TWO NEW UNIMODAL DESCENT POLYNOMIALS

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Abstract. The descent polynomials of separable permutations and derangements are both demonstrated to be unimodal. Actually, we prove that the γ-coefficients of the first are positive with an interpretation parallel to the classical Eulerian polynomial, while the second is spiral, a property stronger than unimodality. Furthermore we conjecture that they are both real-rooted.

1. Introduction

Many polynomials with combinatorial meanings have been shown to be unimodal, see the recent survey of Brändén [3] or [9]. Recall that a polynomial $h(t) = \sum_{i=0}^{d} h_i t^i$ of degree d is said to be *unimodal* if the coefficients are increasing and then decreasing, i.e., there is a certain index c such that

$$h_0 \leq h_1 \leq \cdots \leq h_c \geq h_{c+1} \geq \cdots \geq h_d.$$

Let $p(t) = a_r t^r + a_{r+1} t^{r+1} + \cdots + a_s t^s$ be a real polynomial with $a_r \neq 0$ and $a_s \neq 0$. It is called *palindromic* (or symmetric) of *darga* n if $n = r + s$ and $a_{r+i} = a_{s-i}$ for all i (see [18, 22]). For example, the darga of $1+t$ and t are 1 and 2, respectively. Any palindromic polynomial $p(t) \in \mathbb{Z}[t]$ can be written uniquely [2] as

$$p(t) = \sum_{k=r}^{\lfloor \frac{n}{2} \rfloor} \gamma_k t^k (1+t)^{n-2k},$$

where $\gamma_k \in \mathbb{Z}$. If $\gamma_k \geq 0$ then we say that it is *γ-positive of darga n*. It is clear that the γ-positivity implies palindromic and unimodality.

Let $[n] := \{1, 2, \ldots, n\}$ and denote by $\mathfrak{S}_n$ the set of all permutations of $[n]$. For a permutation $\pi \in \mathfrak{S}_n$, an index $i \in [n]$ is a *decent* (resp. *double descent*) of π if $\pi_i > \pi_{i+1}$ (resp. $\pi_{i-1} > \pi_i > \pi_{i+1}$), where $\pi_0 = \pi_{n+1} = +\infty$. Denote by $\mathrm{des}(\pi)$ and $\mathrm{dd}(\pi)$ the number of descents and double descents of π, respectively. It is well known [3] that the *Eulerian polynomials* [16] or the descent polynomial on $\mathfrak{S}_n$ is γ-positive of darga $n-1$. Moreover,

$$A_n(t) := \sum_{\pi \in \mathfrak{S}_n} t^{\mathrm{des}(\pi)} = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \gamma_{n,k}^{A} t^k (1+t)^{n-1-2k}, \tag{1.1}$$

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where $\gamma^A_{n,k} = \#\{\pi \in \mathfrak{S}_n : \mathrm{dd}(\pi) = 0, \mathrm{des}(\pi) = k\}$.

A permutation π is said to contain the permutation σ if there exists a subsequence of (not necessarily consecutive) entries of π that has the same relative order as σ, and in this case σ is said to be a pattern of π, otherwise, π is said to avoid σ. The set of permutations avoiding patterns $\sigma_1, \ldots, \sigma_r$ in $\mathfrak{S}_n$ is denoted by $\mathfrak{S}_n(\sigma_1, \ldots, \sigma_r)$. The *Narayana polynomial* of order n can be defined as the descent polynomial over $\mathfrak{S}_n(231)$, and it is also γ-positive of darga $n-1$, see [14, Proposition 11.14] for an equivalent statement:

$$N_n(t) := \sum_{\pi \in \mathfrak{S}_n(231)} t^{\mathrm{des}(\pi)} = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \gamma^N_{n,k} t^k (1+t)^{n-1-2k}, \tag{1.2}$$

where $\gamma^N_{n,k} = \#\{\pi \in \mathfrak{S}_n(231) : \mathrm{dd}(\pi) = 0, \mathrm{des}(\pi) = k\}$.

This work was partially motivated (see Remark 2.2) by the second author's recent proof [8] of a conjecture of Gessel [2, 13, 20], which states that for $n \geq 1$, there exist *nonnegative* integers $\gamma_{n,i,j}$, $0 \leq i, j$, $j + 2i \leq n-1$, such that

$$\sum_{\sigma \in \mathfrak{S}_n} s^{\mathrm{ides}(\sigma)} t^{\mathrm{des}(\sigma)} = \sum_{i,j \geq 0} \gamma_{n,i,j} (st)^i (1+st)^j (s+t)^{n-1-j-2i}, \tag{1.3}$$

where $\mathrm{ides}(\sigma)$ denotes the number of descents of σ^{-1}. Note that we recover Eulerian polynomial and its γ-decomposition (1.1) by setting $s = 1$ in (1.3).

A permutation avoiding patterns 2413 and 3142 is called a *separable permutation* (see Proposition 2.8). The number of separable permutations are counted by the *large Schröder numbers* (see [15,21]). The first few numbers are $1, 2, 6, 22, 90, 394, 1806$, see `oeis:A006318`. For interested reader, see [12] for recent work of McNamara and Steingrímsson on separable permutations from topological point of view.

Our first main result is the following γ-positivity for the descent polynomial of separable permutations.

Theorem 1.1. *We have*

$$S_n(t) := \sum_{\pi \in \mathfrak{S}_n(2413,3142)} t^{\mathrm{des}(\pi)} = \sum_{k \geq 0}^{\lfloor \frac{n-1}{2} \rfloor} \gamma^S_{n,k} t^k (1+t)^{n-1-2k}, \tag{1.4}$$

where

$$\gamma^S_{n,k} = \#\{\pi \in \mathfrak{S}_n(3142, 2413) : \mathrm{dd}(\pi) = 0, \mathrm{des}(\pi) = k\}. \tag{1.5}$$

In particular, the polynomial $S_n(t)$ is γ-positive for $n \geq 1$ and a fortiori, palindromic and unimodal.

For example, the first decompositions of $S_n(t)$ read as follows:

$$\begin{aligned}
&S_1(t) = 1,\\
&S_2(t) = 1 + t,\\
&S_3(t) = 1 + 4t + t^2 = (1 + t)^2 + 2t,\\
&S_4(t) = 1 + 10t + 10t^2 + t^3 = (1 + t)^3 + 7t(1 + t),\\
&S_5(t) = 1 + 20t + 48t^2 + 20t^3 + t^4 = (1 + t)^4 + 16t(1 + t)^2 + 10t^2,\\
&S_6(t) = 1 + 35t + 161t^2 + 161t^3 + 35t^4 + t^5 = (1 + t)^5 + 30t(1 + t)^3 + 61t^2(1 + t).
\end{aligned}$$

The palindrome $S_n(t) = t^{n-1}S_n(1/t)$ follows from the involution

$$\pi_1\pi_2\cdots\pi_n \mapsto \pi_n\pi_{n-1}\cdots\pi_1$$

and the fact that $\mathfrak{S}_n(2413, 3142)$ is invariant under this involution. Note that both (1.1) and (1.2) can be proved using the *modified Foata-Strehl action* (see [2] or [9]) on $\mathfrak{S}_n$, but since $\mathfrak{S}_n(2413, 3142)$ is not invariant under this action, it is unclear how Theorem 1.1 could be deduced by the same manner.

A *derangement* is a fixed-point free permutation. Consider the descent polynomial of derangements:

$$D_n(t) := \sum_{\pi\in\mathfrak{D}_n} t^{\mathrm{des}(\pi)},$$

where $\mathfrak{D}_n$ is the set of derangements in $\mathfrak{S}_n$. The first few values of $D_n(t)$ are listed as follows:

$$\begin{aligned}
D_2(t) &= t,\\
D_3(t) &= 2t,\\
D_4(t) &= 4t + 4t^2 + t^3,\\
D_5(t) &= 8t + 24t^2 + 12t^3,\\
D_6(t) &= 16t + 104t^2 + 120t^3 + 24t^4 + t^5,\\
D_7(t) &= 32t + 382t^2 + 896t^3 + 480t^4 + 54t^5.
\end{aligned}$$

The following spiral property implies the unimodality of $D_n(t)$, which can be considered as our second main result.

Theorem 1.2. *Let $D_n(t) = \sum_{k\geq 1} d_{n,k}t^k$. Then, for $n \geq 1$ and $1 \leq k \leq n - 1$*

$$d_{2n,2n-k} < d_{2n,k} < d_{2n,2n-k-1} \quad \text{and} \quad d_{2n+1,k} < d_{2n+1,2n-k} < d_{2n+1,k+1} \tag{1.6}$$

except that $d_{4,1} = d_{4,2} = 4$. In particular, the polynomial $D_n(t)$ is unimodal for $n \geq 2$.

The rest of this paper is organized as follows. Section 2 provides two alternative descriptions of separable permutations, which we take the liberty to name as "Schröder words" and "di-sk trees". Utilizing these new models and a crucial bijection, a proof of Theorem 1.1 is given in Section 3. We shall prove Theorem 1.2 in Section 4 and supply two alternative

FIGURE 1. The direct sum and skew sum operations.

proofs of the γ-positivity for $S_n(t)$ in Section 5. The first one builds on a different action on the di-sk trees, while the second one is based on further analysis on the generating function for $S_n(t)$. Finally we conclude with some remarks and conjectures.

2. SEPARABLE PERMUTATIONS, SCHRÖDER WORDS AND DI-SK TREES

We construct two alternative models for separable permutations. For the first model we need to introduce two operations. The final output is a one-line expression we call Schröder word, next we "anti-telescope" it to form the second model that we call direct-skew tree (abbreviated as di-sk in the sequel). This one extra dimension we gained by switching to planar trees will make our further investigation a lot clearer.

2.1. **Direct sum and skew sum.**

We need two operations defined on all permutations. The *direct sum* of the permutations $\pi \in \mathfrak{S}_k$ and $\sigma \in \mathfrak{S}_l$, is a permutation in $\mathfrak{S}_{k+l}$ denoted by $\pi \oplus \sigma$, point-wisely it satisfies

$$(\pi \oplus \sigma)(i) = \begin{cases} \pi(i), & \text{for } i \in [1, k]; \\ \sigma(i-k) + k, & \text{for } i \in [k+1, k+l]. \end{cases}$$

And similarly the *skew sum* of π and σ, denoted by $\pi \ominus \sigma$,

$$(\pi \ominus \sigma)(i) = \begin{cases} \pi(i) + l, & \text{for } i \in [1, k]; \\ \sigma(i-k), & \text{for } i \in [k+1, k+l]. \end{cases}$$

If we use permutation matrix to represent both permutations, then the direct sum and the skew sum are forming block anti-diagonal matrix and block diagonal matrix, respectively (see Fig. 1). The following observation follows directly from the definition and it was the motivation for studying these two operations and separable permutations.

Proposition 2.1. *The direct sum preserves both descents and inverse descents, while the skew sum increases both descents and inverse descents by* 1. *More precisely, we have*

$$\operatorname{des}(\pi \oplus \sigma) = \operatorname{des}(\pi) + \operatorname{des}(\sigma), \qquad \operatorname{ides}(\pi \oplus \sigma) = \operatorname{ides}(\pi) + \operatorname{ides}(\sigma);$$
$$\operatorname{des}(\pi \ominus \sigma) = \operatorname{des}(\pi) + \operatorname{des}(\sigma) + 1, \qquad \operatorname{ides}(\pi \ominus \sigma) = \operatorname{ides}(\pi) + \operatorname{ides}(\sigma) + 1.$$

Remark 2.2. In an effort to find combinatorial interpretation of $\gamma_{n,i,j}$ in (1.3), we restricted our attention to the terms without $s+t$, whose coefficients are $\gamma_{n,i,n+1-2i}$. This means we need to consider operations on permutations that change both des and ides by the same amount. In view of Proposition 2.1, this leads us naturally to the operations $\oplus$ and $\ominus$.

The readers are invited to check the following computations using $\oplus$ and $\ominus$ and make sure they understand the definition. Here and throughout the rest of the paper, we will use one line notation for permutations unless otherwise stated.

Example 2.3.

$$\begin{gathered}123 \oplus 21 = 12354, \quad 21 \oplus 123 = 21345, \\ 123 \ominus 21 = 34521, \quad 21 \ominus 123 = 54123, \\ (12 \ominus 231) \ominus 3142 = 12 \ominus (231 \ominus 3142) = 896753142, \\ (1 \oplus 1) \ominus 1 = 12 \ominus 1 = 231, \quad 1 \oplus (1 \ominus 1) = 1 \oplus 21 = 132.\end{gathered}$$

As suggested by Example 2.3, we see that both operations $\oplus$ and $\ominus$ are associative but not commutative in general, nor do they associate with each other (i.e., for permutations π, σ and τ we typically have $(\pi \oplus \sigma) \ominus \tau \neq \pi \oplus (\sigma \ominus \tau)$), so when situation like this arises, it is necessary to add parenthesis to indicate in what order we are executing these operations. By convention, operations are taking place from left to right unless there are parenthesis.

2.2. **Sweeping-algorithm.** Bose, Buss and Lubiw [4] showed that it is possible to determine in polynomial time whether a given separable permutation is a pattern in a larger permutation. But in practice, for a given permutation π, it is not so easy to see if it is separable or not. We recall an algorithm, called *sweeping-algorithm*, that we believe first appeared in [15], to verify a given permutation is separable or not and in case of yes, it "decomposes" π into a bunch of 1, $\oplus$, $\ominus$ and parentheses with some conditions.

Definition 2.4 (Sweeping-algorithm)**.** Starting with a permutation $\pi = a_1 a_2 \cdots a_n$ in $\mathfrak{S}_n$. View each a_i as a block B_i.

(1) Read π from left to right and find the least j such that the two adjacent blocks B_j and B_{j+1} contain elements a_j and a_{j+1}, respectively, that are consecutive integers (increasing or decreasing), then form a new block $(B_j \oplus B_{j+1})$ for increasing, $(B_j \ominus B_{j+1})$ for decreasing. If no such j exists, then π is actually non-separable (see Proposition 2.8 below).
(2) We repeat this process until no new blocks can be formed. As long as there are two adjacent blocks satisfying (1), the process would continue and end with a single block, which corresponds to the last executed operator and the outermost pair of parentheses.
(3) We replace all the numbers with 1 (since their order has been coded by $\oplus$, $\ominus$ and parenthesis) and call the final expression, denoted by $\mathrm{sw}(\pi)$, a *Schröder word*.

We try out our construction with one example, and the corresponding block decomposition for its permutation matrix is depicted in Fig. 2 for better visualization, where black dots stand for entry 1 in the matrix and empty slots are all 0s.

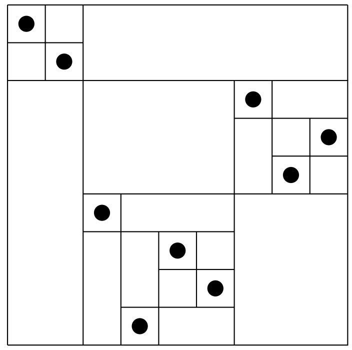


FIGURE 2. Block decomposition of $\pi = 984132756$ using permutation matrix.

Example 2.5. Take $\pi = 984132756$, we need eight sweeps in this example:

$$\begin{array}{rl}
\text{sweep 1:} & (9\ominus 8)4132756\\
\text{sweep 2:} & (9\ominus 8)41(3\ominus 2)756\\
\text{sweep 3:} & (9\ominus 8)4\big(1\oplus(3\ominus 2)\big)756\\
\text{sweep 4:} & (9\ominus 8)\big(4\ominus\big(1\oplus(3\ominus 2)\big)\big)756\\
\text{sweep 5:} & (9\ominus 8)\big(4\ominus\big(1\oplus(3\ominus 2)\big)\big)7(5\oplus 6)\\
\text{sweep 6:} & (9\ominus 8)\big(4\ominus\big(1\oplus(3\ominus 2)\big)\big)\big(7\ominus(5\oplus 6)\big)\\
\text{sweep 7:} & (9\ominus 8)\big(\big(4\ominus\big(1\oplus(3\ominus 2)\big)\big)\oplus\big(7\ominus(5\oplus 6)\big)\big)\\
\text{sweep 8:} & \big((9\ominus 8)\ominus\big(\big(4\ominus\big(1\oplus(3\ominus 2)\big)\big)\oplus\big(7\ominus(5\oplus 6)\big)\big)\big)\\
\text{final:} & \big((1\ominus 1)\ominus\big(\big(1\ominus\big(1\oplus(1\ominus 1)\big)\big)\oplus\big(1\ominus(1\oplus 1)\big)\big)\big) = \mathrm{sw}(\pi).
\end{array}$$

Remark 2.6. Note that associativity on single operator gives us $(\pi\oplus\sigma)\oplus\tau = \pi\oplus(\sigma\oplus\tau)$ and $(\pi\ominus\sigma)\ominus\tau = \pi\ominus(\sigma\ominus\tau)$. Since we always sweep from left to right in our decomposition, it is not hard to see that we will always first get block $(B_1\oplus B_2)\oplus B_3$ before we can form $B_1\oplus(B_2\oplus B_3)$, similarly for operator $\ominus$. More precisely, when a new block is formed from three consecutive smaller blocks B_1, B_2, B_3 in such a way that B_2 and B_3 first combine, then together they combine with B_1, we have the following restriction:

$$\begin{array}{rl}
\textbf{impossible:} & (B_1\oplus(B_2\oplus B_3)),\quad (B_1\ominus(B_2\ominus B_3));\\
\textbf{possible:} & (B_1\oplus(B_2\ominus B_3)),\quad (B_1\ominus(B_2\oplus B_3)).
\end{array} \tag{2.1}$$

Definition 2.7. For $n \geq 1$, let $\mathcal{W}_n$ be the set of all Schröder words of length n, namely, one-line expressions composed of n copies of 1, $n-1$ operators $\oplus$ or $\ominus$, and $n-1$ pairs of parentheses that satisfy restriction (2.1).

Note that we can naturally associate each pair of parenthesis in a Schröder word with a unique operator $\oplus$ or $\ominus$ it is "parenting".

For more information on these two operators and on separable permutations in general we refer to [1, 10, 19]. The following result is well known, see [4] and [10, page 57]. For the reader's convenience we sketch a proof via our sweeping-algorithm.

Proposition 2.8. *For $n \geq 1$, the sweeping-algorithm $\pi \mapsto \mathrm{sw}(\pi)$ is a bijection between $\mathfrak{S}_n(2413, 3142)$ and $\mathcal{W}_n$ such that*

$$i \in \mathrm{DES}(\pi) := \{i \in [n-1] : \pi_i > \pi_{i+1}\} \quad \Leftrightarrow \quad \textit{the } i\textit{th operator in } \mathrm{sw}(\pi) \textit{ is } \ominus.$$

Proof. We first show that the algorithm is well defined, in other words, if the sequence of blocks $B_1B_2\dots B_r$ has no two consecutive integers in two adjacent blocks B_j and B_{j+1}, then there is a pattern 2413 or 3142 in the starting permutation π.

We proceed by induction on $n \geq 1$. For $n \leq 4$, the algorithm is well defined except the two permutations 2413 and 3142. Assume that $n > 4$ and the result is true for all $m < n$. Note that each block is an interval $\{j, j+1, \dots, \ell\}$ with $j \leq \ell$. We order the blocks according to their minima and relable the blocks according to their rank in this ordering. Let $\boxed{a}$ be the block with rank a.

- If $B_1 = \boxed{1}$ or $B_r = \boxed{1}$, then we consider the subsequence $B_2 \dots B_r$ or $B_1B_2\dots B_{r-1}$. By the induction hypothesis, we are done.
- Otherwise, there are two neighbours for block $\boxed{1}$, say $\boxed{a}\ \boxed{1}\ \boxed{b}$, where we assume $b > a \geq 3$ (the case $a > b$ is completely analogous due to symmetry). We first find a unique c that meets the following criterion, if no such c exists we simply take $c = a$:
  - i) $a < c < b$;
  - ii) $\boxed{c}$ is to the left of $\boxed{a}$;
  - iii) c is the largest integer satisfying both i) and ii).

  We then have two claims:

Claim 1: All blocks $\boxed{2}, \boxed{3}, \cdots, \boxed{a\text{-}1}, \boxed{a+1}, \cdots, \boxed{c\text{-}1}$ are to the left of $\boxed{a}$.

Claim 2: If for some $d > b$, $\boxed{d}$ is to the left of $\boxed{a}$, then $\boxed{d}$ is to the left of all blocks $\boxed{1}$ through $\boxed{c}$.

Clearly, if Claim 1 is false, then the pattern 3142 appears in the starting permutation π; if Claim 2 is false, we have 2413 in π. Otherwise Claim 1 and 2 both hold and in this case blocks $\boxed{1}$ through $\boxed{c}$ form a complete interval and by the induction hypothesis, we are done.

Now suppose we are given a Schröder word in $\mathcal{W}_n$, we describe how we can find its preimage under the sweeping-algorithm. Simply note that each pair of parentheses that contain l copies of 1 should produce l consecutive integers $(i, i+1, \cdots, i+l-1)$ for some $i, 1 \leq i \leq n$, and for two consecutive pairs of parentheses, we know which should contain bigger integers by looking at the operator concatenating them. For instance, in Example 2.5, we can reproduce "sweep 8" from "final" by first looking at the second $\ominus$ from left, this is the last operator being executed in this expression. And we realize that the two greatest numbers, namely 8 and 9 should be put in the first parenthesis, and the first $\ominus$ tells us 9 goes before 8, etc. And finally just drop all the operators and parentheses, we arrive at a separable permutation in $\mathfrak{S}_n(2413, 3142)$. Therefore our map is invertible for any expression in $\mathcal{W}_n$, so we get both injectivity and surjectivity.

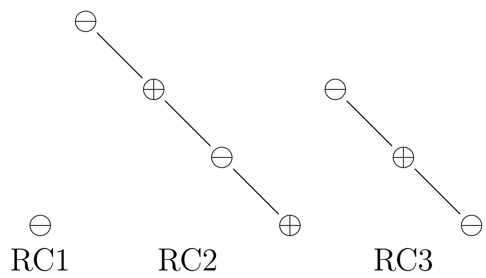


FIGURE 3. The right chains for $\pi = 984132756$.

Finally it is readily seen that if $i \in \mathrm{DES}(\pi)$ then the two blocks containing π_i and π_{i+1}, respectively, must be concatenated by $\ominus$ according to the sweeping-algorithm and vice versa. $\square$

2.3. **The di-sk trees.** We first recall some standard vocabularies for trees [11, Section 2.3.1].

Definition 2.9. An (unlabelled) binary tree is defined recursively as follows. The empty set $\emptyset$ is a binary tree. Otherwise a binary tree has a *root* vertex v, a *left subtree* T_1, and a *right subtree* T_2, both of which are binary trees. We also call the root of T_1 (if T_1 is nonempty) the *left child* and the root of T_2 (if T_2 is nonempty) the *right child* of the vertex v. For the set of all binary trees with $n-1$ vertices (or *nodes*), we denote it as $\mathfrak{T}_n$.

In what follows we use the *in-order* to compare nodes on trees (see [17, page 5]), namely, starting with the root node, we recursively traverse the left subtree to parent then to the right subtree if any.

Definition 2.10. Given a binary tree T, its *right chain* is any maximal chain composed of right children except the first node, which is either the root or a left child. The *length* of a right chain is the number of nodes on this chain. And the number of its right chains is denoted as $r(T)$. Similary, the number of right chains with even (resp. odd) length is denoted as $r_e(T)$ (resp. $r_o(T)$). The order between two right chains is then decided by comparing their first nodes.

Example 2.11. For our running permutation $\pi = 984132756$, the tree $T(\pi)$ has three right chains, with length being $1, 4$ and 3 in this order, so $r_o(T(\pi)) = 2$ and $r_e(T(\pi)) = 1$. See Fig. 3.

Remark 2.12. Note that the right chain is decided only by the structure of the tree, so Definition 2.10 extends naturally to labelled trees.

Definition 2.13. A binary tree is called a *di-sk tree* if it is labelled with $\oplus$ and $\ominus$ and the labelling satisfies the following "right chain condition":

$$\text{The labelling on each right chain should alternate.} \tag{2.2}$$

The number of nodes in the di-sk tree T that are labelled as $\ominus$ is denoted by $n_\ominus(T)$. The set of all di-sk trees with $n-1$ nodes is denoted as $\mathfrak{D}\mathfrak{T}_n$.

$$\big((1\ominus 1)\ominus\big(\big(1\ominus\big(1\oplus(1\ominus 1)\big)\big)\oplus\big(1\ominus(1\oplus 1)\big)\big)\big)$$

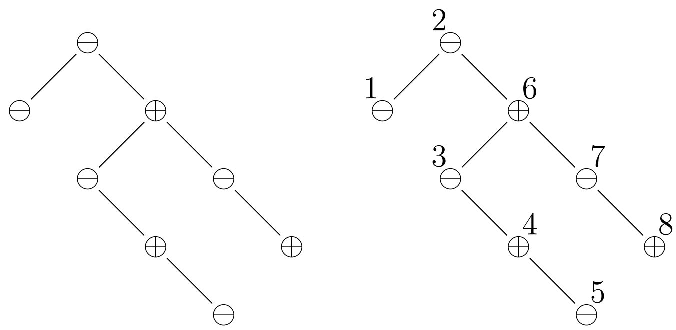


FIGURE 4. Schröder word, di-sk tree, and in-order tranversal.

Now we describe how to go from Schröder words to di-sk trees. One example using our running permutation $\pi = 984132756$ is illustrated in Fig. 4. Given a Schröder word $w = (w_L \oplus w_R)$ (resp. $(w_L \ominus w_R)$) in $\mathcal{W}_n$, we convert it to be the labelled binary tree $T(w) = (T(w_L), \oplus, T(w_R))$ (resp. $(T(w_L), \ominus, T(w_R))$), where w_L and w_R have been converted to be the left subtree $T(w_L)$ and the right subtree $T(w_R)$ of the root $\oplus$ (resp. $\ominus$). It is clear that the condition (2.1) on w is equivalent to the condition (2.2) on $T(w)$. Thus, we can call them both right chain condition from now on. The following result is clear from this construction.

Proposition 2.14. *The map $w \mapsto T(w)$ is a bijection from $\mathcal{W}_n$ to $\mathfrak{DT}_n$ such that*

$$\text{The } i\text{th operator in } w \text{ is } \ominus \quad \Leftrightarrow \quad \text{the } i\text{th node in } T(w) \text{ is } \ominus. \tag{2.3}$$

Combining Propositions 2.8 and 2.14 we obtain the main result of this section.

Theorem 2.15. *For $n \geq 1$, the map $\pi \mapsto T(\pi)$ is a bijection from $\mathfrak{S}_n(2413, 3142)$ to $\mathfrak{DT}_n$ such that*

$$i \in \mathrm{DES}(\pi) \quad \Leftrightarrow \quad \text{the } i\text{th node of } T(\pi) \text{ is } \ominus.$$

Moreover, for $0 \leq k \leq \lfloor (n-1)/2 \rfloor$, it induces a bijection between the following two subsets:

$$\begin{aligned}
\mathfrak{S}^S_{n,k} :=& \{\pi \in \mathfrak{S}_n(3142, 2413) \,:\, \mathrm{dd}(\pi) = 0, \mathrm{des}(\pi) = k\},\\
\mathfrak{DT}^2_{n,k} :=& \{T \in \mathfrak{DT}_n \,:\, T \text{ has no consecutive } \ominus\text{, its first node is } \oplus \text{ and } n_\ominus(T) = k\}.
\end{aligned}$$

In view of Theorem 2.15, we immediately get another expression for $S_n(t)$.

Corollary 2.16. *For $n \geq 1$,*

$$S_n(t) = \sum_{T \in \mathfrak{DT}_n} t^{n_\ominus(T)}. \tag{2.4}$$

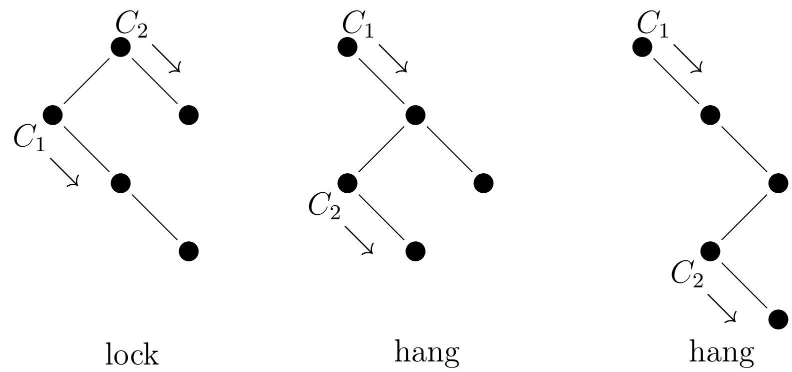


FIGURE 5. Two connection types between two chains C_1 of length 3 and C_2 of length 2.

3. PROOF OF THEOREM 1.1

In this section, we first set up a bijection between two subsets of di-sk trees, and then use this bijection together with a properly chosen weight on di-sk trees to prove Theorem 1.1. To keep the length of our description at minimum, we make some conventions on our terminology. We simply say "chain" when we mean "right chain", and by "odd chain" we mean "right chain with odd length". For a given chain, we call its first node "terminal" and refer to its other nodes as "non-terminal". We use $|C|$ to represent the length of a given chain C.

Recall that we view any binary tree as chains hinged together by left edges. Given two chains C_1 and C_2, C_1 goes before C_2 in order, we distinguish here two different ways they can be connected by left edge. We recommend using Fig. 5 for visualization.

- **Type A (lock):** The terminals of C_1 and C_2 are connected.
- **Type B (hang):** The terminal of C_2 is connected to a non-terminal of C_1.

By "level" we mean the hierarchy observed by looking at the way how all chains are hinged (either locked or hanged) to each other to form one tree. More precisely, we have the following definition.

Definition 3.1. For a given (unlabelled) binary tree, we define the *level* of its chains recursively. This definition extends to di-sk trees naturally.

(1) The first chain is always at level 0;
(2) If a chain is **locked** to a chain at level i, then this chain is defined to be also at level i;
(3) If a chain is **hanged** to a chain at level i, then this chain is defined to be at level $i+1$.

Remark 3.2. By our definition, all chains at level 0 must be locked to each other, so their terminals will form a single left chain. While for $i \geq 1$, chains at level i could break into several groups, inside each group, chains are locked one by one, then together as a subtree they are hanged to some chain at level $i-1$. In other words, the terminals of all chains at level i could form more than one left chains. Because of this observation, when we say two chains are at the "same level", we implicitly require that they are inside the same group, so that their terminals are on the same left chain.

Definition 3.3. Besides $\mathfrak{DT}^2_{n,k}$, we now consider another subset of di-sk trees, namely for $n \geq 1$, $0 \leq k \leq \lfloor (n-1)/2 \rfloor$, we define

$$\mathfrak{DT}^1_{n,k} := \{T \in \mathfrak{DT}_n \,:\, r_o(T) = n-1-2k \text{ and all odd chains start with } \oplus\}.$$

Here $r_o(T)$ is the number of odd right chains on the di-sk tree T.

We are ready to state our main results for this section.

Theorem 3.4. *Let $\gamma^S_{n,k}$ be defined by (1.4). Then, $\gamma^S_{n,k} = |\mathfrak{DT}^1_{n,k}|$.*

Proof. According to Proposition 2.16, weighting each descent by t, each ascent by 1 in a permutation is equivalent to weighting each $\ominus$ by t and each $\oplus$ by 1 on a di-sk tree. As a direct result of this weighting and the right chain condition, each chain of length i will give us $t^{i/2}$ for i even and $t^{(i-1)/2}(1+t)$ for i odd, where the factor $(1+t)$ comes from two different labellings for each odd chain, namely one that begins with $\oplus$ (choose 1) and one that begins with $\ominus$ (choose t). Then forcing the labelling on odd chains to begin with $\oplus$ is essentially taking one representative, namely t^k from the expansion of each term $t^k(1+t)^{n-1-2k}$. Therefore the subset $\mathfrak{DT}^1_{n,k}$ is enumerated by $\gamma^S_{n,k}$. □

Remark 3.5. The weight we placed on the di-sk trees in the proof above can also be described using group action. We will develope another group action in Section 5.1.

Theorem 3.6. *There is a bijection between $\mathfrak{DT}^1_{n,k}$ and $\mathfrak{DT}^2_{n,k}$. Consequently, Theorem 1.1 is true.*

Proof. We quickly give the proof of the second part here, and postpone the bijection part until we have made enough preparation. We have

$$\gamma^S_{n,k} = |\mathfrak{DT}^1_{n,k}| = |\mathfrak{DT}^2_{n,k}| = |\mathfrak{S}^S_{n,k}|,$$

wherein the three equalities from left to right follow respectively from Theorem 3.4, the first part of this theorem and Theorem 2.15. □

We begin by constructing a map $\psi_{n,k}$ from $\mathfrak{DT}^2_{n,k}$ to $\mathfrak{DT}^1_{n,k}$ and then show it is indeed a bijection. For brevity, from now on we will fix some $n \geq 1$ and $0 \leq k \leq \lfloor \frac{n-1}{2} \rfloor$, and drop all the n, k in the subscripts. It should be understood that the same construction applies for all other n, k. We make a few quick observations on the map ψ and the two sets $\mathfrak{DT}^2$ and $\mathfrak{DT}^1$.

(1) The map ψ is simply identity on the intersection $\mathfrak{DT}^2 \cap \mathfrak{DT}^1$.
(2) For any $T \in \mathfrak{DT}^2 \cup \mathfrak{DT}^1$, the number of nodes labelled as $\ominus$ in T equals k, and the rest $n-1-k$ nodes are all labelled as $\oplus$. Since $k \leq \lfloor \frac{n-1}{2} \rfloor$, $(n-1-k)-k \geq 0$, so T always has at least as many $\oplus$ as $\ominus$. And the difference between the numbers of $\oplus$ and $\ominus$ is always $n-1-2k$.

(3) For any $T \in \mathfrak{DT}^2 \backslash \mathfrak{DT}^1$, it must have at least one odd chain that begins with $\ominus$. Since even chains simply pair off $\oplus$ and $\ominus$, T must have at least $n-2k$ other odd chains that begin with $\oplus$ to keep the difference between $\oplus$ and $\ominus$ still being $n-1-2k$. In general, we have that $r_o(\pi) > n-1-2k$ and $r_o(\pi)-(n-1-2k)$ is even.

(4) For any $T \in \mathfrak{DT}^1 \backslash \mathfrak{DT}^2$, it must either begin with $\ominus$, or it has a consecutive pair of $\ominus$, or its presents both incidence.

In view of (2), our map ψ should not change the labelling for the nodes but presumably could change the position of the nodes. Moreover, observation (3) indicates that di-sk trees in $\mathfrak{DT}^2 \backslash \mathfrak{DT}^1$ have "too many" odd chains, and the number of these extra odd chains is even. The above analysis leads us to the following construction that is "cut-and-paste" in nature. We still need to make two key observations before stating our map. Some cases in the proof might seem redundant, but keep in mind that eventually we will construct a bijection, so we try our best to develop two directions in a parallel way so that it will be relatively easier for us to see why it is a bijection.

Lemma 3.7. *Given a di-sk tree $T \in \mathfrak{DT}^2 \backslash \mathfrak{DT}^1$, if it has an odd chain C at level i which begins with $\ominus$, then we can find a unique odd chain C^* at the same level as C that begins with $\oplus$, we call it the "adjoint" of C. See Fig. 6 to compare 6 different cases. Note that the dash line means this portion of the tree could be of any type, including the empty set case.*

Proof. We split the proof into 6 cases according to the level of C, the length of C^* or C, and the labelling of N.

I. The chain C is at level 0 and $|C^*| = 1$. We examine one-by-one the chains at level 0 that appear before C in order, starting from the closest one, say chain C_1. If C_1 is odd, then it must begin with and end with $\oplus$, because otherwise there will be two consecutive $\ominus$, namely the tail of C_1 and the head of C. In this case we are done and $C^* := C_1$ is the adjoint for C. If C_1 is even, then it must begin with $\ominus$ and end with $\oplus$ for the same reason (no consecutive $\ominus$), and C_1 cannot be the first chain because $T \in \mathfrak{DT}^2 \backslash \mathfrak{DT}^1$ so it begins with $\oplus$. Therefore we must have another chain locked to C_1, say C_2, we carry out the same analysis on C_2. This "scanning" will terminate when we spot an odd chain, say C_j for the first time, then $C_1, C_2, \cdots, C_{j-1}$ must all be even, and C_j must begin with $\oplus$, we set $C^* := C_j$ and we are done.

II. The chain C is at level 0 and $|C^*| > 1$. We carry out exactly the same procedure as last case. Actually this procedure does not depend on the value of $|C^*|$. The reason we separate this case will become clear when we prove the analogous lemma for $\mathfrak{DT}^1 \backslash \mathfrak{DT}^2$.

III. The chain C is at level i for $i \geq 1$, $N = \oplus$ and $|C^*| = 1$. Here and in the sequel, N is the node in a chain at level $i-1$, from which the subtree containing C, say T_C is hanged. Since $N = \oplus$, the parent of N must be labelled $\ominus$ due to the right chain condition. This will force the first node of T_C to be labelled $\oplus$ (no consecutive $\ominus$). Then we simply carry out the "scanning" on T_C and reduce this case to the case I above.

IV. The chain C is at level i for $i \geq 1$, $N = \oplus$ and $|C^*| > 1$. We carry out exactly the same procedure as last case.

V. The chain C is at level i for $i \geq 1$, $N = \ominus$ and $|C^*| = 1$. Then the parent of N must be labelled $\oplus$ due to the right chain condition. So unlike cases III and IV, the subtree T_C does not have the "initial condition" (i.e., begins with $\oplus$) that we can use in our argument. Therefore, instead of "searching down", we actually look for its adjoint C^* in the portion between C and N. The first step is to find the unique chain that is closest to C while satisfying the following: i) at the same level as C; ii) comes after C in order; iii) begins with $\ominus$. Denote the terminal of this chain by N_*, in the case that this chain does not exist, simply set $N := N_*$. By the condition we put on N_*, it is easy to see that all chains between C and N_*, at level i will begin with $\oplus$. In particular, the chain locked to N_* must be odd (no consecutive $\ominus$), this is our adjoint C^*.

VI. The chain C is at level i for $i \geq 1$, $N = \ominus$ and $|C^*| > 1$. We carry out exactly the same procedure as last case.

□

Not surprisingly, we have a parallel lemma for $\mathfrak{DT}^1 \backslash \mathfrak{DT}^2$. We also provide Fig. 7 for better illustration. Note that pair of consecutive $\ominus$ cannot exist in the same chain due to the right chain condition.

Lemma 3.8. *Given a di-sk tree $T \in \mathfrak{DT}^1 \backslash \mathfrak{DT}^2$, for each instance it presents that is against the restriction on $\mathfrak{DT}^2$ (see observation (4)), we claim that we can find a unique even chain that we denote by L.*

Proof. Analogous to Lemma 3.7, we prove case-by-case, each case corresponds to a case in the proof of Lemma 3.7.

1. The di-sk tree T begins with $\ominus$. Then this first chain, say L_1, must be even. If L_1 is locked to a chain that begins with $\oplus$ or if L_1 is the last chain at level 0. Then we simply take $L := L_1$. Otherwise L_1 must be locked to a chain that begins with $\ominus$, which has to be an even chain. We denote this chain by L_2. Then we similarly examine the chain that L_2 is locked to, say L_3. If L_3 begins with $\oplus$ then let $L := L_2$, otherwise we move up and consider L_3, etc. This process will end if we arrive at the last chain at level 0, say L_j; or we arrive at some L_j (begins with $\ominus$), which is locked to L_{j+1} (begins with $\oplus$), and all $L_1, L_2, \cdots, L_{j-1}$ begin with $\ominus$. In either case, we set $L := L_j$.
2. The di-sk tree T has a pair of consecutive $\ominus$ at level 0, one $\ominus$ is the tail of an even chain, say ${}_1L$, the other $\ominus$ is the head (terminal) of the even chain locked to ${}_1L$ from above, say L_1. Then the procedure reduces to the case 1 simply by pretending L_1 is the first chain and finding L in a similar way.
3. The di-sk tree T has a pair of consecutive $\ominus$ that are in two chains at different level such that one $\ominus$ is a non-terminal node in a chain at level $i-1$, and the other $\ominus$ is the first node in the chain (say L_1) at level i. Again, we just proceed like in the first two cases, start from L_1 and find our L. Since N is labelled $\oplus$, this L must exist, the extreme case

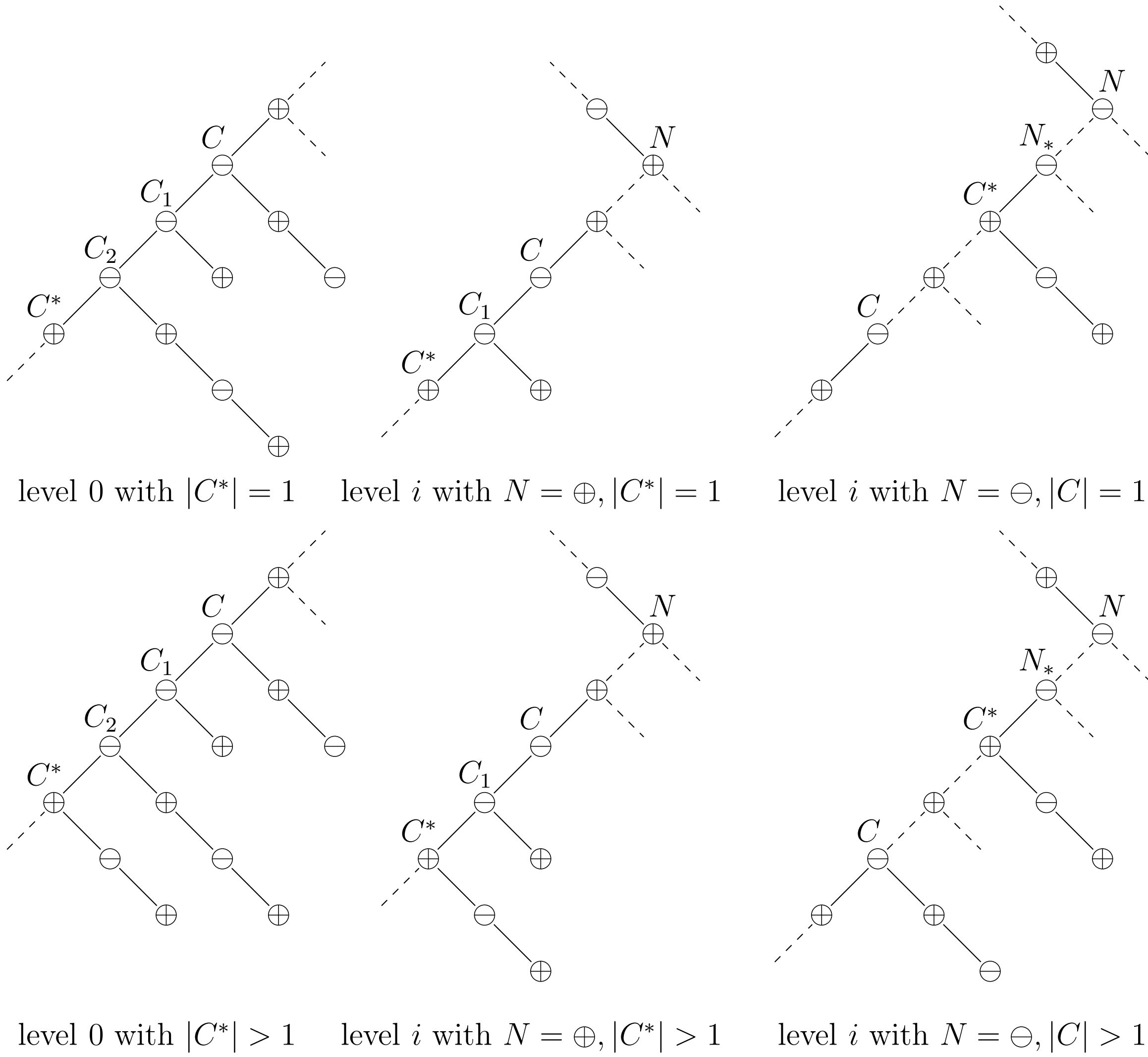


FIGURE 6. Six cases of finding the adjoint for C.

being that L is the chain hanged to N, and all chains at level i on this subtree begin with $\ominus$.

4. The di-sk tree T has a pair of consecutive $\ominus$ at level $i, i \geq 1$, and $N = \oplus$. As in Case 2, we denote the two chains involved by ${}_1L$ and L_1, then proceed like before to find L. And again, we note that since N is labelled $\oplus$, this L must exist, the extreme case being that L is the chain hanged to N, and all chains between ${}_1L$ and N begin with $\ominus$.
5. The di-sk tree T has a pair of consecutive $\ominus$ at level $i, i \geq 1$, $N = \ominus$, and all the chains on the subtree at level i that appear before this pair will begin with $\oplus$. We denote the second $\ominus$ in this pair by N_*, while for the first $\ominus$, the even chain that contains it will be our L. Note that in this case L begins with $\oplus$ rather than $\ominus$.

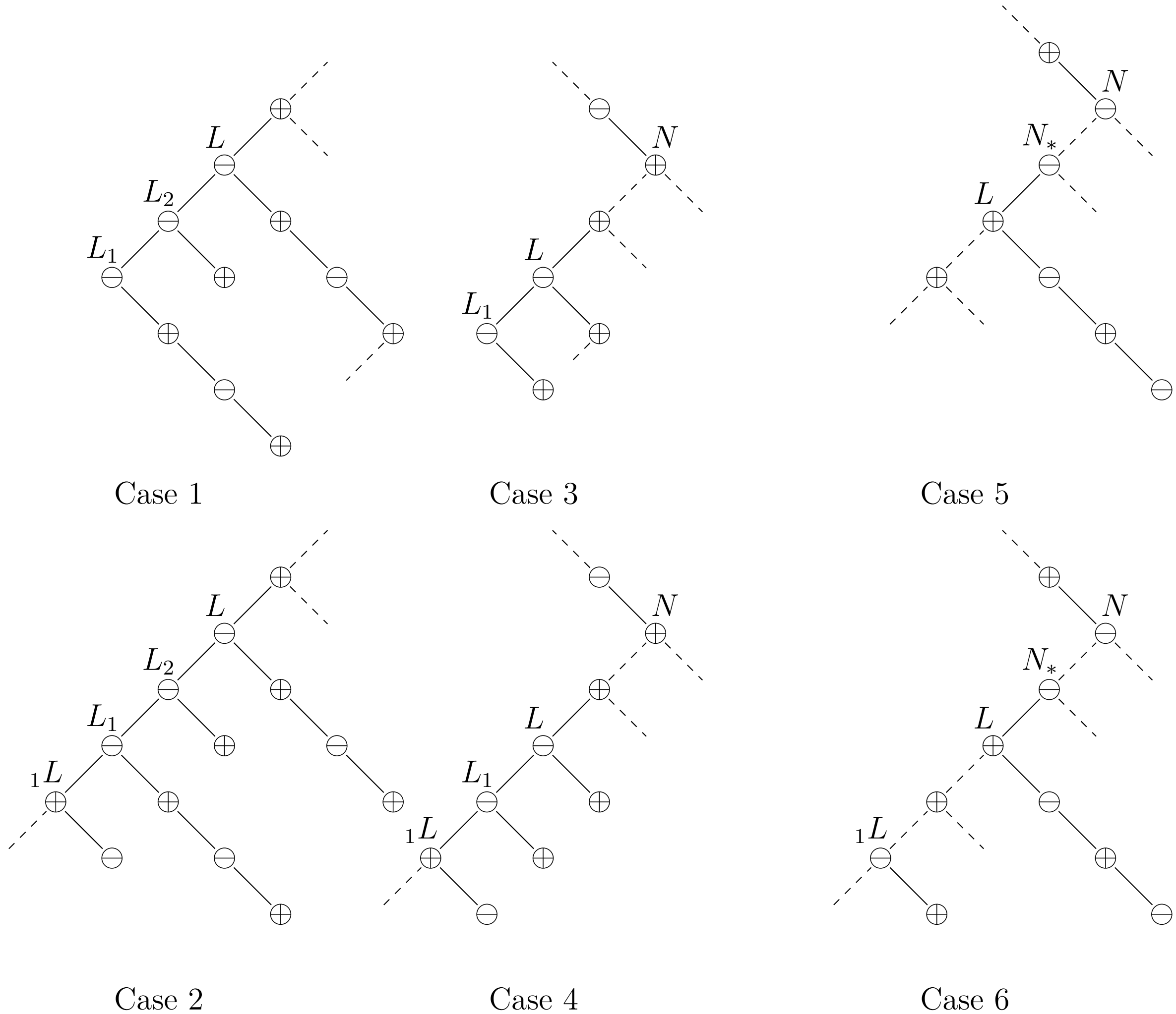


FIGURE 7. Six cases of finding L.

6. The di-sk tree T has a pair of consecutive $\ominus$ at level $i, i \geq 1$, $N = \ominus$, and before this pair, there exist at least one chain that begins with $\ominus$, and we denote the closest one by ${}_1L$. Like in Case 5, we denote the second $\ominus$ in this pair by N_*, while for the first $\ominus$, the even chain that contains it will be our L. Also note that in this case L begins with $\oplus$ rather than $\ominus$.

$\square$

Remark 3.9. Careful readers might be wondering if we have missed the case where for the consecutive pair of $\ominus$, the first is in a chain at level i while the second is in a chain at level $i-1$. We remark here that actually this case is included in Case 5 and 6 as special cases when $N = N_*$.

Lemmas 3.7 and 3.8 are crucial in explaining **where** to apply our "cut and paste", next we describe **how**.

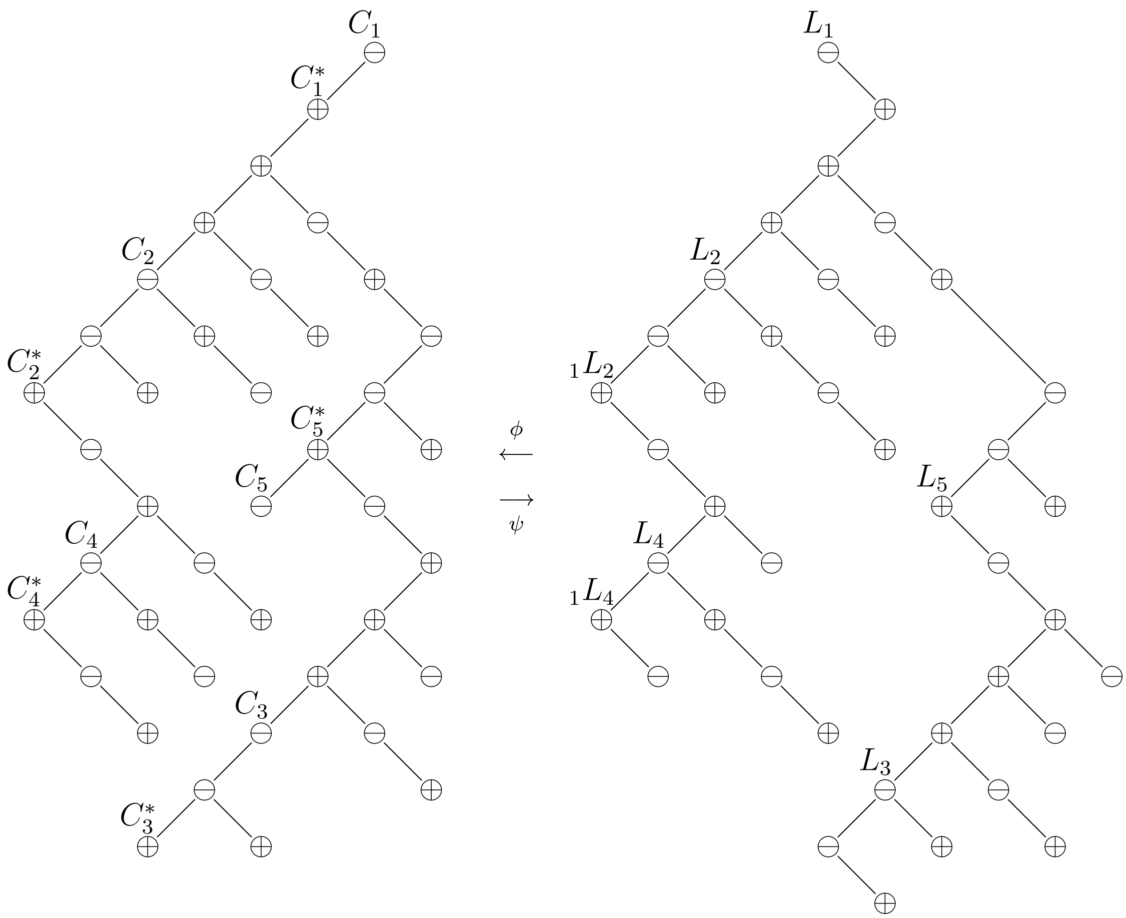
- Operation related to the forward map ψ. Take any $T \in \mathfrak{DT}^2 \backslash \mathfrak{DT}^1$. Starting from level 0, we search for the first odd chain, say C, that begins with $\ominus$, find its adjoint C^* as explained in Lemma 3.7. In Cases I through IV, cut off the last node (labelled $\oplus$) in C^*, together with its left subtree if any, and attach it to the end of C (which is labelled $\ominus$) from right. In Case V and VI, cut off instead the last node (labelled $\ominus$) in C, together with its left subtree if any, and attach it to the end of C^* (which is labelled $\oplus$) from right. In all cases, this operation will effectively turn both C and C^* into even chains, hence decrease $r_o(\pi)$ by 2. Actually in Case I, III and V, one of C and C^* will disappear. And after applying this operation, the new tree still satisfies the right chain condition, therefore we still get a di-sk tree.
- Operation related to the backward map ϕ. Take any $T' \in \mathfrak{DT}^1 \backslash \mathfrak{DT}^2$. Starting from level 0, we search for the first instance that is not allowed in $\mathfrak{DT}^2$, choose accordingly Cases 1 through 6 to find our L. In Cases 1, 3, 5, we cut off the last node in L, lock it (from left) to the first chain at this level, making it the new first chain. While in Cases 2, 4, 6, we cut off the last node in L, attach it (from right) to the end of ${}_1L$. In all cases, this operation will effectively eliminate one instance that is forbidden in $\mathfrak{DT}^2$ and creat an odd chain that begins with $\ominus$. Lastly, after applying this operation, the new tree still satisfies the right chain condition, therefore we still get a di-sk tree.

Let $\psi(T)$ (resp. $\phi(T')$) be the final di-sk tree after we repeatedly apply the first (resp. second) operation at all possible places. Both ψ and ϕ are trivially defined to be the identity map on the intersection $\mathfrak{DT}^2 \cap \mathfrak{DT}^1$. We see that the order in which we apply these operations are essentially irrelevant. We prove this as a lemma, and then we finish our proof of the main theorem.

Lemma 3.10. *Both maps ψ and ϕ as defined above does not depend on the order we apply the needed operations, in other words, one operation is independent from the other.*

Proof. For ψ, simply note that if (C_1, C_1^*) and (C_2, C_2^*) are two pairs of odd chains that begin with $\ominus$ and their adjoints, suppose C_1 appears before C_2, then either they are at different levels; or they are at the same level, and $C_1^* < C_1 < C_2^* < C_2$ for Cases I, II, III, IV, $C_1 < C_1^* < C_2 < C_2^*$ for Cases V and VI. Here "$A < B$" means chain A appears before chain B. Therefore the two operations applied on (C_1, C_1^*) and (C_2, C_2^*) respectively are easily seen to be independent. Similarly for ϕ, since no odd chain begins with $\ominus$ in $\mathfrak{DT}^1$, there cannot be two forbidden incidences that share one common chain. □

Proof of Theorem 3.6. Thanks to Lemma 3.10, it will now suffice to prove these two operations are inverse to each other in all six cases. This should be quite routine to check, especially because we have arranged these cases in Fig. 6 and 7 so that Case I corresponds to Case 1, Case II corresponds to Case 2, etc. It follows that ψ is indeed a bijection and $\psi^{-1} = \phi$. □



$T \in \mathfrak{DT}^2 \backslash \mathfrak{DT}^1$ $\qquad$ $\psi(T) \in \mathfrak{DT}^1 \backslash \mathfrak{DT}^2$

FIGURE 8. Two di-sk trees related via ψ and ϕ.

Such a delicate construction deserve some examples and we offer one here. In Fig. 8 we presents two di-sk trees each with 40 nodes, that are related via our maps ψ and ϕ. It requires applying our operation at five different places on the di-sk tree, covering five different cases. We mark C and its adjoint C^*, as well as L and ${}_1L$ for the reader's convenience.

4. Spiral property for $D_n(t)$

To prove the unimodality of $D_n(t)$, we shall apply the following formula of Désarménien-Foata [5] and Gessel-Reutenauer [7]:

$$\sum_{n\geq 2} \frac{D_n(t)}{(1-t)^{n+1}} z^n = \sum_{r\geq 1} \frac{t^{r-1}}{1-rz}(1-z)^r. \tag{4.1}$$

Lemma 4.1. *The polynomial $D_n(t)$ satisfies the following recurrence relation:*

$$D_n(t) = (-1)^n t^{n-1} + (1 + (n-1)t)D_{n-1}(t) + t(1-t)D'_{n-1}(t). \tag{4.2}$$

Equivalently,

$$d_{n,k} = \begin{cases} 1, & \text{if } n \text{ is even and } k = n-1; \\ 0, & \text{if } n \text{ is odd and } k = n-1; \\ (k+1)d_{n-1,k} + (n-k)d_{n-1,k-1}, & \text{if } k \neq n-1. \end{cases} \tag{4.3}$$

Proof. Extracting the coefficient of z^n in both sides of (4.1) gives

$$\frac{D_n(t)}{(1-t)^{n+1}} = \sum_{r\geq 1} t^{r-1} \left(\sum_{k=0}^{n\wedge r} (-1)^k \binom{r}{k} r^{n-k} \right), \tag{4.4}$$

where $n \wedge r = \min\{n, r\}$. If we set $T_r(n) = \sum_{k=0}^{n\wedge r}(-1)^k \binom{r}{k} r^{n-k}$, then

$$T_r(n) = \begin{cases} rT_r(n-1), & \text{if } 1 \leq r \leq n-1; \\ rT_r(n-1) + (-1)^n \binom{r}{n}, & \text{otherwise.} \end{cases}$$

It then follows from (4.4) that

$$\begin{aligned} t\left(\frac{tD_{n-1}(t)}{(1-t)^n}\right)' &= \sum_{r\geq 1} t^r r T_r(n-1) \\ &= \sum_{r\geq 1} t^r T_r(n) - \sum_{r\geq n} t^r (-1)^n \binom{r}{n} \\ &= \frac{tD_n(t) - (-t)^n}{(1-t)^{n+1}}. \end{aligned}$$

After simplifying we get (4.2). □

Remark 4.2. When $t = 1$, Eq. (4.2) reduces to the well-known recurrence relation

$$d_n = (-1)^n + nd_{n-1}$$

for the number of derangement $d_n = \#\mathfrak{D}_n$. It is also reminiscent of the recurrence

$$A_n(t) = (1 + (n-1)t)A_{n-1}(t) + t(1-t)A'_{n-1}(t) \tag{4.5}$$

for the Eulerian polynomials.

A *desarrangement* is a permutation whose first ascent is even, where an index $i \in [n]$ is an *ascent* of $\pi \in \mathfrak{S}_n$ if $\pi_i < \pi_{i+1}$ (by convention $\pi_{n+1} = +\infty$). For example, 653241 is a desarrangement but 321564 is not. Let $\mathcal{E}_n$ be the set of all desarrangements in $\mathfrak{S}_n$.

A bijective proof of (4.3). By a result of Désarménien and Wachs [6, Corollary 3.3]) we have

$$d_{n,k} = \{\pi \in \mathcal{E}_n : \mathrm{ides}(\pi) = k\}.$$

We say that an index i, $1 \le i \le n-1$, is an *inverse descent* of $\pi \in \mathfrak{S}_n$ if $i+1$ appears to the left of i in π. Clearly, the number of inverse descents of π is $\mathrm{ides}(\pi)$. When n is even, the only desarrangement in $\mathcal{E}_n$ with $n-1$ inverse descents is $n(n-1)\cdots 21$, so $d_{n,n-1} = 1$ in this case. In the n odd case, there is not desarrangement of length n with $n-1$ inverse descents and $d_{n,n-1}$ follows. In the following, we can assume that $1 \le k < n-1$.

Let $\mathfrak{S}_{n-1} \times [n] := \{(\pi, j) : \pi \in \mathfrak{S}_{n-1}, j \in [n]\}$. There is a natural bijection from $\mathfrak{S}_{n-1} \times [n]$ to $\mathfrak{S}_n$ defined by

$$(\pi, j) \mapsto \sigma = \sigma_1 \cdots \sigma_n, \tag{4.6}$$

where $\sigma_n = j$ and for $i \in [n-1]$, $\sigma_i = \pi_i + 1$ if $\pi_i \ge j$, otherwise $\sigma_i = \pi_i$. It is routine to check that

$$\mathrm{ides}(\sigma) = \begin{cases} \mathrm{ides}(\pi), & \text{if } j-1 \text{ is an inverse descent of } \pi; \\ \mathrm{ides}(\pi)+1, & \text{otherwise.} \end{cases}$$

Recurrence relation (4.3) then follows from this property and the fact that in (4.6) if σ is a desarrangement in $\mathcal{E}_n$ with k ($k < n-1$) inverse descents then π is a desarrangement. □

From (4.2) we can readily deduce that $\deg(D_{2n+1}(t)) = 2n-1$ and $D_{2n}(t)$ is a monic polynomial of degree $2n-1$. Moreover, the coefficient of t in $D_n(t)$ is 2^{n-2}.

Proof of Theorem 1.2. It is easy to check that statement (1.6) is true for $n \le 3$. We proceed to prove the statement by induction on n using recurrence (4.3).

Suppose that $m \ge 4$ and statement (1.6) is true for $n = m-1$. We first show that $d_{2m,2m-k} < d_{2m,k} < d_{2m,2m-k-1}$ for $1 \le k \le m-1$. By the recurrence relation (4.3) for $d_{n,k}$, we have

$$d_{2m,2m-k} = (2m-k+1)d_{2m-1,2m-k} + kd_{2m-1,2m-k-1} \tag{4.7}$$

if $k \ne 1$ and

$$d_{2m,k} = (k+1)d_{2m-1,k} + (2m-k)d_{2m-1,k-1} \tag{4.8}$$

$$d_{2m,2m-k-1} = (2m-k)d_{2m-1,2m-k-1} + (k+1)d_{2m-1,2m-k-2}. \tag{4.9}$$

Clearly, $d_{2m,2m-1} = 1 < 2^{2m-2} = d_{2m,1}$. It follows from (4.7) and (4.8) that, for $k \ge 2$,

$$\begin{aligned} d_{2m,k} - d_{2m,2m-k} = (2m-k+1)&(d_{2m-1,k-1} - d_{2m-1,2m-k}) \\ &+ k(d_{2m-1,k} - d_{2m-1,2m-k-1}) + (d_{2m-1,k} - d_{2m-1,k-1}). \end{aligned}$$

By the inductive hypothesis, the difference in every parenthesis in the above expression is positive, which implies that $d_{2m,k} > d_{2m,2m-k}$. Similarly, by (4.8) and (4.9) we have

$$\begin{aligned} d_{2m,2m-k-1} - d_{2m,k} = (2m-k)&(d_{2m-1,2m-k-1} - d_{2m-1,k-1}) \\ &+ (k+1)(d_{2m-1,2m-k-2} - d_{2m-1,k}). \end{aligned}$$

Again, by the inductive hypothesis, we deduce that $d_{2m,2m-k-1} > d_{2m,k}$. This completes the proof of the first part of statement (1.6) for $n = m$. It remains to show the second part of statement (1.6) for $n = m$, which is omitted due to the similarity. This completes the proof of the theorem by induction. □

5. More on separable permutations

5.1. **A second interpretation for $\gamma_{n,k}^S$.** In this section, we will develop a different group action, together with certain "right chain index" (abbreviated as **rc**-index in the sequel) for di-sk trees. We refer readers to [16, section 1.6.3] for introduction and backgrounds on min-max trees and the *cd*-index that are defined for all permutations. Roughly speaking, the *ab*-index and the *cd*-index can be viewed as a refined (or non-commutative) version of the Eulerian polynomial and its γ-decomposition. Our construction of **rc**-index for separable permutations is analogous to the *cd*-index for permutations.

Definition 5.1. Given a separable permutation $\pi \in \mathfrak{S}_n(2413, 3142)$ and its corresponding di-sk tree $T(\pi)$, for $1 \leq i \leq n-1$, ψ_i will reverse ($\oplus$ to $\ominus$ and $\ominus$ to $\oplus$) the labelling for all the nodes on the i-th right chain, counting from left to right. The labelling on all other nodes are fixed by ψ_i. We denote the new di-sk tree thus obtained by $\psi_i T(\pi)$.

Note that there can be at most $n-1$ such right chains. Since there are no restriction on the left child labelling, and that all the right chains are hinged together by left edges, we see these ψ_i act on di-sk trees independently. Hence we get the following result which is similar to [16, section 1.6.3].

Fact 5.2. *The operators ψ_i are commuting involutions and hence generate an abelian group $\mathfrak{G}_\pi$ isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{r(\pi)}$, recall that $r(\pi) = r(T(\pi))$ is the number of right chains in $T(\pi)$ (see Definition 2.10 and Remark 2.12). Hence there are precisely $2^{r(\pi)}$ different di-sk trees $\psi T(\pi)$ for $\psi \in \mathfrak{G}_\pi$.*

Definition 5.3. Given a permutation $\omega \in \mathfrak{S}_n(2413, 3142)$ and an operator $\psi \in \mathfrak{G}_\omega$ we define the permutation $\psi\omega$ by $\psi DT(\omega) = DT(\psi\omega)$. Two permutations $\nu, \omega \in \mathfrak{S}_n(2413, 3142)$ are said to be *DT-equivalent*, denoted $\nu \overset{DT}{\sim} \omega$, if $\nu = \psi\omega$ for some $\psi \in \mathfrak{G}_\omega$. The size of the equivalence class $[\omega]$ is $2^{r(\omega)}$ due to Fact 5.2.

Remark 5.4. Recall that the *ab*-index (see [16, section 1.6.3]) for $\pi = 984132756$ is $u_{D(\pi)} = abbababa$, which is easily seen to match its Schröder word $\mathrm{sw}(\pi)$ if we delete all the 1s and parentheses and replace $\oplus$ and $\ominus$ with a and b, respectively. In other words, our di-sk tree is compatible with the original *ab*-indexing and we will get $S_n(t)$ upon summing over all *ab*-monomials for separable permutations and putting $a = 1$, $b = t$.

Proposition 5.5. *Two separable permutations are DT-equivalent if and only if they have the same tree structure, in other words, they reduce to the same unlabelled binary tree if we delete all the labellings. Consequently, the number of equivalence classes is $|\mathfrak{T}_n| = C_{n-1}$.*

Proof. Since the operators ψ_i only alter the labellings, not the underlying tree structure, we get the "only if" part. Conversely, if two permutations $\nu, \omega \in \mathfrak{S}_n(2413, 3142)$ have the same tree structure for their corresponding di-sk trees, then clearly we can go from one labelling to another by applying $\psi_i, 1 \le i \le n-1$ for at most $n-1$ times, so we have $\nu \overset{DT}{\sim} \omega$. □

Now we proceed to define the **rc**-index for any di-sk tree (or equivalently, any separable permutation). Define $n-1$ noncommutative indeterminates as follows:

$$c_1 = a + b, \quad c_2 = ab + ba, \quad \ldots, \quad c_{n-1} = \underbrace{aba\cdots}_{n-1} + \underbrace{bab\cdots}_{n-1}.$$

We assign a weight for each equivalence class $[\omega]$. For $1 \le i \le r(\omega)$, let $f_i := c_l$, where l is the length of the i-th right chain in $DT(\omega)$. And then let $\Phi_{[\omega]} = \Phi_{[\omega]}(c_1, c_2, \cdots, c_{n-1}) := f_1 f_2 \cdots f_{r(\omega)}$. Our $\Phi_{[\omega]}$ is well-defined due to Proposition 5.5. We call it the rc-weight for this equivalence class $[\omega]$. We get the "global" **rc**-index by summing $\Phi_{[\omega]}$ over all different equivalence classes.

Theorem 5.6. *The equivalence relation* $\overset{DT}{\sim}$ *defined on* $\mathfrak{S}_n(2413, 3142)$ *will give us a unified weighted counting for both the Catalan numbers* C_n *and the large Schröder numbers* S_n*:*

$$\Phi_n = \Phi_n(c_1, c_2, \cdots, c_{n-1}) := \sum \Phi_{[\omega]},$$

where the sum is over all different equivalence classes. Φ_n *is called the* **rc**-index *for* $\mathfrak{S}_n(2413, 3142)$*. In particular, we have* $\Phi_n(1, 1, \cdots, 1) = C_{n-1}$ *and* $\Phi_n(2, 2, \cdots, 2) = S_n$*.*

Proof. The first evaluation is a direct result of Proposition 5.5. For the second one, putting $c_1 = c_2 = \cdots = c_{n-1} = 2$ means each equivalent class $[\omega]$ is counted by the weight $2^{r(\omega)}$, which is exactly the size of this class (Fact 5.2), so summing up we get the total number of di-sk trees, i.e. S_n. □

Analogous to the original cd-index for $\mathfrak{S}_n$, the number of distinct terms in Φ_n is a well studied number $c(n)$, the number of compositions of n. We list the first few values,

$$\begin{aligned}
\Phi_1 &= 1,\\
\Phi_2 &= c_1,\\
\Phi_3 &= c_1^2 + c_2,\\
\Phi_4 &= c_1^3 + c_1c_2 + 2c_2c_1 + c_3,\\
\Phi_5 &= c_1^4 + c_1^2c_2 + 2c_1c_2c_1 + 3c_2c_1^2 + 2c_2^2 + c_1c_3 + 3c_3c_1 + c_4,\\
\Phi_6 &= c_1^5 + c_1^3c_2 + 2c_1^2c_2c_1 + 3c_1c_2c_1^2 + 4c_2c_1^3 + 2c_1c_2^2 + 3c_2c_1c_2 + 5c_2^2c_1 +\\
&\quad c_1^2c_3 + 3c_1c_3c_1 + 6c_3c_1^2 + 2c_2c_3 + 3c_3c_2 + c_1c_4 + 4c_4c_1 + c_5.
\end{aligned}$$

One more thing to notice is that different equivalence classes might have the same rc-weight, essentially because one can either "lock" or "hang" one chain to another, see Fig. 9 for example. And by expanding one monomial in $c_1, c_2, \ldots, c_{n-1}$ we typically *do not* recover the original ab-index for permutations in this equivalence class, which is unfortunate. But it is refined enough to give us another combinatorial interpretation for all the γ-coefficients, which implies the positivity trivially.

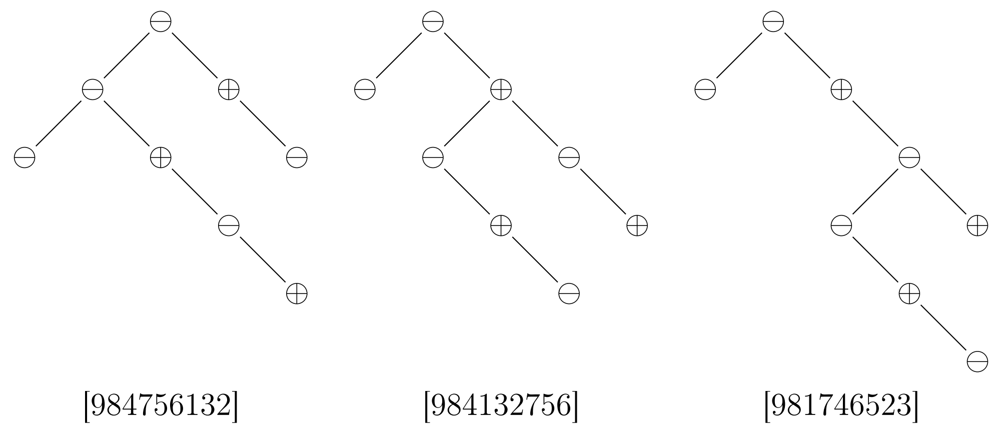


FIGURE 9. Three different di-sk trees have the same weight $c_1c_4c_3$.

Theorem 5.7. *Let* $\gamma^S_{n,k}$ *be defined by* (1.4)*. Then,*

$$\gamma^S_{n,k} = \sum_{\substack{T\in\mathfrak{T}_n \\ r_o(T)=n-1-2k}} 2^{r_e(T)}. \tag{5.1}$$

Proof. Although by expanding rc-weight generally we will not get the original ab-index for each permutation in this equivalence class, it does preserve the number of a's and b's, hence the descent number, which is exactly how we weight each permutation in $S_n(t)$, therefore we have

$$\Phi_n(c_1, c_2, \cdots, c_{n-1})\big|_{a=1,b=t} = S_n(t).$$

On the other hand, upon putting $a = 1, b = t$ in Φ_n, each c_i becomes either $2t^{i/2}$ when i is even or $t^{(i-1)/2}(1+t)$ when i is odd. Now suppose one term $\Phi_{[\omega]} = c_{k_1}c_{k_2}\cdots c_{k_m}$, where $m = r(\omega)$. Then upon evaluation it becomes $2^{r_e(\omega)}t^{(n-1-r_o(\omega))/2}(1+t)^{r_o(\omega)}$, so we establish (5.1). Note that this one compact term actually codify all $2^{r(\omega)}$ separable permutations that are in the same equivalence class $[\omega]$. □

We exemplify this new interpretation on $S_4(t)$ and the five unlabelled binary trees in $\mathfrak{T}_4$ are depicted in Fig. 10.

Example 5.8. When $n = 4$, we have two terms in the γ-decomposition of $S_4(t)$, i.e., $S_4(t) = (1+t)^3 + 7t(1+t)$.

- $k = 0$. We count binary trees with $n-1 = 3$ nodes and $n-1-2k = 3$ right chains of odd length, the only choice is $(1,1,1)$ (right chains listed out with their length from left to right), with no right chain of even length, and the way to hinge these three right chains together is unique, i.e. locked up one by one. So we get $\gamma_{4,0} = 1$.
- $k = 1$. We count binary trees with 3 nodes and only one right chain of odd length, the choices are (3), $(1,2)$ and $(2,1)$. There is only one way for (3), which is weighted

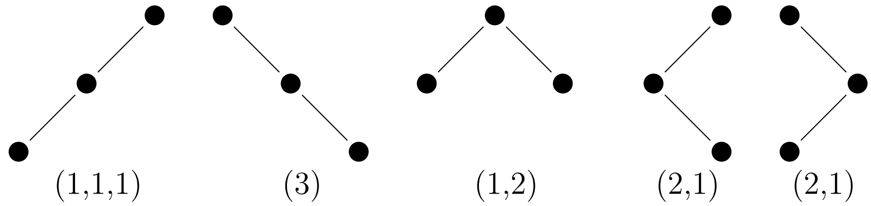


FIGURE 10. Five binary trees in $\mathfrak{T}_4$ and their right chain type.

by 1 (no even right chain), and there are 1 and 2 ways for $(1,2)$ and $(2,1)$, respectively. And all of them are weighted by 2 (each have one right chain). So we get $\gamma_{4,1} = 1 + 2 \cdot (1+2) = 7$.

5.2. **Generating function for $S_n(t)$.** In this section, we compute the generating function for $S_n(t)$ and provide another proof of its γ-positivity. To do this, let

$$S_n^{(1)}(t) := \sum_{T \in \mathfrak{DT}_n^{\oplus}} t^{n_\ominus(T)} \qquad \text{and} \qquad S_n^{(2)}(t) := \sum_{T \in \mathfrak{DT}_n^{\ominus}} t^{n_\ominus(T)},$$

where $\mathfrak{DT}_n^{\oplus}$ and $\mathfrak{DT}_n^{\ominus}$ are the set of all di-sk trees in $\mathfrak{DT}_n$ with root labeled by $\oplus$ and $\ominus$, respectively. For convenience, we set $S_1^{(1)}(t) = S_1^{(2)}(t) = 1$. It follows from (2.4) that

$$S_n(t) = S_n^{(1)}(t) + S_n^{(2)}(t) \tag{5.2}$$

if $n \geq 2$.

Lemma 5.9. *For $n \geq 2$, we have*

$$S_n^{(1)}(t) = \sum_{j=1}^{n-1} S_j(t) S_{n-j}^{(2)}(t) \qquad \textit{and} \qquad S_n^{(2)}(t) = t \sum_{j=1}^{n-1} S_j(t) S_{n-j}^{(1)}(t). \tag{5.3}$$

Proof. By the right chain condition, any di-sk tree in $\mathfrak{DT}_n^{\oplus}$ can be constructed from a root labeled $\oplus$ by attaching a di-sk tree on the left branch and a di-sk tree with a root labeled $\ominus$ on the right branch. This gives the first expression in (5.3). The second expression in (5.3) follows by similar decomposition of a di-sk tree in $\mathfrak{DT}_n^{\ominus}$. $\square$

Theorem 5.10. *For $n \geq 2$, we have the following recurrence relation for $S_n(t)$:*

$$S_n(t) = (1+t) S_{n-1}(t) + t \sum_{j=1}^{n-2} S_j(t) \left(S_{n-j-1}(t) + \sum_{i=1}^{n-j-1} S_i(t) S_{n-j-i}(t) \right). \tag{5.4}$$

Equivalently,

$$S(t,z) = z + (1+t) z S(t,z) + tz S^2(t,z) + t S^3(t,z),$$

where $S(t,z) := \sum_{n \geq 1} S_n(t) z^n$.

Proof. For $n \geq 1$, let

$$S'_n(t) := tS_1^{(1)}(t) + S_1^{(2)}(t).$$

Note that $S'_1(t) = 1 + t$. It follows from (5.3) and (5.2) that, for $n \geq 2$

$$S_n(t) = \sum_{j=1}^{n-1} S_j(t)(S_{n-j}^{(2)}(t) + tS_{n-j}^{(1)}(t)) = \sum_{j=1}^{n-1} S_j(t)S'_{n-j}(t) \tag{5.5}$$

and

$$\begin{aligned} S'_n(t) &= \sum_{j=1}^{n-1} S_j(t)(tS_{n-j}^{(2)}(t) + tS_{n-j}^{(1)}(t)) \\ &= 2tS_{n-1}(t) + t\sum_{j=1}^{n-2} S_j(t)S_{n-j}(t). \end{aligned}$$

Substituting the latter into (5.5), we get (5.4). □

It is not hard to show that if $A(t)$ and $B(t)$ are γ-positive of darga m and n respectively, then $A(t)B(t)$ is γ-positive of darga $m+n$. The γ-positivity of $S_n(t)$ then follows from (5.4) by induction on n. Let $\Gamma_n(x) := \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \gamma_{n,k}^S x^k$ be the γ-polynomial of $S_n(t)$, where $\gamma_{n,k}^S$ is defined by (1.4). Actually, the recurrence relation (5.4) for $S_n(t)$ is equivalent to the following recurrence for its γ-polynomial because

$$S_n(t) = (1+t)^{n-1}\Gamma_n(x), \quad \text{with } x = \frac{t}{(1+t)^2}.$$

Corollary 5.11. *The recurrence relation for* $\Gamma_n(x)$ *is*

$$\Gamma_n(x) = \Gamma_{n-1}(x) + x\sum_{j=1}^{n-2} \Gamma_j(x)\left(\Gamma_{n-j-1}(x) + \sum_{i=1}^{n-j-1} \Gamma_i(x)\Gamma_{n-j-i}(x)\right)$$

with initial value $\Gamma_1(x) = 1$.

6. Concluding remarks and open problems

The combinatorial interpretation for $\gamma_{n,k}^S$ that we established in Theorem 1.1 nicely parallels those for $\gamma_{n,k}^A$ and $\gamma_{n,k}^N$, and note that $\mathfrak{S}_n(231) \subseteq \mathfrak{S}_n(2413, 3142) \subseteq \mathfrak{S}_n$. This in particular will give as by-product, the γ-positivity for the complementary set. Namely, the descent polynomials on permutations that contain at least one of the patterns $(3142, 2413)$, are also γ-positive. Similar result holds for $\mathfrak{S}_n(231)$ and $\mathfrak{S}_n \backslash \mathfrak{S}_n(231)$. This observation raises a natural question: are there any other subsets enjoy the same property? Or more generally, is it possible to characterize all the subsets $S \subseteq \mathfrak{S}_n$ such that the descent polynomials on S and $\mathfrak{S}_n \backslash S$ are both γ-positive?

Regarding Gessel's conjecture, we have the following conjectured inequality.

Conjecture 6.1. Let $\gamma_{n,i,j}$ be defined by (1.3). Then,

$$\gamma_{n,k,n-1-2k} \geq \gamma^S_{n,k}.$$

Let $\widetilde{D}_n(t)$ be the descent polynomial on $\mathfrak{S}_n \backslash \mathfrak{D}_n$. It follows from (4.2) and (4.5) that

$$\widetilde{D}_n(t) = (-t)^{n-1} + (1 + (n-1)t)\widetilde{D}_{n-1}(t) + t(1-t)\widetilde{D}'_{n-1}(t),$$

since $\widetilde{D}_n(t) = A_n(t) - D_n(t)$. By similar discussion as in the proof of Theorem 1.2, we can show that $\widetilde{D}_n(t)$ also has the spiral property, which implies the unimodality.

Finally the two polynomials $S_n(t)$ and $D_n(t)$ seem to be real-rooted basing on computational experiments. We pose this as a conjecture for further investigation.

Conjecture 6.2. The descent polynomials $S_n(t)$ and $D_n(t)$ are real-rooted for each $n \geq 2$.

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