A NOTE ON BOOLEAN LATTICES AND FAREY SEQUENCES III

Andrey O. Matveev

andrey.o.matveev@gmail.com

Abstract

We describe monotone maps between subsequences of the Farey sequences.

1. Introduction

Let $\mathbb{V}(n)$ be an n–dimensional linear space, and $\mathbf{A}$ its proper m–dimensional subspace. We associate with the integers n and m the increasing sequence of irreducible fractions

$$\mathcal{F}\bigl(\mathbb{B}(n),m\bigr) := \Bigl(\tfrac{\dim(\mathbf{B}\cap\mathbf{A})}{\gcd(\dim(\mathbf{B}\cap\mathbf{A}),\dim\mathbf{B})}\Big/\tfrac{\dim\mathbf{B}}{\gcd(\dim(\mathbf{B}\cap\mathbf{A}),\dim\mathbf{B})}: \quad \mathbf{B} \text{ subspace of } \mathbb{V}(n),\ \dim\mathbf{B}>0\Bigr)\ ; \qquad (1)$$

in other words,

$$\mathcal{F}\bigl(\mathbb{B}(n),m\bigr) := \bigl(\tfrac{h}{k}\in\mathcal{F}_n:\ m+k-n\le h\le m\bigr)\ , \qquad (2)$$

where $\mathcal{F}_n$ denotes the Farey sequence of order n. See e.g. [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 15, 16, 17] on the Farey sequences. In particular,

$$\mathcal{F}\bigl(\mathbb{B}(2m),m\bigr) := \bigl(\tfrac{h}{k}\in\mathcal{F}_{2m}:\ k-m\le h\le m\bigr)\ . \qquad (3)$$

Equivalent descriptions of Farey subsequence (2) can be given via the cardinalities of subsets of an n–set, or via the ranks of elements of the Boolean lattice $\mathbb{B}(n)$ of rank n, see [12, 13].

Sequence (2) can be regarded as the intersection

$$\mathcal{F}_n^m\cap\mathcal{G}_n^m =: \mathcal{F}\bigl(\mathbb{B}(n),m\bigr) \qquad (4)$$

of the Farey subsequence

$$\mathcal{F}_n^m := \bigl(\tfrac{h}{k}\in\mathcal{F}_n:\ h\le m\bigr) \qquad (5)$$

introduced in [1], and of the Farey subsequence

$$\mathcal{G}_n^m := \bigl(\tfrac{h}{k}\in\mathcal{F}_n:\ m+k-n\le h\bigr)\ ; \qquad (6)$$

it follows from definitions (5) and (6) that

$$\mathcal{F}_n^m \cup \mathcal{G}_n^m = \mathcal{F}_n \ . \tag{7}$$

Recall that the maps $\mathcal{F}_n \to \mathcal{F}_n$, $\mathcal{F}_n^m \to \mathcal{G}_n^{n-m}$, $\mathcal{G}_n^m \to \mathcal{F}_n^{n-m}$, and $\mathcal{F}(\mathbb{B}(n), m) \to \mathcal{F}(\mathbb{B}(n), n-m)$, such that

$$\tfrac{h}{k} \mapsto \tfrac{k-h}{k} \ , \qquad \left[\begin{smallmatrix} h \\ k \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} -1 & 1 \\ 0 & 1 \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} h \\ k \end{smallmatrix}\right] \ , \tag{8}$$

where $\left[\begin{smallmatrix} h \\ k \end{smallmatrix}\right]$ is a vector presentation of $\frac{h}{k}$, are all order–reversing and bijective.

In analogy to sequences (5) and (6), we define similar subsequences of the sequence $\mathcal{F}\big(\mathbb{B}(n), m\big)$ as follows: given an integer ℓ, $1 \le \ell \le m$,

$$\mathcal{F}\big(\mathbb{B}(n), m\big)^{\ell} := \big(\tfrac{h}{k} \in \mathcal{F}\big(\mathbb{B}(n), m\big) : \ h \le \ell\big) \ ; \tag{9}$$

given an integer ℓ, $m \le \ell \le n-1$,

$$\mathcal{G}\big(\mathbb{B}(n), m\big)^{\ell} := \big(\tfrac{h}{k} \in \mathcal{F}\big(\mathbb{B}(n), m\big) : \ \ell + k - n \le h\big) \ ; \tag{10}$$

thus, $\mathcal{F}\big(\mathbb{B}(n), m\big)^m = \mathcal{G}\big(\mathbb{B}(n), m\big)^m = \mathcal{F}\big(\mathbb{B}(n), m\big)$. It is easy to see, as noted in Section 2, that the sequences $\mathcal{F}\big(\mathbb{B}(n), m\big)^{\ell}$ and $\mathcal{G}\big(\mathbb{B}(n), m\big)^{\ell}$ are both sequences of the form $\mathcal{F}\big(\mathbb{B}(\cdot), \cdot\big)$ for any allowed ℓ's; moreover, if λ is an integer such that $1 \le \lambda \le n - m$, then

$$\mathcal{F}\big(\mathbb{B}(n), m\big)^{\ell} \cap \mathcal{G}\big(\mathbb{B}(n), m\big)^{n-\lambda} = \mathcal{F}\big(\mathbb{B}(\ell + \lambda), \ell\big) \ . \tag{11}$$

In Section 3, we describe monotone maps between (sub)sequences of the Farey sequences of orders $2^s m$, $2^{s+1} m$ and $2^{s+2} m$. In Section 4, these observations are reformulated for the Farey sequences of arbitrary orders. Supplementary information is collected in Appendix 5.

2. The Farey Subsequences $\mathcal{F}\big(\mathbb{B}(n), m\big)^{\ell}$ and $\mathcal{G}\big(\mathbb{B}(n), m\big)^{\ell}$

It is easily verified that the subsequences $\mathcal{F}\big(\mathbb{B}(n), m\big)^{\ell}$ and $\mathcal{G}\big(\mathbb{B}(n), m\big)^{\ell}$ of the sequence $\mathcal{F}\big(\mathbb{B}(n), m\big)$, defined by Eqs. (9) and (10) respectively, are also sequences of the form (2); indeed, they can be redefined as follows:

$$\mathcal{F}\big(\mathbb{B}(n), m\big)^{\ell} := \mathcal{F}_{n-m+\ell}^{\ell} \cap \mathcal{G}_{n-m+\ell}^{\ell} \ , \qquad \mathcal{G}\big(\mathbb{B}(n), m\big)^{\ell} := \mathcal{F}_{n+m-\ell}^{m} \cap \mathcal{G}_{n+m-\ell}^{m} \ ; \tag{12}$$

in particular,

$$\mathcal{F}\big(\mathbb{B}(2m), m\big)^{\ell} := \mathcal{F}_{m+\ell}^{\ell} \cap \mathcal{G}_{m+\ell}^{\ell} \ , \qquad \mathcal{G}\big(\mathbb{B}(2m), m\big)^{\ell} := \mathcal{F}_{3m-\ell}^{m} \cap \mathcal{G}_{3m-\ell}^{m} \ . \tag{13}$$

Remark 1. (i) *If $\ell \in \mathbb{P}$, $1 \le \ell \le m$, then*

$$\mathcal{F}\big(\mathbb{B}(n), m\big)^{\ell} = \mathcal{F}\big(\mathbb{B}(n-m+\ell), \ell\big) \;, \qquad \mathcal{F}\big(\mathbb{B}(2m), m\big)^{\ell} = \mathcal{F}\big(\mathbb{B}(m+\ell), \ell\big) \;. \tag{14}$$

(ii) *If $\ell \in \mathbb{P}$, $m \le \ell \le n-1$, then*

$$\mathcal{G}\big(\mathbb{B}(n), m\big)^{\ell} = \mathcal{F}\big(\mathbb{B}(n+m-\ell), m\big) \;, \qquad \mathcal{G}\big(\mathbb{B}(2m), m\big)^{\ell} = \mathcal{F}\big(\mathbb{B}(3m-\ell), m\big) \;. \tag{15}$$

Proposition 2. *The maps*

$$\mathcal{F}\big(\mathbb{B}(n), m\big)^{\ell} \to \mathcal{F}\big(\mathbb{B}(n), n-\ell\big)^{n-m} \;, \tag{16}$$

$$\mathcal{F}\big(\mathbb{B}(2m), m\big)^{\ell} \to \mathcal{F}\big(\mathbb{B}(2m), 2m-\ell\big)^{m} \;, \tag{17}$$

$$\mathcal{G}\big(\mathbb{B}(n), m\big)^{\ell} \to \mathcal{G}\big(\mathbb{B}(n), n-\ell\big)^{n-m} \;, \tag{18}$$

$$\mathcal{G}\big(\mathbb{B}(2m), m\big)^{\ell} \to \mathcal{G}\big(\mathbb{B}(2m), 2m-\ell\big)^{m} \;, \tag{19}$$

defined by Eq. (8), are order–reversing and bijective.

Proof. Note that $\mathcal{F}\big(\mathbb{B}(n), n-\ell\big)^{n-m} = \mathcal{F}\big(\mathbb{B}(n-m+\ell), n-m\big)$, by Remark 1(i), and apply the order–reversing bijection from Eq. (8) to the sequences $\mathcal{F}\big(\mathbb{B}(n-m+\ell), \ell\big) = \mathcal{F}\big(\mathbb{B}(n), m\big)^{\ell}$ and $\mathcal{F}\big(\mathbb{B}(n-m+\ell), n-m\big)$.

We have $\mathcal{G}\big(\mathbb{B}(n), n-\ell\big)^{n-m} = \mathcal{F}\big(\mathbb{B}(n+m-\ell), n-\ell\big)$, by Remark 1(ii), and Eq. (8) provides the order–reversing bijection between the sequences $\mathcal{F}\big(\mathbb{B}(n+m-\ell), m\big) = \mathcal{G}\big(\mathbb{B}(n), m\big)^{\ell}$ and $\mathcal{F}\big(\mathbb{B}(n+m-\ell), n-\ell\big)$. ∎

Example 3. *Suppose $n := 6$, $m := 4$, $\ell' := 3$ and $\ell'' := 5$.*

$$\begin{aligned}
\mathcal{F}_n &= \big(\tfrac{0}{1} < \tfrac{1}{6} < \tfrac{1}{5} < \tfrac{1}{4} < \tfrac{1}{3} < \tfrac{2}{5} < \tfrac{1}{2} < \tfrac{3}{5} < \tfrac{2}{3} < \tfrac{3}{4} < \tfrac{4}{5} < \tfrac{5}{6} < \tfrac{1}{1}\big) \;,\\
\mathcal{F}_n^m &= \big(\tfrac{0}{1} < \tfrac{1}{6} < \tfrac{1}{5} < \tfrac{1}{4} < \tfrac{1}{3} < \tfrac{2}{5} < \tfrac{1}{2} < \tfrac{3}{5} < \tfrac{2}{3} < \tfrac{3}{4} < \tfrac{4}{5} < \quad < \tfrac{1}{1}\big) \;,\\
\mathcal{G}_n^m &= \big(\tfrac{0}{1} < \quad < \quad < \quad < \tfrac{1}{3} < \quad < \tfrac{1}{2} < \tfrac{3}{5} < \tfrac{2}{3} < \tfrac{3}{4} < \tfrac{4}{5} < \tfrac{5}{6} < \tfrac{1}{1}\big) \;,\\
\mathcal{F}\big(\mathbb{B}(n), m\big) &= \big(\tfrac{0}{1} < \quad < \quad < \quad < \tfrac{1}{3} < \quad < \tfrac{1}{2} < \tfrac{3}{5} < \tfrac{2}{3} < \tfrac{3}{4} < \tfrac{4}{5} < \quad < \tfrac{1}{1}\big) \;,\\
\mathcal{F}\big(\mathbb{B}(n), m\big)^{\ell'} &\\
= \mathcal{F}\big(\mathbb{B}(n-m+\ell'), \ell'\big) &= \big(\tfrac{0}{1} < \quad < \quad < \quad < \tfrac{1}{3} < \quad < \tfrac{1}{2} < \tfrac{3}{5} < \tfrac{2}{3} < \tfrac{3}{4} < \quad < \quad < \tfrac{1}{1}\big) \;,\\
\mathcal{F}\big(\mathbb{B}(n), n-\ell'\big)^{n-m} &\\
= \mathcal{F}\big(\mathbb{B}(n-m+\ell'), n-m\big) &= \big(\tfrac{0}{1} < \quad < \quad < \tfrac{1}{4} < \tfrac{1}{3} < \tfrac{2}{5} < \tfrac{1}{2} < \quad < \tfrac{2}{3} < \quad < \quad < \quad < \tfrac{1}{1}\big) \;,\\
\mathcal{G}\big(\mathbb{B}(n), m\big)^{\ell''} &\\
= \mathcal{F}\big(\mathbb{B}(n+m-\ell''), m\big) &= \big(\tfrac{0}{1} < \quad < \quad < \quad < \quad < \quad < \tfrac{1}{2} < \quad < \tfrac{2}{3} < \tfrac{3}{4} < \tfrac{4}{5} < \quad < \tfrac{1}{1}\big) \;,\\
\mathcal{G}\big(\mathbb{B}(n), n-\ell''\big)^{n-m} &\\
= \mathcal{F}\big(\mathbb{B}(n+m-\ell''), n-\ell''\big) &= \big(\tfrac{0}{1} < \quad < \tfrac{1}{5} < \tfrac{1}{4} < \tfrac{1}{3} < \quad < \tfrac{1}{2} < \quad < \quad < \quad < \quad < \quad < \tfrac{1}{1}\big) \;.
\end{aligned}$$

3. The Farey (Sub)sequences $\mathcal{F}_{2^s m}$, $\mathcal{F}\big(\mathbb{B}(2^{s+1}m), 2^s m\big)$, $\mathcal{F}\big(\mathbb{B}(2^{s+2}m), 2^{s+1}m\big)$, and Monotone Maps

In this section, we consider monotone maps between (sub)sequences of the sequences $\mathcal{F}_{2^s m}$, $\mathcal{F}\big(\mathbb{B}(2^{s+1}m), 2^s m\big)$ and $\mathcal{F}\big(\mathbb{B}(2^{s+2}m), 2^{s+1}m\big)$, where $m \in \mathbb{P}$, $s \in \mathbb{N}$. The observations made for these maps can be used as an instructive induction step in an analysis of composite maps between Farey (sub)sequences that are described in Section 4.

Recall that for the sequences $\mathcal{F}\big(\mathbb{B}(n), m\big)$ and, in particular, for the sequences $\mathcal{F}\big(\mathbb{B}(2m), m\big)$, it is convenient to treat the subsequences

$$\mathcal{F}^{\leq\frac{1}{2}}\big(\mathbb{B}(n), m\big) := \big(\tfrac{h}{k} \in \mathcal{F}\big(\mathbb{B}(n), m\big) : \ \tfrac{h}{k} \leq \tfrac{1}{2}\big) \tag{20}$$

and

$$\mathcal{F}^{\geq\frac{1}{2}}\big(\mathbb{B}(n), m\big) := \big(\tfrac{h}{k} \in \mathcal{F}\big(\mathbb{B}(n), m\big) : \ \tfrac{h}{k} \geq \tfrac{1}{2}\big) \tag{21}$$

separately [12, Thm. 5, Lem. 3, Cor. 4]:

- The maps

$$\mathcal{F}_m \to \mathcal{F}^{\leq\frac{1}{2}}\big(\mathbb{B}(2m), m\big) , \qquad \left[\begin{smallmatrix} h \\ k \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} h \\ k \end{smallmatrix}\right] , \tag{22}$$

$$\mathcal{F}_m \to \mathcal{F}^{\geq\frac{1}{2}}\big(\mathbb{B}(2m), m\big) , \qquad \left[\begin{smallmatrix} h \\ k \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} -1 & 1 \\ 0 & 1 \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} h \\ k \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 2 \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} h \\ k \end{smallmatrix}\right] , \tag{23}$$

$$\mathcal{F}^{\leq\frac{1}{2}}\big(\mathbb{B}(2m), m\big) \to \mathcal{F}_m , \qquad \left[\begin{smallmatrix} h \\ k \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} 1 & 0 \\ -1 & 1 \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} h \\ k \end{smallmatrix}\right] , \tag{24}$$

and

$$\mathcal{F}^{\geq\frac{1}{2}}\big(\mathbb{B}(2m), m\big) \to \mathcal{F}_m , \qquad \left[\begin{smallmatrix} h \\ k \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} 2 & -1 \\ 1 & 0 \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} h \\ k \end{smallmatrix}\right] , \tag{25}$$

are order–preserving and bijective. The maps

$$\mathcal{F}_m \to \mathcal{F}^{\leq\frac{1}{2}}\big(\mathbb{B}(2m), m\big) , \qquad \left[\begin{smallmatrix} h \\ k \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} -1 & 1 \\ 0 & 1 \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} h \\ k \end{smallmatrix}\right] = \left[\begin{smallmatrix} -1 & 1 \\ -1 & 2 \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} h \\ k \end{smallmatrix}\right] , \tag{26}$$

$$\mathcal{F}_m \to \mathcal{F}^{\geq\frac{1}{2}}\big(\mathbb{B}(2m), m\big) , \qquad \left[\begin{smallmatrix} h \\ k \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} h \\ k \end{smallmatrix}\right] , \tag{27}$$

$$\mathcal{F}^{\leq\frac{1}{2}}\big(\mathbb{B}(2m), m\big) \to \mathcal{F}_m , \qquad \left[\begin{smallmatrix} h \\ k \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} -2 & 1 \\ -1 & 1 \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} h \\ k \end{smallmatrix}\right] , \tag{28}$$

and

$$\mathcal{F}^{\geq\frac{1}{2}}\big(\mathbb{B}(2m), m\big) \to \mathcal{F}_m , \qquad \left[\begin{smallmatrix} h \\ k \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} -1 & 1 \\ 1 & 0 \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} h \\ k \end{smallmatrix}\right] , \tag{29}$$

are order–reversing and bijective.

- The maps

$$\mathcal{F}^{\leq\frac{1}{2}}\big(\mathbb{B}(2m), m\big) \to \mathcal{F}^{\leq\frac{1}{2}}\big(\mathbb{B}(2m), m\big) , \qquad \left[\begin{smallmatrix} h \\ k \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} -2 & 1 \\ -3 & 2 \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} h \\ k \end{smallmatrix}\right] , \tag{30}$$

$$\mathcal{F}^{\geq\frac{1}{2}}\big(\mathbb{B}(2m), m\big) \to \mathcal{F}^{\geq\frac{1}{2}}\big(\mathbb{B}(2m), m\big) , \qquad \left[\begin{smallmatrix} h \\ k \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} 1 & 0 \\ 3 & -1 \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} h \\ k \end{smallmatrix}\right] , \tag{31}$$

are order–reversing and bijective.

- The maps

$$\mathcal{F}^{\leq\frac{1}{2}}\big(\mathbb{B}(2m),m\big) \to \mathcal{F}^{\geq\frac{1}{2}}\big(\mathbb{B}(2m),m\big)\;, \qquad \left[\begin{smallmatrix} h\\ k \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} -1 & 1\\ -3 & 2 \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} h\\ k \end{smallmatrix}\right]\;, \tag{32}$$

$$\mathcal{F}^{\geq\frac{1}{2}}\big(\mathbb{B}(2m),m\big) \to \mathcal{F}^{\leq\frac{1}{2}}\big(\mathbb{B}(2m),m\big)\;, \qquad \left[\begin{smallmatrix} h\\ k \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} 2 & -1\\ 3 & -1 \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} h\\ k \end{smallmatrix}\right]\;, \tag{33}$$

are order–preserving and bijective.

Remark 4. *Let $m \in \mathbb{P}$, $s \in \mathbb{N}$.* (i) *Eqs.* (22) *and* (23) *imply that the following maps are order–preserving and injective:*

$$\begin{array}{ll} \mathcal{F}_{2^s m} \to \mathcal{F}^{\leq\frac{1}{2}}\big(\mathbb{B}(2^{s+2}m),2^{s+1}m\big)\;, & \left[\begin{smallmatrix} h\\ k \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} 1 & 0\\ 1 & 1 \end{smallmatrix}\right]^2 \cdot \left[\begin{smallmatrix} h\\ k \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1 & 0\\ 2 & 1 \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} h\\ k \end{smallmatrix}\right]\;, \\ \quad \left[\begin{smallmatrix} 0\\ 1 \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} 0\\ 1 \end{smallmatrix}\right]\;, & \left[\begin{smallmatrix} 1\\ 1 \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} 1\\ 3 \end{smallmatrix}\right]\;; \end{array} \tag{34}$$

$$\begin{array}{ll} \mathcal{F}_{2^s m} \to \mathcal{F}^{\leq\frac{1}{2}}\big(\mathbb{B}(2^{s+2}m),2^{s+1}m\big)\;, & \left[\begin{smallmatrix} h\\ k \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} 1 & 0\\ 1 & 1 \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} 0 & 1\\ -1 & 2 \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} h\\ k \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 & 1\\ -1 & 3 \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} h\\ k \end{smallmatrix}\right]\;, \\ \quad \left[\begin{smallmatrix} 0\\ 1 \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} 1\\ 3 \end{smallmatrix}\right]\;, & \left[\begin{smallmatrix} 1\\ 1 \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} 1\\ 2 \end{smallmatrix}\right]\;; \end{array} \tag{35}$$

$$\begin{array}{ll} \mathcal{F}_{2^s m} \to \mathcal{F}^{\geq\frac{1}{2}}\big(\mathbb{B}(2^{s+2}m),2^{s+1}m\big)\;, & \left[\begin{smallmatrix} h\\ k \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} 0 & 1\\ -1 & 2 \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} 1 & 0\\ 1 & 1 \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} h\\ k \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1 & 1\\ 1 & 2 \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} h\\ k \end{smallmatrix}\right]\;, \\ \quad \left[\begin{smallmatrix} 0\\ 1 \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} 1\\ 2 \end{smallmatrix}\right]\;, & \left[\begin{smallmatrix} 1\\ 1 \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} 2\\ 3 \end{smallmatrix}\right]\;; \end{array} \tag{36}$$

$$\begin{array}{ll} \mathcal{F}_{2^s m} \to \mathcal{F}^{\geq\frac{1}{2}}\big(\mathbb{B}(2^{s+2}m),2^{s+1}m\big)\;, & \left[\begin{smallmatrix} h\\ k \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} 0 & 1\\ -1 & 2 \end{smallmatrix}\right]^2 \cdot \left[\begin{smallmatrix} h\\ k \end{smallmatrix}\right] = \left[\begin{smallmatrix} -1 & 2\\ -2 & 3 \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} h\\ k \end{smallmatrix}\right]\;, \\ \quad \left[\begin{smallmatrix} 0\\ 1 \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} 2\\ 3 \end{smallmatrix}\right]\;, & \left[\begin{smallmatrix} 1\\ 1 \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} 1\\ 1 \end{smallmatrix}\right]\;. \end{array} \tag{37}$$

(ii) *As a consequence, the following maps are order–reversing and injective:*

$$\begin{array}{ll} \mathcal{F}_{2^s m} \to \mathcal{F}^{\leq\frac{1}{2}}\big(\mathbb{B}(2^{s+2}m),2^{s+1}m\big)\;, & \left[\begin{smallmatrix} h\\ k \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} 1 & 0\\ 1 & 1 \end{smallmatrix}\right]^2 \cdot \left[\begin{smallmatrix} -1 & 1\\ 0 & 1 \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} h\\ k \end{smallmatrix}\right] = \left[\begin{smallmatrix} -1 & 1\\ -2 & 3 \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} h\\ k \end{smallmatrix}\right]\;, \\ \quad \left[\begin{smallmatrix} 0\\ 1 \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} 1\\ 3 \end{smallmatrix}\right]\;, & \left[\begin{smallmatrix} 1\\ 1 \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} 0\\ 1 \end{smallmatrix}\right]\;; \end{array} \tag{38}$$

$$\begin{array}{ll} \mathcal{F}_{2^s m} \to \mathcal{F}^{\leq\frac{1}{2}}\big(\mathbb{B}(2^{s+2}m),2^{s+1}m\big)\;, & \left[\begin{smallmatrix} h\\ k \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} 1 & 0\\ 1 & 1 \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} 0 & 1\\ -1 & 2 \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} -1 & 1\\ 0 & 1 \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} h\\ k \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 & 1\\ 1 & 2 \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} h\\ k \end{smallmatrix}\right]\;, \\ \quad \left[\begin{smallmatrix} 0\\ 1 \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} 1\\ 2 \end{smallmatrix}\right]\;, & \left[\begin{smallmatrix} 1\\ 1 \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} 1\\ 3 \end{smallmatrix}\right]\;; \end{array} \tag{39}$$

$$\begin{array}{ll} \mathcal{F}_{2^s m} \to \mathcal{F}^{\geq\frac{1}{2}}\big(\mathbb{B}(2^{s+2}m),2^{s+1}m\big)\;, & \left[\begin{smallmatrix} h\\ k \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} 0 & 1\\ -1 & 2 \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} 1 & 0\\ 1 & 1 \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} -1 & 1\\ 0 & 1 \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} h\\ k \end{smallmatrix}\right] = \left[\begin{smallmatrix} -1 & 2\\ -1 & 3 \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} h\\ k \end{smallmatrix}\right]\;, \\ \quad \left[\begin{smallmatrix} 0\\ 1 \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} 2\\ 3 \end{smallmatrix}\right]\;, & \left[\begin{smallmatrix} 1\\ 1 \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} 1\\ 2 \end{smallmatrix}\right]\;; \end{array} \tag{40}$$

$$\begin{array}{ll} \mathcal{F}_{2^s m} \to \mathcal{F}^{\geq\frac{1}{2}}\big(\mathbb{B}(2^{s+2}m),2^{s+1}m\big)\;, & \left[\begin{smallmatrix} h\\ k \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} 0 & 1\\ -1 & 2 \end{smallmatrix}\right]^2 \cdot \left[\begin{smallmatrix} -1 & 1\\ 0 & 1 \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} h\\ k \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1 & 1\\ 2 & 1 \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} h\\ k \end{smallmatrix}\right]\;, \\ \quad \left[\begin{smallmatrix} 0\\ 1 \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} 1\\ 1 \end{smallmatrix}\right]\;, & \left[\begin{smallmatrix} 1\\ 1 \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} 2\\ 3 \end{smallmatrix}\right]\;. \end{array} \tag{41}$$

Lemma 5. *Let $m \in \mathbb{P}$, $s \in \mathbb{N}$.* (i) *The following maps are order–preserving and injective:*

$$\begin{array}{ll} \mathcal{F}^{\leq\frac{1}{2}}\big(\mathbb{B}(2^{s+1}m),2^s m\big) \to \mathcal{F}^{\leq\frac{1}{2}}\big(\mathbb{B}(2^{s+2}m),2^{s+1}m\big)\;, & \left[\begin{smallmatrix} h\\ k \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} 1 & 0\\ 1 & 1 \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} h\\ k \end{smallmatrix}\right]\;, \\ \qquad \left[\begin{smallmatrix} 0\\ 1 \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} 0\\ 1 \end{smallmatrix}\right]\;, & \left[\begin{smallmatrix} 1\\ 2 \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} 1\\ 3 \end{smallmatrix}\right]\;; \end{array} \tag{42}$$

$$\begin{array}{ll} \mathcal{F}^{\geq\frac{1}{2}}\big(\mathbb{B}(2^{s+1}m),2^s m\big) \to \mathcal{F}^{\leq\frac{1}{2}}\big(\mathbb{B}(2^{s+2}m),2^{s+1}m\big)\;, & \left[\begin{smallmatrix} h\\ k \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} 1 & 0\\ 1 & 1 \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} 2 & -1\\ 3 & -1 \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} h\\ k \end{smallmatrix}\right] = \left[\begin{smallmatrix} 2 & -1\\ 5 & -2 \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} h\\ k \end{smallmatrix}\right]\;, \\ \qquad \left[\begin{smallmatrix} 1\\ 2 \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} 0\\ 1 \end{smallmatrix}\right]\;, & \left[\begin{smallmatrix} 1\\ 1 \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} 1\\ 3 \end{smallmatrix}\right]\;; \end{array} \tag{43}$$

$$\mathcal{F}^{\leq\frac{1}{2}}\big(\mathbb{B}(2^{s+1}m),2^sm\big)\to\mathcal{F}^{\leq\frac{1}{2}}\big(\mathbb{B}(2^{s+2}m),2^{s+1}m\big)\ ,\quad \left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}-1&1\\-3&2\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]=\left[\begin{smallmatrix}-1&1\\-4&3\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\ , \tag{44}$$

$$\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}1\\3\end{smallmatrix}\right]\ ,\qquad \left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]\ ;$$

$$\mathcal{F}^{\geq\frac{1}{2}}\big(\mathbb{B}(2^{s+1}m),2^sm\big)\to\mathcal{F}^{\leq\frac{1}{2}}\big(\mathbb{B}(2^{s+2}m),2^{s+1}m\big)\ ,\quad \left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\ , \tag{45}$$

$$\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}1\\3\end{smallmatrix}\right]\ ,\qquad \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]\ ;$$

$$\mathcal{F}^{\leq\frac{1}{2}}\big(\mathbb{B}(2^{s+1}m),2^sm\big)\to\mathcal{F}^{\geq\frac{1}{2}}\big(\mathbb{B}(2^{s+2}m),2^{s+1}m\big)\ ,\quad \left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}0&1\\-1&2\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\ , \tag{46}$$

$$\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]\ ,\qquad \left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}2\\3\end{smallmatrix}\right]\ ;$$

$$\mathcal{F}^{\geq\frac{1}{2}}\big(\mathbb{B}(2^{s+1}m),2^sm\big)\to\mathcal{F}^{\geq\frac{1}{2}}\big(\mathbb{B}(2^{s+2}m),2^{s+1}m\big)\ ,\quad \left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}0&1\\-1&2\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}2&-1\\3&-1\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]=\left[\begin{smallmatrix}3&-1\\4&-1\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\ , \tag{47}$$

$$\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]\ ,\qquad \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}2\\3\end{smallmatrix}\right]\ ;$$

$$\mathcal{F}^{\leq\frac{1}{2}}\big(\mathbb{B}(2^{s+1}m),2^sm\big)\to\mathcal{F}^{\geq\frac{1}{2}}\big(\mathbb{B}(2^{s+2}m),2^{s+1}m\big)\ ,\quad \left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}0&1\\-1&2\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}-1&1\\-3&2\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]=\left[\begin{smallmatrix}-3&2\\-5&3\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\ , \tag{48}$$

$$\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}2\\3\end{smallmatrix}\right]\ ,\qquad \left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]\ ;$$

$$\mathcal{F}^{\geq\frac{1}{2}}\big(\mathbb{B}(2^{s+1}m),2^sm\big)\to\mathcal{F}^{\geq\frac{1}{2}}\big(\mathbb{B}(2^{s+2}m),2^{s+1}m\big)\ ,\quad \left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}0&1\\-1&2\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\ , \tag{49}$$

$$\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}2\\3\end{smallmatrix}\right]\ ,\qquad \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]\ .$$

(ii) *The following maps are order–reversing and injective:*

$$\mathcal{F}^{\leq\frac{1}{2}}\big(\mathbb{B}(2^{s+1}m),2^sm\big)\to\mathcal{F}^{\leq\frac{1}{2}}\big(\mathbb{B}(2^{s+2}m),2^{s+1}m\big)\ ,\quad \left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}-2&1\\-3&2\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]=\left[\begin{smallmatrix}-2&1\\-5&3\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\ , \tag{50}$$

$$\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]\ ,\qquad \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}1\\3\end{smallmatrix}\right]\ ;$$

$$\mathcal{F}^{\geq\frac{1}{2}}\big(\mathbb{B}(2^{s+1}m),2^sm\big)\to\mathcal{F}^{\leq\frac{1}{2}}\big(\mathbb{B}(2^{s+2}m),2^{s+1}m\big)\ ,\quad \left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}2&-1\\3&-1\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}1&0\\3&-1\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]=\left[\begin{smallmatrix}-1&1\\-1&2\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\ , \tag{51}$$

$$\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]\ ,\qquad \left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}1\\3\end{smallmatrix}\right]\ ;$$

$$\mathcal{F}^{\leq\frac{1}{2}}\big(\mathbb{B}(2^{s+1}m),2^sm\big)\to\mathcal{F}^{\leq\frac{1}{2}}\big(\mathbb{B}(2^{s+2}m),2^{s+1}m\big)\ ,\quad \left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}-1&1\\-3&2\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}-2&1\\-3&2\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]=\left[\begin{smallmatrix}-1&1\\-1&2\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\ , \tag{52}$$

$$\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}1\\3\end{smallmatrix}\right]\ ,\qquad \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]\ ;$$

$$\mathcal{F}^{\geq\frac{1}{2}}\big(\mathbb{B}(2^{s+1}m),2^sm\big)\to\mathcal{F}^{\leq\frac{1}{2}}\big(\mathbb{B}(2^{s+2}m),2^{s+1}m\big)\ ,\quad \left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}1&0\\3&-1\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]=\left[\begin{smallmatrix}1&0\\4&-1\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\ , \tag{53}$$

$$\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}1\\3\end{smallmatrix}\right]\ ,\qquad \left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]\ ;$$

$$\mathcal{F}^{\leq\frac{1}{2}}\big(\mathbb{B}(2^{s+1}m),2^sm\big)\to\mathcal{F}^{\geq\frac{1}{2}}\big(\mathbb{B}(2^{s+2}m),2^{s+1}m\big)\ ,\quad \left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}0&1\\-1&2\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}-2&1\\-3&2\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]=\left[\begin{smallmatrix}-3&2\\-4&3\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\ , \tag{54}$$

$$\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]\ ,\qquad \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}2\\3\end{smallmatrix}\right]\ ;$$

$$\mathcal{F}^{\geq\frac{1}{2}}\big(\mathbb{B}(2^{s+1}m),2^sm\big)\to\mathcal{F}^{\geq\frac{1}{2}}\big(\mathbb{B}(2^{s+2}m),2^{s+1}m\big)\ ,\quad \left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}0&1\\-1&2\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}2&-1\\3&-1\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}1&0\\3&-1\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]=\left[\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\ , \tag{55}$$

$$\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]\ ,\qquad \left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}2\\3\end{smallmatrix}\right]\ ;$$

$$\mathcal{F}^{\leq\frac{1}{2}}\big(\mathbb{B}(2^{s+1}m),2^{s}m\big)\to\mathcal{F}^{\geq\frac{1}{2}}\big(\mathbb{B}(2^{s+2}m),2^{s+1}m\big)\ ,\quad \left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}0&1\\-1&2\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}-1&1\\-3&2\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}-2&1\\-3&2\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]=\left[\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\ , \tag{56}$$

$$\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}2\\3\end{smallmatrix}\right]\ ,\qquad \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]\ ;$$

$$\mathcal{F}^{\geq\frac{1}{2}}\big(\mathbb{B}(2^{s+1}m),2^{s}m\big)\to\mathcal{F}^{\geq\frac{1}{2}}\big(\mathbb{B}(2^{s+2}m),2^{s+1}m\big)\ ,\quad \left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}0&1\\-1&2\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}1&0\\3&-1\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]=\left[\begin{smallmatrix}3&-1\\5&-2\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\ , \tag{57}$$

$$\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}2\\3\end{smallmatrix}\right]\ ,\qquad \left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]\ .$$

Proof. (i): (42) ⇐ (22); (43) ⇐ (33) & (22); (44) ⇐ (32) & (22); (45) ⇐ (22); (46) ⇐ (23); (47) ⇐ (33) & (23); (48) ⇐ (32) & (23); (49) ⇐ (23).

(ii): (50) ⇐ (30) & (42); (51) ⇐ (31) & (51); (52) ⇐ (30) & (44); (53) ⇐ (31) & (45); (54) ⇐ (30) & (46); (55) ⇐ (31) & (47); (56) ⇐ (30) & (48); (57) ⇐ (31) & (49). ☐

Let $\mathcal{C}$ be an increasing sequence of irreducible fractions $\frac{h}{k}$ written in vector form $\left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]$. If $\boldsymbol{S}\in\mathrm{SL}(2,\mathbb{Z})$, then we denote by $\boldsymbol{S}\cdot\mathcal{C}$ the sequence of vectors $\boldsymbol{S}\cdot\left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]$:

$$\boldsymbol{S}\cdot\mathcal{C}:=\left(\boldsymbol{S}\cdot\left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]:\ \left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\in\mathcal{C}\right)\ . \tag{58}$$

Proposition 6. *Let $m\in\mathbb{P}$, $s\in\mathbb{N}$.* (i) *The following maps between subsequences of the sequence $\mathcal{F}\big(\mathbb{B}(2^{s+2}m),2^{s+1}m\big)$ are order–preserving and bijective:*

$$\left[\begin{smallmatrix}1&0\\2&1\end{smallmatrix}\right]\cdot\mathcal{F}_{2^sm}\to\left[\begin{smallmatrix}0&1\\-1&3\end{smallmatrix}\right]\cdot\mathcal{F}_{2^sm}\ ,\quad \left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}-2&1\\-7&3\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\ ,\quad \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}1\\3\end{smallmatrix}\right]\ ,\quad \left[\begin{smallmatrix}1\\3\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]\ ; \tag{59}$$

$$\left[\begin{smallmatrix}1&0\\2&1\end{smallmatrix}\right]\cdot\mathcal{F}_{2^sm}\to\left[\begin{smallmatrix}1&1\\1&2\end{smallmatrix}\right]\cdot\mathcal{F}_{2^sm}\ ,\quad \left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}-1&1\\-3&2\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\ ,\quad \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]\ ,\quad \left[\begin{smallmatrix}1\\3\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}2\\3\end{smallmatrix}\right]\ ; \tag{60}$$

$$\left[\begin{smallmatrix}1&0\\2&1\end{smallmatrix}\right]\cdot\mathcal{F}_{2^sm}\to\left[\begin{smallmatrix}-1&2\\-2&3\end{smallmatrix}\right]\cdot\mathcal{F}_{2^sm}\ ,\quad \left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}-5&2\\-8&3\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\ ,\quad \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}2\\3\end{smallmatrix}\right]\ ,\quad \left[\begin{smallmatrix}1\\3\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]\ ; \tag{61}$$

$$\left[\begin{smallmatrix}0&1\\-1&3\end{smallmatrix}\right]\cdot\mathcal{F}_{2^sm}\to\left[\begin{smallmatrix}1&0\\2&1\end{smallmatrix}\right]\cdot\mathcal{F}_{2^sm}\ ,\quad \left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}3&-1\\7&-2\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\ ,\quad \left[\begin{smallmatrix}1\\3\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]\ ,\quad \left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}1\\3\end{smallmatrix}\right]\ ; \tag{62}$$

$$\left[\begin{smallmatrix}0&1\\-1&3\end{smallmatrix}\right]\cdot\mathcal{F}_{2^sm}\to\left[\begin{smallmatrix}1&1\\1&2\end{smallmatrix}\right]\cdot\mathcal{F}_{2^sm}\ ,\quad \left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}4&-1\\5&-1\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\ ,\quad \left[\begin{smallmatrix}1\\3\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]\ ,\quad \left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}2\\3\end{smallmatrix}\right]\ ; \tag{63}$$

$$\left[\begin{smallmatrix}0&1\\-1&3\end{smallmatrix}\right]\cdot\mathcal{F}_{2^sm}\to\left[\begin{smallmatrix}-1&2\\-2&3\end{smallmatrix}\right]\cdot\mathcal{F}_{2^sm}\ ,\quad \left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}-1&1\\-3&2\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\ ,\quad \left[\begin{smallmatrix}1\\3\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}2\\3\end{smallmatrix}\right]\ ,\quad \left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]\ ; \tag{64}$$

$$\left[\begin{smallmatrix}1&1\\1&2\end{smallmatrix}\right]\cdot\mathcal{F}_{2^sm}\to\left[\begin{smallmatrix}1&0\\2&1\end{smallmatrix}\right]\cdot\mathcal{F}_{2^sm}\ ,\quad \left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}2&-1\\3&-1\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\ ,\quad \left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]\ ,\quad \left[\begin{smallmatrix}2\\3\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}1\\3\end{smallmatrix}\right]\ ; \tag{65}$$

$$\left[\begin{smallmatrix}1&1\\1&2\end{smallmatrix}\right]\cdot\mathcal{F}_{2^sm}\to\left[\begin{smallmatrix}0&1\\-1&3\end{smallmatrix}\right]\cdot\mathcal{F}_{2^sm}\ ,\quad \left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}-1&1\\-5&4\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\ ,\quad \left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}1\\3\end{smallmatrix}\right]\ ,\quad \left[\begin{smallmatrix}2\\3\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]\ ; \tag{66}$$

$$\left[\begin{smallmatrix}1&1\\1&2\end{smallmatrix}\right]\cdot\mathcal{F}_{2^sm}\to\left[\begin{smallmatrix}-1&2\\-2&3\end{smallmatrix}\right]\cdot\mathcal{F}_{2^sm}\ ,\quad \left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}-4&3\\-7&5\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\ ,\quad \left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}2\\3\end{smallmatrix}\right]\ ,\quad \left[\begin{smallmatrix}2\\3\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]\ ; \tag{67}$$

$$\left[\begin{smallmatrix}-1&2\\-2&3\end{smallmatrix}\right]\cdot\mathcal{F}_{2^sm}\to\left[\begin{smallmatrix}1&0\\2&1\end{smallmatrix}\right]\cdot\mathcal{F}_{2^sm}\ ,\quad \left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}3&-2\\8&-5\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\ ,\quad \left[\begin{smallmatrix}2\\3\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]\ ,\quad \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}1\\3\end{smallmatrix}\right]\ ; \tag{68}$$

$$\left[\begin{smallmatrix}-1&2\\-2&3\end{smallmatrix}\right]\cdot\mathcal{F}_{2^sm}\to\left[\begin{smallmatrix}0&1\\-1&3\end{smallmatrix}\right]\cdot\mathcal{F}_{2^sm}\ ,\quad \left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}2&-1\\3&-1\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\ ,\quad \left[\begin{smallmatrix}2\\3\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}1\\3\end{smallmatrix}\right]\ ,\quad \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]\ ; \tag{69}$$

$$\left[\begin{smallmatrix}-1&2\\-2&3\end{smallmatrix}\right]\cdot\mathcal{F}_{2^sm}\to\left[\begin{smallmatrix}1&1\\1&2\end{smallmatrix}\right]\cdot\mathcal{F}_{2^sm}\ ,\quad \left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}5&-3\\7&-4\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\ ,\quad \left[\begin{smallmatrix}2\\3\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]\ ,\quad \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}2\\3\end{smallmatrix}\right]\ . \tag{70}$$

(ii)(a) *The following maps between subsequences of the sequence $\mathcal{F}\big(\mathbb{B}(2^{s+2}m),2^{s+1}m\big)$ are* *order–reversing and bijective:*

$$\left[\begin{smallmatrix}1&0\\2&1\end{smallmatrix}\right]\cdot\mathcal{F}_{2^sm}\to\left[\begin{smallmatrix}0&1\\-1&3\end{smallmatrix}\right]\cdot\mathcal{F}_{2^sm}\ ,\quad \left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}-2&1\\-3&2\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\ ,\quad \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]\ ,\quad \left[\begin{smallmatrix}1\\3\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}1\\3\end{smallmatrix}\right]\ ; \tag{71}$$

$$\left[\begin{smallmatrix}1&0\\2&1\end{smallmatrix}\right]\cdot\mathcal{F}_{2^sm}\to\left[\begin{smallmatrix}1&1\\1&2\end{smallmatrix}\right]\cdot\mathcal{F}_{2^sm}\ ,\quad \left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}-5&2\\-7&3\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\ ,\quad \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}2\\3\end{smallmatrix}\right]\ ,\quad \left[\begin{smallmatrix}1\\3\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]\ ; \tag{72}$$

$$\left[\begin{smallmatrix}1&0\\2&1\end{smallmatrix}\right]\cdot\mathcal{F}_{2^sm}\to\left[\begin{smallmatrix}-1&2\\-2&3\end{smallmatrix}\right]\cdot\mathcal{F}_{2^sm}\ ,\quad \left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}-1&1\\0&1\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\ ,\quad \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]\ ,\quad \left[\begin{smallmatrix}1\\3\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}2\\3\end{smallmatrix}\right]\ ; \tag{73}$$

$$\left[\begin{smallmatrix}0&1\\-1&3\end{smallmatrix}\right]\cdot\mathcal{F}_{2^sm}\to\left[\begin{smallmatrix}1&0\\2&1\end{smallmatrix}\right]\cdot\mathcal{F}_{2^sm}\ ,\quad \left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}-2&1\\-3&2\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\ ,\quad \left[\begin{smallmatrix}1\\3\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}1\\3\end{smallmatrix}\right]\ ,\quad \left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]\ ; \tag{74}$$

$$\left[\begin{smallmatrix}0&1\\-1&3\end{smallmatrix}\right]\cdot\mathcal{F}_{2^sm}\to\left[\begin{smallmatrix}1&1\\1&2\end{smallmatrix}\right]\cdot\mathcal{F}_{2^sm}\ ,\quad \left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}-1&1\\0&1\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\ ,\quad \left[\begin{smallmatrix}1\\3\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}2\\3\end{smallmatrix}\right]\ ,\quad \left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]\ ; \tag{75}$$

$$\left[\begin{smallmatrix}0&1\\-1&3\end{smallmatrix}\right]\cdot\mathcal{F}_{2^sm}\to\left[\begin{smallmatrix}-1&2\\-2&3\end{smallmatrix}\right]\cdot\mathcal{F}_{2^sm}\ ,\quad \left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}4&-1\\7&-2\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\ ,\quad \left[\begin{smallmatrix}1\\3\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]\ ,\quad \left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}2\\3\end{smallmatrix}\right]\ ; \tag{76}$$

$$\left[\begin{smallmatrix}1&1\\1&2\end{smallmatrix}\right]\cdot\mathcal{F}_{2^sm}\to\left[\begin{smallmatrix}1&0\\2&1\end{smallmatrix}\right]\cdot\mathcal{F}_{2^sm}\ ,\quad \left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}-3&2\\-7&5\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\ ,\quad \left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}1\\3\end{smallmatrix}\right]\ ,\quad \left[\begin{smallmatrix}2\\3\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]\ ; \tag{77}$$

$$\left[\begin{smallmatrix}1&1\\1&2\end{smallmatrix}\right]\cdot\mathcal{F}_{2^sm}\to\left[\begin{smallmatrix}0&1\\-1&3\end{smallmatrix}\right]\cdot\mathcal{F}_{2^sm}\ ,\quad \left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}-1&1\\0&1\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\ ,\quad \left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]\ ,\quad \left[\begin{smallmatrix}2\\3\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}1\\3\end{smallmatrix}\right]\ ; \tag{78}$$

$$\left[\begin{smallmatrix}1&1\\1&2\end{smallmatrix}\right]\cdot\mathcal{F}_{2^sm}\to\left[\begin{smallmatrix}-1&2\\-2&3\end{smallmatrix}\right]\cdot\mathcal{F}_{2^sm}\ ,\quad \left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}1&0\\3&-1\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\ ,\quad \left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]\ ,\quad \left[\begin{smallmatrix}2\\3\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}2\\3\end{smallmatrix}\right]\ ; \tag{79}$$

$$\left[\begin{smallmatrix}-1&2\\-2&3\end{smallmatrix}\right]\cdot\mathcal{F}_{2^sm}\to\left[\begin{smallmatrix}1&0\\2&1\end{smallmatrix}\right]\cdot\mathcal{F}_{2^sm}\ ,\quad \left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}-1&1\\0&1\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\ ,\quad \left[\begin{smallmatrix}2\\3\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}1\\3\end{smallmatrix}\right]\ ,\quad \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]\ ; \tag{80}$$

$$\left[\begin{smallmatrix}-1&2\\-2&3\end{smallmatrix}\right]\cdot\mathcal{F}_{2^sm}\to\left[\begin{smallmatrix}0&1\\-1&3\end{smallmatrix}\right]\cdot\mathcal{F}_{2^sm}\ ,\quad \left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}2&-1\\7&-4\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\ ,\quad \left[\begin{smallmatrix}2\\3\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]\ ,\quad \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}1\\3\end{smallmatrix}\right]\ ; \tag{81}$$

$$\left[\begin{smallmatrix}-1&2\\-2&3\end{smallmatrix}\right]\cdot\mathcal{F}_{2^sm}\to\left[\begin{smallmatrix}1&1\\1&2\end{smallmatrix}\right]\cdot\mathcal{F}_{2^sm}\ ,\quad \left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}1&0\\3&-1\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\ ,\quad \left[\begin{smallmatrix}2\\3\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}2\\3\end{smallmatrix}\right]\ ,\quad \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]\ . \tag{82}$$

(ii)(b) *The following maps of subsequences of the sequence* $\mathcal{F}\big(\mathbb{B}(2^{s+2}m),2^{s+1}m\big)$ *onto themselves are order–reversing and bijective:*

$$\left[\begin{smallmatrix}1&0\\2&1\end{smallmatrix}\right]\cdot\mathcal{F}_{2^sm}\to\left[\begin{smallmatrix}1&0\\2&1\end{smallmatrix}\right]\cdot\mathcal{F}_{2^sm}\ ,\quad \left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}-3&1\\-8&3\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\ ,\quad \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}1\\3\end{smallmatrix}\right]\ ,\quad \left[\begin{smallmatrix}1\\4\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}1\\4\end{smallmatrix}\right]\ ,\quad \left[\begin{smallmatrix}1\\3\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]\ ; \tag{83}$$

$$\left[\begin{smallmatrix}0&1\\-1&3\end{smallmatrix}\right]\cdot\mathcal{F}_{2^sm}\to\left[\begin{smallmatrix}0&1\\-1&3\end{smallmatrix}\right]\cdot\mathcal{F}_{2^sm}\ ,\quad \left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}1&0\\5&-1\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\ ,\quad \left[\begin{smallmatrix}1\\3\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]\ ,\quad \left[\begin{smallmatrix}2\\5\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}2\\5\end{smallmatrix}\right]\ ,\quad \left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}1\\3\end{smallmatrix}\right]\ ; \tag{84}$$

$$\left[\begin{smallmatrix}1&1\\1&2\end{smallmatrix}\right]\cdot\mathcal{F}_{2^sm}\to\left[\begin{smallmatrix}1&1\\1&2\end{smallmatrix}\right]\cdot\mathcal{F}_{2^sm}\ ,\quad \left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}-4&3\\-5&4\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\ ,\quad \left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}2\\3\end{smallmatrix}\right]\ ,\quad \left[\begin{smallmatrix}3\\5\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}3\\5\end{smallmatrix}\right]\ ,\quad \left[\begin{smallmatrix}2\\3\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]\ ; \tag{85}$$

$$\left[\begin{smallmatrix}-1&2\\-2&3\end{smallmatrix}\right]\cdot\mathcal{F}_{2^sm}\to\left[\begin{smallmatrix}-1&2\\-2&3\end{smallmatrix}\right]\cdot\mathcal{F}_{2^sm}\ ,\quad \left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}5&-3\\8&-5\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\ ,\quad \left[\begin{smallmatrix}2\\3\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]\ ,\quad \left[\begin{smallmatrix}3\\4\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}3\\4\end{smallmatrix}\right]\ ,\quad \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}2\\3\end{smallmatrix}\right]\ . \tag{86}$$

Proof. (i):

(59) : $\left[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right]^2\cdot\mathcal{F}_{2^sm}\to\left[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}0&1\\-1&2\end{smallmatrix}\right]\cdot\mathcal{F}_{2^sm}\ ,\quad \left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}0&1\\-1&2\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right]^{-2}\cdot\left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\ ;$

(60) : $\left[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right]^2\cdot\mathcal{F}_{2^sm}\to\left[\begin{smallmatrix}0&1\\-1&2\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right]\cdot\mathcal{F}_{2^sm}\ ,\quad \left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}0&1\\-1&2\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right]^{-2}\cdot\left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\ ;$

(61) : $\left[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right]^2\cdot\mathcal{F}_{2^sm}\to\left[\begin{smallmatrix}0&1\\-1&2\end{smallmatrix}\right]^2\cdot\mathcal{F}_{2^sm}\ ,\quad \left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}0&1\\-1&2\end{smallmatrix}\right]^2\cdot\left[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right]^{-2}\cdot\left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\ ;$

(62) : $\left[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}0&1\\-1&2\end{smallmatrix}\right]\cdot\mathcal{F}_{2^sm}\to\left[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right]^2\cdot\mathcal{F}_{2^sm}\ ,\quad \left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right]^2\cdot\left[\begin{smallmatrix}0&1\\-1&2\end{smallmatrix}\right]^{-1}\cdot\left[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right]^{-1}\cdot\left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\ ;$

(63) : $\left[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}0&1\\-1&2\end{smallmatrix}\right]\cdot\mathcal{F}_{2^sm}\to\left[\begin{smallmatrix}0&1\\-1&2\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right]\cdot\mathcal{F}_{2^sm}\ ,\quad \left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}0&1\\-1&2\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}0&1\\-1&2\end{smallmatrix}\right]^{-1}\cdot\left[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right]^{-1}\cdot\left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\ ;$

(64) : $\left[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}0&1\\-1&2\end{smallmatrix}\right]\cdot\mathcal{F}_{2^sm}\to\left[\begin{smallmatrix}0&1\\-1&2\end{smallmatrix}\right]^2\cdot\mathcal{F}_{2^sm}\ ,\quad \left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}0&1\\-1&2\end{smallmatrix}\right]^2\cdot\left[\begin{smallmatrix}0&1\\-1&2\end{smallmatrix}\right]^{-1}\cdot\left[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right]^{-1}\cdot\left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\ ;$

(65) : $\left[\begin{smallmatrix}0&1\\-1&2\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right]\cdot\mathcal{F}_{2^sm}\to\left[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right]^2\cdot\mathcal{F}_{2^sm}\ ,\quad \left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right]^2\cdot\left[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right]^{-1}\cdot\left[\begin{smallmatrix}0&1\\-1&2\end{smallmatrix}\right]^{-1}\cdot\left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\ ;$

(66) : $\left[\begin{smallmatrix}0&1\\-1&2\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right]\cdot\mathcal{F}_{2^sm}\to\left[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}0&1\\-1&2\end{smallmatrix}\right]\cdot\mathcal{F}_{2^sm}\ ,\quad \left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}0&1\\-1&2\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right]^{-1}\cdot\left[\begin{smallmatrix}0&1\\-1&2\end{smallmatrix}\right]^{-1}\cdot\left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\ ;$

(67) : $\left[\begin{smallmatrix}0&1\\-1&2\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right]\cdot\mathcal{F}_{2^sm}\to\left[\begin{smallmatrix}0&1\\-1&2\end{smallmatrix}\right]^2\cdot\mathcal{F}_{2^sm}\ ,\quad \left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}0&1\\-1&2\end{smallmatrix}\right]^2\cdot\left[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right]^{-1}\cdot\left[\begin{smallmatrix}0&1\\-1&2\end{smallmatrix}\right]^{-1}\cdot\left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\ ;$

(68) : $\left[\begin{smallmatrix}0&1\\-1&2\end{smallmatrix}\right]^2\cdot\mathcal{F}_{2^sm}\to\left[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right]^2\cdot\mathcal{F}_{2^sm}\ ,\quad \left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right]^2\cdot\left[\begin{smallmatrix}0&1\\-1&2\end{smallmatrix}\right]^{-2}\cdot\left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\ ;$

(69) : $\left[\begin{smallmatrix}0&1\\-1&2\end{smallmatrix}\right]^2\cdot\mathcal{F}_{2^sm}\to\left[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}0&1\\-1&2\end{smallmatrix}\right]\cdot\mathcal{F}_{2^sm}\ ,\quad \left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}0&1\\-1&2\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}0&1\\-1&2\end{smallmatrix}\right]^{-2}\cdot\left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\ ;$

(70) : $\left[\begin{smallmatrix}0&1\\-1&2\end{smallmatrix}\right]^2\cdot\mathcal{F}_{2^sm}\to\left[\begin{smallmatrix}0&1\\-1&2\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right]\cdot\mathcal{F}_{2^sm}\ ,\quad \left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}0&1\\-1&2\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}0&1\\-1&2\end{smallmatrix}\right]^{-2}\cdot\left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\ .$

(ii)(a):

$$(71):\quad \begin{bmatrix}1&0\\1&1\end{bmatrix}^2\cdot\mathcal{F}_{2^s m}\to\begin{bmatrix}1&0\\1&1\end{bmatrix}\cdot\begin{bmatrix}0&1\\-1&2\end{bmatrix}\cdot\mathcal{F}_{2^s m}\ ,\quad \begin{bmatrix}h\\k\end{bmatrix}\mapsto\begin{bmatrix}1&0\\1&1\end{bmatrix}\cdot\begin{bmatrix}0&1\\-1&2\end{bmatrix}\cdot\begin{bmatrix}-1&1\\0&1\end{bmatrix}\cdot\begin{bmatrix}1&0\\1&1\end{bmatrix}^{-2}\cdot\begin{bmatrix}h\\k\end{bmatrix}\ ;$$

$$(72):\quad \begin{bmatrix}1&0\\1&1\end{bmatrix}^2\cdot\mathcal{F}_{2^s m}\to\begin{bmatrix}0&1\\-1&2\end{bmatrix}\cdot\begin{bmatrix}1&0\\1&1\end{bmatrix}\cdot\mathcal{F}_{2^s m}\ ,\quad \begin{bmatrix}h\\k\end{bmatrix}\mapsto\begin{bmatrix}0&1\\-1&2\end{bmatrix}\cdot\begin{bmatrix}1&0\\1&1\end{bmatrix}\cdot\begin{bmatrix}-1&1\\0&1\end{bmatrix}\cdot\begin{bmatrix}1&0\\1&1\end{bmatrix}^{-2}\cdot\begin{bmatrix}h\\k\end{bmatrix}\ ;$$

$$(73):\quad \begin{bmatrix}1&0\\1&1\end{bmatrix}^2\cdot\mathcal{F}_{2^s m}\to\begin{bmatrix}0&1\\-1&2\end{bmatrix}^2\cdot\mathcal{F}_{2^s m}\ ,\quad \begin{bmatrix}h\\k\end{bmatrix}\mapsto\begin{bmatrix}0&1\\-1&2\end{bmatrix}^2\cdot\begin{bmatrix}-1&1\\0&1\end{bmatrix}\cdot\begin{bmatrix}1&0\\1&1\end{bmatrix}^{-2}\cdot\begin{bmatrix}h\\k\end{bmatrix}\ ;$$

$$(74):\quad \begin{bmatrix}1&0\\1&1\end{bmatrix}\cdot\begin{bmatrix}0&1\\-1&2\end{bmatrix}\cdot\mathcal{F}_{2^s m}\to\begin{bmatrix}1&0\\1&1\end{bmatrix}^2\cdot\mathcal{F}_{2^s m}\ ,\quad \begin{bmatrix}h\\k\end{bmatrix}\mapsto\begin{bmatrix}1&0\\1&1\end{bmatrix}^2\cdot\begin{bmatrix}-1&1\\0&1\end{bmatrix}\cdot\begin{bmatrix}0&1\\-1&2\end{bmatrix}^{-1}\cdot\begin{bmatrix}1&0\\1&1\end{bmatrix}^{-1}\cdot\begin{bmatrix}h\\k\end{bmatrix}\ ;$$

$$(75):\quad \begin{bmatrix}1&0\\1&1\end{bmatrix}\cdot\begin{bmatrix}0&1\\-1&2\end{bmatrix}\cdot\mathcal{F}_{2^s m}\to\begin{bmatrix}0&1\\-1&2\end{bmatrix}\cdot\begin{bmatrix}1&0\\1&1\end{bmatrix}\cdot\mathcal{F}_{2^s m}\ ,\quad \begin{bmatrix}h\\k\end{bmatrix}\mapsto\begin{bmatrix}0&1\\-1&2\end{bmatrix}\cdot\begin{bmatrix}1&0\\1&1\end{bmatrix}\cdot\begin{bmatrix}-1&1\\0&1\end{bmatrix}\cdot\begin{bmatrix}0&1\\-1&2\end{bmatrix}^{-1}\cdot\begin{bmatrix}1&0\\1&1\end{bmatrix}^{-1}\cdot\begin{bmatrix}h\\k\end{bmatrix}\ ;$$

$$(76):\quad \begin{bmatrix}1&0\\1&1\end{bmatrix}\cdot\begin{bmatrix}0&1\\-1&2\end{bmatrix}\cdot\mathcal{F}_{2^s m}\to\begin{bmatrix}0&1\\-1&2\end{bmatrix}^2\cdot\mathcal{F}_{2^s m}\ ,\quad \begin{bmatrix}h\\k\end{bmatrix}\mapsto\begin{bmatrix}0&1\\-1&2\end{bmatrix}^2\cdot\begin{bmatrix}-1&1\\0&1\end{bmatrix}\cdot\begin{bmatrix}0&1\\-1&2\end{bmatrix}^{-1}\cdot\begin{bmatrix}1&0\\1&1\end{bmatrix}^{-1}\cdot\begin{bmatrix}h\\k\end{bmatrix}\ ;$$

$$(77):\quad \begin{bmatrix}0&1\\-1&2\end{bmatrix}\cdot\begin{bmatrix}1&0\\1&1\end{bmatrix}\cdot\mathcal{F}_{2^s m}\to\begin{bmatrix}1&0\\1&1\end{bmatrix}^2\cdot\mathcal{F}_{2^s m}\ ,\quad \begin{bmatrix}h\\k\end{bmatrix}\mapsto\begin{bmatrix}1&0\\1&1\end{bmatrix}^2\cdot\begin{bmatrix}-1&1\\0&1\end{bmatrix}\cdot\begin{bmatrix}1&0\\1&1\end{bmatrix}^{-1}\cdot\begin{bmatrix}0&1\\-1&2\end{bmatrix}^{-1}\cdot\begin{bmatrix}h\\k\end{bmatrix}\ ;$$

$$(78):\quad \begin{bmatrix}0&1\\-1&2\end{bmatrix}\cdot\begin{bmatrix}1&0\\1&1\end{bmatrix}\cdot\mathcal{F}_{2^s m}\to\begin{bmatrix}1&0\\1&1\end{bmatrix}\cdot\begin{bmatrix}0&1\\-1&2\end{bmatrix}\cdot\mathcal{F}_{2^s m}\ ,\quad \begin{bmatrix}h\\k\end{bmatrix}\mapsto\begin{bmatrix}1&0\\1&1\end{bmatrix}\cdot\begin{bmatrix}0&1\\-1&2\end{bmatrix}\cdot\begin{bmatrix}-1&1\\0&1\end{bmatrix}\cdot\begin{bmatrix}1&0\\1&1\end{bmatrix}^{-1}\cdot\begin{bmatrix}0&1\\-1&2\end{bmatrix}^{-1}\cdot\begin{bmatrix}h\\k\end{bmatrix}\ ;$$

$$(79):\quad \begin{bmatrix}0&1\\-1&2\end{bmatrix}\cdot\begin{bmatrix}1&0\\1&1\end{bmatrix}\cdot\mathcal{F}_{2^s m}\to\begin{bmatrix}0&1\\-1&2\end{bmatrix}^2\cdot\mathcal{F}_{2^s m}\ ,\quad \begin{bmatrix}h\\k\end{bmatrix}\mapsto\begin{bmatrix}0&1\\-1&2\end{bmatrix}^2\cdot\begin{bmatrix}-1&1\\0&1\end{bmatrix}\cdot\begin{bmatrix}1&0\\1&1\end{bmatrix}^{-1}\cdot\begin{bmatrix}0&1\\-1&2\end{bmatrix}^{-1}\cdot\begin{bmatrix}h\\k\end{bmatrix}\ ;$$

$$(80):\quad \begin{bmatrix}0&1\\-1&2\end{bmatrix}^2\cdot\mathcal{F}_{2^s m}\to\begin{bmatrix}1&0\\1&1\end{bmatrix}^2\cdot\mathcal{F}_{2^s m}\ ,\quad \begin{bmatrix}h\\k\end{bmatrix}\mapsto\begin{bmatrix}1&0\\1&1\end{bmatrix}^2\cdot\begin{bmatrix}-1&1\\0&1\end{bmatrix}\cdot\begin{bmatrix}0&1\\-1&2\end{bmatrix}^{-2}\cdot\begin{bmatrix}h\\k\end{bmatrix}\ ;$$

$$(81):\quad \begin{bmatrix}0&1\\-1&2\end{bmatrix}^2\cdot\mathcal{F}_{2^s m}\to\begin{bmatrix}1&0\\1&1\end{bmatrix}\cdot\begin{bmatrix}0&1\\-1&2\end{bmatrix}\cdot\mathcal{F}_{2^s m}\ ,\quad \begin{bmatrix}h\\k\end{bmatrix}\mapsto\begin{bmatrix}1&0\\1&1\end{bmatrix}\cdot\begin{bmatrix}0&1\\-1&2\end{bmatrix}\cdot\begin{bmatrix}-1&1\\0&1\end{bmatrix}\cdot\begin{bmatrix}0&1\\-1&2\end{bmatrix}^{-2}\cdot\begin{bmatrix}h\\k\end{bmatrix}\ ;$$

$$(82):\quad \begin{bmatrix}0&1\\-1&2\end{bmatrix}^2\cdot\mathcal{F}_{2^s m}\to\begin{bmatrix}0&1\\-1&2\end{bmatrix}\cdot\begin{bmatrix}1&0\\1&1\end{bmatrix}\cdot\mathcal{F}_{2^s m}\ ,\quad \begin{bmatrix}h\\k\end{bmatrix}\mapsto\begin{bmatrix}0&1\\-1&2\end{bmatrix}\cdot\begin{bmatrix}1&0\\1&1\end{bmatrix}\cdot\begin{bmatrix}-1&1\\0&1\end{bmatrix}\cdot\begin{bmatrix}0&1\\-1&2\end{bmatrix}^{-2}\cdot\begin{bmatrix}h\\k\end{bmatrix}\ .$$

(ii)(b):

$$(83):\quad \begin{bmatrix}1&0\\1&1\end{bmatrix}^2\cdot\mathcal{F}_{2^s m}\to\begin{bmatrix}1&0\\1&1\end{bmatrix}^2\cdot\mathcal{F}_{2^s m}\ ,\quad \begin{bmatrix}h\\k\end{bmatrix}\mapsto\begin{bmatrix}1&0\\1&1\end{bmatrix}^2\cdot\begin{bmatrix}-1&1\\0&1\end{bmatrix}\cdot\begin{bmatrix}1&0\\1&1\end{bmatrix}^{-2}\cdot\begin{bmatrix}h\\k\end{bmatrix}\ ;$$

$$(84):\quad \begin{bmatrix}1&0\\1&1\end{bmatrix}\cdot\begin{bmatrix}0&1\\-1&2\end{bmatrix}\cdot\mathcal{F}_{2^s m}\to\begin{bmatrix}1&0\\1&1\end{bmatrix}\cdot\begin{bmatrix}0&1\\-1&2\end{bmatrix}\cdot\mathcal{F}_{2^s m}\ ,\quad \begin{bmatrix}h\\k\end{bmatrix}\mapsto\begin{bmatrix}1&0\\1&1\end{bmatrix}\cdot\begin{bmatrix}0&1\\-1&2\end{bmatrix}\begin{bmatrix}-1&1\\0&1\end{bmatrix}\cdot\begin{bmatrix}0&1\\-1&2\end{bmatrix}^{-1}\cdot\begin{bmatrix}1&0\\1&1\end{bmatrix}^{-1}\cdot\begin{bmatrix}h\\k\end{bmatrix}\ ;$$

$$(85):\quad \begin{bmatrix}0&1\\-1&2\end{bmatrix}\cdot\begin{bmatrix}1&0\\1&1\end{bmatrix}\cdot\mathcal{F}_{2^s m}\to\begin{bmatrix}0&1\\-1&2\end{bmatrix}\cdot\begin{bmatrix}1&0\\1&1\end{bmatrix}\cdot\mathcal{F}_{2^s m}\ ,\quad \begin{bmatrix}h\\k\end{bmatrix}\mapsto\begin{bmatrix}0&1\\-1&2\end{bmatrix}\cdot\begin{bmatrix}1&0\\1&1\end{bmatrix}\cdot\begin{bmatrix}-1&1\\0&1\end{bmatrix}\cdot\begin{bmatrix}1&0\\1&1\end{bmatrix}^{-1}\cdot\begin{bmatrix}0&1\\-1&2\end{bmatrix}^{-1}\cdot\begin{bmatrix}h\\k\end{bmatrix}\ ;$$

$$(86):\quad \begin{bmatrix}0&1\\-1&2\end{bmatrix}^2\cdot\mathcal{F}_{2^s m}\to\begin{bmatrix}0&1\\-1&2\end{bmatrix}^2\cdot\mathcal{F}_{2^s m}\ ,\quad \begin{bmatrix}h\\k\end{bmatrix}\mapsto\begin{bmatrix}0&1\\-1&2\end{bmatrix}^2\begin{bmatrix}-1&1\\0&1\end{bmatrix}\cdot\begin{bmatrix}0&1\\-1&2\end{bmatrix}^{-2}\cdot\begin{bmatrix}h\\k\end{bmatrix}\ .$$

□

Example 7. *Suppose $m := 1$, $s := 1$, and consider the sequences*

$$\begin{aligned}
\mathcal{F}_{2^{s+2}m} &= \left(\tfrac{0}{1} < \tfrac{1}{8} < \tfrac{1}{7} < \tfrac{1}{6} < \tfrac{1}{5} < \tfrac{1}{4} < \tfrac{2}{7} < \tfrac{1}{3} < \tfrac{3}{8} < \tfrac{2}{5} < \tfrac{3}{7} < \tfrac{1}{2} < \tfrac{4}{7} < \tfrac{3}{5} < \tfrac{5}{8} < \tfrac{2}{3} < \tfrac{5}{7} < \tfrac{3}{4} < \tfrac{4}{5} < \tfrac{5}{6} < \tfrac{6}{7} < \tfrac{7}{8} < \tfrac{1}{1}\right),\\
\mathcal{F}\big(\mathbb{B}(2^{s+2}m),2^{s+1}m\big) &= \left(\tfrac{0}{1} < \ < \ < \ < \tfrac{1}{5} < \tfrac{1}{4} < \ < \tfrac{1}{3} < \ < \tfrac{2}{5} < \tfrac{3}{7} < \tfrac{1}{2} < \tfrac{4}{7} < \tfrac{3}{5} < \ < \tfrac{2}{3} < \ < \tfrac{3}{4} < \tfrac{4}{5} < \ < \ < \ < \tfrac{1}{1}\right),\\
\mathcal{F}_{2^{s+1}m} &= \left(\tfrac{0}{1} < \ < \ < \ < \ < \tfrac{1}{4} < \ < \tfrac{1}{3} < \ < \ < \ < \tfrac{1}{2} < \ < \ < \ < \tfrac{2}{3} < \ < \tfrac{3}{4} < \ < \ < \ < \ < \tfrac{1}{1}\right),\\
\mathcal{F}\big(\mathbb{B}(2^{s+1}m),2^{s}m\big) &= \left(\tfrac{0}{1} < \ < \ < \ < \ < \ < \ < \tfrac{1}{3} < \ < \ < \ < \tfrac{1}{2} < \ < \ < \ < \tfrac{2}{3} < \ < \ < \ < \ < \ < \ < \tfrac{1}{1}\right),\\
&\mathcal{F}_{2^s m}\\
= \mathcal{F}\big(\mathbb{B}(2^{s}m),2^{s-1}m\big) &= \left(\tfrac{0}{1} < \ < \ < \ < \ < \ < \ < \ < \ < \ < \ < \tfrac{1}{2} < \ < \ < \ < \ < \ < \ < \ < \ < \ < \ < \tfrac{1}{1}\right),\\
\mathcal{F}_{2^{s-1}m} &= \left(\tfrac{0}{1} < \ < \ < \ < \ < \ < \ < \ < \ < \ < \ < \ < \ < \ < \ < \ < \ < \ < \ < \ < \ < \ < \tfrac{1}{1}\right).
\end{aligned}$$

It follows from Eq. (59) *that the map*

$$\mathcal{F}_{2^{s+2}m}\supset\begin{bmatrix}1&0\\2&1\end{bmatrix}\cdot\mathcal{F}_{2^s m}=\begin{bmatrix}1&0\\2&1\end{bmatrix}\cdot\left(\begin{bmatrix}0\\1\end{bmatrix},\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}1\\1\end{bmatrix}\right)=\left(\begin{bmatrix}0\\1\end{bmatrix},\begin{bmatrix}1\\4\end{bmatrix},\begin{bmatrix}1\\3\end{bmatrix}\right)$$
$$\to\begin{bmatrix}0&1\\-1&3\end{bmatrix}\cdot\mathcal{F}_{2^s m}=\begin{bmatrix}0&1\\-1&3\end{bmatrix}\cdot\left(\begin{bmatrix}0\\1\end{bmatrix},\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}1\\1\end{bmatrix}\right)=\left(\begin{bmatrix}1\\3\end{bmatrix},\begin{bmatrix}2\\5\end{bmatrix},\begin{bmatrix}1\\2\end{bmatrix}\right)\subset\mathcal{F}_{2^{s+2}m}\ ,\quad \begin{bmatrix}h\\k\end{bmatrix}\mapsto\begin{bmatrix}-2&1\\-7&3\end{bmatrix}\cdot\begin{bmatrix}h\\k\end{bmatrix}\ ,$$

is order–preserving and bijective. Eqs. (71) *and* (84) *imply that the maps*

$$\mathcal{F}_{2^{s+2}m}\supset\left(\begin{bmatrix}0\\1\end{bmatrix},\begin{bmatrix}1\\4\end{bmatrix},\begin{bmatrix}1\\3\end{bmatrix}\right)\to\left(\begin{bmatrix}1\\3\end{bmatrix},\begin{bmatrix}2\\5\end{bmatrix},\begin{bmatrix}1\\2\end{bmatrix}\right)\subset\mathcal{F}_{2^{s+2}m}\ ,\qquad \begin{bmatrix}h\\k\end{bmatrix}\mapsto\begin{bmatrix}-2&1\\-3&2\end{bmatrix}\cdot\begin{bmatrix}h\\k\end{bmatrix}\ ,$$

and

$$\mathcal{F}_{2^{s+2}m} \supset \left(\left[\begin{smallmatrix}1\\3\end{smallmatrix}\right], \left[\begin{smallmatrix}2\\5\end{smallmatrix}\right], \left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]\right) \to \left(\left[\begin{smallmatrix}1\\3\end{smallmatrix}\right], \left[\begin{smallmatrix}2\\5\end{smallmatrix}\right], \left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]\right), \qquad \left[\begin{smallmatrix}h\\k\end{smallmatrix}\right] \mapsto \underset{\substack{\uparrow\\ \text{involutory}}}{\left[\begin{smallmatrix}1 & 0\\5 & -1\end{smallmatrix}\right]} \cdot \left[\begin{smallmatrix}h\\k\end{smallmatrix}\right], \qquad \left[\begin{smallmatrix}2\\5\end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix}2\\5\end{smallmatrix}\right],$$

are order–reversing and bijective.

4. Farey (Sub)sequences and Monotone Maps

The following statement, that uses the approach sketched in Remark 4(i), is based on the properties of maps (22) and (23) recalled in Section 3.

Lemma 8. *Let m and n be positive integers, $n \geq 2m$. Suppose $s := \lfloor \log_2(n/m) \rfloor$.*

For any ordered collection

$$(\boldsymbol{M}_1, \ldots, \boldsymbol{M}_s) \in \left\{\left[\begin{smallmatrix}1 & 0\\1 & 1\end{smallmatrix}\right], \left[\begin{smallmatrix}0 & 1\\-1 & 2\end{smallmatrix}\right]\right\}^s , \tag{87}$$

of length s, whose entries are elements $\left[\begin{smallmatrix}1 & 0\\1 & 1\end{smallmatrix}\right]$ or $\left[\begin{smallmatrix}0 & 1\\-1 & 2\end{smallmatrix}\right]$ of $\mathrm{SL}(2,\mathbb{Z})$, the map

$$\begin{aligned} &\mathcal{F}_m \to \mathcal{F}_n , \\ &\left[\begin{smallmatrix}h\\k\end{smallmatrix}\right] \mapsto \prod_{i=1}^{s} \boldsymbol{M}_{s-i+1} \cdot \left[\begin{smallmatrix}h\\k\end{smallmatrix}\right] \quad \begin{cases} \leq \frac{1}{2} , & \text{if } \boldsymbol{M}_s = \left[\begin{smallmatrix}1 & 0\\1 & 1\end{smallmatrix}\right] , \\ \geq \frac{1}{2} , & \text{if } \boldsymbol{M}_s = \left[\begin{smallmatrix}0 & 1\\-1 & 2\end{smallmatrix}\right] , \end{cases} \end{aligned} \tag{88}$$

is order–preserving and injective.

Since the matrix product from Eq. (88) is an element of $\mathrm{SL}(2,\mathbb{Z})$, the image of any Farey sequence retains the two generic properties of such sequences:

Remark 9. *If $\frac{h_j}{k_j} < \frac{h_{j+1}}{k_{j+1}} < \frac{h_{j+2}}{k_{j+2}}$ are three consecutive fractions in the subsequence $\prod_{i=1}^{s} \boldsymbol{M}_{s-i+1} \cdot \mathcal{F}_m$ of the Farey sequence $\mathcal{F}_n$, considered in Lemma 8, then $k_j h_{j+1} - h_j k_{j+1} = 1$, and $\frac{h_{j+1}}{k_{j+1}} = \frac{h_j + h_{j+2}}{\gcd(h_j+h_{j+2}, k_j+k_{j+2})} \Big/ \frac{k_j + k_{j+2}}{\gcd(h_j+h_{j+2}, k_j+k_{j+2})}$.*

We conclude this section with a straightforward extension of Proposition 6, see also Example 7.

Theorem 10. *Let m and n be positive integers, $n \geq 2m$. Suppose $s := \lfloor \log_2(n/m) \rfloor$. Given two ordered collections of matrices*

$$(\boldsymbol{M}_1, \ldots, \boldsymbol{M}_s),\ (\boldsymbol{N}_1, \ldots, \boldsymbol{N}_s) \in \left\{\left[\begin{smallmatrix}1 & 0\\1 & 1\end{smallmatrix}\right], \left[\begin{smallmatrix}0 & 1\\-1 & 2\end{smallmatrix}\right]\right\}^s , \tag{89}$$

suppose

$$\mathbf{M} := \prod_{i=1}^{s} \boldsymbol{M}_{s-i+1} , \quad \mathbf{N} := \prod_{i=1}^{s} \boldsymbol{N}_{s-i+1} . \tag{90}$$

The map

$$\begin{aligned} \mathcal{F}_n \supset \mathbf{M}\cdot\mathcal{F}_m &\to \mathbf{N}\cdot\mathcal{F}_m \subset \mathcal{F}_n\ , \\ \left[\begin{smallmatrix} h\\ k\end{smallmatrix}\right] &\mapsto \mathbf{N}\cdot\mathbf{M}^{-1}\cdot\left[\begin{smallmatrix} h\\ k\end{smallmatrix}\right]\ , \end{aligned} \tag{91}$$

between subsequences $\mathbf{M}\cdot\mathcal{F}_m$ *and* $\mathbf{N}\cdot\mathcal{F}_m$ *of the Farey sequence* $\mathcal{F}_n$*, is order–preserving and bijective.*

The map

$$\begin{aligned} \mathcal{F}_n \supset \mathbf{M}\cdot\mathcal{F}_m &\to \mathbf{N}\cdot\mathcal{F}_m \subset \mathcal{F}_n\ , \\ \left[\begin{smallmatrix} h\\ k\end{smallmatrix}\right] &\mapsto \mathbf{N}\cdot\left[\begin{smallmatrix} -1 & 1\\ 0 & 1\end{smallmatrix}\right]\cdot\mathbf{M}^{-1}\cdot\left[\begin{smallmatrix} h\\ k\end{smallmatrix}\right]\ , \end{aligned} \tag{92}$$

is order–reversing and bijective; in particular, the map

$$\begin{aligned} \mathcal{F}_n \supset \mathbf{M}\cdot\mathcal{F}_m &\to \mathbf{M}\cdot\mathcal{F}_m\ , \\ \left[\begin{smallmatrix} h\\ k\end{smallmatrix}\right] &\mapsto \underbrace{\mathbf{M}\cdot\left[\begin{smallmatrix} -1 & 1\\ 0 & 1\end{smallmatrix}\right]\cdot\mathbf{M}^{-1}}_{\text{involutory}}\cdot\left[\begin{smallmatrix} h\\ k\end{smallmatrix}\right]\ , \end{aligned} \tag{93}$$

with the fixed point $\mathbf{M}\cdot\left[\begin{smallmatrix} 1\\ 2\end{smallmatrix}\right]$ *in the case of* $m > 1$*, is order–reversing and bijective.*

5. Appendix: Miscellany

5.1. Farey (Sub)sequences: List of Notation

Sequence	*Definition / Description*
$\mathcal{F}_n$	$\{\frac{0}{1}\}\dot\cup\bigl(\frac{h}{k}\in\mathbb{Q}:\ 1\le h<k\le n,\ \gcd(k,h)=1\bigr)\dot\cup\{\frac{1}{1}\}$
$\mathcal{F}_n^m$, $m\ge 1$	$\bigl(\frac{h}{k}\in\mathcal{F}_n:\ h\le m\bigr)$
$\mathcal{G}_n^m$, $m\le n-1$	$\bigl(\frac{h}{k}\in\mathcal{F}_n:\ m+k-n\le h\bigr)$
$\mathcal{F}(\mathbb{B}(n),m)$	$\mathcal{F}_n^m\cap\mathcal{G}_n^m=\bigl(\frac{h}{k}\in\mathcal{F}_n:\ m+k-n\le h\le m\bigr)$
$\mathcal{F}\bigl(\mathbb{B}(2m),m\bigr)$	$\bigl(\frac{h}{k}\in\mathcal{F}_{2m}:\ k-m\le h\le m\bigr)$
$\mathcal{F}^{\le\frac{1}{2}}(\mathbb{B}(n),m)$	$\bigl(\frac{h}{k}\in\mathcal{F}(\mathbb{B}(n),m):\ \frac{h}{k}\le\frac{1}{2}\bigr)$
$\mathcal{F}^{\ge\frac{1}{2}}(\mathbb{B}(n),m)$	$\bigl(\frac{h}{k}\in\mathcal{F}(\mathbb{B}(n),m):\ \frac{h}{k}\ge\frac{1}{2}\bigr)$
$\mathcal{F}\bigl(\mathbb{B}(n),m\bigr)^\ell$, $1\le\ell\le m$	$\bigl(\frac{h}{k}\in\mathcal{F}\bigl(\mathbb{B}(n),m\bigr):\ h\le\ell\bigr)=\mathcal{F}\bigl(\mathbb{B}(n-m+\ell),\ell\bigr)$
$\mathcal{F}\bigl(\mathbb{B}(2m),m\bigr)^\ell$	$\mathcal{F}\bigl(\mathbb{B}(m+\ell),\ell\bigr)$
$\mathcal{G}\bigl(\mathbb{B}(n),m\bigr)^\ell$, $m\le\ell\le n-1$	$\bigl(\frac{h}{k}\in\mathcal{F}\bigl(\mathbb{B}(n),m\bigr):\ \ell+k-n\le h\bigr)=\mathcal{F}\bigl(\mathbb{B}(n+m-\ell),m\bigr)$
$\mathcal{G}\bigl(\mathbb{B}(2m),m\bigr)^\ell$	$\mathcal{F}\bigl(\mathbb{B}(3m-\ell),m\bigr)$

5.2. The Cardinalities of Subsequences

Let us calculate the number of fractions in several subsequences of the Farey sequences; $\bar{\mu}(\cdot)$ denotes the number–theoretic Möbius function.

We have

$$\begin{aligned}|\mathcal{F}_n^m| - \big|\mathcal{F}\big(\mathbb{B}(n), m\big)\big| &= \underbrace{\tfrac{3}{2} + \sum_{d=1}^{m} \bar{\mu}(d) \left\lfloor\tfrac{m}{d}\right\rfloor \left(\left\lfloor\tfrac{n}{d}\right\rfloor - \tfrac{1}{2}\left\lfloor\tfrac{m}{d}\right\rfloor\right)}_{|\mathcal{F}_n^m|} - \left(2 + \sum_{d=1}^{m} \bar{\mu}(d) \left\lfloor\tfrac{m}{d}\right\rfloor \left\lfloor\tfrac{n-m}{d}\right\rfloor\right) \\ &= -\tfrac{1}{2} + \sum_{d=1}^{m} \bar{\mu}(d) \left\lfloor\tfrac{m}{d}\right\rfloor \left(\left\lfloor\tfrac{n}{d}\right\rfloor - \tfrac{1}{2}\left\lfloor\tfrac{m}{d}\right\rfloor - \left\lfloor\tfrac{n-m}{d}\right\rfloor\right) \ ,\end{aligned} \tag{94}$$

and

$$\begin{aligned}|\mathcal{G}_n^m| - \big|\mathcal{F}\big(\mathbb{B}(n), m\big)\big| &= \underbrace{\tfrac{3}{2} + \sum_{d=1}^{n-m} \bar{\mu}(d) \left\lfloor\tfrac{n-m}{d}\right\rfloor \left(\left\lfloor\tfrac{n}{d}\right\rfloor - \tfrac{1}{2}\left\lfloor\tfrac{n-m}{d}\right\rfloor\right)}_{|\mathcal{G}_n^m|} - \left(2 + \sum_{d=1}^{n-m} \bar{\mu}(d) \left\lfloor\tfrac{m}{d}\right\rfloor \left\lfloor\tfrac{n-m}{d}\right\rfloor\right) \\ &= -\tfrac{1}{2} + \sum_{d=1}^{n-m} \bar{\mu}(d) \left\lfloor\tfrac{n-m}{d}\right\rfloor \left(\left\lfloor\tfrac{n}{d}\right\rfloor - \tfrac{1}{2}\left\lfloor\tfrac{n-m}{d}\right\rfloor - \left\lfloor\tfrac{m}{d}\right\rfloor\right) \ ;\end{aligned} \tag{95}$$

in particular,

$$\begin{aligned}|\mathcal{F}_{2m}^m| - \big|\mathcal{F}\big(\mathbb{B}(2m), m\big)\big| &= |\mathcal{G}_{2m}^m| - \big|\mathcal{F}\big(\mathbb{B}(2m), m\big)\big| \\ &= -\tfrac{1}{2} + \sum_{d=1}^{m} \bar{\mu}(d) \left\lfloor\tfrac{m}{d}\right\rfloor \left(\left\lfloor\tfrac{2m}{d}\right\rfloor - \tfrac{3}{2}\left\lfloor\tfrac{m}{d}\right\rfloor\right) \ .\end{aligned} \tag{96}$$

As a consequence,

$$\begin{aligned}|\mathcal{F}_n| - \big|\mathcal{F}\big(\mathbb{B}(n), m\big)\big| &= \left(|\mathcal{F}_n^m| - \big|\mathcal{F}\big(\mathbb{B}(n), m\big)\big|\right) + \left(|\mathcal{G}_n^m| - \big|\mathcal{F}\big(\mathbb{B}(n), m\big)\big|\right) \\ &= -4 + \sum_{d\geq 1} \bar{\mu}(d) \left\lfloor\tfrac{n}{d}\right\rfloor \left(\left\lfloor\tfrac{m}{d}\right\rfloor + \left\lfloor\tfrac{n-m}{d}\right\rfloor\right) \ ,\end{aligned} \tag{97}$$

and

$$\begin{aligned}|\mathcal{F}_{2m}| - \big|\mathcal{F}\big(\mathbb{B}(2m), m\big)\big| &= 2\left(|\mathcal{F}_{2m}^m| - \big|\mathcal{F}\big(\mathbb{B}(2m), m\big)\big|\right) \\ &= -4 + 2\sum_{d=1}^{m} \bar{\mu}(d) \left\lfloor\tfrac{2m}{d}\right\rfloor \left\lfloor\tfrac{m}{d}\right\rfloor \ .\end{aligned} \tag{98}$$

Quantity	*Formula*
$\lvert\mathcal{F}_n\rvert$	$\frac{3}{2}+\frac{1}{2}\sum_{d=1}^{n}\bar{\mu}(d)\left\lfloor\frac{n}{d}\right\rfloor^2$
$\lvert\mathcal{F}_n^m\rvert=\lvert\mathcal{G}_n^{n-m}\rvert$	$\frac{3}{2}+\sum_{d\ge 1}\bar{\mu}(d)\left\lfloor\frac{m}{d}\right\rfloor\left(\left\lfloor\frac{n}{d}\right\rfloor-\frac{1}{2}\left\lfloor\frac{m}{d}\right\rfloor\right)$
$\lvert\mathcal{F}_{2m}^m\rvert=\lvert\mathcal{G}_{2m}^m\rvert$	$\frac{3}{2}+\sum_{d=1}^{m}\bar{\mu}(d)\left\lfloor\frac{m}{d}\right\rfloor\left(\left\lfloor\frac{2m}{d}\right\rfloor-\frac{1}{2}\left\lfloor\frac{m}{d}\right\rfloor\right)$
$\lvert\mathcal{F}(\mathbb{B}(n),m)\rvert$	$2+\sum_{d\ge 1}\bar{\mu}(d)\left\lfloor\frac{m}{d}\right\rfloor\left\lfloor\frac{n-m}{d}\right\rfloor$
$\lvert\mathcal{F}(\mathbb{B}(2m),m)\rvert$	$2+\sum_{d=1}^{m}\bar{\mu}(d)\left\lfloor\frac{m}{d}\right\rfloor^2$
$\lvert\mathcal{F}_n^m\rvert-\lvert\mathcal{F}(\mathbb{B}(n),m)\rvert$	$-\frac{1}{2}+\sum_{d=1}^{m}\bar{\mu}(d)\left\lfloor\frac{m}{d}\right\rfloor\left(\left\lfloor\frac{n}{d}\right\rfloor-\frac{1}{2}\left\lfloor\frac{m}{d}\right\rfloor-\left\lfloor\frac{n-m}{d}\right\rfloor\right)$
$\lvert\mathcal{G}_n^m\rvert-\lvert\mathcal{F}(\mathbb{B}(n),m)\rvert$	$-\frac{1}{2}+\sum_{d=1}^{m}\bar{\mu}(d)\left\lfloor\frac{n-m}{d}\right\rfloor\left(\left\lfloor\frac{n}{d}\right\rfloor-\frac{1}{2}\left\lfloor\frac{n-m}{d}\right\rfloor-\left\lfloor\frac{m}{d}\right\rfloor\right)$
$\lvert\mathcal{F}_{2m}^m\rvert-\lvert\mathcal{F}(\mathbb{B}(2m),m)\rvert=\lvert\mathcal{G}_{2m}^m\rvert-\lvert\mathcal{F}(\mathbb{B}(2m),m)\rvert$	$-\frac{1}{2}+\sum_{d=1}^{m}\bar{\mu}(d)\left\lfloor\frac{m}{d}\right\rfloor\left(\left\lfloor\frac{2m}{d}\right\rfloor-\frac{3}{2}\left\lfloor\frac{m}{d}\right\rfloor\right)$
$\lvert\mathcal{F}_n\rvert-\lvert\mathcal{F}(\mathbb{B}(n),m)\rvert$	$-4+\sum_{d\ge 1}\bar{\mu}(d)\left\lfloor\frac{n}{d}\right\rfloor\left(\left\lfloor\frac{m}{d}\right\rfloor+\left\lfloor\frac{n-m}{d}\right\rfloor\right)$
$\lvert\mathcal{F}_{2m}\rvert-\lvert\mathcal{F}(\mathbb{B}(2m),m)\rvert$	$-4+2\sum_{d=1}^{m}\bar{\mu}(d)\left\lfloor\frac{2m}{d}\right\rfloor\left\lfloor\frac{m}{d}\right\rfloor$

The recursive expression for the number of fractions in the Farey sequence—the basis of similar recursive expressions for the other quantities, which is the inverse of the formula $\lvert\mathcal{F}_n\rvert=\frac{3}{2}+\frac{1}{2}\sum_{d=1}^{n}\bar{\mu}(d)\left\lfloor\frac{n}{d}\right\rfloor^2$, is as follows:

$$\lvert\mathcal{F}_n\rvert=\frac{1}{2}(n+3)n-\sum_{d=2}^{n}\lvert\mathcal{F}_{\lfloor n/d\rfloor}\rvert\ , \tag{99}$$

see [14, A005728].

5.3. Matrix Products

Let us survey several subproducts of the matrix products mentioned in Remark 4, Lemma 8, Theorem 10, and in the proof of Proposition 6.

If $j\in\mathbb{Z}$ then

$$\begin{bmatrix}1&0\\1&1\end{bmatrix}^j=\begin{bmatrix}1&0\\j&1\end{bmatrix}\ ,\quad \begin{bmatrix}0&1\\-1&2\end{bmatrix}^j=\begin{bmatrix}1-j&j\\-j&1+j\end{bmatrix}\ ;\quad \begin{bmatrix}1&0\\1&1\end{bmatrix}^{-1}=\begin{bmatrix}1&0\\-1&1\end{bmatrix}\ ,\quad \begin{bmatrix}0&1\\-1&2\end{bmatrix}^{-1}=\begin{bmatrix}2&-1\\1&0\end{bmatrix}\ . \tag{100}$$

If $i\in\mathbb{Z}$ then

$$\begin{bmatrix}1&0\\1&1\end{bmatrix}^i\cdot\begin{bmatrix}0&1\\-1&2\end{bmatrix}^j=\begin{bmatrix}1-j&j\\i-j-ij&1+j+ij\end{bmatrix}\ ,\quad \begin{bmatrix}0&1\\-1&2\end{bmatrix}^i\cdot\begin{bmatrix}1&0\\1&1\end{bmatrix}^j=\begin{bmatrix}1-i+ij&i\\-i+j+ij&1+i\end{bmatrix}\ ; \tag{101}$$

we also have

$$\begin{bmatrix}1&0\\1&1\end{bmatrix}^i\cdot\begin{bmatrix}-1&1\\0&1\end{bmatrix}=\begin{bmatrix}-1&1\\-i&1+i\end{bmatrix}\ ,\qquad \begin{bmatrix}0&1\\-1&2\end{bmatrix}^i\cdot\begin{bmatrix}-1&1\\0&1\end{bmatrix}=\begin{bmatrix}-1+i&1\\i&1\end{bmatrix}\ , \tag{102}$$

$$\begin{bmatrix}-1&1\\0&1\end{bmatrix}\cdot\begin{bmatrix}1&0\\1&1\end{bmatrix}^j\cdot=\begin{bmatrix}-1+j&1\\j&1\end{bmatrix}\ ,\qquad \begin{bmatrix}-1&1\\0&1\end{bmatrix}\cdot\begin{bmatrix}0&1\\-1&2\end{bmatrix}^j=\begin{bmatrix}-1&1\\-j&1+j\end{bmatrix}\ , \tag{103}$$

and

$$\left[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right]^i\cdot\left[\begin{smallmatrix}-1&1\\0&1\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}0&1\\-1&2\end{smallmatrix}\right]^j=\left[\begin{smallmatrix}-1&1\\-i-j&1+i+j\end{smallmatrix}\right]\ , \tag{104}$$

$$\left[\begin{smallmatrix}0&1\\-1&2\end{smallmatrix}\right]^i\cdot\left[\begin{smallmatrix}-1&1\\0&1\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right]^j\cdot=\left[\begin{smallmatrix}-1+i+j&1\\i+j&1\end{smallmatrix}\right]\ , \tag{105}$$

$$\left[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right]^i\cdot\left[\begin{smallmatrix}-1&1\\0&1\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right]^j=\left[\begin{smallmatrix}-1+j&1\\-i+j+ij&1+i\end{smallmatrix}\right]\ , \tag{106}$$

$$\left[\begin{smallmatrix}0&1\\-1&2\end{smallmatrix}\right]^i\cdot\left[\begin{smallmatrix}-1&1\\0&1\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}0&1\\-1&2\end{smallmatrix}\right]^j=\left[\begin{smallmatrix}-1+i-ij&1+ij\\i-j-ij&1+j+ij\end{smallmatrix}\right]\ . \tag{107}$$

5.4. Order–reversing and bijective mapping $\frac{h}{k}\mapsto\frac{k-h}{k}$

In the context of Farey sequences, the mapping $\left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}-1&1\\0&1\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]$ determined by the involutory matrix $\left[\begin{smallmatrix}-1&1\\0&1\end{smallmatrix}\right]$ is ubiquitous:

Order–reversing and bijective mapping $\frac{h}{k}\mapsto\frac{k-h}{k}$			
	$\mathcal{F}_n$	$\to$	$\mathcal{F}_n$
	$\mathcal{F}_n^m$	$\to$	$\mathcal{G}_n^{n-m}$
	$\mathcal{G}_n^m$	$\to$	$\mathcal{F}_n^{n-m}$
	$\mathcal{F}_{2m}^m$	$\to$	$\mathcal{G}_{2m}^m$
	$\mathcal{G}_{2m}^m$	$\to$	$\mathcal{F}_{2m}^m$
$\left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}-1&1\\0&1\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]$	$\mathcal{F}(\mathbb{B}(n),m)$	$\to$	$\mathcal{F}(\mathbb{B}(n),n-m)$
	$\mathcal{F}(\mathbb{B}(2m),m)$	$\to$	$\mathcal{F}(\mathbb{B}(2m),m)$
	$\mathcal{F}(\mathbb{B}(n),m)^\ell$	$\to$	$\mathcal{F}(\mathbb{B}(n),n-\ell)^{n-m}$
	$\mathcal{G}(\mathbb{B}(n),m)^\ell$	$\to$	$\mathcal{G}(\mathbb{B}(n),n-\ell)^{n-m}$
	$\mathcal{F}(\mathbb{B}(2m),m)^\ell$	$\to$	$\mathcal{F}(\mathbb{B}(2m),2m-\ell)^m$
	$\mathcal{G}(\mathbb{B}(2m),m)^\ell$	$\to$	$\mathcal{G}(\mathbb{B}(2m),2m-\ell)^m$

Let us assign to an ordered collection of matrices $(\boldsymbol{M}_1,\ldots,\boldsymbol{M}_t)\in\{\left[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right],\left[\begin{smallmatrix}0&1\\-1&2\end{smallmatrix}\right]\}^t$ the collection $(\boldsymbol{P}_1,\ldots,\boldsymbol{P}_t)$ such that

$$\boldsymbol{P}_i:=\begin{cases}\left[\begin{smallmatrix}0&1\\-1&2\end{smallmatrix}\right]\ , & \text{if } \boldsymbol{M}_i=\left[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right]\ ,\\ \left[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right]\ , & \text{if } \boldsymbol{M}_i=\left[\begin{smallmatrix}0&1\\-1&2\end{smallmatrix}\right]\ ,\end{cases} \tag{108}$$

$1\le i\le t$. For any order m, the map

$$\begin{gathered}\prod_{i=1}^{t}\boldsymbol{M}_{t-i+1}\cdot\mathcal{F}_m\to\prod_{i=1}^{t}\boldsymbol{P}_{t-i+1}\cdot\mathcal{F}_m\ ,\\ \left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}-1&1\\0&1\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\ ,\end{gathered} \tag{109}$$

is order–reversing and bijective.

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