Trial for a proof of the Syracuse Conjecture

Mallet Nicolas

May 2015

Abstract The infamous 3x+1 conjecture spread by Lothar Collatz in 1952, despite its elementary formulation, remained unproved for over 60 years. From the heuristical probabilistic approach to the complex mapping of the algorithm, the scientific community has fetched for many methods to try to prove it formally, and thus, mathematicians like Erdös tend to believe that "mathematics are not yet ready for such problems". In this research report, covering domains like algebra and graph theory, it is shown a trial of proof of the conjecture by disproval of its two antitheses: the existence of an ever-growing Syracuse sequence and the existence of a cycle different from the cycle $\{4, 2, 1\}$.

Acknowledgements

These two years I've passed working on this conjecture from May 16th 2013 until May 25th 2015 constituted an amazing experience over a tremendous task I was unsure to accomplish someday. And by "tremendous task", I mean the simple fact of filling this report with traces of research which eventually made up to a possible proof of this conjecture. But the validation of this proof is another task which, I hope, is a number I could also reach out to.

The trigger of this scientific journey was quite anecdotic. It was during my years passed as a pupil at the Lycée Marguerite Yourcenar in Erstein, France. We were given an exercise in class of Maths spécialité (a course which aims at important topics for further studies in mathematics which weren't usually brought up in the main cursus) which, obviously, talked about the Syracuse algorithm. My teacher, Mr. Adjiage, also briefly talked about the conjecture, and thus, I learnt about it, and gave it a try afterwards. I've been through easy calculations at first, ending up with failures at first, and then, I kept on scratching papers with calculations, diagrams, reasonings,... I couldn't let such an easy yet unsolved problem get away with this. It was a fascinating discovery, and consequently, I'd like to thank Mr. Adjiage Grégory not only for having me know about this conjecture, but also for the lessons about linear algebra and the induction process which helped me construct my reasoning.

I'm also very thankful to Mr. Saïdi Abdelkader, not only for the lectures of Analysis II I had in my second semester, but also for letting me know about the arXiv website and the way I could communicate my works further on to a research unit or an university which studies this conjecture. It is very useful for one beginning his studies in Mathematics and still far from my objective (Doctorate Degree).

Of course, I'd like to thank the other teachers in mathematics I had during my school life and during my first year in university. Thanks to them, I've been initiated to more advanced, more rigorous and more accurate notations. My drafts finally were granted a more formal presentation. Moreover, I've been introduced to polynomial divisions, and the proofs by construction really helped me for my work about the cycles and the sequences. For these reasons, I acknowledge my gratitude over Dr. Hans-Werner Henn, Mr. Hoang-Duc Auguste, Mr. Steiner Christophe. I hope that this report could properly make honor to their teachings.

And naturally, I'd like to thank my English teachers for having given me their advices on how could I improve my English. It also let me an opportunity to try and express myself in English about a scientific topic, which will be really useful for my professional project.

As for those who have supported me, have believed in me for my work, whether I've met them in real life, like my family and my classmates, which also stood by my side, helped me destress when my mind was too saturated to see clearly what I'm doing ; or in forums, like the forum Cours Et Devoirs on jeuxvideo.com. I'd like to thank them for their support during all this time I've been working on it with the hopes of making it to the end.

And finally, I shall thank the readers for taking their time considering this work. Their critics on my expression, or maybe my method, will most probably help me for further works as my mathematical curiosity will someday ask me to focus on some other fascinating problems to solve.

Contents

Chapter 1

Primary study of the conjecture

1.1 Analysis of the Syracuse algorithm

Definition 1.1 We call the Syracuse algorithm the following algorithm: let N be an integer. If N is odd, multiply it by 3 and add 1. If N is even, divide it by 2. When $N = 1$, the algorithm stops. [2]

The algorithm is made of two transformations: one that we call the even transformation $E(X) : N \to N/2$ and one that we call the odd transformation $O(X) : N \to 3N + 1$. We observe that the conditions that apply to the transformations will tell more about their repetitivity. The transformation $O(X)$ applies only to an odd number, that is only when $N = 2x + 1, x \in \mathbb{Z}$. We apply the transformation to this form: $2x+1 \to 3(2x+1)+1 = 6x+4 = 2(3x + 2)$. As we have $x \in \mathbb{Z}$, by product and addition, we have $3x + 2 \in \mathbb{Z}$. So $O(N)$ is even, so we're not authorized to apply the odd transformation twice in a row.

In other terms, when an odd number follows the operations of the algorithm, it will undergo this chain of transformation:

$$N \xrightarrow{O} 3N + 1 \xrightarrow{E} \frac{3N + 1}{2} \xrightarrow{E} \dots \xrightarrow{E} \frac{3N + 1}{2^{ord_2(3N+1)}}$$

The algorithm is based on two operations that maps a single number to a single number. They both admit a reciprocal operation that retrieves the argument from the image. For the transformation $O : N \to 3N + 1$, we have $O^{-1} : N \to \dfrac{N-1}{3}$. For the transformation $E : N \to N/2$, we have $E^{-1} : N \to 2N$. We can of course verify it:

$$N \to 3N + 1 \to (3N + 1)/2 \to \dots \to (3N + 1)/2^o rd_2(3N + 1) \qquad (1.1)$$

The initial condition caused by the parity test in order to determine which transformation among O^{-1} and E^{-1} we apply to N will have no importance here as it is. What will matter is the multiplicity of the initial transformations. We have seen before that the transformation $O(X)$ cannot be iterated twice in a row, whereas the even transformation reiterates until the transformed number is even.

In the invert algorithm, we'll therefore start with a number and decide to apply an arbitrary number of transformations E^{-1} until we apply once the transformation O^{-1}, then we repeat again and again. In the context of our conjecture, we decide the number we start with is 1, and our goal would be to reach all the natural non-null numbers with only those transformations.

Definition 1.2 We call the Syracuse sequence of an integer N the ordered sequence of all numbers obtained while applying the Syracuse algorithm. We define:

$$(u_n(N)) : \begin{cases} u_0 = N \\ u_{n+1}(N) = \begin{cases} 3u_n + 1 & \text{if} \quad n \quad \text{odd} \\ \dfrac{u_n}{2} & \text{if} \quad n \quad \text{even} \end{cases} \end{cases}$$

Example The Syracuse sequence of $N = 13$ is

$$u_n(13) = \{13, 40, 20, 10, 5, 16, 8, 4, 2, 1\}$$

The Syracuse sequence of an integer describes also the list of all numbers it reaches via the transformations. For lack of determinism, we'll admit the notation $T(X)$ describing the next transformation applied to X independently from its parity.

$$u_{n+1}(N) = T(u_n(N)) \quad \text{with} \quad T : X \to \begin{cases} 3X + 1 & \text{if} \quad n \quad \text{odd} \\ \dfrac{X}{2} & \text{if} \quad n \quad \text{even} \end{cases} \tag{1.2}$$

Since we stop the algorithm when the Syracuse sequence of a number N contains 1, the member equal to 1 is the last member of a Syracuse sequence. We pose there exists a rank for which $u_n(N) = 1$, and all other following terms are undefined. Both two definitions 1 and 2 are describing a property for a Syracuse sequence which satisfies the conjecture: it has a finite number of elements. Indeed, the conjecture is satisfied if and only if we always get to 1 while we apply the algorithm. However, when we get to 1, the algorithm stops. The number of members is therefore countable from the initial number until the term equal to 1 and we call the flight time the cardinal number of a Syracuse sequence without the initial term. The flight time can also be translated as the number of transformations that occurred during the progress of a Syracuse algorithm, that is how much time we applied $T(X)$.

$$N \models \mathcal{S} \Leftrightarrow Card(u_n(N)) \neq \infty$$

Statement 1.1 Any Syracuse sequence of $T^n(N)$ is included in the Syracuse sequence of N.

Proof:

$$\begin{aligned}
&(u_n(N)) = \{T^n(N), n \in N_{\geqslant 0}\}\\
&(u_n(T^k(N))) = \left\{T^n(T^k(N)), n \in N_{\geqslant 0}\right\} = \left\{T^n \circ T^k(N), n \in N_{\geqslant 0}\right\}\\
&= \left\{T^{n+k}(N), n \in N_{\geqslant 0}\right\} = \{T^m(N), m \in N_{\geqslant 0}\}\\
&m = n + k \geqslant 0 + k = k\\
&(u_n(N)) = \{T^p(N), 0 \leqslant p \leqslant k\} \cup (u_n(T^k(N)))
\end{aligned}$$

This elementary statement comes in handy as much as an axiom. Because of the unicity of the image of a number by $T(N')$, then it has the same image than $T^n(N)$, with $N' = T^{n-1}(N)$. By construction, $T^{n+k}(N) = T^k(N')$. We can also extract a Syracuse sequence from another Syracuse sequence therefore.

Definition 1.3 The Jumping Cycle Function (noted JCF or $\eta(N)$) is an application that maps an odd natural number to the next odd number when it follows the operations of the algorithm.

$$\begin{aligned}
\eta : \quad & N \to \mathbb{N}^*\\
& N \mapsto \frac{3N+1}{2^{\alpha = ord_2(3N+1)}}
\end{aligned}$$

The idea behind this function was to materialize the multiple transformations encountered by an odd number until it reaches another number. We compress all the transformations applied on an odd number into one function. Multiple antecedents admit the same image by this function. A simple example to show this example is to take the preimages of an even number and this even number multiplied by a power of 2 by the odd transformation. Let N_1 and N_2 be two numbers described as before. Then we have:

$$N_1 = \frac{2x-1}{3} \qquad N_2 = \frac{2^{\beta+1}x - 1}{3} \qquad \beta \in \mathbb{N}^*$$

We evaluate both of these numbers by the JCF and we obtain:

$$\eta(N_1) = \frac{3 * \frac{2x-1}{3} + 1}{2^\alpha} = \frac{2x-1+1}{2^\alpha} = \frac{2x}{2^\alpha} = x \qquad \alpha = 1$$
$$\eta(N_2) = \frac{3 * \frac{2^{\beta+1}x-1}{3} + 1}{2^\alpha} = \frac{2^{\beta+1}x-1+1}{2^\alpha} = \frac{2^{\beta+1}x}{2^\alpha} = x \qquad \alpha = \beta + 1$$
$$\Rightarrow \eta(N_1) = \eta(N_2)$$

This array of calculus also shows that this function is not an injection. Otherwise, we would have:

$$\begin{aligned} &\eta(N_1) = \eta(N_2) \\ &\Leftrightarrow N_1 = N_2 \\ &\Leftrightarrow \frac{2x-1}{3} = \frac{2^{\beta+1}x-1}{3} \\ &\Leftrightarrow 2x-1 = 2^{\beta+1}x-1 \\ &\Leftrightarrow 2x = 2^{\beta+1}x \\ &\Leftrightarrow 2 = 2^{\beta+1} \\ &\Leftrightarrow 1 = 2^\beta \\ &\Leftrightarrow \beta = 0 \end{aligned}$$

But we supposed that $\beta \in \mathbb{N}^*$, which constitutes a contradiction.

We assume we choose α so that $\eta(N)$ is odd, except when the form of N will determine it after application of the transformation $O(X)$.

Statement 1.2 We note i the numbers of applications of the JCF and we index α with the variable p for each iteration of the JCF. The general formula for the nested composition of the JCF is, with $\alpha_p = ord_2(O \circ \eta^{p-1}(N))$:

$$\eta^i(N) = \frac{3^i N + \sum_{j=0}^{i-1} 3^{i-1-j} * 2^{\sum_{l=0}^{j} \alpha_l}}{2^{\sum_{p=0}^{i} \alpha_p}} \tag{1.3}$$

Proof of the statement 1.2 by induction:

Initialization: For $i = 1$,

$$\eta^1(N) = \frac{3N+1}{2^{\alpha_1}} = \frac{3^1N + 3^0 2^0}{2^{\alpha_1}} = \frac{3^i N + \sum_{j=0}^{1-1=0} 3^{1-1-0} 2^{\sum_{l=0}^{j} \alpha_l}}{2^{\sum_{p=0}^{i} \alpha_p}}$$

Induction step: We pose $\mathcal{P}_i$: "$\eta^i(N) = \dfrac{3^i N + \sum_{j=0}^{i-1} 3^{i-1-j} * 2^{\sum_{l=0}^{j} \alpha_l}}{2^{\sum_{p=0}^{i} \alpha_p}}$". We suppose $\exists i \in \mathbb{N}^*, i \models \mathcal{P}_i$. We prove that $\mathcal{P}_i \vdash \mathcal{P}_{i+1}$. We transform the induction hypothesis $P_i \rightarrow \eta(P_i)$.

$$\begin{aligned}
&\eta(\eta^i(N)) = \eta(\frac{3^i N + \sum_{j=0}^{i-1} 3^{i-1-j} * 2^{\sum_{l=0}^{j} \alpha_l}}{2^{\sum_{p=0}^{i} \alpha_p}}) \\
&\Leftrightarrow \eta^{i+1}(N) = \frac{3 * \dfrac{3^i N + \sum_{j=0}^{i-1} 3^{i-1-j} * 2^{\sum_{l=0}^{j} \alpha_l}}{2^{\sum_{p=0}^{i} \alpha_p}}}{2^{\alpha_{i+1}}} + 1 \\
&\Leftrightarrow \eta^{i+1}(N) = \frac{3^{i+1} N + \sum_{j=0}^{i-1} 3^{i-j} * 2^{\sum_{l=0}^{j} \alpha_l} + 3^{i-i} 2^{\sum_{p=0}^{i} \alpha_p}}{2^{\sum_{p=0}^{i} \alpha_p + \alpha_{i+1}}} \\
&\Leftrightarrow \eta^{i+1}(N) = \frac{3^{i+1} N + \sum_{j=0}^{i} 3^{i-j} * 2^{\sum_{l=0}^{j} \alpha_l}}{2^{\sum_{p=0}^{i+1} \alpha_p}} \\
&\Leftrightarrow \mathcal{P}_{i+1}
\end{aligned}$$

1.2 Propositional study of the conjecture

Statement 0.0 *(The Syracuse Conjecture)* All the natural non-null numbers end up reaching out to the trivial cycle $4, 2, 1$ as the algorithm goes on.

Definition 1.4 The oneness property is the property of a number which Syracuse sequence contains the trivial cycle $4, 2, 1$ In other words, it satisfies the Syracuse conjecture. In our demonstration, we'll note:

$$O : \text{"The Syracuse sequence ends up with 1"}$$

And we have $1 \in (u_n(N)) \Leftrightarrow \exists! i \in \mathbb{N}^*, \eta^i(N) = 1 \Leftrightarrow N \models \mathcal{O}$.

Example. $N = 13$ satisfies the oneness property. Indeed, $u_9(13) = 1$. So $13 \models \mathcal{O}$.

The name of this property has been inspired by the work of Paul Bourke on his website, who has given this name to this property for the first time, although here, we're not focusing in the value of the oneness as Dr. Bourke defines it originally. We're only interested in whether the Syracuse sequence contains 1 or not. [1] This property validates the conjecture for the numbers verifying it. When the conjecture says that all numbers end up reaching out to 1, it actually asks to verify the oneness property for all the non-null natural numbers.

$$S \Leftrightarrow \forall N \in \mathbb{N}^*, N \models \mathcal{O}$$

We suppose that i is unique in this logic relation, because $\eta^i(N) = 1$ means that during the algorithm, we have reached 1. Let's suppose there was another one, with a different value. We call it i'. As we have stated that $i, i' \in \mathbb{N}^*$, it is submitted to a relation order among the other elements of the set. This relation order is composed of three operators: ¿,=,¡. By hypothesis, $i \neq i'$. There are two cases left to check.

If $i' > i$, then it means literally that we have continued the iteration of the algorithm when we weren't asked to, that is after we have reached $\eta^i(N)$. In the case of the algorithm, we can have only one member in a Syracuse sequence that is equal to 1, which brings a contradiction.

If $i' < i$, then again, we have two members of the Syracuse sequence of one number that are equal to 1. The first member of this sequence to have reached one is $\eta^{i'}(N)$. We should have already stopped the algorithm at the moment we reached that number. That means $\eta^i(N)$ must be undefined. But that makes a contradiction to the fact we supposed that $\eta^i(N) = 1$.

That implies that $O \models$"The algorithm stops." and $\neg\mathcal{O} \models$ "The algorithm doesn't end.", and therefore $O \models (i \neq +\infty)$ and $\neg\mathcal{O} \models (i = +\infty)$. The equality relation is binary, either a number is equal to some other or it is unequal (eventually, the latter will bring the question of superiority/inferiority). Therefore, the converse is also correct for both the proposition and its opposite.

$$(i < +\infty) \models \mathcal{O}; (i = +\infty) \models \neg\mathcal{O}$$

The proposition of oneness for all natural numbers is obviously not admitted to be true (yet). So there can exist counter-examples that don't satisfy this property. We can conceive two antitheses related to this property: the infinite growth and the existence of another cycle.

Statement 0.1 If a Syracuse sequence contains terms that are always greater to whichever number in function to their rank, then this Syracuse sequence doesn't satisfy the Syracuse conjecture. We note $N \models \mathcal{G}$ or $(u_n(N)) \models \mathcal{G}$ a number which Syracuse sequence knows an infinite growth.

$$N \models \mathcal{G} \Rightarrow N \models \neg\mathcal{O} \Rightarrow N \models \neg\mathcal{S}$$

Proof : We refer to Definition 2.1 in the part 2. of this report. $(u_n(N)) \models \mathcal{G} \Leftrightarrow \forall A > 0, \exists I \in N, i \geqslant I \Rightarrow T^i(N) > A$. We suppose $N \models \mathcal{G} \wedge O$. Since $N \models \mathcal{G}$, we have $\forall A > 0, \exists I \in N, i \geqslant I \Rightarrow T^i(N) > A$. But $N \models \mathcal{O}$, so $+\infty \notin (u_n(N))$. Indeed, $T(+\infty)$ is undefined. So $(u_n(N)) - 4, 2, 1$ is finite. It admits a finite maximum. $\max_n u_n(N) = A \geqslant u_n(N) = T^n(N)$. So $\exists A > 0, \forall I \in N, T^I(N) \leqslant A \Rightarrow N \models \neg\mathcal{G}$. Contradiction.

Statement 0.2 If a Syracuse sequence contains a cycle other than the trivial cycle $4, 2, 1$, then this Syracuse sequence doesn't satisfy the Syracuse conjecture. We note $N \models C$ or $(u_n(N)) \models C$ a number which Syracuse sequence contains such a cycle.

$$N \models C \Rightarrow N \models \neg\mathcal{O} \Rightarrow N \models \neg\mathcal{S}$$

Proof : We refer to Definition 3.1 in the part 3 of this report.

$(u_n(N)) \models C \Leftrightarrow \{u_0 \dots u_{\theta-1}, \{u_\theta \dots u_{\theta+s-1}\}\}, \{u_\theta \dots u_{\theta+s-1}\} \neq \{4, 2, 1\}$

$(u_n(N)) \models \neg C \Leftrightarrow \{u_0, u_1, \dots u_{\theta-1}, \{4, 2, 1\}\}$. According to Statement 3.2, if the Syracuse sequence of N admits a cycle, it is unique. Since we suppose that $N \models \mathcal{C} \wedge \mathcal{O}$, then it has two different cycles $\{u_\theta \dots u_{\theta+s-1}\} \neq \{4, 2, 1\}$, which is impossible.

These are the two only antitheses, and we'll show that with those two antitheses disproved, we'll demonstrate the conjecture.

Statement 1.3 The propositions $\mathcal{G}$ and $\mathcal{C}$ are incompatible.

Proof We suppose $\exists N \in N_{\geqslant 1}, N \vDash (\mathcal{G} \wedge \mathcal{C})$. We have to prove $\mathcal{G} \Rightarrow \neg\mathcal{C}$ and $\mathcal{C} \Rightarrow \neg\mathcal{G}$.

$\boxed{\mathcal{G} \Rightarrow \neg\mathcal{C} :}$

$(u_n(N)) \models C \Leftrightarrow (u_n(N)) = \{u_0 \dots u_{\theta-1}, \{u_\theta \dots u_{\theta+s-1}\}\}, \{u_\theta \dots u_{\theta+s-1}\}$
$\neq \{4, 2, 1\}$.

Analogically to the proof of the Statement 0.1, with $\{u_\theta \dots u_{\theta+s-1}\}$ instead of $\{4, 2, 1\}$, $(u_n(N))$ admits a maximum.

So $N \models \neg\mathcal{G}$.

$\boxed{\mathcal{C} \Rightarrow \neg\mathcal{G} :}$

$N \models \mathcal{G} \Leftrightarrow \forall A > 0, \exists I \in N, i \geqslant I \Rightarrow T^i(N) > A$. It means as far as we go in the algorithm, we always have a new maximum. By construction, let $T^{\gamma_m}(N) = \max_{i \leqslant \gamma(m+1)} T^i(N)$ and $T^{\gamma_m}(N) < T^{\gamma_{m+1}}(N)$. We can deduce extracted sequences of $(u_n(N))$ with the formula $(u_n(N)) \cap T^i(N), \gamma_m \leqslant i \leqslant \gamma_{m+1} - 1 = T^i(N), \gamma_m \leqslant i \leqslant \gamma_{m+1} - 1$. All these sequences are finite and since $T^{\gamma_m}(N) < T^{\gamma_{m+1}}(N)$, they are all different. These can't be all cycles of N due to the Statement 3.2 and because $T(T^{\gamma_{m+1}-1}(N)) = T^{\gamma_{m+1}}(N) > T^{\gamma_m}(N)$.

So $N \models \neg\mathcal{C}$.

Definition 1.5 We say a number is involved in the Syracuse algorithm if it admits at least one preimage by the JCF if it is odd or if it can be reached via the 3N+1 operation if it is even.

$$N \models \mathcal{I} \Leftrightarrow \begin{cases} \exists N' \in \mathbb{N}^*, \eta(N') = N \quad \text{if} \quad N \in \overline{1} \\ O^{-1}(N) \in \mathbb{N}^* \quad \text{if} \quad N \in \overline{0} \end{cases}$$

Example:

1. $N' = 31 = 2*15+1$. And $\eta(41) = \dfrac{3*41+1}{4} = \dfrac{124}{4} = 31$. $N = 31$ is involved in the algorithm.

2. $N' = 33 = 2*16+1$. But no number can reach out to it. 33 is not involved in the algorithm.

3. $N' = 34 = 2*17$ is even. And $O(11) = 3*11+1 = 34$. 34 is involved in the algorithm.

4. $N' = 32 = 2*16$ is even. But no number can reach out to it. Indeed, $O^{-1}(32) = \dfrac{32-1}{3} = \dfrac{31}{3} = 10 + \dfrac{1}{3} \notin N$. 32 is not involved in the algorithm.

This property is important for the identification of the numbers we'll use in our conjecture. It has a filter effect which allows us to conceive what kind of numbers we'll be demonstrating the conjecture with. If it happens the numbers that we can demonstrate the conjecture with has a common form, or a behavior corresponding to some conditions fulfilled, it will give us even more tools to demonstrate the conjecture.

Statement 1.4 If a number satisfies the conjecture, then its preimage satisfies the conjecture as well.

$$N \models \mathcal{O} \Rightarrow O^{-1}(N) \models \mathcal{O}$$

Proof: Let $N \in \mathbb{N}^*, N' \in \eta^{-1}(N)$. We suppose that $N \models \mathcal{O}$, with O the oneness property. $N \models \mathcal{O} \Leftrightarrow \exists i \in \mathbb{N}^*, \eta^i(N) = 1$. If $N' \in \eta^{-1}(N)$, then $\eta(N') = N \Rightarrow \eta^i(\eta(N')) = \eta^{i+1}(N') = 1$. We pose $\iota = i+1 \in \mathbb{N}^*$. $\in \iota \in \mathbb{N}^*, \eta^\iota(N') = 1 \Leftrightarrow N' \models \mathcal{O}$.

The converse of this statement $\eta^{-1}(N) \models \mathcal{O} \Rightarrow N \models \mathcal{O}$ can be verified with the definition of a Syracuse sequence and the unicity of the image by the JCF. If we have $\eta^{-1}(N) \models \mathcal{O}$, then we have its Syracuse sequence:

$$(u_n(\eta^{-1}(N))) = \left\{\eta^{-1}(N) \dots \eta(N) \dots 4, 2, 1\right\} = \left\{\eta^{-1}(N) \dots N \dots 4, 2, 1\right\}$$

So the Syracuse sequence of N is contained in the Syracuse sequence of $\eta^{-1}(N)$. We construct it by extraction:

$$(u_n(N)) = \{N \dots 4, 2, 1\} \ni 1$$

So $N \models \mathcal{O}$. However, this statement has to be carefully interpreted. This DOES NOT prove the Syracuse conjecture immediately. If we pose the Syracuse algorithm of 1, we have $u_n(1) = \{1\}$. It is made of one term, which is $u_0(1) = 1$. This is the same to say that $\eta^0(N) = id(N) = N$. But the exponent i is null, so $i = 0 \notin \mathbb{N}^*$. We can only make this interpretation for the exponent i, since we suppose the algorithm stops when $u_k(N) = 1$.

Corollary 1.5 If we demonstrate the conjecture for all the involved natural numbers, we have demonstrated it $\forall N \in \mathbb{N}^*$.

$$S \Leftrightarrow \forall N \models \mathcal{I}, N \models \mathcal{O}$$

According to the Statement 1.4, we've seen that if a number satisfies the conjecture, so does its preimage. However, a non-involved admits no preimage by the JCF and therefore, it is not retrieved in the process of a Syracuse sequence. Yet, this statement allows us to do as if these numbers were not considered, in order to use more common traits and properties on the involved numbers to disprove the antitheses. Now, we need a set for our study which could maybe make use of this important corollary. After some empirical research with the first numbers, I have observed a common behavior for the numbers belonging in the same class modulo 6. I decided therefore to study this conjecture with the numbers written as such : $N = r + 6n$.

1.3 The $r + 6n$ numbers

We decide to study this conjecture while expressing all the numbers in the non-null natural set as a congruence relation modulo 6. The consequent variable scalar for a natural number in product with 6 is noted n. The number forms we're left with are:

$$N \in \{6n, 1 + 6n, 2 + 6n, 3 + 6n, 4 + 6n, 5 + 6n \mid n \in \mathbb{Z}\}$$

But to include this set in the context of our algorithm, these numbers must be positive. So considering that $n \in \mathbb{Z}$ isn't exactly right. For $n < 0$, we have $r + 6n < 0$. Furthermore, if $n = 0$, then 0 would be included in the study of our algorithm. On the other hand, if we omit the case where $n = 0$, then we remove the numbers $\{1, 2, 3, 4, 5\}$, so we couldn't demonstrate the conjecture rigorously. We'll see that another set of number forms $r+24n$ will be needed during the disproval of the first antithesis. The difference between the choices of these two sets resides on the forms of the numbers involved in the conjecture.

Statement 1.6 Among the even numbers, only the even numbers in the class 4 modulo 6 are involved in the algorithm.

$$N \in \overline{4} \Rightarrow N \models \mathcal{I}$$
$$N \in \left\{\overline{0}, \overline{2}\right\} \Rightarrow N \not\models I$$

The proof of this statement is immediate. According to the definition 1.5 (definition of the involvement property), only the numbers that can be reached by the "odd" transformation are involved in the conjecture. Basically, we take the algebraic definition of an odd number and we apply the transformation to it:

$$N = 2x + 1 \xrightarrow{O} 3(2x + 1) = 6x + 3 + 1 = 6x + 4$$
$$x \in \mathbb{Z} \Rightarrow 6x + 4 \in \overline{4} \Rightarrow O(N) \notin \left\{\overline{0}, \overline{2}\right\}$$

All the images of the odd numbers by the "odd" transformation are written as numbers belonging in the class $\overline{4}$.

Statement 1.7 Among the odd numbers, the numbers in the class 3 modulo 6 are not involved in the algorithm.

$$N \in \left\{\overline{1}, \overline{5}\right\} \Rightarrow N \models \mathcal{I}$$
$$N \in \overline{3} \Rightarrow N \not\models I$$

Even though the demonstration isn't as quick as for the previous statement, it can also be done by evaluating all the classes by the JCF.

$$\eta(1 + 6n) = \frac{4 + 18n}{2^\alpha} = \frac{2 + 9n}{2^{\alpha-1}}$$
$$\eta(3 + 6n) = \frac{10 + 18n}{2^\alpha} = \frac{5 + 9n}{2^{\alpha-1}}$$
$$\eta(5 + 6n) = \frac{16 + 18n}{2^\alpha} = \frac{8 + 9n}{2^{\alpha-1}}$$

In order to determine under which form will be the image, we have to execute a congruence test on n. Indeed, we have the addition of two terms:

the first one is a constant which parity is determined, the second one is a product of an odd integer with a variable. If we want to apply another "even" transformation, we need to have both parities similar. But the parity of the second term of the second term will only depend on the variable n. So we need the parity of n to be the same than the parity of the first addend.

$$\begin{array}{lll} \text{For} & \eta(r+6n,) & \overline{9n} = \overline{\dfrac{3r+1}{2}} \Rightarrow \overline{n} = \overline{r'} \quad (mod.2) \\ \text{For} & \eta(1+6n,) & \overline{n} = \overline{2} = \overline{0} \quad (mod.2) \\ \text{For} & \eta(3+6n,) & \overline{n} = \overline{5} = \overline{1} \quad (mod.2) \\ \text{For} & \eta(1+6n,) & \overline{n} = \overline{8} = \overline{0} \quad (mod.2) \end{array}$$

We suppose that the parity of n corresponds to the condition to apply another "even" transformation. So we obtain the following transformation graph 1a.

We need to execute the congruence test one more time, since we're given the same conditions than before. This leads us to the transformation graph 1b.

We can notice we have the same classes modulo 9 than before. There's no need to redo the congruence test a third time, since the results will be redundant. We can draw a map out of all these data. (graph 2a)

But of course, this cycle of "even" operations may come to an end when n' doesn't fulfill the congruence condition. Eventually, we come up with an odd number whenever the parity of both addends don't correspond. We complete the graph as we operate with a different parity for n'. (graph 2b)

We observe with this graph that none of the odd identities are belonging to the class 3 modulo 6. In particular, 3+6n admits no preimage by the JCF. As an odd number, it is not involved in the algorithm.

In consequence to both of these previous statements, we'll need to focus only on half of the numbers of the set $\mathbb{N}^*$.

Corollary 1.8 Proving the Syracuse conjecture for the natural numbers in the classes 1,4,5 modulo 6 will prove it for all non-null natural numbers.

$$S \Rightarrow \forall N \in \left\{\overline{1}, \overline{4}, \overline{5}\right\}, N \models \mathcal{O}$$

Chapter 2

The finite growth of a Syracuse sequence

We imagine the case when a number undergoing the operations of the algorithm keeps on rising infinitely. This kind of number will in consequent never reach out to 1. The idea is to find a number like this, using conditions for the infinite growth phenomenon to occur.

2.1 What defines an infinite growth ?

Definition 2.1 We say a Syracuse sequence models an infinite growth if its initial number keeps on reaching greater numbers as we iterate the algorithm infinitely.

$$(u_n(N)) \models G \Leftrightarrow \forall A > 0, \exists I \in \mathbb{N}, i \geqslant I \Rightarrow T^i(N) > A$$

The consequences of this definition if a Syracuse sequence models are as well close to those of the limit of whatever sequence.

Statement 2.1 When a Syracuse sequence models an infinite growth, the number of odd and even transformations is infinite.

That comes from the simple fact that we order the algorithm to stop when $N = 1$. Because $\mathcal{G}$ constitutes an antithesis of $\mathcal{O}$, then we won't reach 1 and thus, the algorithm continues forever.

Statement 2.2 When a Syracuse sequence models an infinite growth, its maximum is infinite.

$$(u_n(N)) \models \mathcal{G} \Leftrightarrow \max_n u_n(N) = +\infty$$

We suppose we start from a finite number N. As for now, the Syracuse sequence of N is only composed of N, it is the maximum and minimum of this sequence. We'll suppose N was even from the start, to simplify things. Otherwise, if N was odd, then $T(N) = 3N + 1$, which would automatically fix a new maximum right after N.

Because $(u_n(N)) \models \mathcal{G}$, we can state that it has to reach a new maximum. Otherwise, we suppose $\max_n u_n(N)$ is fixed finite. Then the set of the Syracuse sequence of N is bounded from above and below. We can write $(u_n(N)) \in \|1... \max_n u_n(N)\|$.

So $u_{n+1}, u_{n+2}... \in \|1... \max_n u_n(N)\|$. And therefore, we have either $\in k > 0, u_k = 1 \quad ((u_n(N)) \models \mathcal{O})$, or $\exists k > 0, u_{n+k} = u_n \quad ((u_n(N)) \models \mathcal{C})$. Or else, if such a k doesn't exist, then either both of these implications have been produced before the rank n ($k < 0$). Or if $k = 0$, it means either the initial N was one, or $u_n = u_n$ (obviously true by reflexivity of the equality relation). If k doesn't exist at all, then it's a contradiction between the fact that $(u_n(N)) \leqslant \max_n u_n(N)$ and $\lim_{n\to+\infty} u_n(N) = +\infty$. To the latter, any number $A > \max_n u_n(N)$ constitutes a counter-example. And as $\max_n u_n(N)$ is fixed finite, this A number exists as well.

So we have to fix another maximum for $(u_n(N))$. This maximum is naturally greater than the previous maximum. And we can iterate the reasoning by starting another algorithm with starting number that new maximum. Thanks to the statement 1.1, and because $\exists m \in \mathbb{N}^*, u_m(N) = \max_n u_n(N)$, we can extract the Syracuse sequence of $\max_n u_n(N)$. If we keep on going infinitely, then every newer maximum will be greater than the others, until we reach out infinity.

2.2 A conditioned growth

2.2.1 The concept of variation

As we have admitted in the introductive propositional analysis, a number that satisfies both the infinite growth and the cycle simply doesn't exist. In particular, for an application of the JCF, in the case of this disproval where we suppose $N \models \mathcal{G}$, there are two cases possible: either the image is greater than the antecedent, or the image is smaller than the antecedent (since according to Statement 1.3, $\mathcal{G}$ and $\mathcal{C}$ are incompatible).

Since we have to prove an infinite growth, we have to see under which conditions the sequence is increasing. We can use many methods to come out with the same variation, but we'll use the derivative function of the JCF or the quotient of two numbers to show this.

$$\frac{d\eta}{dN} = \frac{\frac{3N+1}{2^\alpha}}{dN} = \frac{3}{2^\alpha}$$
$$\frac{\eta(N)}{N} = \frac{3N+1}{N2^\alpha} = \frac{3}{2^\alpha} + \frac{1}{N2^\alpha}$$

We'll have to assume that for a given variation, N tends to infinity. And therefore, it's safe to say that the variation will be no more than the quotient of a power of 3 by a power of 2. We can justify it with an empiric argument: the conjecture has actually been verified for all numbers inferior or equal to $20 * 2^58 \approx 5,76 * 10^18$ [3]. Knowing that the decimal logarithm of 2^α is positive, it will actually leave $N2^\alpha$ at a decimal logarithm at the same level, if not even greater. So when we compare the inverse of this product to $\frac{3}{2^\alpha}$, we can of course observe that the former has a mere signification in the variation rate. So we can put it away for the rest of the reasoning.

As for the nested composition of the JCF, we parallely obtain that:

$$\frac{d\eta^i}{dN} = \frac{\frac{3^i N + \sum_{j=0}^{i-1} 3^{i-1-j} * 2^{\sum_{l=0}^{j} \alpha_l}}{2^{\sum_{p=0}^{i} \alpha_p}}}{dN} = \frac{3^i}{2^{\sum_{p=0}^{i} \alpha_p}}$$

We can now put this in application with the forms of the precedent part. We observe that we have 4 forms which admit one even transformation: $1 + 6(1 + 4k)$,$1 + 6(3 + 4k)$,$5 + 6(1 + 4k)$ and $5 + 6(3 + 4k)$. So these will admit a variation of $\frac{3}{2}$. Also, 2 forms are admitting two even transformations: $1 + 6(4k)$ and $5 + 6(2 + 4k)$. They admit a variation of $\frac{3}{4}$. Finally, the last two forms can admit three or more even transformations: $1 + 6(2 + 4k)$ and $5 + 6(4k)$. We'll also assume they admit a variation of exactly $\frac{3}{8}$.

2.2.2 The $r + 24n$ numbers

The idea is to observe a structure of N such that a relatively mechanical behavior depending on its form will indicate how much divisions by 2 it will admit before becoming odd again. We study the ruler sequence, which returns the multiplicative order base 2 of n starting from 1.

0, 1, 0, 2, 0, 1, 0, 3, 0, 1, 0, 2, 0, 1, 0, 4, 0, 1, 0, 2, 0, 1, 0, 3, 0, 1, 0, 2, 0, 1, 0, 5, 0, 1, 0, 2, 0, 1, 0, 3, 0, 1, 0, 2, 0, 1, 0, 4, 0, 1, 0, 2, 0, 1, 0, 3, 0, 1, 0, 2, 0, 1, 0, 6, 0, 1, 0, 2, 0, 1, 0, 3, 0, 1, 0, 2, 0, 1, 0, 4, 0, 1, 0, 2, 0, 1, 0, 3, 0, 1, 0, 2, 0, 1, 0, 5, 0, 1, 0, 2, 0, 1, 0, 3, 0, 1, 0, 2, 0, 1, 0... (OEIS A007814 [4])

According to the Statement 1.6, all the even numbers implied in the algorithm are written under the form 4+6n. We observe the sequence starting from the first numbers and continuing while we advance by 6 steps. The first terms are:

2, 1, 4, 1, 2, 1, 3, 1, 2, 1, 6, 1, 2, 1, 3, 1, 2, 1,...

In the case of our algorithm, however, we have to remove the 4+6n that lead to a 3+6n form, which is non-involved, by removing the 2nd, 5th, 8th, 11th,... terms.

2, 4, 1, 1, 3, 2, 1, 1, 2, 3, 1, 1,...

What will matter to us here will only be the members equal to 1, and 2. Eventually, we'll distinguish aside the members superior or equal to 3. We'll prove that one half of the even numbers are divisible by only 2, and one quarter of the even numbers are divisible by only 4.

For this demonstration, we'll proceed by counting during the construction of the sequence. We take the sequence 4,3,6,3,... This sequence is deprived of 1s and 2s. We put a 2 in front and between adjacent elements to obtain 2,4,2,3,2,6,2,3,2,... The number of 2s represents the half of the numbers of this sequence. To get back the sequence representing the ruler sequence for 4+6n numbers, we add a 1 between adjacent elements, so the number of 1s represents the half of the numbers of this sequence. As for the number of 2s, it represents now the half of the half of this sequence, so the quarter of this sequence. We can interpret this as in our proposition to demonstrate.

We can also notice the pattern $\{p_n, q_n, 1, 1\}$, stating that half of the numbers of the sequences are 1s. With $p_n = \begin{cases} 2 & \text{for} \quad n \quad \text{odd} \\ k \geqslant 3 & \text{for} \quad n \quad \text{even} \end{cases}$ and $q_n = \begin{cases} k \geqslant 3 & \text{for} \quad n \quad \text{odd} \\ 2 & \text{for} \quad n \quad \text{even} \end{cases}$, we observe that for n even, we have the pattern $\{2, k, 1, 1\}$ where the 2s represent one quarter of the sequence, and same thing with the other pattern $\{k, 2, 1, 1\}$.

With the help of this result, we can find the division ring we were looking for. If we pick 1+6ns and 5+6ns individually, so we study the oddly and evenly indexed members, we have the two sequences: For $1 + 6n$: 2, 1, 3, 1, 2, 1,... For $5 + 6n$: 4, 1, 2, 1, 3, 1,... We recognize the patterns $\{2, 1, k, 1\}$ for $1 + 6n$ and $\{k, 1, 2, 1\}$ for $5 + 6n$. So we can notice a mechanical behavior depending on the congruence of n modulo 4. Thus, let the eight forms: $1+6(4k), 1+6(1+4k), 1+6(2+4k), 1+6(3+4k), 5+6(4k), 5+6(1+4k), 5+6(2+4k), 5+6(3+4k)$, with:

$$\eta(1+6(4k)) = \frac{4+72k}{2^\alpha} = 1+18k = 1+6(3k)$$
$$\eta(1+6(1+4k)) = \frac{22+72k}{2^\alpha} = 11+36k = 5+6(1+6k)$$
$$\eta(1+6(2+4k)) = \frac{40+72k}{2^\alpha} = \frac{5+9k}{2^{\alpha-3}}$$
$$\eta(1+6(3+4k)) = \frac{58+72k}{2^\alpha} = 29+36k = 5+6(4+6k)$$
$$\eta(5+6(4k)) = \frac{16+72k}{2^\alpha} = \frac{2+9k}{2^{\alpha-3}}$$
$$\eta(5+6(1+4k)) = \frac{34+72k}{2^\alpha} = 17+36k = 5+6(2+6k)$$
$$\eta(5+6(2+4k)) = \frac{52+72k}{2^\alpha} = 13+18k = 1+6(2+3k)$$
$$\eta(5+6(3+4k)) = \frac{70+72k}{2^\alpha} = 35+36k = 5+6(5+6k)$$

But these images doesn't correspond directly to the first one we had yet. We have to study the congruence of k modulo 4 to rearrange these forms and regain the structure $r+6n$.

- $\eta(1+6(4k)) = 1+6(3k)$

$k \equiv r\,[4]$	0	1	2	3
$3k \equiv r'\,[4]$	0	3	2	1
3k $\rightarrow$	4(3k')	3+4(3k')	2+4(3k'+1)	1+4(3k'+2)

  From this congruence table, we deduce the 4 forms that 1+6(4k) can reach via the JCF, this depending on the congruence of k modulo 4. We can reach the 4 forms 1+6n in the set of $r+24n$ forms.

- $\eta(1+6(1+4k)) = 5+6(1+6k)$

$k \equiv r\,[4]$	0	1	2	3
$6k+1 \equiv r'\,[4]$	1	3	1	3
1+6k $\rightarrow$	1+4(6k')	3+4(6k'+1)	1+4(6k'+3)	3+4(6k'+4)

  Here, we observe there are two forms possible for the image of $1+6(1+4k)$: $5+6(1+4k)$ and $5+6(3+4k)$.

- $\eta(1+6(2+4k)) = 5+9k$

  We'll assume later that this form will assume only 3 even operations, because of its variation. That will also mean that k has to be even. Otherwise, if it was odd, we'd need to apply another even operation, and therefore $\alpha = 4$. So instead of studying the congruence of k modulo 4, we'll opt for a modulo 8.

$k \equiv r\,[8]$	0	2	4	6
$9k \equiv r'\,[8]$	0	2	4	6
9k $\rightarrow$ 6(...)	4(3k')	3+4(3k')	2+4(3k'+1)	1+4(3k'+2)

From this table, we see that the possible images 1+6(2+4k) can reach via the JCF are all the 5+6n forms.

- $\eta(1 + 6(3 + 4k)) = 5 + 6(4 + 6k)$

$k \equiv r\,[4]$	0	1	2	3
$6k + 4 \equiv r'\,[4]$	0	2	0	2
4+6k $\rightarrow$	4(6k'+1)	2+4(6k'+2)	4(6k'+3)	2+4(6k'+5)

There are 2 possible images for 1+6(3+4k) by the JCF : 5+6(4k) and 5+6(2+4k).

- $\eta(5 + 6(4k)) = 2 + 9k$

As for $1 + 6(2 + 4k)$, this form will assume only 3 even operations, because of its variation. But this time, k has to be odd. And instead of studying the congruence of k modulo 4, we'll opt for a modulo 8.

$k \equiv r\,[8]$	0	2	4	6
$9k \equiv r'\,[8]$	0	2	4	6
9k $\rightarrow$ 6(...)	1+4(3k')	4(3k'+1)	3+4(3k'+1)	2+4(3k'+2)

From this congruence table, we deduce the 4 forms that 1+6(4k) can reach via the JCF, this depending on the congruence of k modulo 4. We can reach the 4 forms 1+6n in the set of $r + 24n$ numbers.

- $\eta(5 + 6(1 + 4k)) = 5 + 6(2 + 6k)$

$k \equiv r\,[4]$	0	1	2	3
$6k + 2 \equiv r'\,[4]$	2	0	2	0
2+6k $\rightarrow$	2+4(6k')	4(6k'+2)	2+4(6k'+3)	4(6k'+5)

The possible images of 5+6(1+4k) by the JCF are 5+6(2+4k) and 5+6(4k).

- $\eta(5 + 6(2 + 4k)) = 1 + 6(2 + 3k)$

$k \equiv r\,[4]$	0	1	2	3
$3k + 2 \equiv r'\,[4]$	2	1	0	3
2+3k $\rightarrow$	2+4(3k')	1+4(3k'+1)	4(3k'+2)	3+4(3k'+2)

The possible images of 5+6(2+4k) by the JCF are all the four 1+6n forms.

- $\eta(5+6(3+4k)) = 5+6(5+6k)$

$k \equiv r\,[4]$	0	1	2	3
6k+5≡ r'	1	3	1	3
5+6k $\rightarrow$	1+4(6k'+1)	3+4(6k'+2)	1+4(6k'+4)	3+4(6k'+5)

The possible images of 5+6(3+4k) by the JCF are 5+6(1+4k) and 5+6(3+4k). It is another case of auto-recurrency, similarly to 1+6(4k).

As we observe these images, we see they will bring no absolute influence on the next image by the JCF, unless k' is defined under conditions of congruence as before. At first, for the forms whose image admit 4 possible forms, we see that k' is written as $k' = r + 3k$". Which means its parity isn't determined automatically and depends on k'.

As for the forms with two arrival forms, the parity sure is somewhat independent from k'. However, it will depend on the initial k. For example, if we pick the image by the JCF of 5+6(3+4k), we see when k is odd, then we fall back on 5+6(3+4k). But it does not mean that all the times we iterate the JCF, we'll get to $5+6(3+4k)$ back again over and over. Indeed, the congruence of k modulo 4 will determine the next image. If we had $k \equiv 1\,[4]$, then $k_1 = 6k' + 2$, which is even. But when k is even, then the image is $5+6(1+4k)$ and we leave the self-recurrent loop.

Definition 2.2 We call the a-forms the forms which admit one even transformation, b-forms the forms which admit two even transformations, c-forms which admit three or more even transformations. We note a,b,c, their respective number of appearances.

$$\sum_{p=0}^{i} \alpha_p \geqslant a + 2b + 3c$$

So the condition for having an increase of the Syracuse sequence becomes:

$$\frac{a+2b+3c}{n} > 1,59$$

2.3 The idea of recursive routes

2.3.1 Introduction to this idea

Definition 2.3 A recursive route is a closed walk in the graph of the eight odd forms, such that all graph vertices in it are different (meaning we pass by each form once or never). We call l its length and v its variation. A recursive route is said to be increasing if $v > 1$. A recursive route is said to be decreasing if $v < 1$.

This concept will allow us to "calculate" global variations for shorter patterns designing the behavior of a number via the algorithm. We can state that these routes represent all the behaviours possible for any natural numbers.

If we stipulate it is untrue, then there exists a walk which doesn't fulfill all the conditions that correspond to the definition of a recursive route. A walk in the case of our demonstration remains always in the graph of the eight odd forms. We can eventually change our set of study to change the graph in consequence. But in this case, we'd have another graph with a finite number of nodes, for a finite number of forms.

We can imagine a directed graph $G = (V, E)$ with 8 nodes, where each nodes and vertices are all the vertices except himself. We note the vertices $v_1 ... v_8$. It encompasses the case of our proof, since it has the same order $|V| = 8$ and its size, number of edges, is superior to the size of our initial graph: $|E| = 7 * 8 = 56$. We start from v_1 and we pass by the edge v_1v_2. Since we suppose that we can't have a recursive route, then all edges v_kv_1 are unallowed. We choose another edge, let's say v_2v_3. Now all edges v_kv_2 are unallowed. We keep going on and on until v_8. At this rate, all edges $v_kv_1...v_kv_7$ are unallowed, and thus $\forall k \in \|1, 8\|$. And in particular, for $k = 8$, all the edges we could pass by are unallowed. If we supposed this graph represented the algorithm, that would mean that after $v_1v_2v_3v_4v_5v_6v_7v_8$, we can't continue any further. In other words, we stop the algorithm. This means either $v_8 = 1 + 6n$ and that $n = 0$. So the oneness property is satisfied. Otherwise, it is illegitimate.

2.3.2 Research for (a,b,c) triplets which guarantee an increasing recursive route

We have to define a system which will allow us to find the triplets which guarantee an increasing recursive route. We use the different variables: i,a,b,c. First hypothesis: when we apply 3N+1 once, we have to identify the form we're applying it to. And there's one unique form for each operation. We can then say that:

$$a + b + c = n$$

Second hypothesis: We also have $1 \leqslant a \leqslant 4, 1 \leqslant b \leqslant 2, 1 \leqslant c \leqslant 2$. So the variation of the recursive route is written so:

$$\frac{3^n}{2^{a+2b+3c}} > 1$$

This is equivalent to our study to find a threshold for the number of iterations of the even operation. So our second real condition is:

$$\frac{a+2b+3c}{n} < 1,59$$

With these two hypotheses, we make the following system for us for the solving:

$$\begin{cases} a+b+c=n \\ \dfrac{a+2b+3c}{n} < 1,59 \end{cases}$$

Here's the method to solve this:

1. We suppose the value of n between 1 and 8, and we choose a value of any of the variables according to the interval they are included in.

2. We check in the second line of the system if the condition is verified.

Even before we start the big research that awaits us, we can emit some particular cases and conditions.

Statements 2.3

1. For $a = n$, the triplet $(n, 0, 0)$ leads to an increasing recursive route.

2. For $n < 4$, $c = 0$.

3. If (a, b, c) leads to an increasing recursive route, then so do $(a+1, b-1, c)$, $(a+1, b, c-1)$, and $(a, b+1, c-1)$ if they exist.

4. If (a, b, c) doesn't lead to an increasing recursive route, then so do $(a-1, b+1, c)$, $(a-1, b, c+1)$, and $(a, b-1, c+1)$ if they exist.

Proof.

1. As $a \leqslant 4$, this means as well that $n \leqslant 4$. So the following consequence will apply for the routes with less than five jumping cycles. Let's solve the first line:

   $$a = n \to n+b+c = n \Leftrightarrow b+c = 0 \Leftrightarrow (a, b, c) = (n, 0, 0), \text{for}(b, c) \in \mathbb{N}^2_{>0}$$

   That's the only triplet working. The second line is equivalent to:

   $$\frac{a+2b+3c}{n} < 1,59 \Rightarrow \frac{n}{n} = 1 < 1,59$$

We could have expected such results, since the variation of a is $\frac{3}{2} = 1,5 > 1$. But this allows us to shorten our researches.

2. Let's suppose $c > 0$ and study each value of n with that. If $n = 1$, then the only triplet working for the first line is $(0,0,1)$, obviously. $(2)\frac{a+2b+3c}{n} = \frac{3}{1} = 3 > 1,59$ For $n = 1$, there's no solution for the system if $c > 0$.

   If $n = 2$, then $a+2b+3c \leqslant a+2b+3$. We look for the configuration of (a,b) which minimizes $a+2b$, knowing that to fulfill the first condition, we must have $a+b = 1$. Between the two terms, a has the lowest coefficient. So we give him the maximum value he could have and b becomes null. Let's check the second line:
$$(2) \Leftrightarrow \frac{a+2b+3c}{n} \leqslant \frac{1+3}{2} = 2 > 1,59$$
   With this minimal configuration unable to verify the second condition, every other configuration won't work neither. So for $n = 2$, there's no solution for the system if $c > 0$.

   If $n = 3$, then like before, we must minimize $a+2b$ and verify $a+b = 2$. Then again, a becomes 2 and b becomes null. Let's check the second line:
$$(2) \Leftrightarrow \frac{a+2b+3c}{n} \leqslant \frac{2+3}{3} = 1,66 > 1,59$$
   With this minimal configuration unable to verify the second condition, every other configuration won't work neither. So for $n = 3$, there's no solution for the system if $c > 0$.

3. We suppose that (a,b,c) leads to an increasing recursive route. This means that, in particular :
$$\frac{a+2b+3c}{n} < 1,59$$
   And we also have :
$$\frac{a+2b+3c-2}{n} < \frac{a+2b+3c-1}{n} < \frac{a+2b+3c}{n}$$
   From this, we form :
$$\frac{a+2b+3c-1}{n} = \frac{a+2b+3c+2-3}{n} = \frac{a+2(b+1)+3(c-1)}{n}$$
$$\frac{a+2b+3c-1}{n} = \frac{a+2b+3c+1-2}{n} = \frac{(a+1)+2(b-1)+3c}{n}$$
$$\frac{a+2b+3c-2}{n} = \frac{a+2b+3c+1-3}{n} = \frac{(a+1)+2b+3(c-1)}{n}$$

All three entities corresponding to $(a, b+1, c-1), (a+1, b-1, c), (a+1, b, c-1)$. So by transitivity :

$$\frac{a+2(b+1)+3(c-1)}{n} < 1,59$$
$$\frac{(a+1)+2(b-1)+3c}{n} < 1,59$$
$$\frac{(a+1)+2b+3(c-1)}{n} < 1,59$$

Also :

$$\begin{aligned} a+b+c=n \Leftrightarrow a+b+c+1-1=n &\Leftrightarrow (a+1)+(b-1)+c=n \\ &\Leftrightarrow (a+1)+b+(c-1)=n \\ &\Leftrightarrow a+(b+1)+(c-1)=n \end{aligned}$$

Both conditions (1) and (2) are satisfied.

4. Same method than before, except that we suppose

$$\frac{a+2b+3c}{n} > 1,59$$

and

$$\frac{a+2b+3c}{n} < \frac{a+2b+3c+1}{n} < \frac{a+2b+3c+2}{n}$$

The second point is quite an important result compared to the two previous ones, because it allows us to forbid a lot of potential combinations that would lead us to a decreasing recursive route. We will only have to adjust the parameters a and b. It already gets us 8 forms out of 14 forms away to study when $n < 4$.

The third and fourth statement will easily sharpen our method. All we need to do is to find two triplets (a,b,c) and (a+1,b-1,c) or (a,b+1,c-1) so that $v_{(a,b,c)} < 1,59$and $v_{(a+1,b-1,c)} > 1,59$ (or $v_{(a,b+1,c-1)} > 1,59$)

Enough done with the notices and the preliminary settings: let's begin the raw work of all the combinations leading to an increasing recursive route.

When $n = 1$, we're confronted to a case of self-recurrency that concerns only two forms: $1 + 6(4k)$ and $5 + 6(3 + 4k)$. The variation of $1 + 6(4k)$ when applied by the jumping cycle function is $\frac{3}{4} < 1$, whereas the variation of $5 + 6(3 + 4k)$ when applied by the jumping cycle function is $\frac{3}{2} < 1$. The only increasing self-recursive form is $5 + 6(3 + 4k)$.

When $n = 2$, we have to study the combinations (1,1,0) and (0,2,0). Let's operate with them the raw way. If we take the combination (1,1,0), we have:

$$(2) \Leftrightarrow \frac{a + 2b + 3c}{n} \neq \frac{1 + 2}{2} = 1,5 < 1,59$$

If we take the combination (0,2,0), we have:

$$(2) \Leftrightarrow \frac{2b}{n} \neq \frac{4}{2} = 2 > 1,59$$

So for $n = 2$, the only combination leading to an increasing recursive route else than (2,0,0) is (1,1,0).

When $n = 3$, we have to study the combinations (2,1,0), (1,2,0).

$$(2) \Leftrightarrow \frac{a + 2b + 3c}{n} \neq \frac{2 + 2}{3} = 1,33 < 1,59$$

If we take the combination (1,2,0), we have:

$$(2) \Leftrightarrow \frac{a + 2b + 3c}{n} \neq \frac{1 + 4}{3} = 1,66 > 1,59$$

So for $n = 3$, the only combination leading to an increasing recursive route else than (3,0,0) is (2,1,0).

When $n = 4$, c can be non-null. Let's take different values for a, then for b, to deduce c. When $a = 3$, $b + c = 1$. So whether $b = 1$ or $c = 1$. If we take the combination (3,0,1), we obtain:

$$(2) \Leftrightarrow \frac{a + 2b + 3c}{n} \neq \frac{3 + 3}{4} = 1,5 < 1,59$$

As (3,0,1) is the configuration for which $a+2b+3c$ is at its maximum, the same result goes for (3,1,0). When $a = 2$,$b + c = 2 \Rightarrow (b, c) = \{(2, 0), (1, 1), (0, 2)\}$. If we take the combination $(b, c) = (2, 0)$, we obtain: When $(a, b, c) = (2, 1, 1)$, we have:

$$(2) \Leftrightarrow \frac{a + 2b + 3c}{4} \neq \frac{2 + 2 + 3}{4} = 1,75 > 1,59$$

As $a + 2b + c > 1,59$, this combination and any other configuration making it even higher are associated to a decreasing recursive route. When $a = 1$, $b+c = 3$. The possible combinations are (1,2,1) and (1,1,2). When $(a, b, c) = (1, 2, 1)$:

$$(2) \Leftrightarrow \frac{a + 2b + 3c}{4} = \frac{1 + 4 + 3}{2} = 2 > 1,59$$

Using the same reasoning than previously, both combinations are associated to a decreasing recursive route. So for $n = 4$, the combinations that return an increasing recursive route are $(4,0,0), (3,1,0), (3,0,1), (2,2,0)$.

When $n = 5$, a can't be equal to 0, because otherwise, $b + c = 5$, but b and c are both inferior or equal to 2. So $b + c \leqslant 4$. We pick different values of a: When $a = 4$, there are two possibilities left: $b = 1$, or $c = 1$. When $c = 1$, we're choosing the combination (4,0,1), which makes the second line equal to:

$$(2) \Leftrightarrow \frac{a + 2b + 3c}{n} \neq \frac{4 + 3}{5} = 1,4 < 1,59$$

As (4,0,1) was the configuration for which the rate is at its maximum, (4,1,0) will also lead to an increasing recursive route. When $a = 3$, we're left with three choices: (3,1,1),(3,0,2),(3,2,0). If we pick (3,2,0), we obtain:

$$(2) \Leftrightarrow \frac{a + 2b + 3c}{n} \neq \frac{4 + 3}{5} = 1,4 < 1,59$$

If we pick (3,1,1), we obtain:

$$(2) \Leftrightarrow \frac{a + 2b + 3c}{n} \neq \frac{3 + 2 + 3}{5} = 1,6 > 1,59$$

As this configuration gives a lower rate than (3,0,2), the latter will lead to a decreasing recursive route as well. When $a = 2$, the two choices we're permitted to pick are: (2,2,1) and (2,1,2). The minimal configuration here is (2,2,1). If we pick it:

$$(2) \Leftrightarrow \frac{a + 2b + 3c}{n} \neq \frac{2 + 4 + 3}{5} = 1,8 > 1,59$$

So (2,1,2) will lead to a decreasing recursive route as well. When $a = 1$, we're left with the only combination (1,2,2).

$$(2) \Leftrightarrow \frac{a + 2b + 3c}{n} \neq \frac{1 + 4 + 6}{5} = 2,2 > 1,59$$

So for $n = 5$, the only combinations leading to an increasing recursive route are (4,1,0), (4,0,1) and (3,2,0).

When $n = 6$, we have $a > 1$. Otherwise, for the same reason than before, b or c would be greater than 2. When $a = 4$, we can have the following configurations: (4,2,0), (4,1,1), (4,0,2). If we choose (4,1,1), we will have:

$$(2) \Leftrightarrow \frac{a + 2b + 3c}{n} \neq \frac{4 + 2 + 3}{6} = 1,5 < 1,59$$

As $a + 2b + 3c$ is superior when we choose (4,1,1) rather than (4,2,0), the latter will also lead to an increasing recursive route. If we pick (4,0,2), we will have:

$$(2) \Leftrightarrow \frac{a + 2b + 3c}{n} \neq \frac{4 + 6}{6} = 1,66 > 1,59$$

When $a = 3$, we have the following configurations: (3,2,1) and (3,1,2). When we take (3,2,1), we obtain:

$$(2) \Leftrightarrow \frac{a + 2b + 3c}{n} \neq \frac{3 + 4 + 3}{6} = 1,66 > 1,59$$

According to the Statement 2.3.4, (3,1,2) will lead to a decreasing recursive route as well. When $a = 2$, the only configuration possible is (2,2,2), which gives:

$$(2) \Leftrightarrow \frac{a + 2b + 3c}{n} \neq \frac{2 + 4 + 6}{6} = 2 > 1,59$$

So for $n = 6$, the combinations leading to an increasing recursive route are (4,1,1) and (4,2,0).

When $n = 7$, it's the same to say we remove 1 to one of the variables of the (4,2,2) combination. So we have (4,2,1), (4,1,2), (3,2,2). When we pick (4,2,1):

$$(2) \Leftrightarrow \frac{a + 2b + 3c}{n} = \frac{4 + 4 + 3}{7} = 1,57 < 1,59$$

When we pick (4,1,2):

$$(2) \Leftrightarrow \frac{a + 2b + 3c}{n} = \frac{4 + 2 + 6}{7} = 1,71 > 1,59$$

According to the Statement 2.3.4, both of them leads to a decreasing recursive route. So for $n = 7$, the only combination leading to an increasing recursive route is (4,2,1).

When $n = 8$, we have the simplest case, because the only possible triplet working for the first line is (4,2,2). We just have to check in the second line:

$$(2) \Leftrightarrow \frac{4 + 2 * 2 + 3 * 2}{8} = \frac{14}{8} = 1,75 > 1,59$$

So, for $n = 8$, the system doesn't find any solution for it, which means during our research, we have to make sure to find our initial form on or before the 7th place.

To conclude this work, we can list the set of combinations that lead to an increasing recursive route:

- (1,0,0)

- (2,0,0)
- (2,1,0)
- (3,0,0)
- (2,1,0)
- (4,0,0)
- (3,1,0)
- (3,0,1)
- (2,2,0)
- (4,1,0)
- (4,0,1)
- (3,2,0)
- (4,1,1)
- (4,2,0)
- (4,2,1)

By the way, we can also notice that for n¡6, if b¿0, then c=0, and conversely. That is yet another filtering we can use for the second part of the work. With a global observation of these triplets, we can formulate the following statements which will help in simplifying our research.

Statements 2.4

1. $max_c = 1$
2. $max_l = 7$
3. If 1+6(1+4k) is in a recursive route, we have to choose to include 5+6(3+4k) right after it, or not to include it at all.

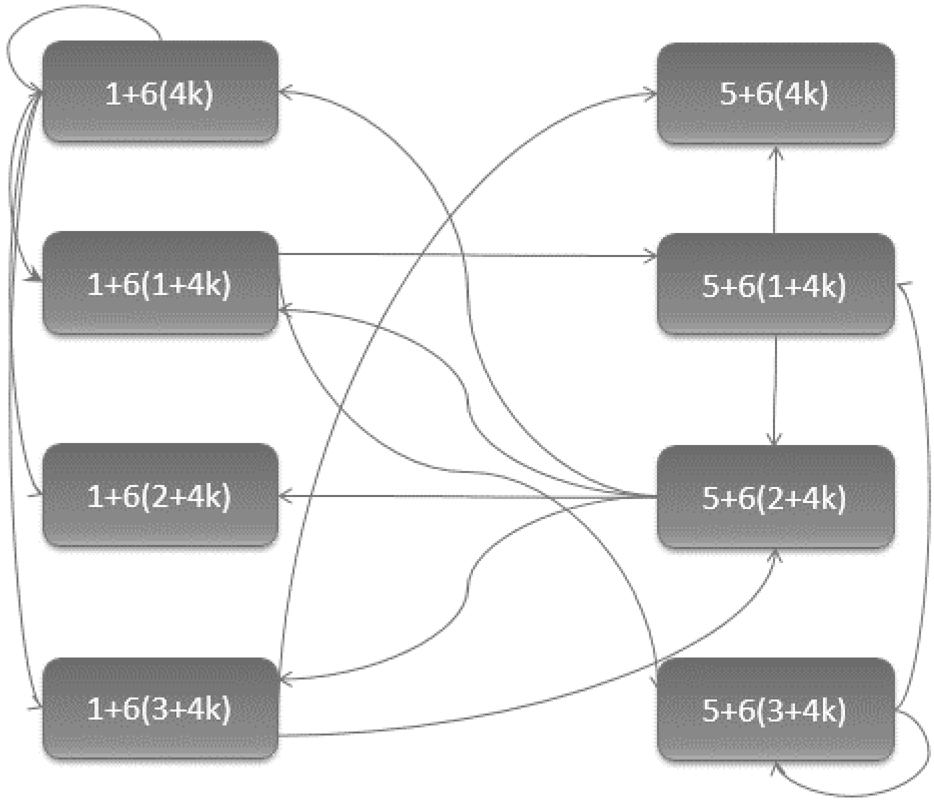

Figure 2.1: Possible images for each form

2.3.3 Research for concrete increasing recursive routes

That was the first part of the work. Now we need to find the linking between the forms when applied to the algorithm. To do so, we'll have to write the recursive routes possible, using an initial form and the different combinations we have found above. To do so, we'll build a diagram. We have to include on this diagram all the $1+6n$ and $5+6n$ forms and arrows to show their potential images using our congruence tables. According to the possible congruences we have found on Part 2.2.2, we deduce this graph:

Now, to proceed to the research, we have to choose one form from which we start an algorithm. We pass by every single possibility after applying the jumping cycle function once. It is translated in the diagram by going from a chosen form by a form across the arrow. Using our previous criterion, some choices won't be permitted. That was also the point on making these rules: it will fasten and shorten the research a lot. Therefore, at each step, we have to remind the combinations left to choose. Once we get back to our form using our rules, we get an increasing recursive route. We repeat this process on and on until every single route is studied.

If we begin by 1+6(4k), we begin with a variation $v = \frac{3}{4}$ and $b = 1$. So we forbid ourselves to use a c-form unless we want a 6-step increasing recursive route. According to the diagram, we're given four possibilities:

- $1 + 6(4k)$: It is a case of self-recurrency: the form goes from itself to itself in a 1-step recursive route. But it is a decreasing recursive route, because the combination is (0,1,0), which corresponds not to the list we established above.

- $1 + 6(1 + 4k)$: $a = 1$, so $(a, b, c) = (1, 1, 0)$. Using this form, we're given two possibilities: 5+6(1+4k) and 5+6(3+4k). If we choose 5+6(3+4k), we will undoubtedly fall on 5+6(1+4k), but we can choose to increment 1 or 2 in a. We decide to have $a = 2$, for the moment. After 5+6(1+4k), we're given two choices:

  - 5+6(4k): $(a, b, c) = (2, 1, 1)$. The combinations left to take are: (4,1,1), (4,2,1). So we're obliged to take 5+6(3+4k) before getting to 5+6(1+4k) to take the last a-form on the way. The only image possible for 5+6(4k) is 5+6(2+4k). If we pick 5+6(2+4k), we won't be able to return to 1+6(4k), for a isn't enough.
  - 5+6(2+4k): $(a, b, c) = (2, 2, 0)$. The available configurations are: (2,2,0), (3,2,0), (4,2,0), (4,2,1). Its images being 1+6n, we can get back to 1+6(4k) with the combination (2,2,0). 1+6(1+4k) has already been taken.
    - 1+6(2+4k): $(a, b, c) = (2, 2, 1)$. The only configuration possible is (4,2,1), implying the obligation to take 5+6(3+4k), so $a = 3$, actually. But we can't select 1+6(3+4k), because 5+6(2+4k) has already been taken and 5+6(4k) is forbidden. So $a \neq 4$, which leads us to a dead end.
    - 1+6(3+4k): $(a, b, c) = (3, 2, 0)$. Its only image is 5+6(4k), so actually, $c = 1$. The only possible combination for this is (4,2,1). So we're taking 5+6(3+4k) with us. The only image possible after this is 1+6(4k), our initial form, which works with the combination (4,2,1).

- 1+6(2+4k): $(a, b, c) = (0, 1, 1)$. The configurations we're allowed now are: (4,1,1), (4,2,1). Its potential images are 5+6(1+4k), 5+6(3+4k), 5+6(2+4k). But if we use 5+6n with n odd, we renounce to have $a = 4$, because we couldn't use 1+6(1+4k) afterwards. Otherwise, we'd fall on a form which was already taken. As for 5+6(2+4k): $(a, b, c) = (0, 2, 1)$.

Our only solution is (4,2,1). So a must be maximised again. However, 1+6(3+4k) won't lead to any permitted image.

- 1+6(3+4k): $(a, b, c) = (1, 1, 0)$. The configurations we can still obtain are: (1,1,0), (2,1,0), (3,1,0), (2,2,0), (4,1,0), (3,2,0), (4,1,1), (4,2,0), (4,2,1). Let's study the two images it has:
  - 5+6(4k): $(a, b, c) = (1, 1, 1)$. Only 2 configurations work at that stage: (4,1,1) and (4,2,1). The images left to use are 5+6(1+4k), 5+6(2+4k), 5+6(3+4k). As we must have $a = 4$, we'll need to pass by $1 + 6(1 + 4k) \to 5 + 6(3 + 4k) \to 5 + 6(1 + 4k)$, so we can't allow ourselves to use either 5+6(1+4k) nor 5+6(3+4k). But we can't pick 1+6(1+4k) directly, so we have to try with 5+6(2+4k). $(a, b, c) = (1, 2, 1)$. Our only choice is 1+6(1+4k), because a isn't equal to 4. However, after the obligatory route $1+6(1+4k) \to 5+6(3+4k) \to 5+6(1+4k)$, we're facing a dead end: 5+6(4k) and 5+6(2+4k) have already been taken.
  - 5+6(2+4k): $(a, b, c) = (1, 2, 0)$. The combinations we can still use are: (2,2,0), (3,2,0), (4,2,0), (4,2,1). Its images are 1+6(4k), 1+6(1+4k), 1+6(2+4k). 1+6(4k), the initial form, is unallowed, because a is not enough.
    * 1+6(1+4k): Passing by the obligatory route $1 + 6(1 + 4k) \to 5 + 6(3 + 4k) \to 5 + 6(1 + 4k)$, $(a, b, c) = (4, 2, 0)$. Our only form possible is 5+6(4k), but it will lead to a 5+6n form, whereas all the possible forms have already been taken.
    * 1+6(2+4k): $(a, b, c) = (1, 2, 1)$. Our only combination allowed is (4,2,1) and our only choice is the obligatory route $1+6(1+4k) \to 5 + 6(3 + 4k) \to 5 + 6(1 + 4k)$. Eventually, 5+6(4k) is unallowed, and 5+6(2+4k) has already been taken.

Increasing recursive routes found :

- 1+6(4k)→1+6(1+4k)→5+6(1+4k)→5+6(2+4k)→1+6(4k)

If we begin by 1+6(1+4k), we can allow ourselves to study the paths that go after 5+6(1+4k). We eventually add one to a if necessary to obtain an increasing recursive route. When we choose then 5+6(1+4k), we're given two possibilities:

- 5+6(4k): $(a, b, c) = (2, 0, 1)$. The configurations left to choose are: (3,0,1), (4,0,1), (4,1,1), (4,2,1). The only allowed form is 5+6(2+4k). $(a, b, c) = (2, 1, 1)$. Like for 1+6(4k), we're left with (4,1,1), (4,2,1).

Unlike for 1+6(4k), we can potentially choose 1+6(4k),1+6(3+4k). But the latter leads to two forms that are already taken. And the former only leads to 1+6(3+4k), because a isn't enough to go directly to the initial form.

- 5+6(2+4k): $(a, b, c) = (2, 1, 0)$. The possible configurations we have are: (2,1,0), (3,1,0), (2,2,0), (4,1,0), (3,2,0), (4,1,1), (4,2,0), (4,2,1). We have four possibilities afterwards: 1+6n. We can get to the initial form directly, using 5+6(3+4k) or not.
  - 1+6(4k): $(a, b, c) = (2, 2, 0)$. The combinations left are: (2,2,0), (3,2,0), (4,2,0), (4,2,1). Again, we can go to the initial form directly, with or without 5+6(3+4k).
    - 1+6(2+4k): $(a, b, c) = (2, 2, 1)$. We can only manage to get (4,2,1). We're left with the form 5+6(4k) which is unallowed. The rest has already be chosen. 1+6(3+4k): $(a, b, c) = (3, 2, 0)$. The form leads to 5+6(4k), for 5+6(2+4k) has already been taken. The latter leads to 1+6(1+4k), our initial form, undoubtedly. We get to the only combination possible, which is (4,2,1) if we choose to pick 5+6(3+4k) from the start.
  - 1+6(2+4k): $(a, b, c) = (2, 1, 1)$. The configurations left to choose are: (4,1,1), (4,2,1). The forms we haven't taken yet are 1+6(4k), 1+6(1+4k), 1+6(3+4k), 5+6(4k). But we can only reach 5+6(4k) which is unallowed. So taking 1+6(2+4k) leads to no solution.
  - 1+6(3+4k): $(a, b, c) = (3, 1, 0)$. Our only option is 5+6(4k), whose images left to use are 1+6(4k),1+6(1+4k). Returning to 1+6(1+4k), our initial form, is possible, taking 1+6(4k) or not, but only if we picked 5+6(3+4k) from the start.

Increasing recursive routes found :

- 1+6(1+4k)→5+6(1+4k)→5+6(2+4k)→1+6(1+4k)
- 1+6(1+4k)→5+6(1+4k)→5+6(2+4k)→1+6(4k)→1+6(1+4k)
- 1+6(1+4k)→5+6(3+4k)→5+6(1+4k)→5+6(2+4k)→1+6(1+4k)
- 1+6(1+4k)→5+6(3+4k)→5+6(1+4k)→5+6(2+4k)→1+6(4k) →1+6(1+4k)

If we begin with 1+6(2+4k), we begin once again with a c-form. So then again, our possible configurations are: (3,0,1),(4,0,1),(4,1,1),(4,2,1). So a has to be superior or equal to 3. However, after taking whether the obligatory route or 1+6(3+4k), even though $a \neq 3$ for the former choice, we can only choose 5+6(2+4k), a b-form, because 5+6(4k) is forbidden as a c-form. So $b = 1$ and a has to be equal to 4. Now if we take the a-forms that are left, we have the choice between a forbidden form and a form that has already been taken. So, in clear, we have no choice at all.

Increasing recursive routes found: None

If we begin by 1+6(3+4k), two choices are given to us: 5+6(4k) and 5+6(2+4k).

- 5+6(4k): $(a, b, c) = (1, 0, 1)$. The combinations we can choose are: (3,0,1), (4,0,1), (4,1,1), (4,2,1). The images left untaken and usable are: 5+6(1+4k), 5+6(2+4k), 5+6(3+4k). We also have a necessity to have all the 4 a-forms, because the only forms leading to 1+6(3+4k) are 1+6(4k) and 5+6(2+4k). So, b=1 and the only combination we're allowed now are (4,1,1) and (4,2,1). Because if we pick any 5+6n with n odd left, we won't be able to pick 1+6(1+4k) afterwards, since we'll get back to 5+6(1+4k) either way, which was already taken.

- 5+6(2+4k): $(a, b, c) = (1, 1, 0)$. We're left with the combinations: (1,1,0), (2,1,0), (3,1,0), (2,2,0), (4,1,0), (3,2,0), (4,1,1), (4,2,0), (4,2,1). The forms we can select afterwards are 1+6n with n an integer. We can directly get back to 1+6(3+4k), using the configuration (1,1,0). Let's check the other forms:

  - 1+6(4k): $(a, b, c) = (1, 2, 0)$. We can now choose: (2,2,0), (3,2,0), (4,2,0), (4,2,1). The possible images are 1+6(1+4k), 1+6(2+4k), 1+6(3+4k). Now, we can't return to 1+6(3+4k), our initial form, because a isn't enough.

    * 1+6(1+4k): $(a, b, c) = (2, 2, 0)$. Passing by this form, there's an obligatory path: $1+6(1+4k) \to 5+6(3+4k) \to 5+6(1+4k) \to 5+6(4k)$. Indeed, 5+6(2+4k) has already been taken. But since all the other forms are taken afterwards, there's no solution with this form.

    * 1+6(2+4k): $(a, b, c) = (1, 2, 1)$. The only configuration left is (4,2,1). To pick all the other a-forms, we can only choose $1+6(1+4k) \to 5+6(3+4k) \to 5+6(1+4k)$. However, after picking 5+6(1+4k), we're facing a dead end, for 5+6(4k) is unallowed because of 1+6(2+4k) and 5+6(2+4k) has already been taken.

  - 1+6(1+4k): $(a, b, c) = (2, 1, 0)$. We're left with the potential configurations: (3,1,0), (2,2,0), (4,1,0), (3,2,0), (4,1,1), (4,2,0), (4,2,1). After following the 5+6(1+4k) obligatory route, we're forced to pick 5+6(4k), so $c = 1$. We're actually left with (4,1,1) and (4,2,1). We suppose that we take 5+6(3+4k) to have $a = 4$. We can then get back directly to 1+6(3+4k) with the configuration (4,1,1). Other than that, we can also use 1+6(4k) to get then to the initial form with the configuration (4,2,1).
  - 1+6(2+4k): $(a, b, c) = (1, 1, 1)$. The configurations we can still choose are: (4,1,1), (4,2,1). So a must be equal to 4. But we fall on either 5+6(4k), which is forbidden, and 5+6(2+4k), which has already been taken.

Increasing recursive routes found :

- 1+6(3+4k)→5+6(2+4k)→1+6(3+4k)

If we begin with 5+6(4k), our first variation is v=3/8. That already limits the available configurations a lot: (3,0,1),(4,0,1),(4,1,1),(4,2,1). However, there are 3 forms we're available to choose:

- 5+6(1+4k): $(a, b, c) = (1, 0, 1)$. By taking this specific form, we renounce to two a-forms: 1+6(1+4k) and 5+6(3+4k). So $\max_a = 2$. This cancels every possibility of having an increasing recursive route.

- 5+6(2+4k): $(a, b, c) = (0, 1, 1)$. Our possible configurations are (4,1,1) and (4,2,1), so we must have $a = 4$, implying we have to take every a-forms, these are the obligatory route and the 1+6(3+4k). But after these forms, we have to choose between 5+6(4k) and 5+6(2+4k), two forms which were already taken, with a being inferior or equal to 3.

- 5+6(3+4k): $(a, b, c) = (2, 0, 1)$. This way, we renounce to 1+6(1+4k), because it will lead to two forms that were already taken. So the maximum value that a can be equal to is 3. Moreover, after this form, only one possibility is given to us, which is 5+6(2+4k), a b-form. So the only combinations possible are (4,1,1), (4,2,1), implying that a must be equal to 4, which was shown impossible beforehand.

Increasing recursive routes found: None

If we begin by 5+6(3+4k), we will undoubtedly fall on 5+6(1+4k), because there are two choices: 5+6(1+4k) and 5+6(3+4k). The latter is a self-recurrency case: we can repeatedly go from this form to itself over and over again directly. And this recursive route is increasing, because $(a, b, c) = (1, 0, 0)$. Let's study the former one:

- 5+6(1+4k): when we choose it, $(a, b, c) = (2, 0, 0)$. The combinations left to choose are: (3,0,0), (2,1,0), (4,0,0), (3,1,0), (3,0,1), (2,2,0), (4,1,0), (4,0,1), (3,2,0), (4,1,1), (4,2,0), (4,2,1). We have two possibilities, whether 5+6(4k) or 5+6(2+4k).
  - If we choose 5+6(2+4k), $(a, b, c) = (2, 1, 0)$. The configurations we can still select are: (2,1,0), (3,1,0), (2,2,0), (4,1,0), (3,2,0), (4,1,1), (4,2,0), (4,2,1). This form gives us four possibilities not taken yet: 1+6(4k), 1+6(1+4k), 1+6(2+4k), 1+6(3+4k).
    - 1+6(4k): $b = 2$. The triplets we can still take are: (2,2,0), (3,2,0), (4,2,0), (4,2,1). Three possibilities are given to us, because 1+6(4k) being a self-recurrent form, picking it again is forbidden:
      - 1+6(1+4k): $(a, b, c) = (3, 2, 0)$. The only form allowed is our initial form: 5+6(3+4k) ; because 5+6(1+4k) has already been taken. This coincides with the combination (3,2,0), so we obtain an increasing recursive form.
      - 1+6(2+4k): $(a, b, c) = (2, 2, 1)$. We have to manage to choose the combination (4,2,1) and the two other forms which increments a are 1+6(3+4k) and 1+6(1+4k). The former is unallowed, because on the first hand, we can't take 5+6(4k) or else $c = 2$, which leads to no increasive recursive form; on the other hand, 5+6(2+4k) is already taken. So there's no way $a = 4$, so this form leads to a decreasing recursive route.
      - 1+6(3+4k): $(a, b, c) = (3, 2, 0)$. What we can still choose for a configuration is (4,2,0) and (4,2,1). But in reality, only (4,2,1) is possible, because the only path possible is to take 5+6(4k) (same reasoning than above). Then $c = 1$. From there, we pick 1+6(1+4k) with $(a, b, c) = (4, 2, 1)$, whose only destination is 5+6(3+4k), our initial form.
    - 1+6(1+4k): (a,b,c)=(3,1,0), and the only form possible is 5+6(3+4k), our initial form, because 5+6(1+4k) is already taken.
    - 1+6(2+4k): our current combination is (2,1,1). What we can still choose are (4,1,1), (4,2,1). So a must be equal to 4. However, we can't choose 1+6(3+4k), because whether we have $c = 2$, which can't be, or we pick a form that is already taken. So a can't be equal to 4.

    * 1+6(3+4k): no forms are allowed afterwards.

  – If we choose 5+6(4k), $(a, b, c) = (2, 0, 1)$. The configurations we can still select are: (3,0,1), (4,0,1), (4,1,1), (4,2,1). The forms not taken yet are: 5+6(2+4k), 5+6(3+4k). We can't choose 5+6(3+4k), because it leads to a decreasing recursive form. So we have to choose 5+6(2+4k) with (a,b,c)=(2,1,1). So we need to gather all 4 a-forms, but we can't actually. Otherwise, we'd have to pick 1+6(1+4k) whose images are 5+6(3+4k) and 5+6(1+4k), both already taken.

Increasing recursive routes found :

- 5+6(3+4k)→5+6(3+4k)
- 5+6(3+4k)→5+6(1+4k)→5+6(2+4k)→1+6(4k)→1+6(1+4k) →5+6(3+4k)
- 5+6(3+4k)→5+6(1+4k)→5+6(2+4k)→1+6(4k)→1+6(3+4k) →5+6(4k)→1+6(1+4k)→5+6(3+4k)
- 5+6(3+4k)→5+6(1+4k)→5+6(2+4k)→1+6(1+4k)→5+6(3+4k)
- 5+6(3+4k)→5+6(1+4k)→5+6(2+4k)→1+6(3+4k)→5+6(2+4k) →1+6(1+4k)→5+6(3+4k)

If we begin by 5+6(1+4k), the main part of the work is somehow done, because we have seen earlier that 5+6(3+4k)'s only path by the jumping cycle function is 5+6(1+4k). So we can take the previous increasing recursive routes we have found (except its auto-recurrency), and do the following changes:

- The initial form is changed into 5+6(1+4k) removing the 5+6(3+4k) form in first position.
- Before the last 5+6(1+4k), we add a step through 5+6(3+4k).

So we obtain:

- 5+6(1+4k)→5+6(2+4k)→1+6(4k)→1+6(1+4k)→5+6(3+4k) →5+6(1+4k)
- 5+6(1+4k)→5+6(2+4k)→1+6(4k)→1+6(3+4k)→5+6(4k) →1+6(1+4k)→5+6(3+4k)→5+6(1+4k)

- 5+6(1+4k)→5+6(2+4k)→1+6(1+4k)→5+6(3+4k)→5+6(1+4k)
- 5+6(1+4k)→5+6(2+4k)→1+6(3+4k)→5+6(2+4k)→1+6(1+4k) →5+6(3+4k)→5+6(1+4k)

If we begin by 5+6(2+4k), we start with a variation rate equal to $v = \frac{3}{4}$. $b = 1$. Its images are 1+6n, with n an integer. Let's study all of these possibilities:

- 1+6(4k): $(a, b, c) = (0, 2, 0)$. The configurations we're left with are: (2,2,0), (3,2,0), (4,2,0), (4,2,1). The images for this form are 1+6n, so we have 3 possibilities afterwards, because 1+6n can't go back to itself in the context of our research: 1+6(1+4k), 1+6(2+4k), 1+6(3+4k).
  - 1+6(1+4k): $(a, b, c) = (1, 2, 0)$. Our set of configurations remains the same. There is no restrictions about choosing 5+6(3+4k) on our road or not, because we could have $a = 2$, or $a = 3$. And when we get to 5+6(1+4k), two possibilities are given to us: 5+6(2+4k), our initial form, working with both configurations (2,2,0) and (3,2,0), and 5+6(4k). For this latter form, our only image possible is 1+6(3+4k), because returning to our initial form with a (2,2,1) or a (3,2,1) combination leads to a decreasing recursive route. However, taking 1+6(3+4k) gets us back to 5+6(2+4k) with a (4,2,1) combination.
  - 1+6(2+4k): $(a, b, c) = (0, 2, 1)$. The only combination possible is (4,2,1). We will have to do all so that all the a-forms are selected. However, after 1+6(3+4k) and the obligatory route $1+6(1+4k) \rightarrow 5+6(3+4k) \rightarrow 5+6(1+4k)$, our only choice is 5+6(2+4k), as we already chose a c-form, consequently making 5+6(4k) forbidden. So either $a = 1$ or $a = 3$, but a will never be equal to 4.
  - 1+6(3+4k): $(a, b, c) = (1, 2, 0)$. We have the same set of configurations as for 1+6(4k). Concerning the images, we're forced to choose 5+6(4k), otherwise, we'd return to our initial form with an insufficient a coefficient. After this, our only chance is to pick the obligatory route $1 + 6(1 + 4k) \rightarrow 5 + 6(3 + 4k) \rightarrow 5 + 6(1 + 4k)$, to have (4,2,1), then to get back to 5+6(2+4k). But we can only choose between 5+6(1+4k) and 5+6(3+4k), so we can't have an increasing recursive route like this.
- 1+6(1+4k): $(a, b, c) = (1, 1, 0)$. We can have these configurations now: (2,1,0), (3,1,0), (2,2,0), (4,1,0), (3,2,0), (4,1,1), (4,2,0), (4,2,1). As we

get to 5+6(1+4k), through the obligatory route, we can return to our initial form, with the combinations (2,1,0) and (3,1,0). Or else, we have 5+6(4k): $(a, b, c) = (3, 1, 1)$. Our set of configurations is now reduced to (4,1,1), (4,2,1). And after that, we can't neither return to 5+6(2+4k) nor pick another form that has already been taken.

- 1+6(2+4k): $(a, b, c) = (0, 1, 1)$. The configurations we have left are: (4,1,1), (4,2,1). So we have to do everything so that $a = 4$. But if we pass by either the form 1+6(3+4k) or the obligatory route $1 + 6(1 + 4k) \rightarrow 5 + 6(3 + 4k) \rightarrow 5 + 6(1 + 4k)$, we will have to choose between 5+6(4k) or 5+6(2+4k). Alas, the former is unallowed, because a c-form has already been taken and a is not enough to allow us to return to 5+6(2+4k).

- 1+6(3+4k): $(a, b, c) = (1, 1, 0)$. With this, we can return to 5+6(2+4k) directly with the combination (1,1,0). Otherwise, we can still choose the following configurations: (2,1,0), (3,1,0), (2,2,0), (4,1,0), (3,2,0), (4,1,1), (4,2,0), (4,2,1). Our other choice is to pick 5+6(4k). With this, our present configuration is (1,1,1), leaving us only (4,1,1), (4,2,1) afterwards. But similarly than before, we won't be able to pick 1+6(1+4k).

Increasing recursive routes found :

- 5+6(2+4k)→1+6(4k)→1+6(1+4k)→5+6(3+4k)→5+6(1+4k)
  →5+6(2+4k)
- 5+6(2+4k)→1+6(4k)→1+6(1+4k)→5+6(1+4k)→5+6(2+4k)
- 5+6(2+4k)→1+6(1+4k)→5+6(3+4k)→5+6(1+4k)→5+6(2+4k)
- 5+6(2+4k)→1+6(1+4k)→5+6(1+4k)→5+6(2+4k)
- 5+6(2+4k)→1+6(3+4k)→5+6(2+4k)

2.3.4 Conclusion to this research

The reason why we've made this research was to prove the necessity of conditioning the k factor to ensure the increase of a Syracuse sequence. If we didn't, then we could still suppose there exists an auto-regulative mechanism which fixes the k factor so that it follows some increasing recursive routes an infinite number of times, which would of course lead to the infinite growth. With what we have obtained while searching for increasing recursive routes, we'll prove the statement that will initiate the conclusive step of this disproval.

Statement 2.5 A number $N = r + 6(r' + 4k)$ with $(r, r') \neq \{(1, 2), (5, 0)\}$ achieves successfully an increasing recursive route iff k is written under a congruence form that satisfies all the conditions induced by each of the steps of the route.

As we have noticed before, if the starting form is 1+6(2+4k) or 5+6(4k), there exists no increasing recursive route. So trying to prove this statement with this form is pointless. We'll need to prove this important statement, otherwise building that candidate N will not be possible. And to do so, we'll take all of the 6 forms in question and compare their increasing recursive routes to the rest.

If the starting form is $1 + 6(4k_0)$, then there exists only one increasing recursive route. And for each step, it is always possible to pick another form.

If the starting form is $1+6(1+4k_0)$, then when we arrive to $5+6(1+4k_n)$, we need to have k_n even. Otherwise, for k_n odd, $\eta(5 + 6(1 + 4k_n)) = 5 + 6(4k_{n+1})$. But after this form, there exists no increasing recursive route.

If the starting form is $1+6(3+4k_0)$, the only increasing recursive route is $1+6(3+4k_0) \to 5+6(2+4k_1) \to 1+6(3+4k_2)$. k_0 has to be odd. Otherwise, we get $\eta(1 + 6(3 + 4k_0)) = 5 + 6(4k_1)$, which leads to no increasing recursive route.

If the starting form is $5 + 6(1 + 4k_0)$, then to achieve the first step $5 + 6(1+4k_0) \to 5+6(2+4k_1)$, k_0 has to be even. Otherwise, $\eta(5+6(1+4k_0)) = 5 + 6(4k_1)$. And afterwards, there's no increasing recursive route.

If the starting form is $5 + 6(2 + 4k_0)$, then we must have $k_0 / \equiv 0[4]$. Otherwise, $\eta(5 + 6(2 + 4k_0)) = 1 + 6(2 + 4k_1)$. And afterwards, there is no increasing recursive route.

If the starting form is $5 + 6(3 + 4k_0)$, since it includes the increasing recursive routes of 5+6(1+4k), with one more step. So it has the same condition than for 5+6(1+4k).

There's also a way to draw the following graphs materializing all the recursive routes possible and mark the points of bifurcation.

The graphs in the following pages are annoted with this legend:

- Green: Leads to an increasing recursive route
- Red: Bifurcates to a decreasing recursive route
- Clear blue: Form whose images is made of forms that are already taken

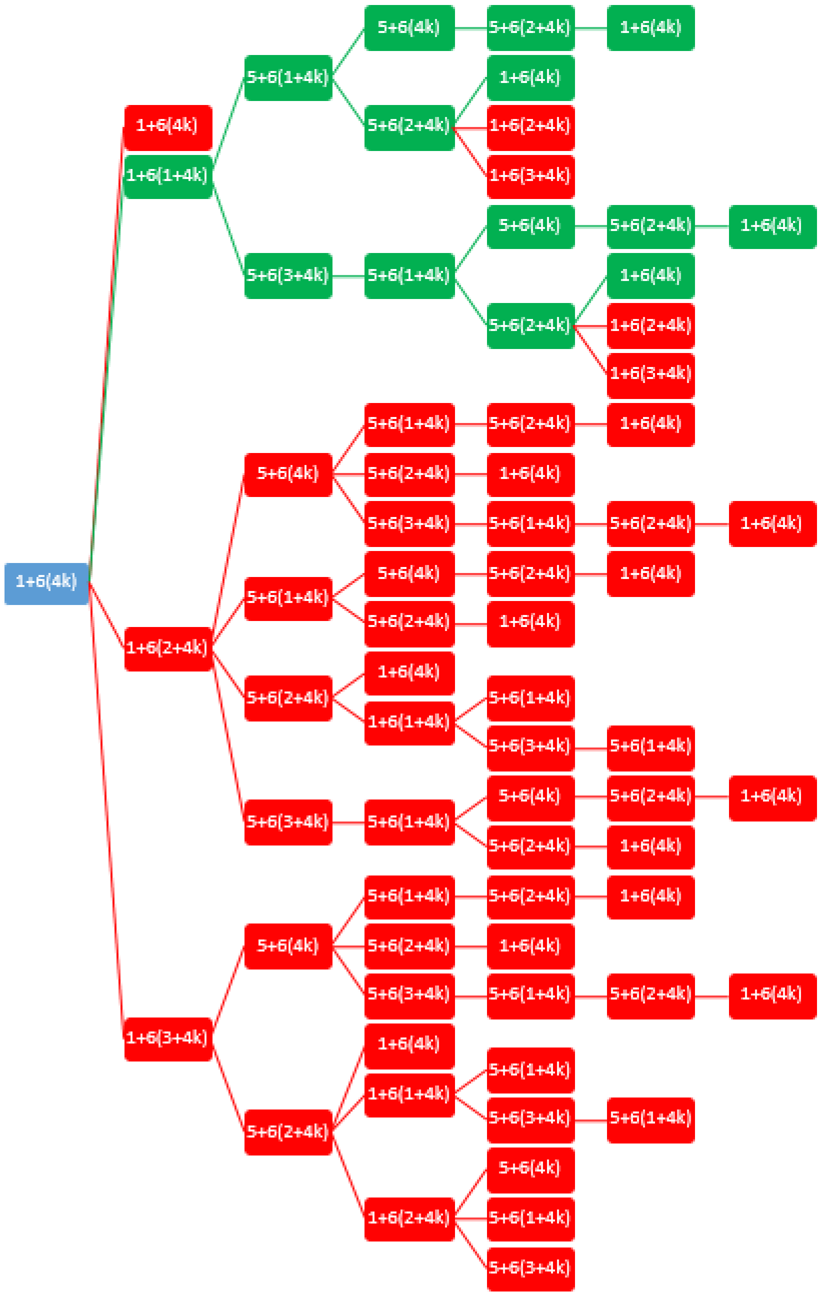

Figure 2.2: Diagram of the recursive routes possible for 1+6(4k)

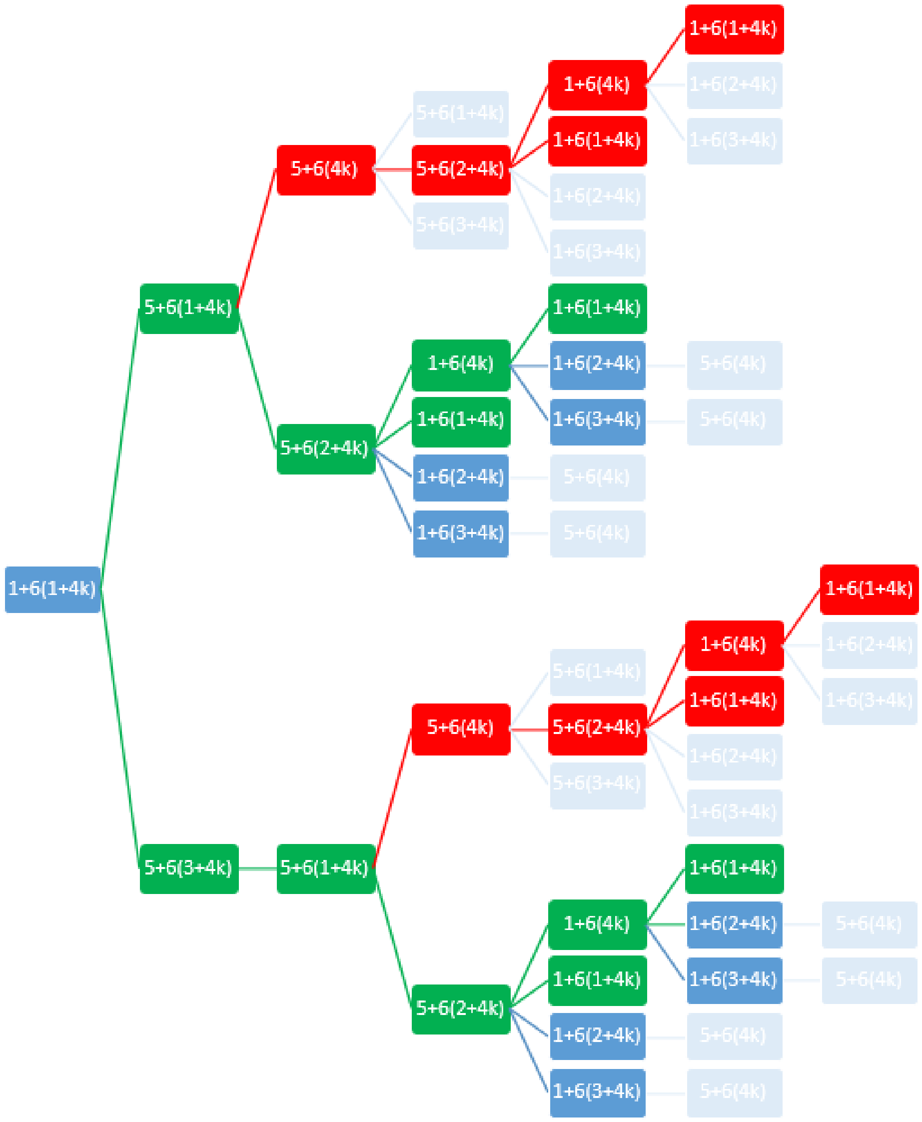

Figure 2.3: Diagram of the recursive routes possible for 1+6(1+4k)

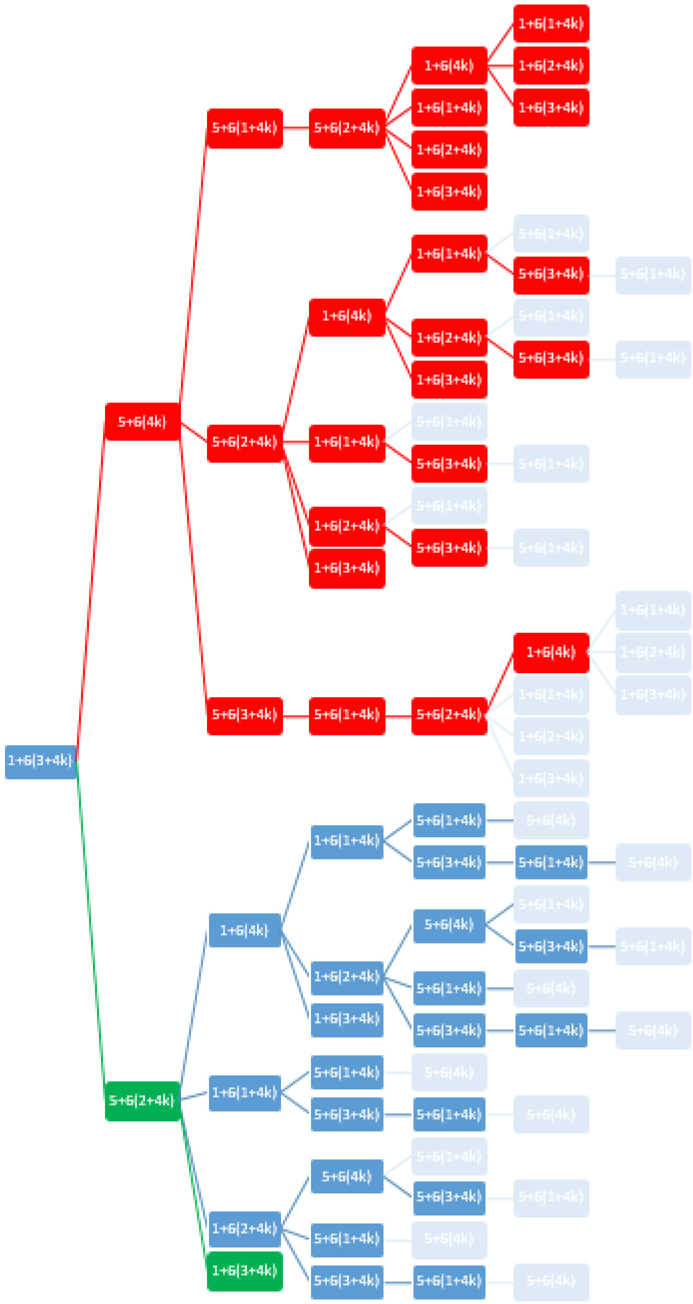

Figure 2.4: Diagram of the recursive routes possible for 1+6(3+4k)

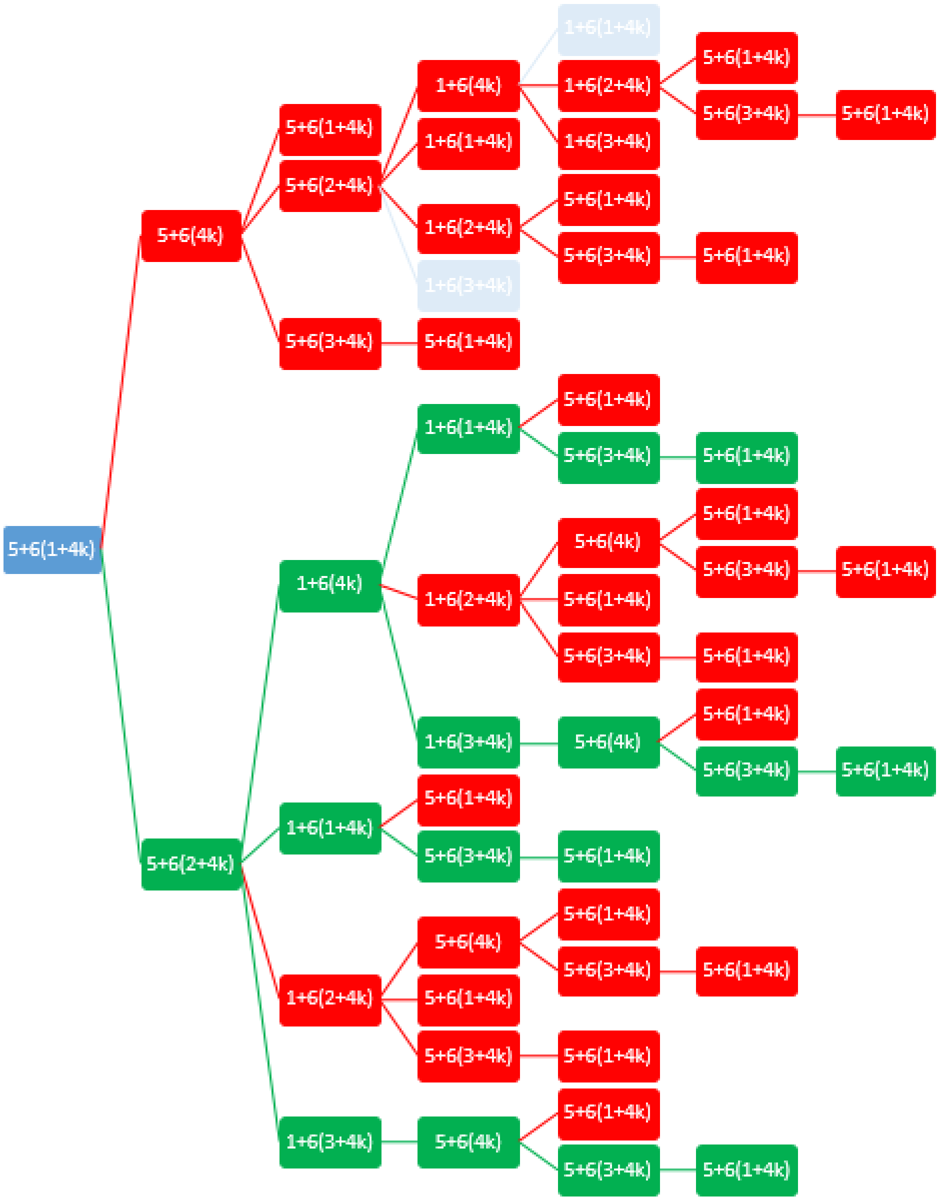

Figure 2.5: Diagram of the recursive routes possible for 5+6(1+4k)

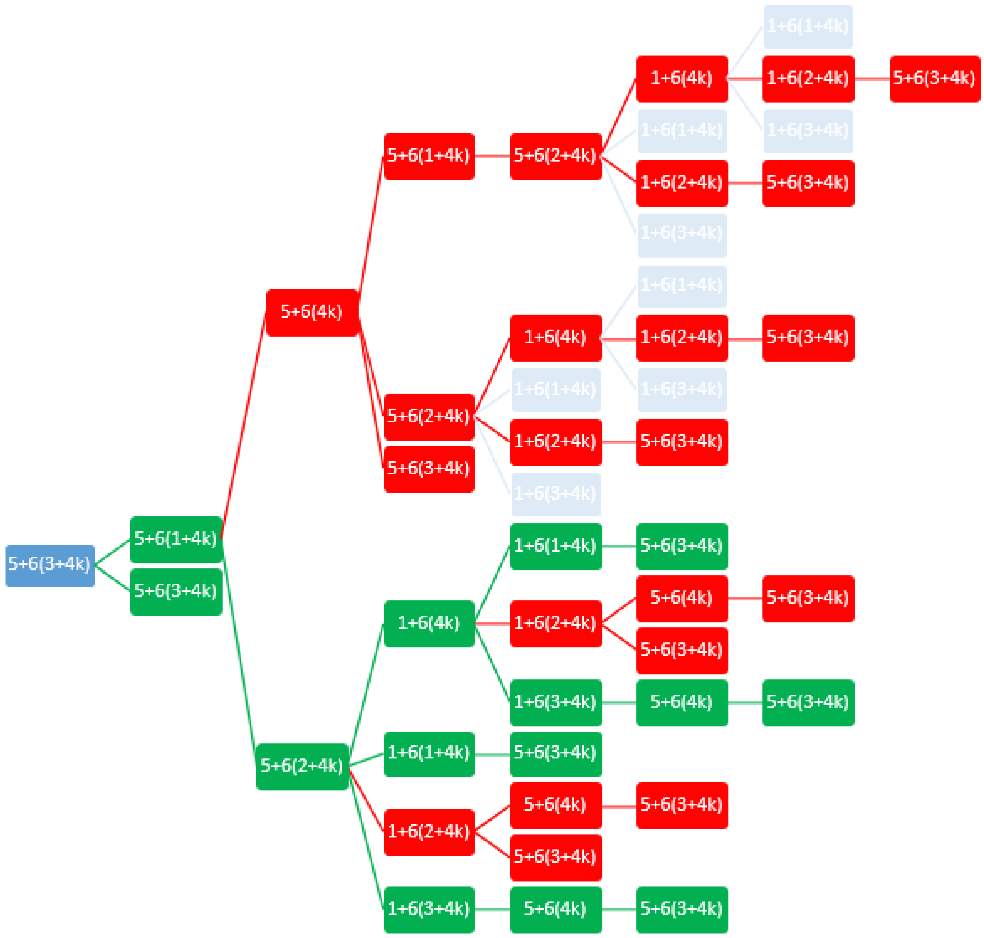

Figure 2.6: Diagram of the recursive routes possible for 5+6(2+4k)

Figure 2.7: Diagram of the recursive routes possible for 5+6(3+4k)

2.4 The final expression of n_0

This investigation for conditions will finally allow us to find the expression of N so that this N number admits an infinite growth. At first, we suppose N follows one single recursive route of length l.We acknowledged at first we have to write N as such in the set we're studying the conjecture. We pose:

$$N = r_N + 6n_0$$

But in the case of the study in this chapter, we have written n_0 as in $r' + 4n'$, to determine exactly the form of $\eta(N)$ in the sets of forms $r + 6n$ and $r + 24n$.

$$n_0 = r_0 + 4k_0$$

We have seen that depending on the congruence of k modulo 4, the class of $\eta(N) = r + 24n$ is determined. We suppose that N follows successfully an increasing recursive route. Then k_0 is written as such its congruence matches with the arrival form.

$$k_0 = r_1 + m_1k_1$$

According to the congruence tables we have established in Part 2.2.1, we know that the condition to get to a specified form is the congruence of k_1. So we also write that condition for k_1 the same way as k_0.

$$k_1 = r_2 + m_2k_2$$

We can keep on repeating this on and on to have:

$$k_{l-1} = r_l + m_lk_l$$

And we successively replace in the expression of k_0 the variable k_1 by its conditioned expression, then k_2, and so on. Eventually, until k_l, we obtain the form k_0 has to take if we want N to finish the increasing recursive route.

$$k_0 = k_l * \prod_{i=1}^{l} m_i + \sum_{j=1}^{l} (r_j * \prod_{p=1}^{l-j} m_p)$$

We can prove by induction this works for whichever value of l. The initialization phase is immediate. For $l = 1$,

$$k_0 = k_1 * \prod_{i=1}^{1} m_i + \sum_{j=1}^{1} (r_j * \prod_{p=0}^{1-1} m_p) = k_1m_1 + r_1 * 1 = m_1k_1 + r_1$$

Now, we suppose there exists a certain value of l for which $k_0 = k_l * \prod_{i=1}^{l} m_i + \sum_{j=1}^{l}(r_j * \prod_{p=1}^{l-j} m_p)$. If we make a step forward, we can write $k_l = m_{l+1}k_{l+1} + r_{l+1}$. We replace it in the formula of k_0.

$$\begin{aligned}
&k_0 = (m_{l+1}k_{l+1} + r_{l+1}) * \textstyle\prod_{i=1}^{l} m_i + \sum_{j=1}^{l}(r_j * \prod_{p=1}^{l-j} m_p) \\
&= m_{l+1}k_{l+1} * \textstyle\prod_{i=1}^{l} m_i + r_{l+1} * \prod_{i=1}^{l} m_i + \sum_{j=1}^{l}(r_j * \prod_{p=1}^{l-j} m_p) \\
&= k_{l+1} * \textstyle\prod_{i=1}^{l+1} m_i + \sum_{j=1}^{l+1}(r_j * \prod_{p=1}^{l-j} m_p)
\end{aligned}$$

For reasons of simplification, we'll suppose by minimization that $r_j = 0$. Indeed, we know that r_j is the residue of the congruence relation of k_{p-1} modulo m_p. Of course, $m_p > 1$ because $m_p \in \{2, 4\}$. So the residue is defined, and positive. Our aim is to prove the only N possible is $+\infty$. So if we start with a minimized value and find $+\infty$, then this N has $+\infty$ as a minimal value. All other possible values are superior to $+\infty$. So in fact, the only number possible is $+\infty$. This will easily simplify our works.

$$k_0 = k_l * \prod_{i=1}^{l} m_i$$

We can do absolutely the same reasoning if we want to follow another increasing recursive route of length l'.

$$k_l = k_{l+l'} * \prod_{i=l+1}^{l+l'} m_i + \sum_{j=l+1}^{l+l'} (r_j * \prod_{p=l+1}^{l+l'-j} m_p)$$

$$k_0 = k_{l+l'} * \prod_{i=l+1}^{l+l'} m_i * \prod_{i=1}^{l} m_i = k_{l+l'} * \prod_{i=l+1}^{l+l'} m_i$$

It is immediate therefore (thanks to our minimization) that for a number ω of recursive routes followed with length l_p, we have:

$$k_0 = k_{\sum_{p=1}^{\omega} l_p} * \prod_{i=0}^{\sum_{p=1}^{\omega} l_p} m_i$$

But as we supposed in our antithesis, the number of iterations of the JCF is infinite and we have said that we always have to follow some increasing recursive routes if we want an infinite growth. So actually $\omega = +\infty$.

$$k_0 = k_{\sum_{p=1}^{+\infty} l_p} * \prod_{i=0}^{\sum_{p=1}^{+} \infty l_p} m_i$$

But $l_p > 0$. So $\sum_{p=1}^{+\infty} l_p = +\infty$.

$$k_0 = k_{\sum_{p=1}^{+\infty} l_p} * \prod_{i=0}^{+\infty} m_i$$

We'll just let $k_{\sum_{p=1}^{+\infty} l_p}$ like that. That allows us to keep the interpretation that even if we applied the JCF an infinite number of times, we're still not done yet. If we stopped, that would mean we have reached 1, which would contradict the antithesis. We keep in mind however, that $k_{\sum_{p=1}^{+\infty} l_p} > 0$. Finally, m_i is the modulo of the congruence relation of k_{i+1}. It is a power of 2 superior to 1. So we have:

$$\prod_{i=1}^{+\infty} m_i = +\infty \Rightarrow k_0 = k_{\sum_{p=1}^{+\infty} l_p} * +\infty = +\infty$$

We have determined the minimum value of k_0 which is infinity. It is its only value. Now, back to this candidate we were looking for:

$$\min_{k_0} N = r_N + 6(r_0 + \infty) = r_N + \infty = +\infty \Rightarrow N = +\infty$$

Thus, the only number possible that can admit an infinite growth is $+\infty$. But we're not allowed to pick this number as a starting number. Furthermore, it is actually unreachable. If there was a number that actually reached it, it would be written as the image of infinity by the reciprocal "odd" transformation (or not forcibly, we have to suppose an infinite odd number and an infinite even number) and multiple reciprocal "even" operations. So we'd have:

$$+\infty \xrightarrow{O^{-1}} \frac{+\infty - 1}{3} = \frac{+\infty}{3} = +\infty \xrightarrow{E^{-1}} 2 * (+\infty) = +\infty \xrightarrow{E^{-1}} ...$$

$$+\infty \xrightarrow{E^{-1}} 2 * (+\infty) = +\infty \xrightarrow{E^{-1}} 2 * (+\infty) = +\infty \xrightarrow{E^{-1}} ...$$

So the only number that reaches infinity... is infinity itself. And therefore, we'd have to pick infinity as a starting number, but we're unallowed to do so.

So the first antithesis is disproved.

Chapter 3

The non-existence of another cycle

The other antithesis aims at finding another number undergoing an infinite number of operations of the algorithm, but which Syracuse sequence keeps on repeating the same numbers over and over again. Therefore, if 1 isn't in the cycle, it won't appear at all in the Syracuse sequence.

3.1 What defines a cycle?

Definition 3.1 We call a cycle a finite subsequence of a Syracuse sequence which starts from a member and finishes at the term that returns the starting member. As it is a subsequence in an ordered sequence, it is ordered as well, and we note it:

$$(u_n(N)) = \{u_0, u_1, \dots u_{\theta-1}, \{u_\theta \dots u_{\theta+s-1}\}\}$$

where s is the number of elements in the cycle and θ is the rank of the starting member of the cycle.

With this interpretation, we can write the Syracuse sequence of a number which satisfies the oneness property in another way:

$$N \models \mathcal{O} \Leftrightarrow (u_n(N)) = \{u_0, u_1, \dots u_{\theta-1}, \{4, 2, 1\}\}$$

Statement 3.1 Any cycle divisible by multiple equivalent subsequences is equivalent to one of these subsequences. We say the cycle is reducible.

$$\{u_0 \dots u_{\theta-1}, \{u_\theta \dots u_{\theta+s-1}, u_{\theta+s} \dots u_{\theta+2s-1}\}\} \equiv \{u_0 \dots u_{\theta-1}, \{u_\theta \dots u_{\theta+s-1}\}\}$$

Proof. We pose $u_n(N) = \{u_0 \dots u_{\theta-1}, \{u_\theta \dots u_{\theta+s-1}\}\}$ the Syracuse sequence of some number N. Then by construction, we can deduce the term following $u_{\theta+s-1}$. $u_{\theta+s} = T(u_{\theta+s-1})$. But by definition, we also have $u_(\theta+s) = u_\theta$. By injectivity of $T(X)$, $T(u_{\theta+s}) = T(u_\theta) \Leftrightarrow u_{\theta+s+1} = u_\theta$. We can continue forth until $u_{\theta+2s-1}$. Thus, $u_n(N) \ni \{u_\theta \dots u_{\theta+s-1}, u_{\theta+s} \dots u_{\theta+2s-1}\}$. And we have $u_{\theta+2s-1} = u_{\theta+s-1}$. By injectivity of $T(X)$, $T(u_{\theta+2s-1}) = T(u_{\theta+s-1}) = u_{\theta+s} = u_\theta$. So the subsequence $\{u_\theta \dots u_{\theta+2s-1}\}$ is also a cycle of $u_n(N)$. Both of these cycles span the other terms of the sequence. We can write the sequence as spanned by these cycles:

$$
\begin{aligned}
u_n(N) &= \{u_0, u_1, \dots u_{\theta-1}, \{u_\theta \dots u_{\theta+s-1}\}\} \\
u_n(N) &= \{u_0, u_1, \dots u_{\theta-1}, u_\theta \dots u_{\theta+s-1}, u_\theta \dots u_{\theta+s-1}, \dots\} \\
u_n(N) &= \{u_0, u_1, \dots u_{\theta-1}, \{u_\theta \dots u_{\theta+s-1}, u_{\theta+s} \dots u_{\theta+2s-1}\}\} \\
u_n(N) &= \{u_0, u_1, \dots u_{\theta-1}, u_\theta \dots u_{\theta+s-1}, u_{\theta+s} \dots u_{\theta+2s-1}, \dots\} \\
u_n(N) &= \{u_0, u_1, \dots u_{\theta-1}, u_\theta \dots u_{\theta+s-1}, u_\theta \dots u_{\theta+s-1}, \dots\}
\end{aligned}
$$

So

$$\{u_0 \dots u_{\theta-1}, \{u_\theta \dots u_{\theta+s-1}, u_{\theta+s} \dots u_{\theta+2s-1}\}\} \equiv \{u_0 \dots u_{\theta-1}, \{u_\theta \dots u_{\theta+s-1}\}\}$$

Concretely, if we assumed the algorithm didn't stop when $N = 1$, we could say that the cycle $\{4, 2, 1, \dots 4, 2, 1\}$ could be divisible by the subsequence $\{4, 2, 1\}$.

Because of the Statement 3.1., we can give some more details on a cycle reducible by a subsequence. We have that the length of this subsequence divides the length of the cycle. As these subsequences are all equivalent, so are their length. And if a cycle has to be written as the succession of these subsequences, then its length being the sum of their length, it is a multiple of their length.

That gives another consequence, maybe less practical: if the length of a cycle is a prime number, then it is irreducible in the case of our conjecture. Because the length s' of this subsequence divides the length s of the cycle, we have $s' \in Div(s)$. But if we suppose that $s \in P$, then $Div(s) = \{-s, -1, 1, s\}$. The length of a subsequence is positive, and thus, we have two possibilities: either $s' = s$ or $s' = 1$. If $s' = s$, then obviously, because $T(X)$ maps to an unique image for whichever number, both cycle and subsequence are the same. If $s' = 1$, then the subsequence induces particularly that $T(u_\theta) = u_\theta$. We resolve this equation in $\mathbb{Z}$ (since it won't admit a solution for neither-odd-nor-even number, a.k.a decimals, hyperreal numbers and complex numbers):

1. For $u_\theta = 2x$, $T(2x) = 2x \xrightarrow{T \equiv E} 2x/2 = 2x \Rightarrow x = 2x \Rightarrow x = 0$

2. For $u_\theta = 2x+1$, $T(2x+1) = 2x+1 \xrightarrow{T\equiv O} 3(2x+1)+1 = 2x+1 \Rightarrow$ $6x+4 = 2x+1 \Rightarrow 4x+3 = 0 \Rightarrow x = -\dfrac{3}{4}$

So the only number in the kernel of $T(X) - X$ is 0, since $-\dfrac{1}{2}$ isn't an integer. So such a subsequence would be $\{0\}$ and the original sequence would be only made of zeros, by construction. But in the case of our conjecture, we only afford non-null natural numbers. So 0 is excluded, and since there are no other subsequences left, then the sequence is forcibly irreducible. Finally, we can make a reinterpretation of the condition for a subsequence to reduce a cycle: $s'|s \Leftrightarrow s \equiv 0\,[s']$.

Statement 3.2 If the Syracuse sequence of N admits a cycle different from $\{4, 2, 1\}$, then it contains an unique cycle.

Proof. Supposing the subsequences $\{u_\theta \ldots u_{\theta+s-1}\}$ and $\{u_{\theta'} \ldots u_{\theta'+s'-1}\}$ with $\theta \leqslant \theta'$ are both cycles of the same Syracuse sequence $(u_n(N))$. We distinguish multiple cases:

- $\theta = \theta'$: It is obvious that $u_\theta = u_{\theta'}$. We'll need to compare the variables s and s'.
  - For $s = s'$, it is obvious that $\{u_{\theta'} \ldots u_{\theta'+s'-1}\} = \{u_\theta \ldots u_{\theta+s-1}\}$.
  - For $s > s'$, then the subsequence $\{u_{\theta'} \ldots u_{\theta'+s'-1}\}$ is contained in $\{u_\theta \ldots u_{\theta+s-1}\}$. By definition, we should have $T(u_{\theta+s'-1}) = u_\theta$ and $T(u_{\theta+s-1}) = u_\theta$. There are two cases left:
    - ∗ If $\{u_\theta \ldots u_{\theta+s-1}\}$ is reducible by $\{u_\theta \ldots u_{\theta+s'-1}\}$, then both cycles are equivalent.
    - ∗ If not, then $u_{\theta+s-1} \neq u_{\theta+s'-1}$. So by injectivity of $T(X)$, $T(u_{\theta+s-1}) \neq T(u_{\theta+s'-1})$. But both of these expressions are equal to u_θ. That makes a contradiction with the transitivity of the equality.
  - For $s < s'$: same method.
- $\theta < \theta'$: This time, we need to distinguish the case if $u_\theta = u_{\theta'}$ or not.
  - For $u_\theta = u_{\theta'}$, we'll need to study the reducibility of $\{u_\theta \ldots u_{\theta+s-1}\}$. For this, we need to suppose if $u_{\theta'} \in \{u_\theta \ldots u_{\theta+s-1}\}$. If so is the case, then it has to be written such that $\theta' = \theta + d, d \mid s$, so that we can write:

$$\{u_\theta \ldots u_{\theta+s-1}\} = \{u_\theta \ldots u_{\theta+d-1}, u_{\theta+d} \ldots u_{\theta+2d-1}, u_{\theta+2d} \ldots u_{\theta+s-1}\}$$

. Basically, we could have two equivalent cycles if $s'|s$. If not, then we would have two different values for $u_{\theta'}$ again because of the bijectivity of $T(X)$.

- For $u_\theta \neq u_{\theta'}$, then we'll need to study the belonging of $u_{\theta'}$ in $\{u_\theta \dots u_{\theta+s-1}\}$.
  * If we suppose that $u_{\theta'} \notin \{u_\theta \dots u_{\theta+s-1}\}$ (so its value doesn't correspond to the value of one of the members), then there's no way it can be another cycle of the Syracuse sequence of N. Indeed, if we span the sequence with the use of $\{u_\theta \dots u_{\theta+s-1}\}$, we have $u_n(N) = \{u_0 \dots u_{\theta-1}, u_\theta \dots u_{\theta+s-1}, u_\theta \dots u_{\theta+s-1} \dots\}$, with $u_{\theta'} \notin u_n(N)$. However, we have $\{u_{\theta'} \dots u_{\theta'+s'-1}\}$ is a sequence of $u_n(N)$.
  * If we suppose that $u_{\theta'} \in \{u_\theta \dots u_{\theta+s-1}\}$, then $s' = s$. Otherwise, analogically to what we have explained before, $u_{\theta'}$ would admit more than one value. If so, there would be two cycles: $\{u_\theta \dots u_{\theta+s-1}\}$ and $\{u'_\theta \dots u_{\theta'+s-1}\}$. But both of the cycles are the same. We can write them as the span of the Syracuse sequence of N.

$$\begin{aligned}
&u_n(N) = \{u_0, u_1, \dots u_{\theta-1}, u_\theta \dots u_{\theta+s-1}, u_\theta \dots u_{\theta+s-1}, \dots\} \\
&u_n(N) = \{u_0 \dots u_{\theta-1}, u_\theta \dots u_{\theta'} \dots u_{\theta'+s-1}, u'_\theta \dots u_{\theta'+s-1} \dots\} \\
&= \{u_0 \dots u_{\theta-1}, u_\theta \dots u_{\theta'} \dots u_{\theta+s-1}, u_\theta \dots \\
&\qquad u_{\theta'+s-1}, u'_\theta \dots u_{\theta+s-1}, u_\theta \dots u_{\theta'+s-1} \dots\}
\end{aligned}$$

  We can actually simplify the second expression like the first one, and we see that both cycles are the same.

This statement ensures the stability of our second antithesis, and that motivates its disproval, because if a cycle wasn't unique, it would have a mere signification in the progress of the algorithm. The effect of a cycle is to lock the set of reachable numbers to a finite set of infinitely appearing numbers. If a cycle wasn't the trivial cycle $\{4, 2, 1\}$, then it wouldn't include the number 1, making the oneness property unsatisfied, thus giving a counter-example to the conjecture. However, if a cycle wasn't unique, we'd switch from cycles to cycles, thus having a chance to reach 1.

Statement 3.3 If the Syracuse sequence of N admits a cycle different from $\{4, 2, 1\}$, then the s-1 successive applications of $T(N)$ is a projector when $i \geqslant \theta$.

Proof. We suppose $N \models \mathcal{C}$. Then $u_n(N) = \{u_0 \dots u_{\theta-1}, \{u_\theta \dots u_{\theta+s-1}\}\}$
By definition of a cycle, $T(u_{\theta+s-1}) = u_\theta \Leftrightarrow T \circ T(u_{\theta+s-2}) = u_\theta$
$\Leftrightarrow T^2(u_{\theta+s-2}) = u_\theta$. By recurrency, we have $T^z(u_{\theta+s-1-z}) = u_\theta, \forall z \in [1, s-1]$. More specifically, for $z = s-1$, $T^{s-1}(u_\theta) = u_\theta$.

This is a quasi-immediate consequence of the statement 3.1 right above. If the cycle was $\{4, 2, 1\}$, then each member of this cycle would appear only once, since we're told to stop the algorithm when N=1. However, if it's a cycle other than $\{4, 2, 1\}$, then the cycle won't contain 1 in it. In consequent, we're never asked to stop the algorithm. So the variable i is infinite. So the Syracuse sequence of such a number contains an infinite number of members. Particularly, it contains the numbers $T^i(u_0)$ with $i \in [0, +\infty[$. θ being finite, it is contained in this interval, as well as $\theta + zs$. So the infinite-dimensional family $(u_(\theta + zs))$, which members have all the same value (that is u_θ). In other words, the number u_θ appears an infinite number of time. The same reasoning can be done for $u_{\theta+1}, u_{\theta+2}, \dots$

Ideally, in the research of our cycle, we suppose that our starting member of the cycle is also the initial number of the algorithm. If it happened we'd find a cycle which starting member isn't the initial number of the algorithm, then we can check its image by the corresponding transformation. If it isn't the starting member of the cycle, then nevertheless, its next images by the transformations $O(X)$ and $E(X)$ still lead to the cycle after a number of iterations equal to the one for the starting number minus 1, since the transformations are bijective. For reasons of simplification, we'll find the starting member of a cycle.

3.2 The snake biting its own tail

Assuming we have defined a function in our introductive part that materializes the transition from one odd number to another odd number, we can make use of it to find some equations to solve in order to find a candidate number that comes back to itself after some iterations of the JCF.

But before we get started, we have to understand that though a number admits one image by the JCF, it admits an infinite set of elements as its preimage. We'll need to find a way to describe the set of all numbers that reach out to a number by the JCF. Knowing of course that even if we'd find a number that reaches out to another number, it has to be involved in the conjecture.

Let's assume the number wasn't involved in the algorithm. Then by definition, it admits no preimage by neither the JCF nor the "odd" transformation. The idea of a cycle is a subsequence of numbers, each one of them the image of the previous one by one of the transformations. So each one of them belongs to the preimage of the next one. A non-involved number would enter in contradiction to these principles.

But of course, we need a function that allows us to retrieve the preimage of a number by the JCF.

Definition 3.2 The reversal function (noted $\mu_k(N)$) is the function that maps a number to one of its possible antecedents by the JCF.

$$\begin{aligned}\mu_k : (\mathbb{N}*)^2 &\to \mathbb{R}\\ (N,k) &\mapsto O^{-1} \circ (E^{-1})^k(N) = \frac{2^k N - 1}{3}\end{aligned}$$

In contrast to the JCF, this function is bijective, and not only surjective. Indeed, if we pick any natural number and we apply the reciprocal odd and even transformations a finite number of times, we can determine the kth greatest number which return this natural This function for now has no restriction on its image, meaning it can be non-natural for now. We can illustrate with a few examples:

- $\mu_1(17) = \dfrac{2*17-1}{3} = \dfrac{34-1}{3} = \dfrac{33}{3} = 11 \in \mathbb{N}^*$

  $\mu_2(17) = \dfrac{4*17-1}{3} = \dfrac{68-1}{3} = \dfrac{67}{3} = 22,3 \notin \mathbb{N}^*$
- $\mu_1(13) = \dfrac{2*13-1}{3} = \dfrac{30-1}{3} = \dfrac{29}{3} = 9,\overline{6} \notin \mathbb{N}^*$

  $\mu_2(13) = \dfrac{4*13-1}{3} = \dfrac{52-1}{3} = \dfrac{51}{3} = 17 \in \mathbb{N}^*$

So we must refine it in a concept that is even more accurate with the circumstances of the conjecture.

Definition 3.3 The ascendancy of N is the infinite ordered set of all involved natural numbers which image by the JCF is N. The mth ascendant of N is the mth element in the ascendancy of N. We note:

$$A_N = \{a_N^1, a_N^2, \ldots\} = \{N' \models I \mid N' \in \eta^{-1}(N)\}$$

This new concept defines a more "pragmatical" form for the numbers involved in the algorithm which return a same number. This implies that this number has to be written under the form 1+6n or 5+6n. We acknowledge this is an ordered set, since the more the exponent k increases, the bigger the ascendant becomes.

$$k' > k \Leftrightarrow 2^{k'} > 2^k \Leftrightarrow 2^{k'} - 1 > 2^k - 1 \Leftrightarrow \frac{2^{k'} - 1}{3} > \frac{2^k - 1}{3}$$

Such a set has different properties.

Statement 3.5 The preimage of a number by the JCF in a Syracuse sequence (or therefore, an image of this number by the reversal function) belongs to its ascendancy.

$$\eta^{k-1}(N) \in A_{\eta^k(N)}$$

Proof. We pose $(u_n(N)) = \{N \dots \eta^1(N) \dots \eta^{k-1}(N) \dots \eta^k(N) \dots\}$.

$\eta^{k-1}(N) \in A_{\eta^k(N)} \Leftrightarrow \exists m \in N_{\leqslant 1}, \mu_m(\eta^k(N)) = \eta^{k-1}(N) \models I$. From $\eta^{k-1}(N)$ to $\eta^k(N)$, there has been α_k even operations.

$\mu_{\alpha_k}(\eta^k(N)) = \mu_{\alpha_k}(\eta(\eta^{k-1}(N))) = (\mu_{\alpha_k} \circ \eta)(\eta^{k-1}(N)) = id(\eta^{k-1}(N)) = \eta^{k-1}(N) \in \mathbb{N}_{\geqslant 1}$. According to Definition 1.5,

$$N \models I \Leftrightarrow \begin{cases} \exists N' \in N*, \eta(N') = N & \text{if} \quad N \in \overline{1} \\ O^{-1}(N) \in N* & \text{if} \quad N \in \overline{0} \end{cases}$$

So $\eta(\eta^{k-2}(N)) = \eta^{k-2+1}(N) = \eta^{k-1}(N) \models I$.

This statement, though a bit evident, will however help in proving a property that will make things a bit more difficult.

Statement 3.6 The belonging to the ascendancy of N is not a transitive property.

A counter-example is a Syracuse sequence containing the ordered terms :

$$\eta^{k-2}(N) = 17, \eta^{k-1}(N) = 13, \eta^k(N) = 5$$

. According to the previous property, we have: $\eta^{k-2}(N) = 17 \in A_{\eta^{k-1}}(N) = A_{13}\eta^{k-1}(N) = 13 \in A_{\eta^k(N)} = A_5$ But, when we calculate the ascendancy of 5:

$$A_5 = \frac{2^3 * 5 - 1}{3}, \frac{2^5 * 5 - 1}{3}, \dots = 13, 53, \dots$$

The set is ordered, and $17 \in]13, 53[\notin A_5$. In particular, $\eta^{k-2}(N) \notin A_5$. In particular, our main idea is to start from a number to get back to one of its ascendants. With this Statement 3.6, we're only restricted to one occurrence of the reversal function. If this wasn't the case, we'd have to verify an infinite number of equations. And finally, one last statement that's inspired from the previous statement, underlining the fact we have only one occurrence of the reversal function.

Statement 3.7 Supposing $N, N', n \in \mathbb{N}, N \neq N', A_N, A_{N'}$ their ascendancies. Then if a number is the ascendant of N, it cannot belong to the ascendancy of N'.

$$(n \in A_N) \supset (n \notin A_{N'})$$

Proof. Supposing $n \in A_N$ and $n \in A_{N'}$, with $N \neq N'$. Then $\exists m, m' \in \mathbb{N}_{\geqslant 1}, n = \mu_m(N) = \mu_{m'}(N')$. $\eta(\mu_m(N)) = \eta(\mu_{m'}(N')) \Leftrightarrow (\eta \circ \mu_m)(N) = (\eta \circ \mu_{m'})(N') \Leftrightarrow id_N = id_{N'} \Leftrightarrow N = N'$. That purely contradicts the hypothesis that $N \neq N'$.

With this even stronger result, we know that there exists absolutely no numbers in the ascendancy of a number which would be retrievable thanks to the ascendancy of some other number. For instance, the ascendancy of 1 will only contain powers of 2 applied to the reciprocal odd operation. Depending on the form of N, we'll calculate the ascendancy of N. To do so, we have to achieve successive reciprocal even operations, then verify when the reciprocal odd operation is possible in order to get a natural number.

Statement 3.8 The ascendancy of $1 + 6n$ and $5 + 6n$ can be directly calculated with the set of formulae:

$$\begin{aligned} A_{1+6n} &= \left\{\sum_{p=0}^{m} 4^p + 2^{m+1}n \mid m \in \overline{0} \cap \mathbb{N}^*, (\epsilon_n, \epsilon_m) \neq \{(0,1),(1,2),(2,0)\}\right\} \\ A_{5+6n} &= \left\{5 * 2^m + 3(2^{m+1}n) \mid m \in \overline{1} \cap \mathbb{N}*, \epsilon_n \neq \epsilon_m\right\} \end{aligned} \tag{3.1}$$

with ϵ_n and ϵ_m the residue of n and m modulo 3.

Proof. Because $A_{1+6n} \subset \left\{\dfrac{2^k - 1}{3} + 2^{k+1}n \mid k \in \mathbb{N}*\right\}, \exists x \in A_{1+6n}, \exists m \in \mathbb{N}*, x = \dfrac{2^m - 1}{3} + 2^{m+1}n$. Furthermore, $x \in A_{1+6n} \Rightarrow x \in \mathbb{N}* \Leftrightarrow \dfrac{2^m - 1}{3} + 2^{m+1}n \in \mathbb{N}*$. Since $2^{m+1}n \in N*$, we must have $\dfrac{2^m}{3} - \dfrac{1}{3} \in \mathbb{N}* \Leftrightarrow 2^m \equiv 1\,[3]$. $2 \equiv 2\,[3] \Rightarrow 2^2 = 4 \equiv 1\,[3] \Rightarrow 2^{2k} \equiv 1\,[3]\,, k \in \mathbb{Z}$. So for m even, $x \in \mathbb{N}*$. Finally, $x \in A_{1+6n} \Rightarrow x \models I \Leftrightarrow x \notin \overline{3}$. We observe for which values of m we have $x = \dfrac{2^m - 1}{3} + 2^{m+1}n \notin \overline{3}$. For m even, $2^m \equiv 4\,[6] \Rightarrow 2^{m+1}n \equiv 2n\,[6]$. The congruence of $2^{m+1}n$ modulo 6 is independent from m. Also, $\dfrac{2^m - 1}{3} = \sum_{p=0}^{m} 4^p$. We pose $1 \equiv 1\,[6] \Rightarrow \sum_{p=0}^{m=1} 4^p \equiv 5\,[6] \Rightarrow \sum_{p=0}^{m=2} 4^p \equiv 3\,[6]$. And $\forall a \in \mathbb{N}*, 4^a + 4^{a+1} + 4^{a+2} \equiv 4 + 4 + 4 \equiv 6 * 2 \equiv 0\,[6]$. So in fact, $\sum_{p=0}^{m \equiv 0[3]} 4^p \equiv 1\,[6]\,, \sum_{p=0}^{m \equiv 1[3]} 4^p \equiv 5\,[6]\,, \sum_{p=0}^{m \equiv 2[3]} 4^p \equiv 3\,[6]$ We pose ϵ_n and ϵ_m the residue of n and m modulo 3. So $x \equiv 1 - 2\epsilon_m + 2\epsilon_n\,[6]$. We have to verify

$1 + 2\epsilon_n - 2\epsilon_m \neq 3\,[6]$ for $\epsilon_n, \epsilon_m \in 0, 1, 2.$ $\Leftrightarrow 2\epsilon_n - 2\epsilon_m \neq 2\,[6] \Leftrightarrow \epsilon_n - \epsilon_m \neq 1\,[3] \Leftrightarrow \epsilon_n \neq 1 + \epsilon_m\,[3]$ or $\epsilon_m \neq \epsilon_n - 1\,[3] \Leftrightarrow \epsilon_n \neq 1, 2, 0\,[3]$ or $\epsilon_m \neq 2, 0, 1\,[3]$. So we have $(\epsilon_n, \epsilon_m) \neq \{(0, 1), (1, 2), (2, 0)\}$.

Same reasoning for $A_{5+6n} \subset \{\mu_m(5 + 6n) \mid m \in \mathbb{N}*\}$ We calculate $\mu_k(5 + 6n) = O^{-1} \circ E^{-k}(5 + 6n) = \dfrac{2^k(5 + 6n) - 1}{3} = \dfrac{5 * 2^k - 1}{3} + 2^(k + 1)n$ Because $A_{5+6n} \subset \left\{\dfrac{5 * 2^k - 1}{3} + 5 * 2^(k + 1)n \mid k \in \mathbb{N}*\right\}, \forall x \in A_{5+6n}, \exists m \in \mathbb{N}*, x = \dfrac{5 * 2^k - 1}{3} + 5 * 2^{m+1}n$. Since $5 * 2^{m+1}n \in \mathbb{N}*$, we must have $\dfrac{5 * 2^m}{3} - \dfrac{1}{3} \in \mathbb{N}* \Leftrightarrow 5 * 2^m \equiv 1\,[3]$. $2 \equiv 2\,[3] \Rightarrow 2^2 = 4 \equiv 1\,[3] \Rightarrow 2^2k \equiv 1\,[3] \Rightarrow 5 * 2^2k \equiv 2\,[3] \Rightarrow 5 * 2^(2k + 1) \equiv 1\,[3]\,, k \in \mathbb{Z}$. So for m odd, $x \in \mathbb{N}*$. Finally, $x \in A_{5+6n} \Rightarrow x \models I \Leftrightarrow x \notin \overline{3}$ We observe for which values of m we have $x = \dfrac{5 * 2^m - 1}{3} + 5 * 2^{m+1}n \notin \overline{3}$ For m odd, $2^m \equiv 2\,[6] \Rightarrow 2^{m+1}n \equiv 4n\,[6] \Rightarrow 5 * 2^{m+1} \equiv 2n\,[6]$. The congruence of $5 * 2^{m+1}$ n modulo 6 is independent from m. Also, we study the possible residues of $\dfrac{5 * 2^m - 1}{3}$ modulo 6 by studying the possible residues of $5 * 2^m - 1$ modulo 18. $2^2 \equiv 4\,[18] \Rightarrow 2^4 \equiv 16\,[18] \Rightarrow 2^6 \equiv 10\,[18] \Rightarrow 2^8 \equiv 4\,[18] \Rightarrow \ldots; 10^2 = 100 = 18 * 5 + 10 \Rightarrow 2^{6k} \equiv 10\,[18] \Rightarrow 2^{6k+2} \equiv 4\,[18] \Rightarrow 2^{6k+4} \equiv 16\,[18] \Rightarrow 2^{6k} - 1 \equiv 9\,[18]\,; 2^{6k+2} - 1 \equiv 3\,[18]\,; 2^{6k+4} - 1 \equiv 15\,[18] \Rightarrow \dfrac{2^{6k} - 1}{3} \equiv 3\,[6]\,; \dfrac{2^{6k+2} - 1}{3} \equiv 1\,[6]\,; \dfrac{2^{6k+4} - 1}{3} \equiv 5\,[6] \Rightarrow \dfrac{2^{6k} - 1}{3} + 5 * 2^{m+1} \equiv 3 + 2n\,[6]\,; \dfrac{2^{6k+2} - 1}{3} + 5 * 2^{m+1} \equiv 1 + 2n\,[6]\,; \dfrac{2^{6k+4} - 1}{3} + 5 * 2^{m+1} \equiv 5 + 2n\,[6]$ In order to have $x \neq 3\,[6]$, we must have $3 - 2\epsilon_m + 2\epsilon_n \neq 3\,[6] \Leftrightarrow -2\epsilon_m + 2\epsilon_n \neq 0\,[6] \Leftrightarrow \epsilon_m \neq \epsilon_n\,[6]$. Since $\epsilon_n, \epsilon_m \in \{0, 1, 2\}$, we have $\epsilon_n \neq \epsilon_m$.

The idea of the snake biting its own tail is to take a number N, define which involved numbers reach out to it via the JCF (therefore calculating its ascendency), and see if one of these involved numbers belongs to the Syracuse sequence of N.

We pose $N' \in A_N$. If $N' \in (u_n(N))$, then $\exists i \in \mathbb{N} \leqslant 1, N' = \eta^i(N)$. And since $N' \in A_N, \exists k \in \mathbb{N} \leqslant 1, N' = \mu_k(N)$. Eventually, we have to solve the equation :

$$\eta^i(N) = \mu_k(N), \mu_k(N) \in A_N \tag{3.2}$$

But as we have seen before, depending on the class of N modulo 6, the general form of the ascendants of N differs. So the equation above needs to be solved with a disjunction concerning the class of N modulo 6.

$$\begin{aligned} \eta^i(1+6n) &= \mu_k(1+6n), \mu_k(1+6n) \in A_1 + 6n \\ \eta^i(5+6n) &= \mu_k(5+6n), \mu_k(5+6n) \in A_5 + 6n \end{aligned} \tag{3.3}$$

We replace $\mu_k(r+6n)$ and $\eta^i(N)$ by the according formulae to obtain these equations.

$$\begin{aligned} \frac{3^i + 6 * 3^i n + \sum_{j=0}^{i-1} 3^{i-1-j} 2^{\sum_{l=0}^{j} \alpha_l}}{2^{\sum_{p=0}^{i} \alpha_p}} &= \sum_{s=0}^{m} 4^s + 2^{m+1} n \\ \frac{5 * 3^i + 6 * 3^i n + \sum_{j=0}^{i-1} 3^{i-1-j} 2^{\sum_{l=0}^{j} \alpha_l}}{2^{\sum_{p=0}^{i} \alpha_p}} &= 5 * 2^m + 3(2^{m+1} n) \end{aligned} \tag{3.4}$$

3.3 Resolution of our equations

The equations take the forms of a summation of terms with many variables. Among the variables, there are:

- i, exponent of many terms and bound of a summation index.
- n, factor of two terms.
- α_p, bound by the variable i. The bigger the variable i, the more the variables α_p.
- m, the order of the ascendant of N

The simplest thing to do is to express n in function to the other variables. In addition to that, if we interpret n concretely, we have that it is the quotient modulo 6 of N.

But $N = r_0 + 6n$. As for the conditions on n so that this belonging condition is fulfilled, we compare n to the other entities:

$$\begin{aligned} &r_0 \in \{1,4,5\} \Rightarrow 1 \leqslant r_0 \leqslant 5 \Rightarrow -5 \leqslant -r_0 \leqslant -1 \\ &6 > 0 \end{aligned}$$

N is a natural integer. So :

$$\begin{aligned} &r_0 + 6n > 0 \\ &6n > -r_0 \\ &6n > -5 \\ &n > -\frac{5}{6} \end{aligned}$$

Finally, $n \in \mathbb{Z} \Rightarrow n > \lfloor -\frac{5}{6} \rfloor = -1 \Rightarrow n \geqslant 0$.

That means that $n \in \mathbb{N}$. We'll see how we can get this condition to our use to solve the equations. Now let's rewrite these equations to have the expression of n:

$$\begin{aligned}
&\frac{3^i + 6*3^i n + \sum_{j=0}^{i-1} 3^{i-1-j} 2^{\sum_{l=0}^{j} \alpha_l}}{2^{\sum_{p=0}^{i} \alpha_p}} = \sum_{s=0}^{m} 4^s + 2^{m+1} n \\
&3^i + 6*3^i n + \sum_{j=0}^{i-1} 3^{i-1-j} 2^{\sum_{l=0}^{j} \alpha_l} = 2^{\sum_{p=0}^{i} \alpha_p} \sum_{s=0}^{m} 4^s + 2^{\sum_{p=0}^{i} \alpha_p + m+1} n \\
&3^i + 6*3^i n + \sum_{j=0}^{i-1} 3^{i-1-j} 2^{\sum_{l=0}^{j} \alpha_l} - 2^{\sum_{p=0}^{i} \alpha_p + m+1} n = 2^{\sum_{p=0}^{i} \alpha_p} \sum_{s=0}^{m} 4^s \\
&(6*3^i - 2^{\sum_{p=0}^{i} \alpha_p + m+1}) n = 2^{\sum_{p=0}^{i} \alpha_p} * \sum_{s=0}^{m} 4^s - 3^i - \sum_{j=0}^{i-1} 3^{i-1-j} 2^{\sum_{l=0}^{j} \alpha_l} \\
&n = \frac{2^{\sum_{p=0}^{i} \alpha_p} \sum_{s=0}^{m} 4^s - 3^i - \sum_{j=0}^{i-1} 3^{i-1-j} 2^{\sum_{l=0}^{j} \alpha_l}}{6*3^i - 2^{\sum_{p=0}^{i} \alpha_p + m+1}}
\end{aligned}$$

The division by $6*3^i - 2^{\sum_{p=0}^{i} \alpha_p + m+1}$ is legitimate, because this expression never equals 0. Otherwise, we'd have:

$$6*3^i = 2^{\sum_{p=0}^{i} \alpha_p + m+1} \Leftrightarrow 3^{i+1} = 2^{\sum_{p=0}^{i} \alpha_p + m}$$

Taking in account the fact that $\sum_{p=0}^{i} \alpha_p + m \in \mathbb{N}$, and so is $i+1 \in \mathbb{N}$, we'd deduce a power of 3 would be even, which is impossible.

$$\begin{aligned}
&\frac{5*3^i + 6*3^i n + \sum_{j=0}^{i-1} 3^{i-1-j} 2^{\sum_{l=0}^{j} \alpha_l}}{2^{\sum_{p=0}^{i} \alpha_p} = 5*2^m + 3(2^{m+1} n)} \\
&5*3^i + 6*3^i n + \sum_{j=0}^{i-1} 3^{i-1-j} 2^{\sum_{l=0}^{j} \alpha_l} = 5*2^{\sum_{p=0}^{i} \alpha_p + m} + 3*2^{\sum_{p=0}^{i} \alpha_p + m+1} n \\
&5*3^i + 6*3^i n + \sum_{j=0}^{i-1} 3^{i-1-j} 2^{\sum_{l=0}^{j} \alpha_l} - 3*2^{\sum_{p=0}^{i} \alpha_p + m+1} n = 5*2^{\sum_{p=0}^{i} \alpha_p + m} \\
&n(6*3^i - 3*2^{\sum_{p=0}^{i} \alpha_p + m+1}) = 5*2^{\sum_{p=0}^{i} \alpha_p + m} - 5*3^i \\
&\qquad\qquad - \sum_{j=0}^{i-1} 3^{i-1-j} 2^{\sum_{l=0}^{j} \alpha_l} \\
&n = \frac{5*2^{\sum_{p=0}^{i} \alpha_p + m} - 5*3^i - \sum_{j=0}^{i-1} 3^{i-1-j} 2^{\sum_{l=0}^{j} \alpha_l}}{6*3^i - 3*2^{\sum_{p=0}^{i} \alpha_p + m+1}}
\end{aligned}$$

We remind that n must be a natural number. This includes the condition that it has to be an integer. In other terms, as we expressed n under the form of a fraction, we must have the numerator divisible by the denominator. We'll prove such divisibility relation is impossible. To accomplish this, we have to notice we can write n as the fraction of two polynomials.

$$n = \frac{-3^i - \sum_{j=0}^{i-1} 3^{i-1-j} 2^{\sum_{l=0}^{j} \alpha_l} + 2^{\sum_{p=0}^{i} \alpha_p} \sum_{s=0}^{m} 4^s}{2*3^{i+1} - 2^{\sum_{p=0}^{i} \alpha_p + m+1}} = \frac{U(3)}{D(3)}$$

Here, the expressions for both polynomials are:

$$U(X) = \sum_{j=0}^{i} a_j X^j \quad ; \quad D(X) = \sum_{j=0}^{i+1} b_j X^j$$
$$U(X) = -X^i - \sum_{j=0}^{i-1} X^{i-1-j} 2^{\sum_{l=0}^{j} \alpha_l} + X^0 2^{\sum_{p=0}^{i} \alpha_p} \sum_{s=0}^{m} 4^s$$
$$D(X) = 2 * X^{i+1} - X^0 2^{\sum_{p=0}^{i} \alpha_p + m + 1}$$

We have to fix i as a constant of course. Not only because we've disproved the first antithesis, but also because otherwise, $\sum_{p=0}^{i} \alpha_p = +\infty$ and then $2^{\sum_{p=0}^{i} \alpha_p} \sum_{s=0}^{m} 4^s = +\infty$. However, coefficients of a polynomial has to be finite. Or else, one polynomial can have an infinite number of different degrees.

Talking about degrees, that's the aim of our method: we need to prove that $D(X)$ can't divide $U(X)$. Let's first interpret this:

$$\exists K(X) \in \mathbb{R}[X], U(X) = K(X)D(X)$$

This equality between two polynomials implies that the degree of $U(X)$ is equal to the degree of $K(X)D(X)$. But the degree of a product of polynomials is equal to the sum of the degrees of the polynomials. We can prove this again quickly while writing two polynomials under their explicit form.

$$\begin{aligned}
&D(X) = \sum_{k=0}^{\deg D} d_k X^k = d_{\deg D} X^{\deg D} + \sum_{k=0}^{\deg D - 1} d_k X^k \\
&K(X) = \sum_{l=0}^{\deg K} k_l X^l = k_{\deg K} X^{\deg K} + \sum_{l=0}^{\deg K - 1} k_l X^l \\
&D(X)K(X) \\
&\quad = (d_{\deg D} X^{\deg D} + \sum_{j=0}^{\deg D - 1} d_j X^j)(k_{\deg K} X^{\deg K} + \sum_{l=0}^{\deg K - 1} k_l X^l) \\
&\quad = d_{\deg D} k_{\deg K} X^{\deg D + \deg K} + \sum_{j=0}^{\deg D - 1} d_j k_{\deg K} X^{j + \deg K} \\
&\quad + \sum_{l=0}^{\deg K - 1} k_l d_{\deg D} X^{l + \deg D} + \sum_{j=0}^{\deg D - 1} d_j X^j \sum_{l=0}^{\deg K - 1} k_l X^l
\end{aligned}$$

The result is a polynomial. We identify the degree as the highest exponent on the variable of the evaluated polynomial. Here, we see that $\deg D(X) * K(X) = \deg D(X) + \deg K(X)$. So we'll need to verify the equality $\deg U(X) = \deg D(X) + \deg K(X)$. But given the expressions of these polynomials from before, we have: $\deg U(X) = i$ and $\deg D(X) = i + 1$. Because the degree of a polynomial is positive, we have: $\deg K(X) > 0$. And $\deg K(X) + \deg D(X) > i + 1$. So if $U(X)$ was divisible by $D(X)$, we'd have $\deg U(X) > i + 1 > i = \deg U(X)$, which is impossible. We can apply this result to all of the 4 equations above, since the only difference remains in the coefficients. We just pose $U(X)$ and $D(X)$ in a generalized expression and we change their coefficients so that they're coinciding with the equations. Eventually, we evaluate these polynomials to retrieve exactly these equations, and we conclude.

$$n = \frac{2^{\sum_{p=0}^{i}\alpha_p}\sum_{s=0}^{m}4^s - 3^i - \sum_{j=0}^{i-1}3^{i-1-j}2^{\sum_{l=0}^{j}\alpha_l}}{2*3^{i+1} - 2^{\sum_{p=0}^{i}\alpha_p+m+1}}$$

We define the coefficients of both polynomials as:

$$a_j = \begin{cases} -1 & \text{for} \quad j = i \\ -2^{\sum_{l=0}^{j}\alpha_l} & \text{for} \quad 0 < j < i \\ 2^{\sum_{p=0}^{i}\alpha_p}\sum_{s=0}^{m}4^s & \text{for} \quad j = 0 \end{cases}$$
$$b_j = \begin{cases} 2 & \text{for} \quad j = i+1 \\ 0 & \text{for} \quad 0 < j < i+1 \\ -2^{\sum_{p=0}^{i}\alpha_p+m+1} & \text{for} \quad j = 0 \end{cases}$$

The terms with the highest exponent for which the coefficient isn't null correspond to the degrees of the polynomials quoted before. So this equation has no solution.

$$n = \frac{5*2^{\sum_{p=0}^{i}\alpha_p+m} - 5*3^i - \sum_{j=0}^{i-1}3^{i-1-j}2^{\sum_{l=0}^{j}\alpha_l}}{2*3^{i+1} - 3*2^{\sum_{p=0}^{i}\alpha_p+m+1}}$$

We define the coefficients of both polynomials as:

$$a_j = \begin{cases} -5 & \text{for} \quad j = i \\ -2^{\sum_{l=0}^{j}\alpha_l} & \text{for} \quad 0 < j < i \\ 5*2^{\sum_{p=0}^{i}\alpha_p+m}\sum_{s=0}^{m} & \text{for} \quad j = 0 \end{cases}$$
$$b_j = \begin{cases} 2 & \text{for} \quad j = i+1 \\ 0 & \text{for} \quad 1 < j < i+1 \quad \text{or} \quad j = 0 \\ -2^{\sum_{p=0}^{i}\alpha_p+m+1} & \text{for} \quad j = 1 \end{cases}$$

The terms with the highest exponent for which the coefficient isn't null correspond to the degrees of the polynomials quoted before. So this equation has no solution. So no equations admit a natural solution. In general, we can conclude that the former equation admits no solution.

So the second antithesis is disproved.

Chapter 4

Proof of the Syracuse conjecture

In principle, when the antitheses are disproved false, then the original hypothesis is true. Yet, we'll make sure we have disproved all the antitheses possible for the Syracuse conjecture and use the tools we're given with our work from the precedent pages to prove the initial conjecture rigorously.

4.1 What we have learnt from the disproval of the antitheses

4.1.1 The disproval of the infinite growth

Thanks to our demonstration in the second part of this report, we have seen there exists no number N which Syracuse sequence knows an infinite growth. So the opposite of everything we've learnt about the infinite growth applies in the case of our conjecture.

First of all, if we remind the Definition 2.1. of an infinite growth, we have compared it to a sequence admitting an infinite limit. Thus, we have written the definition of this limit in respect to the Syracuse sequence.

$$(u_n(N)) \vDash \mathcal{G} \Leftrightarrow \forall A > 0, \exists I \in \mathbb{N}, i \geqslant I \Rightarrow T^i(N) > A$$

If we apply a logical not to this formula, we obtain:

$$\begin{aligned} &\neg((u_n(N)) \vDash \mathcal{G}) \Leftrightarrow \neg(\forall A > 0, \exists I \in \mathbb{N}, i \geqslant I \Rightarrow T^i(N) > A) \\ &\Leftrightarrow (u_n(N)) \nvDash \mathcal{G} \Leftrightarrow \exists A > 0, \forall I \in \mathbb{N}, i < I \Rightarrow T^i(N) \leqslant A \end{aligned}$$

We can simplify the part $\forall I \in \mathbb{N}, i < I$, which is a tautology. Indeed, $i \in \mathbb{N}$ which is a countably infinite set. So $\forall i \in \mathbb{N}, \exists i' \in \mathbb{N}, i' > i$. Therefore, we have a tautology being the trigger of a formula. So this formula, becomes a tautology as well.

$$(u_n(N)) \nvDash \mathcal{G} \Leftrightarrow \exists A > 0, T^i(N) \leqslant A$$

The Syracuse sequence of N is therefore bounded from above by an infinite number of majorants. We'll consider the smallest of them which belongs to this Syracuse sequence. It is its maximal value. We'll note it $\max_n u_n(N)$.

In addition to this, because we study the conjecture for all non-null natural numbers, any Syracuse sequence is bounded from below. The smallest value of N^* is indeed 1. So we also have:

$$T^i(N) \geqslant 1$$

So:

$$(u_n(N)) \nvDash \mathcal{G} \Leftrightarrow 1 \leqslant T^i(N) \leqslant \max_n u_n(N) \tag{4.1}$$

And because $T^i(N) \in \mathbb{N}^*$, then the set made of the images $T^i(N)$ has a finite number of distinct values. This latter result will add up to the results that will follow to show that all the Syracuse sequence have a finite cardinal.

Statement 4.1 All Syracuse sequence admits a finite maximum and a finite number of distinct values.

4.1.2 The disproval of the existence of another cycle

Thanks to our demonstration in the third part of this report, we have seen there exists no number N which Syracuse sequence contains a cycle other than $4, 2, 1$. So everything we've learnt about the consequences of the existence of cycles is false in the case of the conjecture except if the cycle is $4, 2, 1$. Like for the disproval of the infinite growth, we see which laws are applied in the case of our conjecture.

First of all, if a cycle went to appear, we've seen the set of numbers that composes it appears more than once in the process of the algorithm. Now that we have seen there's no cycle other than $4, 2, 1$ possible, that means all the numbers in $\mathbb{N}^*$ appear once or never. This adds a detail to the algorithm.

Statement 4.2 If a number already appeared during the process of the algorithm, it can't be reached again.

If we consider the number $\sigma_(N')$ of appearances of a number a variable initialized to 0 at first in the algorithm, we state that $\forall N' \in \mathbb{N}^*, \sigma_(N') < 2$. Of course, we cannot decrement this variable. This would mean whether that it appeared a negative number of time (if $\sigma_(N') = 0$) or that we deny the appearance of a number and then, meaning that $\nexists i \in \mathbb{N}^*, \eta^i(N) = N'$ (but if $\sigma_(N') = 1, \exists i \in \mathbb{N}^*, \eta^i(N) = N'$). In the algorithm, whenever we find a value of i for which $\eta^i(N) = N'$, we increment $\sigma_(N')$ by 1. And we can't increment it again. Otherwise, $\sigma_(N') = 2$, which is impossible.

Now, if we combine both Statements 4.1 and 4.2, we have the main line of the final part of our demonstration. The cardinal of a Syracuse sequence, or its "flight time", is calculated by making the sum of the numbers $\sigma_(N')$ of appearances in the process of the algorithm of all the numbers between 1 and $\max_n u_n(N)$.

$$Card(u_n(N)) = \sum_{k=1}^{\max_n u_n(N)} \sigma_k \tag{4.2}$$

According to the Statement 4.1, all Syracuse sequence have a finite number of distinct values. So the cardinal of the Syracuse sequence of N is a finite sum of positive integers. According to the Statement 4.2, we have $0 \leqslant \sigma_k \leqslant 1$. We obtain thus, that:

$$\begin{array}{l} 1 \leqslant \sum_{k=1}^{\max_n u_n(N)} \sigma_k \leqslant \sum_{k=1}^{\max_n u_n(N)} 1 = \max_n u_n(N) \\ \Leftrightarrow 1 \leqslant Card(u_n(N)) \leqslant \max_n u_n(N) \end{array}$$

This statement is true for $N \in \mathbb{N}_(\geqslant 1)$. What is important with this statement is the fact that the cardinal of a sequence or a set is an integer. Having it bounded from above and below means the Syracuse sequence of N is a finite countable set. This is the main element that will help in showing we always reach out to the trivial cycle.

4.2 The set of all reachable numbers

We're coming to the ultimate step of our demonstration. Since we have rejected all the antitheses possible for the conjecture, then we supposingly proved the initial conjecture was true for all $n \in \mathbb{N}$. But before we hastily admit it with an impassible confidence, for lack of being rigorous, we shall use the disproval of these antitheses to demonstrate the validity of the conjecture.

Let's get back to the algorithm itself: we've said it was composed of two verification steps: the oneness and the parity. In other terms, we first check if the algorithm stops because $N = 1$, then if it's not over, we check the parity of N to see what operation to apply next.

We have shown the first antithesis was impossible. Every Boolean opposite proposition that comes out of it is therefore supposed true. This antithesis claimed that the growth of a Syracuse sequence is infinite. Now, we know it is finite. Saying the sequence would converge is wrong though, but all we suppose is that there exists a number A for which all members of a Syracuse sequence are inferior to this number. In other terms, the sequence admits a maximum number that bounds it from above.

But the minimum of a Syracuse sequence is 1, when we suppose N a positive integer. Indeed:

$$N \geqslant 1 \Rightarrow 3N + 1 \geqslant 4 > 1$$

We can only apply the even operation when N is even. For $N = 1$, we can't. We suppose that $N \geqslant 2$.

$$N \geqslant 2 \Rightarrow \frac{N}{2} \geqslant \frac{2}{2} = 1$$

So we have a lower bound and an upper bound:

$$\begin{cases} \forall i \in \mathbb{N}, \eta^i(N) \geqslant 1 \\ \forall i \in \mathbb{N}, \exists A > 1, \eta^i(N) \leqslant A \end{cases} \Rightarrow \forall i \in \mathbb{N}, \exists A > 1, 1 \leqslant \eta^i(N) \leqslant A$$

But N is a positive integer, and so does $\eta^i(N)$. So the set of all numbers that the algorithm could reach is a finite set of numbers.

$$\eta^i(N) \in \Omega_{\eta^i(N)} = \left\{1, 2, 4, ..., max\eta^i(N)\right\}, \forall N \in \mathbb{N}^*$$

Furthermore, we have supposed that the second antithesis was wrong. It claimed that there exists another cycle other than $\{4, 2, 1\{$. Such a phenomenon would induce that there's a finite number of elements of reachable numbers which appears infinitely in the algorithm. In particular, they appear multiple times. Now, we can say that every reachable number other than 4, 2, and 1 appears once or never.

Thanks to the disproval of the first antithesis, by construction, we can build a hypothetic set of reachable numbers for all N. We suppose we take one N among the natural integers. The set of reachable numbers for $\eta^1(N)$ can be made up as a set indexed with $\eta^1(N)$, and we set $Card(\omega_{\eta^1(N)}) = w$.

$$\omega_{\eta^1(N)} = \left\{1, 2, 4, ... \max \eta^i(N)\right\} - N$$

We also assume that $Card(\eta^i(N)) = i + \sum_{p=1}^{i} \alpha_p$. Indeed, the function we have established in the beginning of our demonstration models the step when N is odd, until the time N becomes odd back again. Meanwhile, we have reached all the even numbers during the even operations that was induced during a first application of the 3N+1 operation. This number of even operation is α_p.

We can suppose that $N < \max \eta^i(N)$, instead of $N = \max \eta^i(N)$. This will generalize and simplify our interpretation. In this set, we have included $\eta^2(N) \in \mathbb{N}^*$. Otherwise, if it didn't belong to this set, then we'd have: $\eta^2(N) < 1$ or $\eta^2(N) > \max \eta^i(N)$ In other terms, this consequence induces that:

$$\exists i \in \mathbb{N}, \eta^2(N) < 1 \quad \text{or} \quad \eta^2(N) > \max \eta^i(N)$$

Which contradicts the bounding relation we have established a few paragraphs ago.

Thanks to the fact that a number appears once or never, we are sure that N won't be reached by $\eta^1(N)$. Eventually, when we identify $\eta^1(N) = N$, we are confronted to two possibilities: either $N = 1$ or $N \neq 1$. If $N = 1$, the algorithm stops, hopefully. If not, then we keep the algorithm going, and we pick a number that $\eta^2(N)$ can reach. It's a number that belongs to $\omega_(\eta^2(N))$. We know we can't reach neither N nor $\eta^1(N)$ nor all the even numbers we have before getting to $\eta^1(N)$. So:

$$\begin{aligned}
&\omega_(\eta^2(N)) = \{1, 2, 4, ..., \max \eta^i(N)\} - \left\{\eta^k(N) | k < 2\right\}\\
&\quad = (\{1, 2, 4, ..., \max \eta^i(N)\} - N) - \eta^1(N)\\
&Card(\Omega_{\eta^2(N)}) = Card((\{1, 2, 4, ..., \max \eta^i(N)\} - N) - \eta^1(N))\\
&\quad = Card(\{1, 2, 4, ..., \max \eta^i(N)\} - N) - Card(\eta^1(N))\\
&\quad = w - 1 - \alpha_1
\end{aligned}$$

Each time we iterate the algorithm (let's say that we apply an ith operation), the cardinal of the reachable numbers left is depleted by 1 plus the exponent α_i. The two possibilities are still present. Eventually, when we arrive to the step where there will only be powers of 2 left, we will only be able to choose $N = 1$, proving the oneness of N.

Therefore, we show that as the algorithm goes on, we're eventually reaching out to the trivial cycle 4,2,1, and thus for whatever initial natural number N. As we've demonstrated this, we've demonstrated the Syracuse conjecture for all non-null natural N.

Bibliography

[1] Paul Bourke. Decision procedure for "oneness". Website, December 1992.

[2] J. C. Lagarias. *The Ultimate Challenge: The 3x+1 problem*. American Mathematical Society, 2010.

[3] T. Oliveira E Silva. Maximum excursion and stopping time record-holders for the 3x+1 problem: computational results. *Mathematics of Computation*, 68:371–384, 1999.

[4] John Tromp. Oeis a007814. Website, December 1996.