_{\mathcal{A}_2}(1) - \zeta_{\mathcal{A}_2}(1, 3, 1) = 0.$$

On the other hand

$$\frac{1}{p} \left(H(4) + 2H(3, 1) \right) = \frac{1}{p} \left(H(1)H(3) + H(3, 1) - H(1, 3) \right) \\ \equiv - \left(3H(4, 1) + H(3, 2) + 3pH(4, 2) + 6pH(5, 1) \right)$$

by (20). Thus (40) is reduced to

$$H_{p-1}^*(2, 3) \equiv 6H(4, 1) + 2H(3, 2) + 6pH(4, 2) + 12pH(5, 1) \\ - 4p \left[H(5, 1) + 3H(4, 2) + 6H(4, 1, 1) + 6H(3, 2, 1) + 6H(3, 1, 2) + 12H(3, 1, 1, 1) \right] \\ \equiv 6H(4, 1) + 2H(3)H(2) - 2H(2, 3) \equiv 6H(4, 1) - 2H(2, 3)$$

by using the reversal relation and Theorem 8.6. This together with Theorem 4.2 implies (35). Further, (35) shows that

$$\zeta_{\mathcal{A}_2}^*(2, 2, 1) = \zeta_{\mathcal{A}_2}(2, 2, 1) + \zeta_{\mathcal{A}_2}(2, 3) + \zeta_{\mathcal{A}_2}(4, 1) + \zeta_{\mathcal{A}_2}(5) \\ = \zeta_{\mathcal{A}_2}(2, 2, 1) + 3\zeta_{\mathcal{A}_2}(4, 1).$$

Combining this with Theorem 2.11 and (35) we have

$$-6\beta_3^2 = \zeta_{\mathcal{A}_2}(2)\zeta_{\mathcal{A}_2}(2, 1) = 2\zeta_{\mathcal{A}_2}(2, 2, 1) + \zeta_{\mathcal{A}_2}(4, 1) + \zeta_{\mathcal{A}_2}(2, 3) + \zeta_{\mathcal{A}_2}(2, 1, 2) \\ = 2\zeta_{\mathcal{A}_2}^*(2, 2, 1) - 3\zeta_{\mathcal{A}_2}(4, 1) + \zeta_{\mathcal{A}_2}(2, 1, 2) = 2\zeta_{\mathcal{A}_2}^*(2, 2, 1) - 3\zeta_{\mathcal{A}_2}(4, 1) - 3\beta_3^2$$

by (34). This clearly implies (36).

Finally, the second “=” of (37) is an easy consequence the concatenation relations (26) since $\zeta_{\mathcal{A}_2}(3) = \zeta_{\mathcal{A}_2}(1) = 0$. Therefore

$$\zeta_{\mathcal{A}_2}^*(3, 1, 1) = \zeta_{\mathcal{A}_2}(3, 1, 1) + \zeta_{\mathcal{A}_2}(3, 2) + \zeta_{\mathcal{A}_2}(4, 1) + \zeta_{\mathcal{A}_2}(5) \\ = -\zeta_{\mathcal{A}_2}^*(3, 1, 1) - \zeta_{\mathcal{A}_2}(4, 1)$$

by (35). So the last “=” of (37) is verified. This finishes the proof of the theorem. \square

Similarly we can obtain the following results in weight 6. We leave the detailed computation to the interested reader and provide essentially all the other values in Table 5.

Theorem 9.3. *In weight 6 we have*

$$\begin{aligned}
\zeta_{s_2}^*(1, 1, 4) &= \zeta_{s_2}^*(4, 1, 1) = \zeta_{s_2}(1, 1, 4) = \zeta_{s_2}(4, 1, 1), \\
\zeta_{s_2}^*(1, 4, 1) &= \zeta_{s_2}(1, 4, 1) = -2\zeta_{s_2}(4, 1, 1) + 6p\beta_7, \\
\zeta_{s_2}^*(2, 3, 1) &= \zeta_{s_2}(1, 3, 2) = 3\zeta_{s_2}(4, 1, 1) - \frac{65}{4}p\beta_7, \\
\zeta_{s_2}^*(1, 2, 3) &= \zeta_{s_2}(3, 2, 1) = -2\zeta_{s_2}(4, 1, 1) + \frac{17}{4}p\beta_7, \\
\zeta_{s_2}^*(2, 1, 3) &= \zeta_{s_2}(3, 1, 2) = -\zeta_{s_2}(4, 1, 1) + 11p\beta_7, \\
\zeta_{s_2}^*(1, 3, 2) &= \zeta_{s_2}(2, 3, 1) = 3\zeta_{s_2}(4, 1, 1) - \frac{9}{4}p\beta_7, \\
\zeta_{s_2}^*(3, 2, 1) &= \zeta_{s_2}(1, 2, 3) = -2\zeta_{s_2}(4, 1, 1) - \frac{11}{4}p\beta_7, \\
\zeta_{s_2}^*(3, 1, 2) &= \zeta_{s_2}(2, 1, 3) = -\zeta_{s_2}(4, 1, 1) + 18p\beta_7.
\end{aligned}$$

Corollary 9.4. *The FMZV space of weight 6 superbity 2 is given by*

$$\text{FMZ}_{6,2} = \langle \zeta_{s_2}(4, 1, 1), p\beta_7 \rangle.$$

Proof. By Theorem 2.11 and Theorem 9.1 all depth 1 and 2 values are \mathbb{Q} multiples of $p\beta_7$. All depth 3 values are presented in Theorem 9.3. By duality (30) and Theorem 8.7 for weight 7 values (which are all \mathbb{Q} multiples of β_7) we see that all values of larger depths are also linear combinations of $\zeta_{s_2}(4, 1, 1)$ and $p\beta_7$. This completes the proof of the corollary. \square

Using Maple, one can compute FMZVs of depth 2 up to weight 8 which are listed in the next theorem. For essentially complete tables of values of weight up to 6 (inclusive) see Tables 3, Tables 4 and Tables 5.

$\zeta_{s_2}(1, 4)$	$3p\beta_3^2 - 2z311$	$\zeta_{s_2}(4, 1)$	$-3p\beta_3^2 + 2z311$	$\zeta_{s_2}(1, 1, 3)$	$-z311$
$\zeta_{s_2}(2, 3)$	$-3p\beta_3^2 + 4z311$	$\zeta_{s_2}(1, 2, 2)$	$-\frac{3}{2}p\beta_3^2 + 3z311$	$\zeta_{s_2}(2, 1, 2)$	$-3p\beta_3^2$
$\zeta_{s_2}(3, 2)$	$3p\beta_3^2 - 4z311$	$\zeta_{s_2}(2, 2, 1)$	$\frac{9}{2}p\beta_3^2 - 3z311$	$\zeta_{s_2}(1, 3, 1)$	0

TABLE 4. FMZV of superbity 2 in weight 5, where $z311 = \zeta_{s_2}(3, 1, 1)$.

$\zeta_{\mathcal{A}_2}(6)$	$6\beta_7$	$\zeta_{\mathcal{A}_2}(5, 1)$	$-10\beta_7$	$\zeta_{\mathcal{A}_2}(2, 1, 3)$	$18\beta_7 - \alpha$
$\zeta_{\mathcal{A}_2}(1, 5)$	$4\beta_7$	$\zeta_{\mathcal{A}_2}(1, 2, 3)$	$-\frac{11}{4}\beta_7 - 2\alpha$	$\zeta_{\mathcal{A}_2}(2, 2, 2)$	$2\beta_7$
$\zeta_{\mathcal{A}_2}(2, 4)$	$-10\beta_7$	$\zeta_{\mathcal{A}_2}(1, 3, 2)$	$-\frac{65}{4}\beta_7 + 3\alpha$	$\zeta_{\mathcal{A}_2}(2, 3, 1)$	$-\frac{9}{4}\beta_7 + 3\alpha$
$\zeta_{\mathcal{A}_2}(3, 3)$	$-3\beta_7$	$\zeta_{\mathcal{A}_2}(1, 4, 1)$	$6\beta_7 - 2\alpha$	$\zeta_{\mathcal{A}_2}(3, 1, 2)$	$11\beta_7 - \alpha$
$\zeta_{\mathcal{A}_2}(4, 2)$	$4\beta_7$	$\zeta_{\mathcal{A}_2}(1, 1, 4)$	α	$\zeta_{\mathcal{A}_2}(3, 2, 1)$	$\frac{17}{4}\beta_7 - 2\alpha$

TABLE 5. FMZV of superbity 2 in weight 6, where $\alpha = \zeta_{\mathcal{A}_2}(4, 1, 1)$.

Theorem 9.5. *Setting $\zeta_2 = \zeta_{\mathcal{A}_2}$ we have*

$$\begin{aligned}
\text{FMZ}_{1,2} &= \langle 0 \rangle, & \text{FMZ}_{2,2} &= \langle \zeta_2(2) \rangle, & \text{FMZ}_{3,2} &= \langle \zeta_2(1, 2) \rangle, \\
\text{FMZ}_{4,2} &= \langle \zeta_2(2, 2) \rangle, & \text{FMZ}_{5,2} &= \langle \zeta_2(2, 3), \zeta_2(1, 2, 2) \rangle, \\
\text{FMZ}_{6,2} &= \langle \zeta_2(2, 2, 2), \zeta_2(1, 2, 3) \rangle, \\
\text{FMZ}_{7,2} &= \left\langle \begin{array}{l} \zeta_2(\{1\}^5, 2), \zeta_2(\{1\}^3, 4), \zeta_2(1, 6), \\ \zeta_2(2, 3, 1, 1), \zeta_2(3, 1, 1, 2), \zeta_2(3, 3, 1) \end{array} \right\rangle, \\
\text{FMZ}_{8,2} &= \left\langle \begin{array}{l} \zeta_2(\{1\}^6, 2), \zeta_2(\{1\}^4, 4), \zeta_2(1, 1, 6), \zeta_2(1, 2, 5), \\ \zeta_2(5, 1, 1, 1) \end{array} \right\rangle, \\
\text{FMZ}_{9,2} &= \left\langle \begin{array}{l} \zeta_2(\{1\}^7, 2), \zeta_2(\{1\}^5, 4), \zeta_2(\{1\}^3, 6), \zeta_2(1, 8), \zeta_2(1, 2, 6), \\ \zeta_2(1, 3, 2, 3), \zeta_2(2, 1, 3, 3), \zeta_2(2, 3, 1, 3), \zeta_2(1, 2, 5, 1) \end{array} \right\rangle.
\end{aligned}$$

We expect that the bold-faced FMZVs are not really needed for generating the corresponding spaces because of the the following conjectured relations in \mathcal{A}_2 which we have verified numerically for the first 1000 primes.. Setting $\zeta_2 = \zeta_{\mathcal{A}_2}$, then

$$\begin{aligned}
68\zeta_2(2, 3, 1, 1) &= -480\zeta_2(\{1\}^5, 2) + 716\zeta_2(\{1\}^3, 4) - 843\zeta_2(1, 6) \\
34\zeta_2(3, 1, 1, 2) &= -585\zeta_2(\{1\}^5, 2) + 998\zeta_2(\{1\}^3, 4) - 1029\zeta_2(1, 6) \\
68\zeta_2(3, 3, 1) &= 1730\zeta_2(\{1\}^5, 2) - 2480\zeta_2(\{1\}^3, 4) + 2319\zeta_2(1, 6) \\
12\zeta_2(5, 1, 1, 1) &= -73\zeta_2(\{1\}^6, 2) + 12\zeta_2(1, 1, 6) - 3\zeta_2(1, 2, 5) \\
84\zeta_2(3, 2, 1, 3) &= -995\zeta_2(1, 8) + 952\zeta_2(\{1\}^3, 6) - 1288\zeta_2(\{1\}^5, 4) \\
&\quad - 437\zeta_2(\{1\}^7, 2) - 624\zeta_2(1, 2, 6) \\
924\zeta_2(2, 1, 3, 3) &= 2509\zeta_2(1, 8) - 6356\zeta_2(\{1\}^3, 6) + 5432\zeta_2(\{1\}^5, 4) \\
&\quad - 180\zeta_2(\{1\}^7, 2) + 801\zeta_2(1, 2, 6) \\
924\zeta_2(2, 3, 1, 3) &= -9424\zeta_2(1, 8) - 5824\zeta_2(\{1\}^3, 6) + 11368\zeta_2(\{1\}^5, 4) \\
&\quad - 3807\zeta_2(\{1\}^7, 2) + 852\zeta_2(1, 2, 6) \\
36\zeta_2(1, 2, 5, 1) &= 358\zeta_2(1, 8) - 248\zeta_2(\{1\}^3, 6) + 464\zeta_2(\{1\}^5, 4)
\end{aligned}$$

$$+ 5\zeta_2(\{1\}^7, 2) + 147\zeta_2(1, 2, 6).$$

In general, we have the following dimension conjecture in superbity 2 parallel to FMZVs of superbity 1. See Table 6 for numerical support.

Conjecture 9.6. *Let w be any positive integer. Set $d_{0,2} = 1$ and $d_{w,2} = \dim \text{FMZ}_{w,2}$. Then $d_{1,2} = 0, d_{2,2} = 1$ and for all $w \geq 3$ we have*

$$d_{w,2} = d_{w-2,2} + d_{w-3,2}.$$

Now let us look at some numerical data given in Table 6.

w	0	1	2	3	4	5	6	7	8	9	10	11	12
$\dim \text{MZ}_w$	1	0	1	1	1	2	2	3	4	5	7	9	12
$\dim \text{FMZ}_{w,1}$	1	0	0	1	0	1	1	1	2	2	3	4	5
$\dim \text{FMZ}_{w,2}$	1	0	1	1	1	2	2	3	4	5	7	9	12
$\dim \text{FMZ}_{w,3}$	1	1	1	2	2	3	4	5	7	9	12	16	21
$\dim \text{FMZ}_{w,4}$	2	1	1	3	3	5	6	8	11	14	19	25	
$\dim \text{FMZ}_{w,5}$	2	1	2	3	5	7	9	12	16	21			

TABLE 6. Numerically verified conjectural dimensions of MZ_w and $\text{FMZ}_{w,\ell}$ for $\ell \leq 5$. Bold-faced numbers are all one less than they're supposed to be by the conjectured recurrence formula.

Note that at every superbity the conjectured recurrence relation is always $f_w = f_{w-2} + f_{w-3}$ in general. The only exceptions occur at superbities at least 4 but only for very small weights where we have “too few” values. For example, at superbity 4 and weight 2 the dimension is supposed to be 2, but because of the trivial relation

$$\zeta_{\mathcal{A}_4}(2) + 2\zeta_{\mathcal{A}_4}(1, 1) = \zeta_{\mathcal{A}_2}(1)^2 = 0 \in \mathcal{A}_4$$

the dimension is decreased to only 1. For another example, at superbity $\ell = 5$ or 6 we have another trivial relation

$$6\zeta_{\mathcal{A}_\ell}(1, 1, 1) + \zeta_{\mathcal{A}_\ell}(1, 2) + \zeta_{\mathcal{A}_\ell}(2, 1) + \zeta_{\mathcal{A}_\ell}(3) = \zeta_{\mathcal{A}_\ell}(1)^3 = 0.$$

It might be possible to modify the map f_{KZ} in Conjecture 8.2 to prove an isomorphism of at superbity ℓ].

Conjecture 9.7. *For all positive integers w and ℓ , we have*

$$\text{FMZ}_{w,\ell} \cong \frac{\text{MZ}_w}{\zeta(2)\text{MZ}_{w-2}} \oplus \frac{\text{MZ}_{w+1}}{\zeta(2)\text{MZ}_{w-1}} \oplus \cdots \oplus \frac{\text{MZ}_{w+\ell-1}}{\zeta(2)\text{MZ}_{w+\ell-3}}.$$

Clearly the image of $\zeta_{\mathcal{A}_2}(\mathbf{s})$ in the first component should be the symmetrized MZV $\zeta_{\square}^{\mathcal{S}}(\mathbf{s})$, and the second components should involve some variation of $\zeta_{\square}^{\mathcal{S}}(\mathbf{s} + \mathbf{e}_j)$ where $\mathbf{s} + \mathbf{e}_j$ means the j th component of \mathbf{s} is increased by 1. But we have to keep in mind the reversal relation at superbity 2 is given by the following formula

$$(41) \quad (-1)^{|\mathbf{s}|} \zeta_{\mathcal{A}_2}(s_d, \dots, s_1) = \zeta_{\mathcal{A}_2}(\mathbf{s}) + p \sum_{j=1}^d s_j \zeta_{\mathcal{A}_2}(\mathbf{s} + \mathbf{e}_j).$$

10. FINITE EULER SUMS OF SUPERBITY 1 AND 2

In [16] we obtained a few results for FES. More similar results can be found in [7]. We now turn to FES in arbitrary depth. As in Conjecture 8.2 we will link them to a symmetrized version of the Euler sums. For any $s_1, \dots, s_d \in \mathbb{D}$, we may define the symmetrized Euler sums

$$(42) \quad \zeta_{\square}^{\mathcal{S}}(s_1, \dots, s_d) = \sum_{i=0}^d \left(\prod_{j=1}^i (-1)^{s_j} \operatorname{sgn}(s_j) \right) \zeta_{\square}(\mathbf{p}(s_i, \dots, s_1)) \zeta_{\square}(\mathbf{p}(s_{i+1}, \dots, s_d)),$$

$$(43) \quad \zeta_{*}^{\mathcal{S}}(s_1, \dots, s_d) = \sum_{i=0}^d \left(\prod_{j=1}^i (-1)^{s_j} \operatorname{sgn}(s_j) \right) \zeta_{*}(s_i, \dots, s_1) \zeta_{*}(s_{i+1}, \dots, s_d),$$

where ζ_{\square} and ζ_{*} are regularized values given by Theorem 3.9. Similar to Conjecture 8.2 we propose

Conjecture 10.1. *Let $f_w = \dim_{\mathbb{Q}} \text{FES}_{w,1}$ for $w \geq 1$. Then*

$$\sum_{w=1}^{\infty} f_w t^w = \frac{t}{1-t-t^2}.$$

Moreover, $\text{ES}_{w,1}$ has a basis

$$\left\{ \zeta_{\mathcal{A}_1}(\bar{1}, a_1, \dots, a_r) : a_1, \dots, a_r \in \{1, 2\}, a_1 + \dots + a_r = w - 1 \right\}.$$

Conjecture 10.2. *There is an isomorphism*

$$f_{\text{FES}} : \text{FES}_{w,1} \longrightarrow \frac{\text{ES}_w}{\zeta(2)\text{ES}_{w-2}},$$

$$\zeta_{\mathcal{A}_1}(\mathbf{s}) \longmapsto \zeta_{*}^{\mathcal{S}}(\mathbf{s}).$$

We have checked this up to weight 5 under Conjecture 2.6 concerning the \mathcal{A}_1 -Bernoulli numbers and \mathcal{A}_1 -Fermat quotient q_2 .

Since essentially the same proofs for Proposition 8.3 given by Jarossay [10] work for Euler sums with the help of level two Drinfeld associator, one may also replace ζ_{*} by ζ_{\square}

in the conjecture. However, we don't know whether the space of Euler sums ES_n can be generated by symmetrized Euler sums.

Conjecture 10.2 would imply that $\dim_{\mathbb{Q}} \text{FES}_{w,1} = F_w$ where F_w are Fibonacci numbers: $F_0 = 0, F_1 = 1, F_w = F_{w-1} + F_{w-2}$ for all $w \geq 2$. From what we have found so far we get the following table. Let ES_w (resp. $\text{FES}_{w,\ell}$) be the \mathbb{Q} -space spanned by the Euler sums of weight w (resp. finite Euler sums of weight w and superbitity ℓ).

w	0	1	2	3	4	5	6	7
$\dim \text{ES}_w$	1	1	2	3	5	8	13	21
$\dim \text{FES}_{w,1}$	1	1	1	2	3	5	8	13
$\dim \text{FES}_{w,2}$	1	1	2	4	7	12	20	33

TABLE 7. Conjectural dimensions of ES_w and $\text{FES}_{w,\ell}$ for $\ell \leq 2$.

In Table 7, we provide $\dim \text{ES}_w$ for comparison purpose. We also verified $\dim \text{FES}_{w,1}$ and $\dim \text{FES}_{w,2}$ in the table numerically for the first 1000 primes.

As further support of Conjecture 10.1, we have

Theorem 10.3. *For all $w \leq 7$, We have*

$$\text{FES}_w = \langle \zeta_{\mathcal{A}_1}(\bar{1}, a_1, \dots, a_r) : a_1, \dots, a_r \in \{1, 2\}, a_1 + \dots + a_r = w - 1 \rangle.$$

Proof. This is proved with Maple. We found that the double shuffle relations defined in Definition 3.12 are insufficient to produce all the relations. However, with reversal relations we can find the generating set as given in the theorem, for all $w \leq 7$. \square

In the classical setting, we know the double shuffle relations do not generate all the linear relations among Euler sums. For example, in proving the identities $\zeta(\{3\}^n) = 8^n \zeta(\{\bar{2}, 1\}^n)$ for all $n \geq 1$, we need to use the distribution relations (see [22, Remark 3.5]).

Example 10.4. For FESs, the first missing reversal relation (i.e., not provable by the double shuffle relations), which appears in weight two already, is given by

$$\zeta_{\mathcal{A}_1}(1, \bar{1}) + \zeta_{\mathcal{A}_1}(\bar{1}, 1) = 0.$$

In weight 3, we need two reversal relations

$$\zeta_{\mathcal{A}_1}(\bar{2}, 1) = \zeta_{\mathcal{A}_1}(1, \bar{2}), \quad \text{and} \quad \zeta_{\mathcal{A}_1}(1, 1, \bar{1}) = \zeta_{\mathcal{A}_1}(\bar{1}, 1, 1).$$

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