# A FUSS-TYPE FAMILY OF POSITIVE DEFINITE SEQUENCES

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Abstract. We study a two-parameter family $a_n(p,t)$ of deformations of the Fuss numbers. We show a sufficient condition for positive definiteness of $a_n(p,t)$ and prove that some of the corresponding probability measures are infinitely divisible with respect to the additive free convolution.

## 1. Introduction

The aim of the paper is to study a two-parameter family of sequences $a_n(p,t)$, $p,t\in\mathbb{R}$, defined by (17), which can be regarded as deformation of the Fuss numbers. Assuming that $p\geq 0$ we prove that the sequence $a_n(p,t)$ is positive definite if and only if $p\geq 1$ and $g(p)\leq t\leq 2p/(p+1)$, where $g(p)$ is defined by (25). We conjecture that the assumption that $p\geq 0$ is redundant.

The case $t=2p/(p+1)$ is particularly interesting by connections with the work [6] of M. Bousquet-Mélou and G. Schaeffer. They introduced the notion of constellation as a tool for studying factorization problems in the symmetric groups. For $p\geq 2$ a *p-constellation* is a 2-cell decomposition of the oriented sphere into vertices, edges and faces, with faces colored black and white in such a way that:

- all faces adjacent to a given white face are black and vice versa,
- the degree of any black face is $p$,
- the degree of any white face is a multiple of $p$.

A constellation is called *rooted* if one of the edges is distinguished.

The number of rooted $p$-constellations formed of $n$ polygons, counted up to isomorphism, is given by

$$C_p(n):=\binom{np}{n}\frac{(p+1)p^{n-1}}{(np-n+1)(np-n+2)}, \tag{1}$$

$p\geq 2$, $n\geq 1$, see Corollary 2.4 in [6]. Some of these sequences appear in the On-line Encyclopedia of Integer Sequences (OEIS) [26], namely: $C_2=A000257$, $C_3=A069726$, $C_4=A090374$.

We will prove that the probability distribution $\eta(p,t)$ corresponding to positive definite sequence $a_n(p,t)$ is absolutely continuous, except for $\eta(1,1)=\delta_1$, and the support of $\eta(p,t)$ is $[0,p^p(p-1)^{1-p}]$. The density function will be denoted $f_{p,t}(x)$. For $p=2$ and $p=3$ we compute the $R$-transform of $\eta(p,t)$. We prove that $\eta(2,p)$ (resp. $\eta(3,t)$) is infinitely divisible with respect to the additive free convolution if and only if $1\leq t\leq 4/3$ (resp. $1/2\leq t\leq 3/2$).

Finally, let us record some other sequences from OEIS which are related to this work: $A005807$: $2a_n(2,1/2)$ (sums of adjacent Catalan numbers), $A007226$: $2a_n(3,1/2)$

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(studied in [15]), $A007054$: $3a_n(2,4/3)$ (super ballot numbers), $A038629$: $3a_n(2,2/3)$, $A000139$: $2a_n(3,3/2)$, $A197271$: $5a_n(4,8/5)$, $A197272$: $3a_n(5,5/3)$. In Section 4 we also encounter sequences $A022558$ and $A220910$.

## 2. Fuss numbers

The *Fuss-Catalan numbers* $\binom{np+1}{n}\frac{1}{np+1}$ have several combinatorial applications, see [9, 7, 2, 27, 6, 24]. They count for example:

(1) the number of ways of subdividing a convex polygon, with $n(p-1)+2$ vertices, into $n$ disjoint $p+1$-gons by means of nonintersecting diagonals,
(2) the number of sequences $(a_1, a_2, \ldots, a_{np})$, where $a_i \in \{1, 1-p\}$, with all partial sums $a_1 + \ldots + a_k$ nonnegative and with $a_1 + \ldots + a_{np} = 0$,
(3) the number of noncrossing partitions $\pi$ of $\{1, 2, \ldots, n(p-1)\}$, such that $p-1$ divides the cardinality of every block of $\pi$,
(4) the number of $p$-cacti formed of $n$ polygons, see [6].

The generating function:

$$\mathcal{B}_p(z) := \sum_{n=0}^{\infty} \binom{np+1}{n} \frac{z^n}{np+1} \tag{2}$$

satisfies

$$\mathcal{B}_p(z) = 1 + z\mathcal{B}_p(z)^p. \tag{3}$$

Recall also the Lambert's formula for the Taylor expansion of the powers of $\mathcal{B}_p(z)$:

$$\mathcal{B}_p(z)^r = \sum_{n=0}^{\infty} \binom{np+r}{n} \frac{rz^n}{np+r}. \tag{4}$$

These formulas remain true for $p, r \in \mathbb{R}$ and the coefficients $\binom{np+r}{n}\frac{r}{np+r}$ (understood to be 1 for $n=0$ and $\frac{r}{n!}\prod_{i=1}^{n-1}(np+r-i)$ for $n \geq 1$) are called *two-parameter Fuss numbers* or *Raney numbers*, see [9, 13, 22, 12, 8].

In some cases the function $\mathcal{B}_p$ can be written explicitly, for example

$$\begin{aligned}
\mathcal{B}_2(z) &= \frac{2}{1+\sqrt{1-4z}} = \frac{1-\sqrt{1-4z}}{2z}, \\
\mathcal{B}_3(z) &= \frac{3}{3-4\sin^2\alpha}, \\
\mathcal{B}_{3/2}(z) &= \frac{3}{\left(\sqrt{3}\cos\beta - \sin\beta\right)^2},
\end{aligned}$$

where $\alpha = \frac{1}{3}\arcsin\left(\sqrt{27z/4}\right)$, $\beta = \frac{1}{3}\arcsin\left(3z\sqrt{3}/2\right)$, see [16].

Fuss numbers also have applications in free probability and in the theory of random matrices, as moments of the multiplicative free powers of the Marchenko-Pastur distribution [1, 3, 13, 17, 18]. This implies that for $p \geq 1$ the sequence $\binom{np+1}{n}\frac{1}{np+1}$ is positive definite. More generally, the sequence $\binom{np+r}{n}\frac{r}{np+r}$ is positive definite if and only if either $p \geq 0$, $0 \leq r \leq p$, or $p \leq 0$, $p-1 \leq r \leq 0$ or $r = 0$, see [13, 16, 12, 8]. The case $r = 0$ is trivial, as it gives the sequence $1, 0, 0, 0, \ldots$, moments of $\delta_0$. The distributions corresponding to the second case, $p \le 0$, $p-1 \le r \le 0$, are just reflections of those corresponding to $p \ge 0$, $0 \le r \le p$. It is a consequence of the identity

$$\binom{np+r}{n}\frac{r(-1)^n}{np+r} = \binom{n(1-p)-r}{n}\frac{-r}{n(1-p)-r}. \tag{5}$$

For $p > 1$, $r > 0$ we have the following integral representation:

$$\binom{np+r}{n}\frac{r}{np+r} = \int_0^{c(p)} x^n W_{p,r}(x)\,dx,$$

where where $c(p) := p^p(p-1)^{1-p}$, and $W_{p,r}$ can be described as:

$$W_{p,r}(x) = \frac{(\sin(p-1)\phi)^{p-r-1}\sin\phi\sin r\phi}{\pi\,(\sin p\phi)^{p-r}}, \tag{6}$$

where

$$x = \rho(\phi) = \frac{(\sin p\phi)^p}{\sin\phi\,(\sin(p-1)\phi)^{p-1}}, \qquad 0 < \phi < \pi/p. \tag{7}$$

This function is nonnegative if and only if $r \le p$, see [10, 18, 8].

If $p = k/l$ is a rational number, $1 \le l < k$, then $W_{p,r}$ can be expressed in terms of the Meijer $G$-function (see [22, 14]):

$$W_{p,r}(x) = \frac{rp^r}{x(p-1)^{r+1/2}\sqrt{2k\pi}}\, G^{k,0}_{k,k}\left(\frac{x^l}{c(p)^l}\,\middle|\,\begin{matrix}\alpha_1,\ldots,\alpha_k\\ \beta_1,\ldots,\beta_k\end{matrix}\right), \tag{8}$$

$x \in (0, c(p))$ and the parameters $\alpha_j, \beta_j$ are given by:

$$\alpha_j = \begin{cases} \dfrac{j}{l} & \text{if } 1 \le j \le l,\\[2mm] \dfrac{r+j-l}{k-l} & \text{if } l+1 \le j \le k, \end{cases} \tag{9}$$

$$\beta_j = \frac{r+j-1}{k}, \qquad 1 \le j \le k. \tag{10}$$

**Examples:** Let us record formulas for the functions $W_{p,r}$ for $p = 2, 3, 3/2$ and $r = 1, 2$. In these cases $W_{p,r}$ can be expressed as an elementary function, see [21, 22, 14].

$$W_{2,1}(x) = \frac{1}{2\pi}\sqrt{\frac{4-x}{x}}, \tag{11}$$

$$W_{2,2}(x) = \frac{1}{2\pi}\sqrt{x(4-x)}, \tag{12}$$

where $x \in (0, 4)$. $W_{2,1}$ is the density of the Marchenko-Pastur distribution and $W_{2,2}$ is the Wigner's semicircle law translated by 2.

$$W_{3,1}(x) = \frac{3\left(1+\sqrt{1-4x/27}\right)^{2/3} - (4x)^{1/3}}{3^{1/2}\pi(4x)^{2/3}\left(1+\sqrt{1-4x/27}\right)^{1/3}}, \tag{13}$$

$$W_{3,2}(x) = \frac{9\left(1+\sqrt{1-4x/27}\right)^{4/3} - (4x)^{2/3}}{2\pi 3^{3/2}(4x)^{1/3}\left(1+\sqrt{1-4x/27}\right)^{2/3}}, \tag{14}$$

where $x \in (0, 27/4)$.

$$\begin{aligned} W_{3/2,1}(x) = 3^{1/2}&\frac{\left(1+\sqrt{1-4x^2/27}\right)^{1/3} - \left(1-\sqrt{1-4x^2/27}\right)^{1/3}}{2(2x)^{1/3}\pi} \\ +3^{1/2}(2x)^{1/3}&\frac{\left(1+\sqrt{1-4x^2/27}\right)^{2/3} - \left(1-\sqrt{1-4x^2/27}\right)^{2/3}}{4\pi}, \end{aligned} \tag{15}$$

$$\begin{aligned} W_{3/2,2}(x) = &\frac{3^{1/2}(2x)^{5/3}}{8\pi}\left(\left(1+\sqrt{1-4x^2/27}\right)^{1/3} - \left(1-\sqrt{1-4x^2/27}\right)^{1/3}\right) \\ &+\frac{3^{1/2}(2x)^{1/3}(x^2-1)}{4\pi}\left(\left(1+\sqrt{1-4x^2/27}\right)^{2/3} - \left(1-\sqrt{1-4x^2/27}\right)^{2/3}\right), \end{aligned} \tag{16}$$

where $x \in (0, 3\sqrt{3}/2)$. The function $W_{3/2,2}(x)$ is not nonnegative on its domain.

## 3. A family of sequences

For $p, t \in \mathbb{R}$ define sequence $a_n(p,t)$ as an affine combination of $\binom{np+1}{n}\frac{1}{np+1}$ and $\binom{np+2}{n}\frac{2}{np+2}$:

$$a_n(p,t) := \binom{np+1}{n}\frac{t}{np+1} + \binom{np+2}{n}\frac{2(1-t)}{np+2} \tag{17}$$

$$= \binom{np}{n}\frac{n(2p-t-pt)+2}{(np-n+1)(np-n+2)}, \tag{18}$$

in particular $a_0(p,t) = 1$.

The generating function is

$$t\mathcal{B}_p(z) + (1-t)\mathcal{B}_p(z)^2 = \sum_{n=0}^{\infty} a_n(p,t) z^n. \tag{19}$$

For example:

$$t\mathcal{B}_2(z) + (1-t)\mathcal{B}_2(z)^2 = \frac{1 - t + 3tz - 2z - (1-t+tz)\sqrt{1-4z}}{2z^2},$$

$$t\mathcal{B}_3(z) + (1-t)\mathcal{B}_3(z)^2 = \frac{9 - 12t\sin^2\alpha}{\left(3 - 4\sin^2\alpha\right)^2},$$

$$t\mathcal{B}_{3/2}(z) + (1-t)\mathcal{B}_{3/2}(z)^2 = \frac{9 - 6t\sin^2\beta + 6t\sqrt{3}\sin\beta\cos\beta}{\left(\sqrt{3}\cos\beta - \sin\beta\right)^4}$$

where $\alpha = \frac{1}{3}\arcsin\left(\sqrt{27z/4}\right)$, $\beta = \frac{1}{3}\arcsin\left(3z\sqrt{3}/2\right)$.

We are going to study positive definiteness of $a_n(p,t)$. First we observe

**Proposition 3.1.** *If the sequence $a_n(p,t)$ is positive definite then*

$$2p - pt - t^2 + 3t - 3 \ge 0. \tag{20}$$

*In particular $t \neq 2$ and either $p \le -3$ or $p \ge 1$.*

*Proof.* The left hand side is just $a_2(p,t) - a_1(p,t)^2$. □

**Examples.**

**1.** For $p = 1$ we have $a_n(1,t) = 1 + n - nt$. Since $a_2(1,t) - a_1(1,t)^2 = -(t-1)^2$, the sequence $a_n(1,t)$ is positive definite if and only if $t = 1$. Note that $a_n(1,1) = 1$ is the moment sequence of the one-point measure $\delta_1$.

**2.** For $t = 2/(p+1)$ we get

$$a_n\left(p, 2/(p+1)\right) = \binom{np}{n}\frac{2}{np - n + 2}.$$

If $p > 1$ then this is product of two positive definite sequences: $\binom{np}{n}$ (see [16, 25]) and $2/(np - n + 2)$.

**3.** Similarly, for $p > 1$, $t = 2p/(p+1)$ the sequence

$$a_n\left(p, 2p/(p+1)\right) = \binom{np}{n}\frac{2}{(np-n+1)(np-n+2)}.$$

is positive definite. Note that from (1) we have

$$C_p(n) = \frac{(p+1)p^n}{2p} a_n\left(p, \frac{2p}{p+1}\right), \tag{21}$$

so for $p \ge 1$ the sequence $C_p(n)$ is positive definite.

The sequence $a_n(p,t)$ is an affine combination of two sequences: $\binom{np+1}{n}\frac{1}{np+1}$ and $\binom{np+2}{n}\frac{2}{np+2}$. The former is positive definite for $p \ge 1$ and the latter for $p \ge 2$. This implies, that $a_n(p,t)$ is positive definite for $p \ge 2$, $0 \le t \le 1$. We are going to prove something stronger. Note that if $t_1 \le t_2 \le t_3$ and the sequences $a_n(p,t_1)$, $a_n(p,t_3)$ are positive definite then so is $a_n(p,t_2)$ as their convex combination.

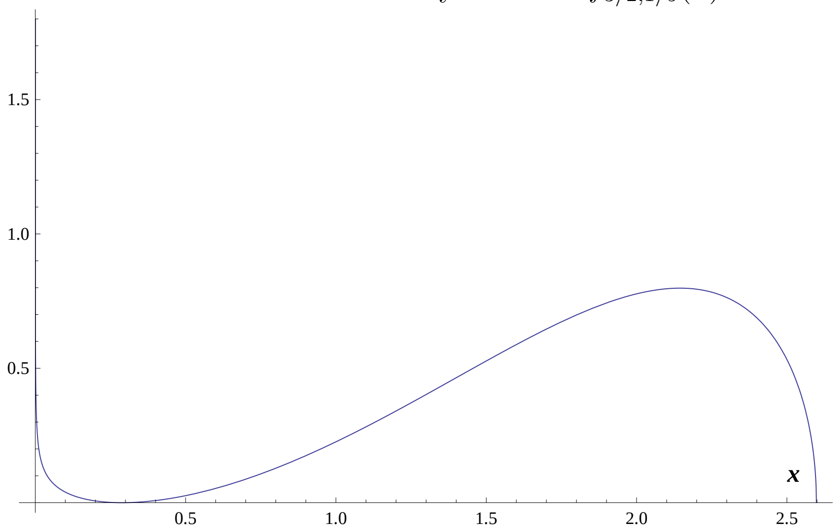
If we assume that $p > 1$ then

$$a_n(p,t) = \int_0^{c(p)} x^n f_{p,t}(x)\,dx,$$

where

$$f_{p,t}(x) = tW_{p,1}(x) + (1-t)W_{p,2}(x).$$

FIGURE 1. The density function $f_{3/2,1/5}(x)$




Then the positive definiteness of $a_n(p,t)$ is equivalent to the fact that $f_{p,t}$ is nonnegative on $(0, c(p))$. For example the function

$$f_{2,t}(x) = \frac{t + x - tx}{2\pi} \sqrt{\frac{4-x}{x}} \tag{22}$$

is nonnegative on $(0,4)$ if and only if $0 \le t \le 4/3$.

By (6) we can write

$$f_{p,t}(x) = \frac{\sin^2\phi\,(\sin(p-1)\phi)^{p-3}\left[t\sin(p-1)\phi + 2(1-t)\sin p\phi\cos\phi\right]}{\pi\,(\sin p\phi)^{p-1}} \tag{23}$$

for $x$ as in (7). Define

$$\begin{aligned} \Psi_{p,t}(\phi) &= t\sin(1-1/p)\phi + 2(1-t)\sin\phi\cos\phi/p \\ &= (2-t)\sin\phi\cos\phi/p - t\cos\phi\sin\phi/p \\ &= (1-t)\sin(1+1/p)\phi + \sin(1-1/p)\phi. \end{aligned} \tag{24}$$

Then the sequence $a_n(p,t)$ is positive definite if and only if $\Psi_{p,t}(\phi) \ge 0$ for $\phi \in [0,\pi]$. For $p \ge 1$ put

$$g(p) := \min\{t \in \mathbb{R} : \Psi_{p,t}(\phi) \ge 0 \text{ for all } 0 < \phi < \pi\}. \tag{25}$$

Since $\Psi_{p,t}(\pi) = t\sin(\pi/p)$ and $\Psi_{p,1}(\phi) = \sin(1-1/p)\phi$, we have $0 \le g(p) \le 1$ for all $p \ge 1$.

**Proposition 3.2.** *The function $g$ is continuous on $[1,\infty)$, $g(1) = 1$, $g(p) = 0$ for $p \ge 2$ and is strictly decreasing on $[1,2]$. In particular $g(3/2) = 1/5$.*

*Proof.* For $p = 1$ we have $\Psi_{1,t}(\phi) = (1-t)\sin 2\phi$, which implies $g(1) = 1$. If $p \ge 2$ then $\Psi_{p,0}(\phi) = 2\sin\phi\cos\phi/p$ is nonnegative for $\phi \in [0,\pi]$, which yields $g(p) = 0$.

Now observe, that for fixed $t, \phi$, with $0 \le t \le 1$, $0 < \phi \le \pi$, the function $p \mapsto \Psi_{p,t}(\phi)$ is strictly increasing on $[1,2]$. Indeed, we can write

$$\Psi_{p,t}(\phi) = 2(1-t)\sin\phi\cos\phi/p + t\sin(\phi - \phi/p)$$

and if $0 < \phi \le \pi$ then both the summands are increasing with $p \in [1,2]$. This implies, that $g(p)$ is strictly decreasing on $[1,2]$.

To prove continuity of $g$ assume that $1 \le p_1 < p_2 \le 2$ and put $t_1 := g(p_1)$, $t_2 := g(p_2)$. Then $t_1 > t_2$, $\Psi_{p_1,t_1}(\phi) \ge 0$ for all $\phi \in [0,\pi]$ and there is $\phi_1$, with $p_1\pi/(1+p_1) < \phi_1 < \pi$, such that $\Psi_{p_1,t_1}(\phi_1) = 0$. Then we have that $\Psi_{p_2,t_1}(\phi) > 0$ for all $\phi \in (0,\pi]$. From the third expression in (24) we have that

$$-c_1 := \sin(1+1/p_1)\phi_1 < 0.$$

If we assume that $(p_2 - p_1)\phi_1 < c_1/2$ then we have

$$|\sin(1+1/p_1)\phi_1 - \sin(1+1/p_2)\phi_1| \le (1/p_1 - 1/p_2)\,\phi_1 < c_1/2$$

and, consequently, $\sin(1+1/p_2)\phi_1 < -c_1/2$.

If we take $t$, with $0 \le t < t_1$, then

$$\Psi_{p_2,t}(\phi_1) = \Psi_{p_2,t}(\phi_1) - \Psi_{p_1,t_1}(\phi_1)$$

$$= (1-t_1)\big(\sin(1+1/p_2)\phi_1 - \sin(1+1/p_1)\phi_1\big) + \big(\sin(1-1/p_2)\phi_1 - \sin(1-1/p_1)\phi_1\big)$$
$$+(t_1 - t)\sin(1+1/p_2)\phi_1 \le (2-t_1)(p_2-p_1)\phi_1 - (t_1-t)c_1/2.$$

Hence, if

$$(2-t_1)(p_2-p_1)\phi_1 < (t_1-t)c_1/2$$

then $\Psi_{p_2,t}(\phi_1) < 0$. This implies that

$$g(p_1) - g(p_2) = t_1 - t_2 \le 2(2-t_1)(p_2-p_1)\phi_1/c_1.$$

and proves continuity of $g$.

For $p = 3/2$ we can write

$$\Psi_{3/2,t}(\phi) = \frac{\sin\phi/3}{4}\left[(1-t)\left(5 - 8\sin^2\phi/3\right)^2 + 5t - 1\right].$$

Note that $\sqrt{5/8} < \sqrt{3}/2 = \sin\pi/3$, so, assuming that $0 \le t \le 1$, $\Psi_{3/2,t}$ attains its minimum on $[0,\pi]$ at $\phi = 3\arcsin\sqrt{5/8}$. This yields $g(3/2) = 1/5$. □

Now we are able to describe the domain of positive definiteness of the sequence $a_n(p,t)$, see Fig 2. The density function for the particular case $p = 3/2$, $t = 1/5$ is illustrated in Fig. 1.

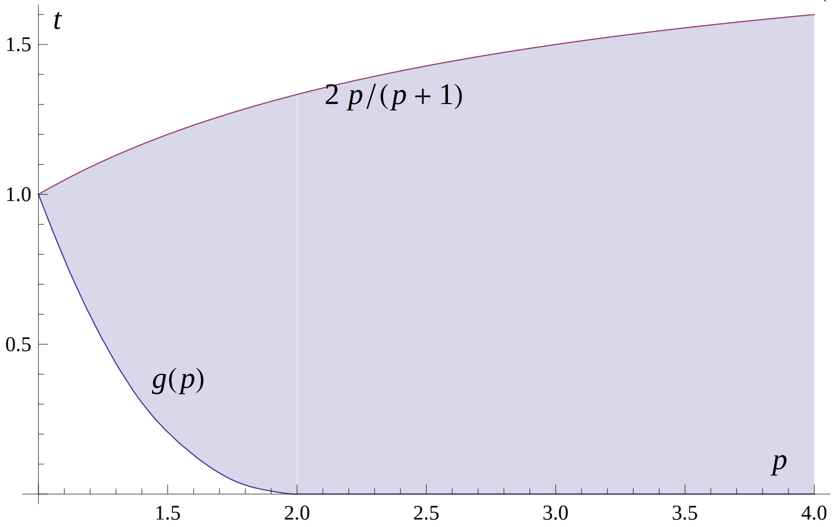
**Theorem 3.3.** *Suppose that $p \ge 0$. Then the sequence $a_n(p,t)$ is positive definite if and only if $p \ge 1$ and*

$$g(p) \le t \le \frac{2p}{1+p}. \tag{26}$$

*Proof.* Fix $p \ge 1$. By the definition of $g(p)$ the sequence $a_n(p,t)$ is positive definite for $t = g(p)$ and not positive definite for $t < g(p)$.

We have already observed, that for $p \ge 1$ the sequence $a_n(p, 2p/(p+1))$ is positive definite. If $t > 2p/(p+1)$ then $n(2p - t - pt) + 2 < 0$ and consequently $a_n(p,t) < 0$ for all $n$ sufficiently large. Alternatively, we have $\Psi'_{p,t}(0) = 2p - pt - t < 0$ in this case, which implies $\Psi_{p,t}(x) < 0$ for some $x \in (0, \pi/p)$. □

Figure 2. Domain of positive definiteness of the sequence $a_n(p,t)$




## 4. Free transforms

Throughout this section we assume that $p \geq 1$ and the sequence $a_n(p,t)$ is positive definite, i.e. $g(p) \leq t \leq 2p/(p+1)$. Denote by $\eta(p,t)$ the corresponding distribution, i.e. $\eta(1,1) = \delta_1$ and $\eta(p,t) = f_{p,t}(x)\,dx$ on $[0, p^p(p-1)^{1-p}]$ for $p > 1$. We are going to study relations of these measures with free probability.

Recall that for a compactly supported probability measure $\mu$ on $\mathbb{R}$, with the moment generating function

$$M_\mu(z) := \sum_{n=0}^{\infty} z^n \int_{\mathbb{R}} x^n\,d\mu(x) = \int_{\mathbb{R}} \frac{1}{1-xz}\,d\mu(x), \tag{27}$$

the $S$- and $R$-transforms are defined by

$$M_\mu\left(\frac{z}{1+z} S_\mu(z)\right) = 1 + z, \tag{28}$$

$$1 + R_\mu\left(z M_\mu(z)\right) = M_\mu(z). \tag{29}$$

Moreover, we have relation

$$R_\mu\left(z S_\mu(z)\right) = z. \tag{30}$$

The coefficients $r_n(\mu)$ in the Taylor expansion $R_\mu(z) = \sum_{n=1}^{\infty} r_n(\mu) z^n$ are called *free cumulants* of $\mu$. It is known that $\mu$ is infinitely divisible with respect to the additive free convolution if and only if the sequence $\{r_{n+2}(\mu)\}_{n=0}^{\infty}$ is positive definite, see [28, 19].

For the distributions $\eta(p,t)$ we have

$$M_{\eta(p,t)}(z) := \sum_{n=0}^{\infty} a_n(p,t) z^n = t\mathcal{B}_p(z) + (1-t)\mathcal{B}_p(z)^2.$$

Now we are going to compute the $S$-transform of $\eta(p,t)$.

**Proposition 4.1.** *For $p>1$, $g(p)\le t\le 2p/(p+1)$ we have*

$$S_{\eta(p,t)}(w)=(2+2w)^{1-p}\frac{\left(\sqrt{(2-t)^2+4(1-t)w}+t\right)^p}{\sqrt{(2-t)^2+4(1-t)w}+2-t}. \tag{31}$$

*Proof.* From (3) we can derive relation

$$\mathcal{B}_p\left(z(1+z)^{-p}\right)=1+z,$$

see [13]. Therefore

$$M_{\eta(p,t)}\left(z(1+z)^{-p}\right)=t(1+z)+(1-t)(1+z)^2.$$

If we substitute

$$t(1+z)+(1-t)(1+z)^2=1+w$$

then

$$z=\frac{\sqrt{(2-t)^2+4(1-t)w}-2+t}{2(1-t)}=\frac{2w}{\sqrt{(2-t)^2+4(1-t)w}+2-t}$$

and

$$1+z=\frac{\sqrt{(2-t)^2+4(1-t)w}-t}{2(1-t)}=\frac{2(1+w)}{\sqrt{(2-t)^2+4(1-t)w}+t},$$

which combining with (28) yields (31). □

Now we are going to compute $R$-transform of $\eta(p,t)$ for $p=2$ and $p=3$. We will denote $r_n(p,t):=r_n(\eta(p,t))$.

4.1. **The case** $p=2$**.** The density function $f_{2,t}$ is given by (22), $0\le t\le 4/3$. From (31) we can compute the $R$-transform for $p=2$:

**Proposition 4.2.** $R_{\eta(2,1)}=z/(1-z)$ *and for* $t\ne 1$

$$R_{\eta(2,t)}(z)=\frac{1-t-2z+3tz-z^2+(t-1-z)\sqrt{1+z(2-4t)+z^2}}{2(t-1)}.$$

*Moreover, $\eta(2,t)$ is infinitely divisible with respect to the additive free convolution if and only if either $t=0$ or $1\le t\le 4/3$.*

*Proof.* First we find $R_{\eta(2,t)}(z)$ by solving equation $S_{\eta(2,t)}\left(R_{\eta(2,t)}(z)\right)R_{\eta(2,t)}(z)=z$, equivalent with (30), with the condition $R_{\eta(2,t)}(0)=0$. In particular $R_{2,0}=2z+z^2$, which implies that $\eta(2,0)$ is infinitely divisible with respect to the additive free convolution.

Now we can find:

$$\begin{aligned}
r_1(2,t)&=2-t,\\
r_2(2,t)&=1+t-t^2,\\
r_3(2,t)&=3t^2-2t^3,\\
r_4(2,t)&=-4t^2+10t^3-5t^4.
\end{aligned}$$

Since

$$r_2(2,t)r_4(2,t)-r_3(2,t)^2=t^2(t-1)(t-2)(t^2-2),$$

for $0<t<1$ the distribution $\eta(2,t)$ is not infinitely divisible with respect to the additive free convolution.

For $t \neq 1$ we have

$$1+R_{\eta(2,t)}(z)=\frac{t-1-2z+3tz-z^2+(t-1-z)\sqrt{1+z(2-4t)+z^2}}{2(t-1)}$$

and $1+R_{\eta(2,1)}(z)=1/(1-z)$. Then for $1<t\leq 3/2$ the function

$$\frac{1+R_{\eta(2,t)}(1/z)}{z}=\frac{(t-1)z^2-2z+3tz-1+\big(z(t-1)-1\big)\sqrt{1+z(2-4t)+z^2}}{2(t-1)z^3}$$

is the Cauchy transform of the probability distribution

$$\frac{(1-tx+x)\sqrt{4t(t-1)-(x-2t+1)^2}}{2\pi(t-1)x^3}\,dx,$$

on the interval

$$x\in\left[2t-1-2\sqrt{t^2-t},2t-1+2\sqrt{t^2-t}\right].$$

Therefore for $1<t\leq 4/3$

$$r_n(2,t)=\int_{2t-1-2\sqrt{t^2-t}}^{2t-1+2\sqrt{t^2-t}} x^n\frac{(1-tx+x)\sqrt{4t(t-1)-(x-2t+1)^2}}{2\pi(t-1)x^3}\,dx, \tag{32}$$

which proves that the sequence $\{r_{n+2}(2,t)\}_{n=0}^{\infty}$ is positive definite. □

**Remark.** Note, that for $\eta(2,0)$ the cumulant sequence is $(2,1,0,0,\ldots)$, so the sequence $\{r_{n+2}(2,0)\}_{n=0}^{\infty}=(1,0,0,\ldots)$ is positive definite. Actually, $\eta(2,0)$, given by (12), is a translation of the Wigner semicircle distribution $\frac{1}{2\pi}\sqrt{4-x^2}\,dx$, $x\in[-2,2]$. The free additive infinite divisibility of $\eta(2,0)$ was overlooked in [16], Corollary 7.1, where $\eta(2,0)$ was denoted $\mu(2,2)$.

**Example 1.** Define a sequence $a_n$ by $a_0:=1$ and $a_n:=3^n\cdot r_n(2,4/3)$ for $n\geq 1$:

$$1,2,5,16,64,304,1632,9552,59520,388720,2632864,\ldots.$$

Applying (32) for $t=4/3$ we obtain

$$a_n=\int_1^9 x^n\frac{\sqrt{(x-1)(9-x)^3}}{2\pi x^3}\,dx. \tag{33}$$

Its generating function is

$$\sum_{n=0}^{\infty}a_nz^n=1+R_{\eta(2,4/3)}(3z)=\frac{1+18z-27z^2+\sqrt{(1-z)(1-9z)^3}}{2}. \tag{34}$$

**Example 2.** Now let us consider the binomial transform of $a_n$:

$$b_n:=\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}a_k.$$

The corresponding density function is that of the sequence $a_n$ translated by $-1$, so

$$b_n=\int_0^8 x^n\frac{\sqrt{x(8-x)^3}}{2\pi(x+1)^3}\,dx. \tag{35}$$

For the generating function we have

$$\sum_{n=0}^{\infty}b_nz^n=\sum_{k=0}^{\infty}a_k(-1)^k\sum_{n=k}^{\infty}\binom{n}{k}(-z)^n$$

$$= \sum_{k=0}^{\infty} a_k \frac{z^k}{(1+z)^{k+1}} = \frac{1}{1+z} \left(1 + R_{\eta(2,4/3)} \left(3z/(1+z)\right)\right),$$

so from (34)

$$\sum_{n=0}^{\infty} b_n z^n = \frac{1 + 20z - 8z^2 + \sqrt{(1-8z)^3}}{2(1+z)^3}. \tag{36}$$

This proves that $b_n$ coincides with $A022558$ in OEIS:

$$1, 1, 2, 6, 23, 103, 512, 2740, 15485, 91245, 555662, \ldots,$$

which counts the permutations of length $n$ which avoid the pattern 1342, see Theorem 2 in [5].

4.2. **The case $p = 3$.**

**Proposition 4.3.**

$$R_{\eta(3,t)}(z) = \frac{z(4 - 7t + 4t^2 - 2z) - (t-1)^2 + (1 - 2t + t^2 - tz)\sqrt{1-4tz}}{2(t+z-1)^2} \tag{37}$$

*and the distribution $\eta(3, t)$ is infinitely divisible with respect to the additive free convolution if and only if $1/2 \le t \le 3/2$.*

*Proof.* The proof is similar as for $p = 2$. First we find $R_{\eta(3,t)}$ by solving the equation

$$S_{\eta(3,t)} \left(R_{\eta(3,t)}(z)\right) R_{\eta(3,t)}(z) = z,$$

with the condition that $R_{\eta(3,t)}(0) = 0$. Then we find out that

$$1 + R_{\eta(3,t)}(z) = \frac{(t-1)^2 + tz(4t-3) + (1 - 2t + t^2 - tz)\sqrt{1-4tz}}{2(t+z-1)^2}$$

is the moment generating function for the density

$$\frac{\left(t - x(t-1)^2\right)\sqrt{4t - x}}{2\pi(tx - x + 1)^2\sqrt{x}}, \qquad x \in [0, 4t], \tag{38}$$

which is positive provided $1/2 \le t \le 3/2$. ☐

**Example.** The sequence $a_n = A220910(n)$:

$$1, 1, 3, 14, 83, 570, 4318, 35068, 299907, 2668994, 24513578, \ldots$$

counts matchings avoiding the pattern 231, see [4] for details. Its generating function equals

$$M(z) = \sum_{n=0}^{\infty} a_n z^n = \frac{1 + 36z + \sqrt{(1-12z)^3}}{2(1+4z)^2} = 1 + R_{\eta(3,3/2)}(2z), \tag{39}$$

so we have $a_n = 2^n \cdot r_n(3, 3/2)$ for $n \ge 1$. Therefore these numbers can be represented as moments:

$$a_n = \int_0^{12} x^n \frac{\sqrt{(12-x)^3}}{2\pi(x+4)^2\sqrt{x}}\, dx. \tag{40}$$

Now we are going to prove a recurrence relation, which was was conjectured by R. J. Mathar (see OEIS, entry A220910, Aug. 04 2013).

**Proposition 4.4.** *For $n \geq 2$ we have*

$$na_n = (8n-34)a_{n-1} + 24(2n-3)a_{n-2}. \tag{41}$$

*Proof.* One can check that the generating function satisfies differential equation:

$$(1-8z-48z^2)M'(z) + (26-24z)M(z) = 27.$$

The coefficient at $z^{n-1}$ on the left hand side is equal to

$$na_n - 8(n-1)a_{n-1} - 48(n-2)a_{n-2} + 26a_{n-1} - 24a_{n-2}$$

for $n \geq 2$, which gives (41). □

Now we will provide two formulas for $a_n = A220910(n)$.

**Proposition 4.5.**

$$a_n = \frac{1-8n}{2}(-4)^n + \binom{2n}{n}\sum_{k=0}^{n} \frac{3^{n+1}(k+1)\prod_{i=0}^{k-1}(n-i)}{8(-3)^k \prod_{i=0}^{k+1}(n-i-1/2)} \tag{42}$$

$$= \frac{(-4)^n(1-8n)}{16}\left[8 - \sum_{k=0}^{n+1}\frac{(-3)^k}{k!}\prod_{i=0}^{k-1}(i-3/2)\right] + \binom{2n}{n}\frac{3^{n+3}}{32(n+1)}. \tag{43}$$

*Proof.* Putting $x = 12t$ in (40) and applying formula (15.6.1) from [20] we get

$$a_n = \frac{9\cdot 12^n}{2\pi}\int_0^1 \frac{t^{n-1/2}(1-t)^{3/2}}{(1+3t)^2}\,dt \tag{44}$$

$$= \frac{27(2n)!3^n}{8n!(n+2)!}\,{}_2F_1(2, n+1/2;\, n+3\,|\,-3)\,. \tag{45}$$

From (15.8.2) in [20] and from the identities

$$\frac{\Gamma(n-3/2)}{\Gamma(n+1/2)} = \frac{4}{(2n-3)(2n-1)}, \qquad \frac{\Gamma(3/2-n)}{\Gamma(5/2)} = \frac{(-2)^{n+1}(2n-1)}{3(2n-1)!!},$$

we have

$${}_2F_1(2, n+1/2;\, n+3\,|\,-3) = \frac{4(n+2)!}{9n!(2n-1)(2n-3)}\,{}_2F_1(2, -n;\, 5/2-n\,|\,-1/3)$$
$$+\frac{(-2)^{n+1}(n+2)!(2n-1)}{3^{n+3/2}(2n-1)!!}\,{}_2F_1(n+1/2, -3/2;\, n-1/2\,|\,-1/3)\,.$$

Since

$${}_2F_1(2, -n;\, 5/2-n\,|\,z) = \sum_{k=0}^{n}(k+1)z^k\prod_{i=0}^{k-1}\frac{n-i}{n-5/2-i}$$

and

$${}_2F_1(n+1/2, -3/2;\, n-1/2\,|\,z) = \frac{(2n-2nz-2z-1)\sqrt{1-z}}{2n-1}$$

(see formula (15.4.9) in [20]), we obtain

$${}_2F_1(2, n+1/2;\, n+3\,|\,-3) = \frac{n!(n+2)!(8n-1)(-4)^{n+1}}{(2n)!3^{n+3}}$$
$$+\frac{4(n+1)(n+2)}{9(2n-1)(2n-3)}\sum_{k=0}^{n}\frac{k+1}{(-3)^k}\prod_{i=0}^{k-1}\frac{n-i}{n-5/2-i},$$

which leads to (42).

For the second formula we apply the identity

$$ {}_2F_1(2,b;c\,|\,z)\,(1-z) = (bz-z-c+2)\,{}_2F_1(1,b;c\,|\,z) + c - 1, $$

see (15.5.11) in [20], to (45) and get

$$ {}_2F_1(2,n+1/2;\,n+3\,|\,-3) = \frac{1-8n}{8}\,{}_2F_1(1,n+1/2;\,n+3\,|\,-3) + \frac{n+2}{4}. $$

Applying formula (123), page 462, from [23]:

$$ {}_2F_1(1,b;\,m+1\,|\,z) = \frac{m!}{z^m(b-1)\ldots(b-m)}\left((1-z)^{m-b} - \sum_{k=0}^{m-1}\frac{z^k}{k!}\prod_{i=0}^{k-1}(b+i-m)\right), $$

with $b = n+1/2$, $m = n+2$, $z = -3$, and using the identity

$$ 4^{n+1}n!(n+1/2-1)\ldots(n+1/2-n-2) = 3(2n)! $$

we get (43). □

## References


[1] N. Alexeev, F. Götze and A. Tikhomirov, *Asymptotic distribution of singular values of powers of random matrices,* Lith. Math. J. **50** (2010), 121-132.

[2] D. Armstrong, *Generalized Noncrossing Partitions and Combinatorics of Coxeter Groups,* Memoirs of the American Mathematical Society, **202** (2009) 159 pp.

[3] T. Banica, S. Belinschi, M. Capitaine and B. Collins, *Free Bessel laws,* Canadian J. Math. **63** (2011), 3–37.

[4] Jonathan Bloom and Sergi Elizalde, *Pattern avoidance in matchings and partitions,* The Electronic Journal of Combinatorics **20** 2 (2013) #5.

[5] M. Bona, *Exact enumeration of 1342-avoiding permutations; A close link with labeled trees and planar maps,* J. Combinatorial Theory, **A80** (1997), 257–272.

[6] M. Bousquet-Mélou and G. Schaeffer, *Enumeration of planar constellations,* Adv. in Appl. Math. **24** (2000), 337-368.

[7] P. H. Edelman, *Chain enumeration and non-crossing partitions,* Discrete Math. **31** (1980), 171–180.

[8] P. J. Forrester and D.-Z. Liu, *Raney distributions and random matrix theory,* J. Stat. Phys. **158** (2015), 1051–1082.

[9] R. L. Graham, D. E. Knuth and O. Patashnik, *Concrete Mathematics. A Foundation for Computer Science,* Addison-Wesley, New York, 1994.

[10] U. Haagerup and S. Möller, *The law of large numbers for the free multiplicative convolution,* Springer Proceedings in Mathematics and Statistics, Vol. 58, ed. T. M. Carlsen et al., Springer, Berlin, 2014.

[11] D. Liu, C. Song and Z. Wang, *On explicit probability densities associated with Fuss-Catalan numbers,* Proc. Amer. Math. Soc. **139**/10 (2011), 3735–3738.

[12] J.-G. Liu and R. Pego, *On generating functions of Hausdorff moment sequences,* to appear in Trans. Am. Math. Soc.

[13] W. Młotkowski, *Fuss-Catalan numbers in noncommutative probability,* Documenta Math. **15** (2010), 939–955.

[14] W. Młotkowski, K. A. Penson and K. Życzkowski, *Densities of the Raney distributions,* Documenta Mathematica **18** (2013), 1573-1596.

[15] W. Młotkowski and K. A. Penson, *The probability measure corresponding to 2-plane trees,* Probability nad Mathematical Statistics, **33**/2 (2013), 255-264.

[16] W. Młotkowski and K. A. Penson, *Probability distributions with binomial moments,* Infinite Dimensional Analysis, Quantum Probability and Related Topics, **17**/2 (2014) 1450014, 32pp.

[17] W. Młotkowski, M. A. Nowak, K. A. Penson and K. Życzkowski *Spectral density of generalized Wishart matrices and free multiplicative convolution,* Phys. Rev. E **92** (2015) 012121.

[18] T. Neuschel, *Plancherel-Rotach formulae for average characteristic polynomials of products of Ginibre random matrices and the Fuss-Catalan distribution,* Random Matrices Theory Appl. **3** (2014), 1450003, 18 pp.
[19] A. Nica and R. Speicher, *Lectures on the Combinatorics of Free Probability*, Cambridge University Press, Cambridge, 2006.
[20] F. W. J. Olver, D. W. Lozier, R. F. Boisvert and C. W. Clark, *NIST Handbook of Mathematical Functions,* Cambridge University Press, Cambridge, 2010.
[21] K. A. Penson and A. I. Solomon, *Coherent states from combinatorial sequences,* pp. 527-530 in *Quantum theory and symmetries*, Kraków 2001, (World Sci. Publ., River Edge, NJ, 2002).
[22] K. A. Penson and K. Życzkowski, *Product of Ginibre matrices: Fuss-Catalan and Raney distributions,* Phys. Rev. E **83** (2011) 061118, 9 pp.
[23] A. P. Prudnikov, Yu. A. Brychkov and O. I. Marichev, *Integrals and Series,* Gordon and Breach, Amsterdam, 1998, Vol. 3: More special functions.
[24] J. H. Przytycki and A. S. Sikora, *Polygon Dissections and Euler, Fuss, Kirkman, and Cayley Numbers,* Journal of Combinatorial Theory, Series A **92**, Issue 1, (2000) 68-76.
[25] T. Simon, *Comparing Fréchet and positive stable laws,* Electron. J. Probab. **19** (2014) no. 16, 1–25.
[26] N. J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences,* (2015), published electronically at: http://oeis.org/.
[27] Alison Schuetz and Gwyneth Whieldon, *Polygonal Dissections and Reversions of Series,* arXiv:1401.7194.
[28] D. V. Voiculescu, K. J. Dykema, A. Nica, *Free random variables,* CRM, Montréal, 1992.

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