DIFFERENCE OPERATORS FOR PARTITIONS AND SOME APPLICATIONS

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Abstract. Motivated by the Nekrasov-Okounkov formula on hook lengths, the first author conjectured that

$$\frac{1}{n!}\sum_{|\lambda|=n} f_\lambda^2 \sum_{\square\in\lambda} h_\square^{2k}$$

is always a polynomial of n for any $k\in\mathbb{N}$, where $h_\square$ denotes the hook length of the box $\square$ in the partition λ and f_λ denotes the number of standard Young tableaux of shape λ. This conjecture was generalized and proved by R. P. Stanley (Ramanujan J., 23 (1–3) : 91–105, 2010). In this paper, we introduce two kinds of difference operators defined on functions of partitions and study their properties. As an application, we obtain a formula to compute

$$\frac{1}{(n+|\mu|)!}\sum_{|\lambda/\mu|=n} f_\lambda f_{\lambda/\mu} F(h_\square^2 : \square\in\lambda)$$

and therefore show that it is indeed a polynomial of n, where μ is any given partition, F is any symmetric function, and $f_{\lambda/\mu}$ denotes the number of standard Young tableaux of shape λ/μ. Our theorems could lead to many classical results on partitions, including marked hook formula, Han-Stanley Theorem, Okada-Panova hook length formula, and Fujii-Kanno-Moriyama-Okada content formula.

1. Introduction

Partitions of positive integers are widely studied in Combinatorics, Number Theory, and Representation Theory. A *partition* is a finite weakly decreasing sequence of positive integers $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_\ell)$. The integer $|\lambda|=\sum_{1\le i\le \ell}\lambda_i$ is called the *size* of the partition λ. A partition λ could be identical with its Young diagram, which is a collection of boxes arranged in left-justified rows with λ_i boxes in the i-th row. The *hook length* of the box $\square$ in the Young diagram, denoted by $h_\square$, is the number of boxes exactly to the right, or exactly above, or the box itself. For example, the Young diagram and hook lengths of the partition $(6,3,3,2)$ are illustrated in Figure 1.

We refer the reader to [25] for the basic knowledge on Young tableaux and symmetric functions. Suppose that λ and μ are two partitions with $\lambda\supseteq\mu$. Denote by f_λ (resp. $f_{\lambda/\mu}$) the number of standard Young tableaux of shape λ (resp. λ/μ). Let $H_\lambda=\prod_{\square\in\lambda}h_\square$ be the product of all hook lengths of boxes in λ. Set $f_\emptyset=1$

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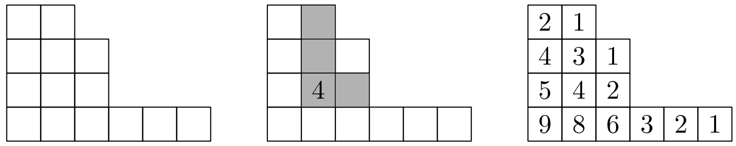

FIGURE 1. The Young diagram of the partition $(6, 3, 3, 2)$ and the hook lengths of corresponding boxes.

and $H_\emptyset = 1$ for the empty partition $\emptyset$. It is well known that (see [7, 10, 15, 25])

$$f_\lambda = \frac{|\lambda|!}{H_\lambda} \qquad \text{and} \qquad \frac{1}{n!}\sum_{|\lambda|=n} f_\lambda^2 = 1. \tag{1.1}$$

Nekrasov and Okounkov [17] obtained the following formula for hook lengths

$$\sum_{n\geq 0}\left(\sum_{|\lambda|=n} f_\lambda^2 \prod_{\square\in\lambda}(t+h_\square^2)\right)\frac{x^n}{n!^2} = \prod_{i\geq 1}(1-x^i)^{-1-t},$$

which was generalized and given a more elementary proof by the first author [10]. Motivated by the above formula, the first author conjectured that

$$\frac{1}{n!}\sum_{|\lambda|=n} f_\lambda^2 \sum_{\square\in\lambda} h_\square^{2k}$$

is always a polynomial of n for any positive integer k, which was generalized and proved by R. P. Stanley [23].

Theorem 1.1 (Han-Stanley)**.** *Let $F = F(z_1, z_2, \ldots)$ be a symmetric function of infinite variables. Then the function of positive integer n*

$$\frac{1}{n!}\sum_{|\lambda|=n} f_\lambda^2 F(h_\square^2 : \square\in\lambda) \tag{1.2}$$

is a polynomial of n, where $F(h_\square^2 : \square\in\lambda)$ means that n of the variables $z_1, z_2, \ldots$ are substituted by $h_\square^2$ for $\square\in\lambda$, and all other variables by 0.

The polynomiality of (1.2) suggested Okada to conjecture an explicit formula [23], which was proved by Panova [21]. Let

$$S(\lambda, r) := \sum_{\square\in\lambda}\prod_{1\leq j\leq r}(h_\square^2 - j^2) \tag{1.3}$$

and

$$K_r := \frac{(2r)!(2r+1)!}{r!(r+1)!^2}. \tag{1.4}$$

The sequence $(K_0 = 1,\ K_1 = 3,\ K_2 = 40,\ K_3 = 1050, \ldots)$ appears as $A204515$ in the *On-Line Encyclopedia of Integer Sequences* [18].

Theorem 1.2 (Okada-Panova)**.** *For each positive integer n we have*

$$n!\sum_{|\lambda|=n}\frac{S(\lambda, r)}{H_\lambda^2} = K_r\binom{n}{r+1}. \tag{1.5}$$

In this paper, we introduce two kinds of difference operators D and D^- on functions of partitions, and obtain more general results by studying the properties on each summand $F(h_\square^2 : \square \in \lambda)$. As will be seen in Lemma 1.5 the constants K_r arise directly from the computation for a single partition λ, without the summation ranging over all partitions of size n.

Definition 1.1. Let $g(\lambda)$ be a function defined on partitions. *Difference operators* D and D^- are defined by

$$Dg(\lambda) = \sum_{\lambda^+} g(\lambda^+) - g(\lambda)$$

and

$$D^- g(\lambda) = |\lambda|\, g(\lambda) - \sum_{\lambda^-} g(\lambda^-),$$

where λ^+ (resp. λ^-) ranges over all partitions obtained by adding (resp. removing) a box to (resp. from) λ. Higher-order difference operators for D are defined by induction $D^0 g := g$ and $D^k g := D(D^{k-1}g)$ $(k \geq 1)$. Also, we write $Dg(\mu) := Dg(\lambda)|_{\lambda=\mu}$ for a fixed partition μ.

Definition 1.2. A function $g(\lambda)$ of partitions is called a *D-polynomial on partitions*, if there exists a positive integer r such that $D^{r+1}\big(g(\lambda)/H_\lambda\big) = 0$ for every partition λ. The minimal r satisfying this condition is called the *degree* of $g(\lambda)$.

Our two main theorems are stated next.

Theorem 1.3. *For each power sum symmetric function $p_\nu(z_1, z_2, \ldots)$ of infinite variables indexed by the partition $\nu = (\nu_1, \nu_2, \ldots, \nu_\ell)$, the function $p_\nu(h_\square^2 : \square \in \lambda)$ of partition λ is a D-polynomial with degree at most $|\nu| + \ell$.*

Theorem 1.4. *Let g be a function of partitions and μ be a given partition. Then we have*

$$\sum_{|\lambda/\mu|=n} f_{\lambda/\mu}\, g(\lambda) = \sum_{k=0}^{n} \binom{n}{k} D^k g(\mu) \tag{1.6}$$

and

$$D^n g(\mu) = \sum_{k=0}^{n} (-1)^{n+k} \binom{n}{k} \sum_{|\lambda/\mu|=k} f_{\lambda/\mu}\, g(\lambda). \tag{1.7}$$

In particular, if there exists some positive integer r such that $D^r g(\lambda) = 0$ for every partition λ, then the left-hand side of (1.6) *is a polynomial of n.*

The proofs of Theorems 1.3 and 1.4 are given in Sections 7 and 3 respectively. Let us give some applications. By Theorem 1.3 and Theorem 1.4 with $\mu = \emptyset$, we derive Han-Stanley Theorem. In Section 8 we prove the following lemma, and show that Okada-Panova formula can be derived by this result and Theorem 1.4 with $\mu = \emptyset$.

Lemma 1.5. *For each positive integer r, the function $S(\lambda, r)$ of partitions is a D-polynomial of degree $r+1$. More precisely,*

$$H_\lambda D^r\Big(\frac{S(\lambda,r)}{H_\lambda}\Big) = K_r|\lambda|, \tag{1.8}$$

$$H_\lambda D^{r+1}\Big(\frac{S(\lambda,r)}{H_\lambda}\Big) = K_r, \tag{1.9}$$

$$H_\lambda D^{r+2}\Big(\frac{S(\lambda,r)}{H_\lambda}\Big) = 0. \tag{1.10}$$

The special case $r = 1$ of Okada-Panova formula is usually called *marked hook formula* [11]:

$$\sum_{|\lambda|=n} \frac{f_\lambda}{H_\lambda} S(\lambda, 1) = 3\binom{n}{2}. \tag{1.11}$$

In Section 5 we obtain a generalization of (1.11).

Theorem 1.6 (Skew marked hook formula)**.** *Let μ be a given partition. For every $n \geq |\mu|$ we have*

$$\sum_{|\lambda|=n,\, \lambda\supset\mu} \frac{H_\mu f_{\lambda/\mu}}{H_\lambda}\Big(S(\lambda,1) - S(\mu,1)\Big) = \frac{3}{2}\,(n-|\mu|)\,(n+|\mu|-1). \tag{1.12}$$

Recall that the *content* of the box $\square = (i, j)$ of a partition is defined by $c_\square = j-i$ (see, for example, [16, 25]). Similar results are obtained for contents in Section 9. Let

$$C(\lambda, r) := \sum_{\square\in\lambda}\prod_{0\leq j\leq r-1}(c_\square^2 - j^2).$$

Lemma 1.7. *For each positive integer r, the function $C(\lambda, r)$ of partitions is a D-polynomial of degree $r+1$. More precisely,*

$$H_\lambda D^r\Big(\frac{C(\lambda,r)}{H_\lambda}\Big) = \frac{(2r)!}{(r+1)!}|\lambda|, \tag{1.13}$$

$$H_\lambda D^{r+1}\Big(\frac{C(\lambda,r)}{H_\lambda}\Big) = \frac{(2r)!}{(r+1)!}, \tag{1.14}$$

$$H_\lambda D^{r+2}\Big(\frac{C(\lambda,r)}{H_\lambda}\Big) = 0. \tag{1.15}$$

Theorem 1.8 (Fujii-Kanno-Moriyama-Okada)**.** *For each positive integer n we have*

$$n!\sum_{|\lambda|=n}\frac{C(\lambda,r)}{H_\lambda^2} = \frac{(2r)!}{(r+1)!}\binom{n}{r+1}.$$

Theorem 1.9 (Skew marked content formula)**.** *Let μ be a given partition. For every $n \geq |\mu|$ we have*

$$\sum_{|\lambda|=n,\, \lambda\supset\mu} \frac{H_\mu f_{\lambda/\mu}}{H_\lambda}\Big(C(\lambda,1) - C(\mu,1)\Big) = \frac{1}{2}\,(n-|\mu|)\,(n+|\mu|-1). \tag{1.16}$$

2. Difference operators for partitions

The difference operators D and D^- defined in Section 1 are our fundamental tools for studying hook length formulas. This section is devoted to establish some basic properties. It is obvious that D and D^- are linear operators.

Lemma 2.1. *Let λ be a partition and g_1, g_2 two functions of partitions. The following identities hold for all $a_1, a_2 \in \mathbb{R}$:*

$$D(a_1g_1 + a_2g_2)(\lambda) = a_1Dg_1(\lambda) + a_2Dg_2(\lambda),$$
$$D^-(a_1g_1 + a_2g_2)(\lambda) = a_1D^-g_1(\lambda) + a_2D^-g_2(\lambda).$$

Lemma 2.2. *For each partition λ we have*

$$D\Big(\frac{1}{H_\lambda}\Big) = 0.$$

Proof. Let $n = |\lambda|$. Consider the following two sets related to standard Young tableaux (written as "SYT" for simplicity)

$$A = \{(i, T) : 1 \le i \le n + 1, T \text{ is an SYT of shape } \lambda\},$$
$$B = \{(\lambda^+, T^+) : |\lambda^+/\lambda| = 1, T^+ \text{ is an SYT of shape } \lambda^+\}.$$

Let $(i, T) \in A$. First we increase every entry which are greater than or equal to i by one in T. Then, we use the Robinson-Schensted-Knuth algorithm [15] to insert the integer i into T to get a new SYT T^+. Let λ^+ be the shape of T^+. We have $|\lambda^+/\lambda| = 1$, so that $(\lambda^+, T^+) \in B$. It is easy to see that this is a bijection between sets A and B. The cardinalities of A and B are $(n+1)f_\lambda$ and $\sum_{\lambda^+} f_{\lambda^+}$ respectively. Hence

$$(n + 1)f_\lambda = \sum_{\lambda^+} f_{\lambda^+}.$$

This implies that

$$D\Big(\frac{1}{H_\lambda}\Big) = \sum_{\lambda^+} \frac{1}{H_{\lambda^+}} - \frac{1}{H_\lambda} = \frac{1}{(n + 1)!}\Big(\sum_{\lambda^+} f_{\lambda^+} - (n + 1)f_\lambda\Big) = 0. \qquad \square$$

Lemma 2.3. *Let $g(\lambda)$ be a function of partitions. Then $D^-g(\lambda) = 0$ for every partition λ if and only if*

$$g(\lambda) = \frac{a}{H_\lambda}$$

for some constant a.

Proof. By the definition of SYTs it is obvious that $f_\lambda = \sum_{\lambda^-} f_{\lambda^-}$. Thus,

$$D^-\Big(\frac{a}{H_\lambda}\Big) = \frac{a|\lambda|}{H_\lambda} - \sum_{\lambda^-} \frac{a}{H_{\lambda^-}} = \frac{a}{(|\lambda| - 1)!}\Big(f_\lambda - \sum_{\lambda^-} f_{\lambda^-}\Big) = 0. \tag{2.1}$$

On the other hand, $D^-g(\lambda) = 0$ implies $|\lambda|\, g(\lambda) = \sum_{\lambda^-} g(\lambda^-)$. Let $a = g(\emptyset)$ where $\emptyset$ is the empty partition. By induction and (2.1) we obtain $g(\lambda) = \frac{a}{H_\lambda}$. $\square$

Notice that it is not easy to determine the function $g(\lambda)$ under the condition $Dg(\lambda) = 0$ for every partition λ. For example, by (1.8) and Lemma 2.4 we have

$$D\left(\frac{\sum_{\square\in\lambda} (h_\square^2 - 1) - 3\binom{n}{2}}{H_\lambda}\right) = 0.$$

Lemma 2.4. *Let $n = |\lambda|$. Then we have*

$$D\Big(\frac{\binom{n}{r}}{H_\lambda}\Big) = \frac{\binom{n}{r-1}}{H_\lambda} \qquad \textit{and} \qquad D^-\Big(\frac{\binom{n}{r}}{H_\lambda}\Big) = \frac{r\binom{n}{r}}{H_\lambda}.$$

Proof. By Lemmas 2.2 and 2.3 we have

$$D\Big(\frac{\binom{n}{r}}{H_\lambda}\Big) = \sum_{\lambda^+} \frac{\binom{n+1}{r}}{H_{\lambda^+}} - \frac{\binom{n}{r}}{H_\lambda} = \frac{\binom{n+1}{r} - \binom{n}{r}}{H_\lambda} = \frac{\binom{n}{r-1}}{H_\lambda},$$

$$D^-\Big(\frac{\binom{n}{r}}{H_\lambda}\Big) = \frac{n\binom{n}{r}}{H_\lambda} - \sum_{\lambda^-} \frac{\binom{n-1}{r}}{H_{\lambda^-}} = \frac{n\binom{n}{r} - n\binom{n-1}{r}}{H_\lambda} = \frac{r\binom{n}{r}}{H_\lambda}. \qquad \square$$

Lemma 2.5. *For each function g defined on partitions we have*

$$D\Big(\frac{g(\lambda)}{H_\lambda}\Big) = \sum_{\lambda^+} \frac{g(\lambda^+) - g(\lambda)}{H_{\lambda^+}},$$

and

$$D^-\Big(\frac{g(\lambda)}{H_\lambda}\Big) = \sum_{\lambda^-} \frac{g(\lambda) - g(\lambda^-)}{H_{\lambda^-}}.$$

Proof. By Lemmas 2.2 and 2.3 we have

$$D\Big(\frac{g(\lambda)}{H_\lambda}\Big) = \sum_{\lambda^+} \frac{g(\lambda^+)}{H_{\lambda^+}} - \frac{g(\lambda)}{H_\lambda} = \sum_{\lambda^+} \frac{g(\lambda^+) - g(\lambda)}{H_{\lambda^+}},$$

$$D^-\Big(\frac{g(\lambda)}{H_\lambda}\Big) = |\lambda|\frac{g(\lambda)}{H_\lambda} - \sum_{\lambda^-} \frac{g(\lambda^-)}{H_{\lambda^-}} = \sum_{\lambda^+} \frac{g(\lambda) - g(\lambda^-)}{H_{\lambda^-}}. \qquad \square$$

Lemma 2.6 (Leibniz's rule)**.** *Let $g_1, g_2, \cdots, g_r$ be functions defined on partitions. We have*

$$D\Big(\frac{\prod_{1\le j\le r} g_j(\lambda)}{H_\lambda}\Big) = \sum_{\lambda^+}\sum_{(*)} \frac{1}{H_{\lambda^+}}\Big(\prod_{k\in A}\big(g_k(\lambda^+) - g_k(\lambda)\big)\prod_{l\in B} g_l(\lambda)\Big)$$

and

$$D^-\Big(\frac{\prod_{1\le j\le r} g_j(\lambda)}{H_\lambda}\Big) = -\sum_{\lambda^-}\sum_{(*)} \frac{1}{H_{\lambda^-}}\Big(\prod_{k\in A}\big(g_k(\lambda^-) - g_k(\lambda)\big)\prod_{l\in B} g_l(\lambda)\Big),$$

where $[r] := \{1, 2, \cdots, r\}$ and the sum $(*)$ ranges over all pairs $(A, B) \subset [r]\times[r]$ such that $A \cup B = [r]$, $A \cap B = \emptyset$ and $A \neq \emptyset$. In particular,

$$D\Big(\frac{g_1(\lambda)g_2(\lambda)}{H_\lambda}\Big) = g_1(\lambda)D\Big(\frac{g_2(\lambda)}{H_\lambda}\Big) + g_2(\lambda)D\Big(\frac{g_1(\lambda)}{H_\lambda}\Big) + \sum_{\lambda^+}\frac{1}{H_{\lambda^+}}\big(g_1(\lambda^+) - g_1(\lambda)\big)\big(g_2(\lambda^+) - g_2(\lambda)\big)$$

and

$$D^-\Big(\frac{g_1(\lambda)g_2(\lambda)}{H_\lambda}\Big) = g_1(\lambda)D^-\Big(\frac{g_2(\lambda)}{H_\lambda}\Big) + g_2(\lambda)D^-\Big(\frac{g_1(\lambda)}{H_\lambda}\Big) - \sum_{\lambda^-}\frac{1}{H_{\lambda^-}}\big(g_1(\lambda) - g_1(\lambda^-)\big)\big(g_2(\lambda) - g_2(\lambda^-)\big).$$

Proof. By Lemma 2.5 we have

$$\begin{aligned} D\Big(\frac{\prod_{1\le j\le r} g_j(\lambda)}{H_\lambda}\Big) &= \sum_{\lambda^+} \frac{1}{H_{\lambda^+}}\Big(\prod_{1\le j\le r} g_j(\lambda^+) - \prod_{1\le j\le r} g_j(\lambda)\Big) \\ &= \sum_{\lambda^+} \frac{1}{H_{\lambda^+}}\Big(\prod_{1\le j\le r} \big(g_j(\lambda) + (g_j(\lambda^+) - g_j(\lambda))\big) - \prod_{1\le j\le r} g_j(\lambda)\Big) \\ &= \sum_{\lambda^+}\sum_{(*)} \frac{1}{H_{\lambda^+}}\Big(\prod_{k\in A} \big(g_k(\lambda^+) - g_k(\lambda)\big) \prod_{l\in B} g_l(\lambda)\Big). \end{aligned}$$

The proof for D^- is similar. □

For higher-order difference operators, we have the following result.

Lemma 2.7. *Suppose that k is a nonnegative integer. Let $n = |\lambda|$. Then we have*

$$D^k\Big(\binom{n}{r} g(\lambda)\Big) = \sum_{i=0}^{k} \binom{k}{i}\binom{n+i}{r-k+i} D^i g(\lambda). \tag{2.2}$$

Proof. First we have

$$\begin{aligned} D\Big(\binom{n+j}{r} g(\lambda)\Big) &= \sum_{\lambda^+} \binom{n+1+j}{r} g(\lambda^+) - \binom{n+j}{r} g(\lambda) \\ &= \binom{n+1+j}{r} Dg(\lambda) + \binom{n+j}{r-1} g(\lambda). \end{aligned} \tag{2.3}$$

We prove (2.2) by induction. Identity (2.2) is true when $k = 0, 1$ by (2.3). Assume that the lemma is true for some $k \ge 1$, then

$$\begin{aligned} &D\Big(D^k(\binom{n}{r} g(\lambda))\Big) \\ &= \sum_{i=0}^{k} \binom{k}{i} D(\binom{n+i}{r-k+i} D^i g(\lambda)) \\ &= \sum_{i=0}^{k} \binom{k}{i}\Big(\binom{n+1+i}{r-k+i} D^{i+1} g(\lambda) + \binom{n+i}{r-k+i-1} D^i g(\lambda)\Big) \\ &= \sum_{i=1}^{k+1} \binom{k}{i-1}\binom{n+i}{r-k+i-1} D^i g(\lambda) + \sum_{i=0}^{k} \binom{k}{i}\binom{n+i}{r-k+i-1} D^i g(\lambda) \\ &= \sum_{i=0}^{k+1} \binom{k+1}{i}\binom{n+i}{r-k+i-1} D^i g(\lambda). \end{aligned}$$

□

Lemma 2.8. *The two difference operators D and D^- are noncommutative, and satisfy*

$$DD^- - D^-D = D.$$

Proof. If $(\lambda^+)^- \ne \lambda$, then $(\lambda^+)^- = \lambda \cup \{\square_1\} \setminus \{\square_2\}$ for some boxes $\square_1 \ne \square_2$. This means that we can switch the order of adding $\square_1$ and removing $\square_2$ and get the same partition $(\lambda \setminus \{\square_2\}) \cup \{\square_1\} \in \{(\lambda^-)^+ : (\lambda^-)^+ \ne \lambda\}$. Consequently,

$$\{(\lambda^+)^- : (\lambda^+)^- \ne \lambda\} = \{(\lambda^-)^+ : (\lambda^-)^+ \ne \lambda\}. \tag{2.4}$$

For a given partition, the number of ways to add a box minus the number of ways to remove a box always equals to 1. Thus

$$\sum_{(\lambda^+)^-} g((\lambda^+)^-) - \sum_{(\lambda^-)^+} g((\lambda^-)^+) = g(\lambda).$$

By definition of D and D^-, we have

$$\begin{aligned} DD^-g(\lambda) &= \sum_{\lambda^+} D^-g(\lambda^+) - D^-g(\lambda) \\ &= |\lambda^+| \sum_{\lambda^+} g(\lambda^+) - \sum_{(\lambda^+)^-} g\big((\lambda^+)^-\big) - |\lambda| g(\lambda) + \sum_{\lambda^-} g(\lambda^-) \end{aligned}$$

and

$$\begin{aligned} D^-Dg(\lambda) &= |\lambda| Dg(\lambda) - \sum_{\lambda^-} Dg(\lambda^-) \\ &= |\lambda| \sum_{\lambda^+} g(\lambda^+) - |\lambda| g(\lambda) - \sum_{(\lambda^-)^+} g\big((\lambda^-)^+\big) + \sum_{\lambda^-} g(\lambda^-). \end{aligned}$$

The above three identities yield

$$DD^-g(\lambda) - D^-Dg(\lambda) = \sum_{\lambda^+} g(\lambda^+) - g(\lambda) = Dg(\lambda). \quad \square$$

3. Telescoping sum for partitions

Lemma 3.1. *For each partition μ and each function g of partitions, let*

$$A(n) := \sum_{|\lambda/\mu|=n} f_{\lambda/\mu} g(\lambda)$$

and

$$B(n) := \sum_{|\lambda/\mu|=n} f_{\lambda/\mu} Dg(\lambda).$$

Then

$$A(n) = A(0) + \sum_{k=0}^{n-1} B(k).$$

Proof. By the definition of the operator D,

$$\sum_{\lambda^+} g(\lambda^+) = g(\lambda) + Dg(\lambda).$$

Summing the above equality over all SYTs T of shape λ/μ with $|\lambda/\mu| = n$, we have

$$\sum_{|\lambda/\mu|=n+1} \sum_{\mathrm{sh}(T)=\lambda/\mu} g(\lambda) = \sum_{|\lambda/\mu|=n} \sum_{\mathrm{sh}(T)=\lambda/\mu} g(\lambda) + \sum_{|\lambda/\mu|=n} \sum_{\mathrm{sh}(T)=\lambda/\mu} Dg(\lambda),$$

or

$$A(n+1) = A(n) + B(n),$$

where $\mathrm{sh}(T)$ denotes the shape of the SYT T. By iteration we obtain

$$\begin{aligned} A(n+1) &= A(n) + B(n) \\ &= A(n-1) + B(n-1) + B(n) \\ &= \cdots \\ &= A(0) + \sum_{k=0}^{n} B(k). \end{aligned}$$

□

Example. Let $g(\lambda) = 1/H_\lambda$. Then $Dg(\lambda) = 0$ by Lemma 2.2. The two quantities defined in Lemma 3.1 are:

$$A(n) = \sum_{|\lambda/\mu|=n} \frac{f_{\lambda/\mu}}{H_\lambda} \qquad \text{and} \qquad B(n) = 0.$$

Consequently,

$$\sum_{|\lambda/\mu|=n} \frac{f_{\lambda/\mu}}{H_\lambda} = \frac{1}{H_\mu}. \tag{3.1}$$

In particular, we derive the second identity in (1.1) by letting $\mu = \emptyset$.

Now we are ready to give the proof of Theorem 1.4.

Proof of Theorem 1.4. First, we prove (1.6) by induction. The case $n = 0$ is trivial. Assume that (1.6) is true for some nonnegative integer n. Then by Lemma 3.1 we have

$$\begin{aligned} \sum_{|\lambda/\mu|=n+1} f_{\lambda/\mu} g(\lambda) &= \sum_{|\lambda/\mu|=n} f_{\lambda/\mu} g(\lambda) + \sum_{|\lambda/\mu|=n} f_{\lambda/\mu} Dg(\lambda) \\ &= \sum_{k=0}^{n} \binom{n}{k} D^k g(\mu) + \sum_{k=0}^{n} \binom{n}{k} D^{k+1} g(\mu) \\ &= \sum_{k=0}^{n+1} \binom{n+1}{k} D^k g(\mu). \end{aligned}$$

Identity (1.7) can be proved by the Möbius inversion formula [22]. □

4. Shifted parts of partitions

Suppose that $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ is a partition with size n. Let

$$\varphi_\lambda(z) = \prod_{i=1}^{n} (z + n + \lambda_i - i),$$

where $\lambda_i = 0$ for $i \geq \ell + 1$. In this section, we will prove the following result.

Theorem 4.1. *Suppose that z is a formal parameter. For each partition λ we have*

$$D\Big(\frac{\varphi_\lambda(z)}{H_\lambda}\Big) = \frac{z\varphi_\lambda(z+1)}{H_\lambda}.$$

Theorem 4.1 has several direct corollaries.

Corollary 4.2. *Suppose that z is a formal parameter and r is a nonnegative integer. For each partition λ we have*

$$D^{r+1}\Big(\frac{\varphi_\lambda(z)}{H_\lambda}\Big) = \frac{z(z+1)\cdots(z+r)\varphi_\lambda(z+r+1)}{H_\lambda}.$$

In particular, $\varphi_\lambda(-r)$ is a D-polynomial with degree at most r, or equivalently,

$$D^{r+1}\Big(\frac{\varphi_\lambda(-r)}{H_\lambda}\Big) = 0.$$

By Corollary 4.2 and Theorem 1.4 we obtain

Corollary 4.3. *Suppose that r is a nonnegative integer and μ is a given partition. Then we have*

$$\sum_{|\lambda/\mu|=n} f_{\lambda/\mu}\frac{\varphi_\lambda(-r)}{H_\lambda} = \sum_{k=0}^{r}\binom{n}{k}D^k\Big(\frac{\varphi_\mu(-r)}{H_\mu}\Big) \tag{4.1}$$

is a polynomial of n with degree at most r.

To prove Theorem 4.1, we need the following lemma proved by the first author in [12].

Lemma 4.4 ((2.2) of [12])**.** *Suppose that $\lambda_i > \lambda_{i+1}$. Then*

$$\frac{H_\lambda}{H_{\lambda'}} = \frac{\prod_{j=1}^{n}(i-\lambda_i+1+\lambda_j-j)}{\prod_{j=1}^{n-1}(i-\lambda_i+{\lambda'}_j-j)},$$

where λ' is obtained from λ by removing a box from the i-th row.

Proof of Theorem 4.1. Let

$$\phi(z) = D\Big(\frac{\varphi_\lambda(z)}{H_\lambda}\Big) - \frac{z\varphi_\lambda(z+1)}{H_\lambda}.$$

It is easy to see that $\phi(z)$ is a polynomial of z with degree at most $n+1 = |\lambda|+1$. Furthermore,

$$[z^{n+1}]\phi(z) = \sum_{\lambda^+}\frac{1}{H_{\lambda^+}} - \frac{1}{H_\lambda} = 0$$

and

$$[z^n]\phi(z) = \sum_{\lambda^+}\frac{\binom{n+2}{2}}{H_{\lambda^+}} - \frac{1}{H_\lambda} - \frac{\binom{n+1}{2}+n}{H_\lambda} = 0.$$

This means that $\phi(z)$ is a polynomial of z with degree at most $n-1$. To show that $\phi(z)=0$, we just need to find n distinct roots for $\phi(z)$. Let $z_i = i-\lambda_i-n-1$ for $1\le i\le n$. Actually we will show that $\phi(z_i)=0$.

If $\lambda_i = \lambda_{i-1}$, we know the factor $z+n+1+\lambda_i-i$ lies in $\varphi_{\lambda^+}(z)$ since we can not add a box in i-th row to λ and thus $\varphi_{\lambda^+}(z_i)=0$. Similarly $\varphi_\lambda(z_i)=\varphi_\lambda(z_i+1)=0$, which means that $\phi(z_i)=0$.

If $\lambda_i+1\le\lambda_{i-1}$, we can add a box in i-th row to λ. First we also have $\varphi_\lambda(z_i+1) = 0$ since $z_i+n+1+\lambda_i-1=0$. To show $\phi(z_i)=0$, we just need to show $D(\frac{\varphi_\lambda(z_i)}{H_\lambda})=0$, or equivalently,

$$\sum_{\lambda^+}\frac{H_\lambda}{H_{\lambda^+}}\varphi_{\lambda^+}(z_i) = \varphi_\lambda(z_i).$$

It is easy to see that only one term on the left side of last identity is not 0. Thus we just need to show that

$$\frac{H_\lambda}{H_{\lambda^*}}\varphi_{\lambda^*}(z_i) = \varphi_\lambda(z_i),$$

where λ^* is obtained by adding a box to λ in i-th row. But the last identity is equivalent to Lemma 4.4. We finish the proof.

□

5. An Application of Theorem 1.4 and Lemma 2.8

In this section, we derive some D-polynomials arising from the work of K. Carde, J. Loubert, A. Potechin and A. Sanborn [4]. Furthermore, the degrees of such D-polynomials can be explicitly determined. As an application of Theorem 1.4 and Lemma 2.8, we obtain the skew marked hook length formula (see Theorem 1.6). Let z be a formal parameter and $\rho(h, z)$ be the function defined on each positive integer h (see [4, 11]):

$$\begin{aligned}\rho(h,z) &:= \frac{(1+\sqrt{z})^h + (1-\sqrt{z})^h}{(1+\sqrt{z})^h - (1-\sqrt{z})^h}\cdot h\sqrt{z}\\ &= \frac{h\sum_{k\geq 0}\binom{h}{2k}z^k}{\sum_{k\geq 0}\binom{h}{2k+1}z^k}\\ &= 1 + \frac{h^2-1}{3}z - \frac{(h^2-1)(h^2-4)}{45}z^2 + \frac{(h^2-1)(h^2-4)(2h^2-11)}{945}z^3 + \cdots .\end{aligned}$$

Definition 5.1. The functions $L_k(\lambda)$ of partitions are defined by the following generating function

$$\prod_{\square\in\lambda}\rho(h_\square, z) = \sum_{k\geq 0} L_k(\lambda) z^k.$$

For example, we have

$$L_0(\lambda) = 1 \quad \text{and} \quad L_1(\lambda) = \frac{1}{3}\sum_{\square\in\lambda}(h_\square^2 - 1) = \frac{S(\lambda,1)}{3}.$$

Theorem 5.1. *For each partition λ we have*

$$D^{2r+1}\left(\frac{L_r(\lambda)}{H_\lambda}\right) = 0, \qquad (r\geq 0) \tag{5.1}$$

$$D^{2r}\left(\frac{L_r(\lambda)}{H_\lambda}\right) = \frac{(2r-1)!!}{H_\lambda}, \qquad (r\geq 1) \tag{5.2}$$

$$D^{2r-1}\left(\frac{L_r(\lambda)}{H_\lambda}\right) = \frac{(2r-1)!!}{H_\lambda}|\lambda|. \qquad (r\geq 1) \tag{5.3}$$

Recall the following result obtained in [4].

Lemma 5.2 (Carde-Loubert-Potechin-Sanborn)**.** *For each partition λ we have*

$$\sum_{\lambda^+} w(\lambda^+) = w(1)w(\lambda) + \sum_{\lambda^-} w(\lambda^-),$$

where

$$w(\lambda) = \prod_{\square\in\lambda}\frac{\rho(h_\square, z)}{h_\square\sqrt{z}}.$$

Lemma 5.2 implies

$$\sum_{\lambda^+} \frac{\prod_{\square\in\lambda^+} \rho(h(\square), z)}{H_{\lambda^+}} - \frac{\prod_{\square\in\lambda} \rho(h(\square), z)}{H_\lambda} = z \sum_{\lambda^-} \frac{\prod_{\square\in\lambda^-} \rho(h(\square), z)}{H_{\lambda^-}}.$$

Comparing the coefficients of z^k, we obtain

$$D\Big(\frac{L_k(\lambda)}{H_\lambda}\Big) = \frac{|\lambda| L_{k-1}(\lambda)}{H_\lambda} - D^-\Big(\frac{L_{k-1}(\lambda)}{H_\lambda}\Big). \tag{5.4}$$

Lemma 5.3. *For each partition λ and each integer $r \geq 1$ we have*

$$D^r\Big(\frac{L_k(\lambda)}{H_\lambda}\Big) = |\lambda| D^{r-1}\Big(\frac{L_{k-1}(\lambda)}{H_\lambda}\Big) + (r-1) D^{r-2}\Big(\frac{L_{k-1}(\lambda)}{H_\lambda}\Big) - D^- D^{r-1}\Big(\frac{L_{k-1}(\lambda)}{H_\lambda}\Big).$$

Proof. The lemma is true when $r = 1$ by (5.4). Assume that it is true for some $r \geq 1$. By Lemmas 2.7 and 2.8 we have

$$\begin{aligned} D^{r+1}\Big(\frac{L_k(\lambda)}{H_\lambda}\Big) &= D\Big(|\lambda| D^{r-1}\Big(\frac{L_{k-1}(\lambda)}{H_\lambda}\Big) + (r-1) D^{r-2}\Big(\frac{L_{k-1}(\lambda)}{H_\lambda}\Big) \\ &\qquad - D^- D^{r-1}\Big(\frac{L_{k-1}(\lambda)}{H_\lambda}\Big)\Big) \\ &= |\lambda| D^{r}\Big(\frac{L_{k-1}(\lambda)}{H_\lambda}\Big) + r D^{r-1}\Big(\frac{L_{k-1}(\lambda)}{H_\lambda}\Big) - D^- D^{r}\Big(\frac{L_{k-1}(\lambda)}{H_\lambda}\Big). \quad \square \end{aligned}$$

Proof of Theorem 5.1. Identity (5.1) is proved by induction on r. When $r = 0$, we have $D(\frac{L_0(\lambda)}{H_\lambda}) = D(\frac{1}{H_\lambda}) = 0$ by Lemma 2.2. Assume that (5.1) is true for some $r \geq 0$. So that

$$D^{2r+1}\Big(\frac{L_r(\lambda)}{H_\lambda}\Big) = D^{2r+2}\Big(\frac{L_r(\lambda)}{H_\lambda}\Big) = 0.$$

By Lemma 5.3 we obtain

$$\begin{aligned} D^{2r+3}\Big(\frac{L_{r+1}(\lambda)}{H_\lambda}\Big) &= |\lambda| D^{2r+2}\Big(\frac{L_r(\lambda)}{H_\lambda}\Big) + (2r+2) D^{2r+1}\Big(\frac{L_r(\lambda)}{H_\lambda}\Big) \\ &\qquad - D^- D^{2r+2}\Big(\frac{L_r(\lambda)}{H_\lambda}\Big) \\ &= 0. \end{aligned}$$

For (5.2) and (5.3) we proceed in the same manner. By Lemma 5.3, we have

$$\begin{aligned} D^{2r+2}\Big(\frac{L_{r+1}(\lambda)}{H_\lambda}\Big) &= |\lambda| D^{2r+1}\Big(\frac{L_r(\lambda)}{H_\lambda}\Big) + (2r+1) D^{2r}\Big(\frac{L_r(\lambda)}{H_\lambda}\Big) \\ &\qquad - D^- D^{2r+1}\Big(\frac{L_r(\lambda)}{H_\lambda}\Big) \\ &= (2r+1) D^{2r}\Big(\frac{L_r(\lambda)}{H_\lambda}\Big) \\ &= (2r+1) \cdot \frac{(2r-1)!!}{H_\lambda} \\ &= \frac{(2r+1)!!}{H_\lambda}, \end{aligned}$$

and

$$\begin{aligned}D^{2r+1}\Big(\frac{L_{r+1}(\lambda)}{H_\lambda}\Big) &= |\lambda| D^{2r}\Big(\frac{L_r(\lambda)}{H_\lambda}\Big) + 2rD^{2r-1}\Big(\frac{L_r(\lambda)}{H_\lambda}\Big) \\ &\quad - D^- D^{2r}\Big(\frac{L_r(\lambda)}{H_\lambda}\Big) \\ &= |\lambda|\frac{(2r-1)!!}{H_\lambda} + (2r-1)!!\frac{2r|\lambda|}{H_\lambda} - D^- \frac{(2r-1)!!}{H_\lambda} \\ &= (2r+1)!!\frac{|\lambda|}{H_\lambda}.\end{aligned}$$

The case $r = 1$ is guaranteed by Lemma 5.3. □

By Theorems 5.1 and 1.4 we obtain

Theorem 5.4. *Let μ be a given partition and r a nonnegative integer. Then*

$$\sum_{|\lambda/\mu|=n} f_{\lambda/\mu} \frac{L_r(\lambda)}{H_\lambda} = \sum_{0\le k\le 2r} \binom{n}{k} D^k\Big(\frac{L_r(\mu)}{H_\mu}\Big)$$

is a polynomial of n with degree at most $2r$. In particular, let $\mu = \emptyset$, we have

$$\sum_{|\lambda|=n} f_\lambda \frac{L_r(\lambda)}{H_\lambda} = \sum_{0\le k\le 2r} d_k \binom{n}{k}$$

where $d_k = D^k\big(\frac{L_r(\lambda)}{H_\lambda}\big)\big|_{\lambda=\emptyset}$.

Proof of Theorem 1.6. Let $r = 1$ in Theorem 5.4, we obtain

$$\begin{aligned}\sum_{|\lambda/\mu|=n} f_{\lambda/\mu}\frac{L_1(\lambda)}{H_\lambda} &= \frac{L_1(\mu)}{H_\mu} + nD\Big(\frac{L_1(\mu)}{H_\mu}\Big) + \binom{n}{2} D^2\Big(\frac{L_1(\mu)}{H_\mu}\Big) \\ &= \frac{L_1(\mu)}{H_\mu} + n\frac{|\mu|}{H_\mu} + \binom{n}{2}\frac{1}{H_\mu},\end{aligned}$$

and

$$\sum_{|\lambda/\mu|=n} f_{\lambda/\mu} H_\mu \frac{L_1(\lambda) - L_1(\mu)}{H_\lambda} = n|\mu| + \binom{n}{2}$$

by (3.1). This is equivalent to (1.12). □

6. A family of D-polynomials

For a partition λ, the *outer corners* (see [3]) are the boxes which can be removed to get a new partition λ^-. Let $(\alpha_1, \beta_1), \ldots, (\alpha_m, \beta_m)$ be the coordinates of outer corners such that $\alpha_1 > \alpha_2 > \cdots > \alpha_m$. Let $y_j = \beta_j - \alpha_j$ be the contents of outer corners for $1 \le j \le m$. We set $\alpha_{m+1} = \beta_0 = 0$ and call $(\alpha_1, \beta_0), (\alpha_2, \beta_1), \ldots, (\alpha_{m+1}, \beta_m)$ the *inner corners* of λ. Let $x_i = \beta_i - \alpha_{i+1}$ be the contents of inner corners for $0 \le i \le m$ (see Figure 2). It is easy to verify that x_i and y_j satisfy the following relation:

$$x_0 < y_1 < x_1 < y_2 < x_2 < \cdots < y_m < x_m. \tag{6.1}$$

According to Olshanski [19] we define

$$q_k(\lambda) := \sum_{0\le i\le m} {x_i}^k - \sum_{1\le j\le m} {y_j}^k \tag{6.2}$$

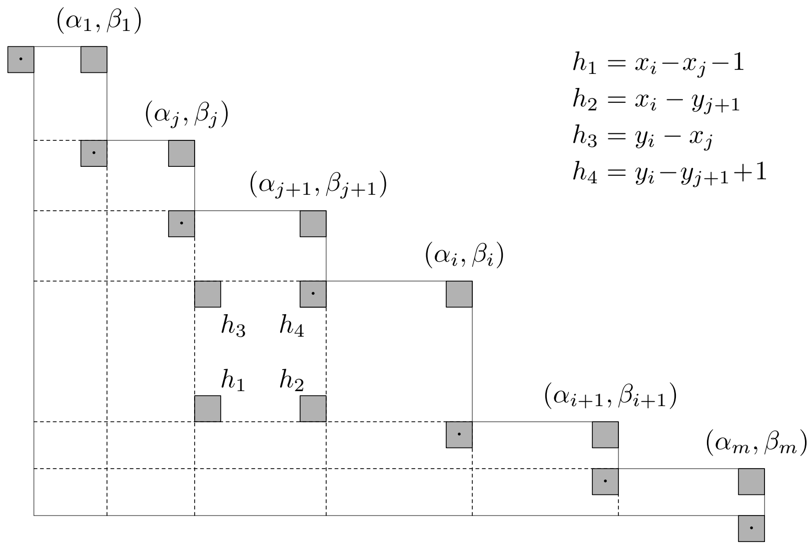

FIGURE 2. A partition and its corners. The outer corners are labelled with (α_i, β_i) $(i = 0, 1, \ldots, m)$. The inner corners are indicated by the dot symbol "·".

for each $k \geq 0$. The first three values of $(q_k(\lambda))_{k\geq 0}$ can be evaluated explicitly. For each partition λ we have

$$q_0(\lambda) = 1, \quad q_1(\lambda) = 0 \quad \text{and} \quad q_2(\lambda) = 2\,|\lambda|. \tag{6.3}$$

Let us prove (6.3). First we have $q_0(\lambda) = (m+1) - m = 1$. By definition of x_i and y_j, we obtain

$$\sum_{0\leq i\leq m} x_i = \sum_{1\leq j\leq m} y_j = \sum_{1\leq i\leq m} \beta_i - \sum_{1\leq j\leq m} \alpha_j.$$

Thus

$$q_1(\lambda) = \sum_{0\leq i\leq m} x_i - \sum_{1\leq j\leq m} y_j = 0.$$

We also have

$$\begin{aligned} q_2(\lambda) &= \sum_{0\leq i\leq m} x_i^2 - \sum_{1\leq j\leq m} y_j^2 \\ &= \sum_{0\leq i\leq m} (\beta_i - \alpha_{i+1})^2 - \sum_{1\leq j\leq m} (\beta_j - \alpha_j)^2 \\ &= \sum_{1\leq i\leq m} 2\beta_i(\alpha_i - \alpha_{i+1}). \end{aligned}$$

From the Young diagram of λ (see Figure 2) it is easy to see that $\sum_{1\leq i\leq m} \beta_i(\alpha_i - \alpha_{i+1})$ is equal to the number of boxes in λ, which is $|\lambda|$. Hence $q_2(\lambda) = 2\,|\lambda|$.

For each partition $\nu = (\nu_1, \nu_2, \ldots, \nu_\ell)$ we define the function $q_\nu(\lambda)$ of partitions by

$$q_\nu(\lambda) := q_{\nu_1}(\lambda) q_{\nu_2}(\lambda) \cdots q_{\nu_\ell}(\lambda). \tag{6.4}$$

Theorem 6.1. *Let ν be a partition. Then $q_\nu(\lambda)$ is a D-polynomial with degree at most $|\nu|/2$. Furthermore, there exist some $b_\delta \in \mathbb{Q}$ such that*

$$D\Big(\frac{q_\nu(\lambda)}{H_\lambda}\Big) = \sum_{|\delta|\le|\nu|-2} b_\delta \frac{q_\delta(\lambda)}{H_\lambda} \tag{6.5}$$

for every partition λ.

Notice that (6.5) could also be obtained by carefully reading [19] or [20]. But for completeness and since it is not explicitly given in [19] or [20], we will include a proof here. First some lemmas are needed. For each $k = 0, 1, \ldots, m$, denote by $\square_k = (\alpha_{k+1}+1, \beta_k+1)$ and $\lambda^{k+} = \lambda \cup \{\square_k\}$.

Lemma 6.2. *Let g be a function defined on integers. Then we have*

$$\begin{aligned}\sum_{\square\in\lambda^{k+}} g(h_\square) - \sum_{\square\in\lambda} g(h_\square) = g(1) &+ \sum_{0\le i\le k-1} \big(g(x_k - x_i) - g(x_k - y_{i+1})\big)\\ &+ \sum_{k+1\le i\le m} \big(g(x_i - x_k) - g(y_i - x_k)\big)\end{aligned}$$

and

$$\frac{\prod_{\square\in\lambda^{k+}} g(h_\square)}{\prod_{\square\in\lambda} g(h_\square)} = g(1) \prod_{0\le i\le k-1} \frac{g(x_k - x_i)}{g(x_k - y_{i+1})} \prod_{k+1\le i\le m} \frac{g(x_i - x_k)}{g(y_i - x_k)}.$$

In particular, we have

$$\frac{H_{\lambda^{k+}}}{H_\lambda} = \frac{\prod\limits_{\substack{0\le i\le m\\ i\ne k}} (x_k - x_i)}{\prod\limits_{1\le j\le m} (x_k - y_j)}.$$

Proof. When adding the box $\square_k$ to λ, it is easy to see that the hook lengths of boxes which are in the same row or the same column with $\square_k$ increase by 1. The hook lengths of other boxes don't change. Thus we have

$$\begin{aligned}\sum_{\square\in\lambda^{k+}} g(h_\square) - \sum_{\square\in\lambda} g(h_\square) &= \sum_{1\le i\le\alpha_{k+1}} \big(g(h_{(i,\beta_k+1)}(\lambda^{k+})) - g(h_{(i,\beta_k+1)}(\lambda))\big)\\ &+ \sum_{1\le j\le\beta_k} \big(g(h_{(\alpha_{k+1}+1,j)}(\lambda^{k+})) - g(h_{(\alpha_{k+1}+1,j)}(\lambda))\big) + g(h_{\square_k}(\lambda^{k+})),\end{aligned}$$

where $h_\square(\lambda)$ (resp. $h_\square(\lambda^{k+})$) denotes the hook length of the box $\square$ in λ (resp. λ^{k+}). On the other hand, the hook lengths of

$$(1, \beta_k+1), (2, \beta_k+1), \cdots, (\alpha_{k+1}, \beta_k+1)$$

in λ and λ^{k+} are

$$x_k - x_i - 1, x_k - x_i - 2, \cdots, x_k - y_{i+1} + 1, x_k - y_{i+1} \quad (0 \le i \le k-1)$$

and

$$x_k - x_i, x_k - x_i - 1, \cdots, x_k - y_{i+1} + 2, x_k - y_{i+1} + 1 \quad (0 \le i \le k-1)$$

respectively. Hence we obtain

$$\sum_{1\le i\le\alpha_{k+1}} \big(g(h_{(i,\beta_k+1)}(\lambda^{k+})) - g(h_{(i,\beta_k+1)}(\lambda))\big) = \sum_{0\le i\le k-1} \big(g(x_k - x_i) - g(x_k - y_{i+1})\big).$$

Similarly,

$$\sum_{1\le j\le \beta_k}\big(g(h_{(\alpha_{k+1}+1,j)}(\lambda^{k+}))-g(h_{(\alpha_{k+1}+1,j)}(\lambda))\big) = \sum_{k+1\le i\le m}\big(g(x_i-x_k)-g(y_i-x_k)\big).$$

Thus we obtain the first identity in the lemma. The second follows from replacing $g(h)$ by $\ln(g(h))$. In particular, $g(h)=h$ implies the third identity. □

Lemma 6.3. *Let g be a function defined on integers. Define*

$$g_1(\lambda) := \sum_{0\le i\le m} g(x_i) - \sum_{1\le j\le m} g(y_j)$$

which is a function of partitions. Then

$$D\Big(\frac{g_1(\lambda)}{H_\lambda}\Big) = \sum_{0\le i\le m}\frac{g(x_i+1)+g(x_i-1)-2g(x_i)}{H_{\lambda^{i+}}}.$$

In particular, let $g(z)=z^k$ so that $g_1(\lambda)=q_k(\lambda)$, then we obtain

$$D\Big(\frac{q_k(\lambda)}{H_\lambda}\Big) = \sum_{0\le i\le m}\frac{2}{H_{\lambda^{i+}}}\sum_{1\le j\le k/2}\binom{k}{2j}{x_i}^{k-2j}.$$

Proof. Denote by $X=\{x_0,x_1,\ldots,x_m\}$ and $Y=\{y_1,y_2,\ldots,y_m\}$. Four cases are to be considered. (i) If $\beta_i+1<\beta_{i+1}$ and $\alpha_{i+1}+1<\alpha_i$. Then it is easy to see that the contents of inner corners and outer corners of λ^{i+} are $X\cup\{x_i-1,x_i+1\}\setminus\{x_i\}$ and $Y\cup\{x_i\}$ respectively. (ii) If $\beta_i+1=\beta_{i+1}$ and $\alpha_{i+1}+1<\alpha_i$, so that $y_{i+1}=x_i+1$. Hence the contents of inner corners and outer corners of λ^{i+} are $X\cup\{x_i-1\}\setminus\{x_i\}$ and $Y\cup\{x_i\}\setminus\{x_i+1\}$ respectively. (iii) If $\beta_i+1<\beta_{i+1}$ and $\alpha_{i+1}+1=\alpha_i$, so that $y_i=x_i-1$. Then the contents of inner corners and outer corners of λ^{i+} are $X\cup\{x_i+1\}\setminus\{x_i\}$ and $Y\cup\{x_i\}\setminus\{x_i-1\}$ respectively. (iv) If $\beta_i+1=\beta_{i+1}$ and $\alpha_{i+1}+1=\alpha_i$. Then $y_i+1=x_i=y_{i+1}-1$. The contents of inner corners and outer corners of λ^{i+} are $X\setminus\{x_i\}$ and $Y\cup\{x_i\}\setminus\{x_i-1,x_i+1\}$ respectively. Thus we always have

$$g_1(\lambda^{i+})-g_1(\lambda) = g(x_i+1)+g(x_i-1)-2g(x_i). \tag{6.6}$$

Therefore

$$D\Big(\frac{g_1(\lambda)}{H_\lambda}\Big) = \sum_{0\le i\le m}\frac{g_1(\lambda^{i+})-g_1(\lambda)}{H_{\lambda^{i+}}} = \sum_{0\le i\le m}\frac{g(x_i+1)+g(x_i-1)-2g(x_i)}{H_{\lambda^{i+}}}$$

by Lemma 2.5. □

Lemma 6.4. *Let k be a nonnegative integer. Then there exist some $b_\nu\in\mathbb{Q}$ such that*

$$\sum_{0\le i\le m}\frac{H_\lambda}{H_{\lambda^{i+}}}{x_i}^k = \sum_{|\nu|\le k} b_\nu q_\nu(\lambda)$$

for every partition λ.

Proof. Let

$$g(z) = \prod_{1\le j\le m}(1-y_jz) - \sum_{0\le i\le m}\frac{H_\lambda}{H_{\lambda^{i+}}}\prod_{\substack{0\le j\le m\\ j\ne i}}(1-x_jz).$$

Then by Lemma 6.2 we obtain

$$\begin{aligned} g(\frac{1}{x_t}) &= \prod_{1\leq j\leq m}(1-\frac{y_j}{x_t}) - \frac{H_\lambda}{H_{\lambda^{t+}}}\prod_{\substack{0\leq j\leq m\\ j\neq t}}(1-\frac{x_j}{x_t}) \\ &= \prod_{1\leq j\leq m}(1-\frac{y_j}{x_t}) - \frac{\prod\limits_{1\leq j\leq m}(x_t-y_j)}{\prod\limits_{\substack{0\leq j\leq m\\ j\neq t}}(x_t-x_j)}\cdot\prod_{\substack{0\leq j\leq m\\ j\neq t}}(1-\frac{x_j}{x_t}) \\ &= 0. \end{aligned}$$

This means that $g(z)$ has at least $m+1$ roots, so that $g(z)=0$ since $g(z)$ is a polynomial of z with degree at most m. Therefore we obtain

$$\sum_{0\leq i\leq m}\frac{H_\lambda}{H_{\lambda^{i+}}}\cdot\frac{1}{1-x_iz} = \frac{\prod_{1\leq j\leq m}(1-y_jz)}{\prod_{0\leq j\leq m}(1-x_jz)},$$

which means that

$$\begin{aligned} \sum_{0\leq i\leq m}\frac{H_\lambda}{H_{\lambda^{i+}}}\big(\sum_{k\geq 0}(x_iz)^k\big) &= \exp\big(\sum_{1\leq j\leq m}\ln(1-y_jz) - \sum_{0\leq i\leq m}\ln(1-x_iz)\big) \\ &= \exp\big(\sum_{k\geq 1}\frac{q_k(\lambda)}{k}z^k\big). \end{aligned}$$

Comparing the coefficients of z^k on both sides, we obtain

$$\sum_{0\leq i\leq m}\frac{H_\lambda}{H_{\lambda^{i+}}}{x_i}^k = \sum_{|\nu|\leq k} b_\nu q_\nu(\lambda)$$

for some $b_\nu\in\mathbb{Q}$. Notice that b_ν are independent of λ. This achieves the proof. □

Proof of Theorem 6.1. Let k be an integer. By Lemma 6.3 we have

$$H_\lambda D\big(\frac{q_k(\lambda)}{H_\lambda}\big) = \sum_{0\leq i\leq m}\frac{H_\lambda}{H_{\lambda^{i+}}}\sum_{1\leq j\leq k/2}2\binom{k}{2j}{x_i}^{k-2j}.$$

Then there exist some $b_\delta\in\mathbb{Q}$ such that

$$D\big(\frac{q_k(\lambda)}{H_\lambda}\big) = \sum_{|\delta|\leq k-2} b_\delta\frac{q_\delta(\lambda)}{H_\lambda}$$

for every partition λ by Lemma 6.4. In other words, (6.5) is true for $\nu=(k)$.

From (6.6) with $g(z)=z^k$ we actually obtain

$$q_k(\lambda^{i+}) - q_k(\lambda) = \sum_{1\leq j\leq k/2}2\binom{k}{2j}{x_i}^{k-2j},$$

which is a polynomial of x_i with degree at most $k-2$. Then by Lemmas 2.6 and 6.4 there exist some $b_\delta\in\mathbb{Q}$ such that

$$H_\lambda D\big(\frac{q_\nu(\lambda)}{H_\lambda}\big) = \sum_{|\delta|\leq|\nu|-2} b_\delta q_\delta(\lambda)$$

for every partition λ. □

7. Hook lengths and D-polynomials

In this section, we prove Theorem 1.3. Let r be a fixed nonnegative integer. We will show that $S(\lambda, r)$ defined in (1.3) can be written as a symmetric polynomial on $\{x_0, x_1, \ldots, x_m\}$ and $\{y_1, y_2, \ldots, y_m\}$, as stated next.

Theorem 7.1. *There exist some rational numbers $b_\nu := b_\nu(r)$ indexed by integer partitions ν such that*

$$S(\lambda, r) = \sum_{|\nu| \le 2r+2} b_\nu q_\nu(\lambda) \tag{7.1}$$

for every partition λ.

Keep the same notations as in Section 6 (see Figure 2). Let

$$B_{ij} = \{(i', j') \in \lambda : \alpha_{i+1} + 1 \le i' \le \alpha_i, \beta_j + 1 \le j' \le \beta_{j+1}\}$$

so that

$$\lambda = \bigcup_{0 \le j < i \le m} B_{ij}.$$

The multiset of hook lengths of B_{ij} are

$$\bigcup_{a = x_i - y_{j+1}}^{x_i - x_j - 1} \{a, a-1, a-2, \ldots, a - (x_i - y_i - 1)\}.$$

Let $F_0(n)$ be a function defined on integers. Define

$$F_1(n) := \sum_{k=1}^{n} F_0(k) \qquad \text{and} \qquad F_2(n) := \sum_{k=1}^{n} F_1(k).$$

Hence

$$\begin{aligned}
\sum_{\square \in B_{ij}} F_0(h_\square) &= \sum_{a = x_i - y_{j+1}}^{x_i - x_j - 1} \sum_{b=0}^{x_i - y_i - 1} F_0(a - b) \\
&= \sum_{a = x_i - y_{j+1}}^{x_i - x_j - 1} (F_1(a) - F_1(a - x_i + y_i)) \\
&= \sum_{a = x_i - y_{j+1}}^{x_i - x_j - 1} F_1(a) - \sum_{a = x_i - y_{j+1}}^{x_i - x_j - 1} F_1(a - x_i + y_i) \\
&= F_2(x_i - x_j - 1) + F_2(y_i - y_{j+1} - 1) \\
&\qquad - F_2(x_i - y_{j+1} - 1) - F_2(y_i - x_j - 1)
\end{aligned}$$

and thus

$$\begin{aligned}
\sum_{\square \in \lambda} F_0(h_\square) &= \sum_{0 \le j < i \le m} \sum_{\square \in B_{ij}} F_0(h_\square) \\
&= \sum_{0 \le j < i \le m} \big(F_2(x_i - x_j - 1) + F_2(y_i - y_{j+1} - 1) \\
&\qquad - F_2(x_i - y_{j+1} - 1) - F_2(y_i - x_j - 1)\big).
\end{aligned} \tag{7.2}$$

For each $n \geq 1$ the polynomial $P_n(z)$ of real number z is defined by

$$P_n(z) := \frac{z^{n+1}}{n+1} + \frac{z^n}{2} + \frac{1}{n+1} \sum_{1 \leq j \leq n/2} \binom{n+1}{2j} z^{n-2j+1} (-1)^{j+1} B_{2j},$$

where B_{2j} are the classical Bernoulli numbers [5, 6, 14]. Let k be a positive integer. According to Euler-MacLaurin formula,

$$P_n(k) = 1^n + 2^n + \cdots + k^n.$$

Consequently, $P_n(k) = P_n(k+1) - (k+1)^n$. It is easy to obtain the following identity:

$$P_n(-k-1) = (-1)^{n+1} P_n(k). \qquad (n \geq 1) \tag{7.3}$$

For simplicity we rewrite

$$P_n(z) = \frac{z^n}{2} + \sum_{0 \leq j \leq n/2} \zeta_j(n) z^{n-2j+1}. \tag{7.4}$$

Let $G_0(j) = \prod_{1 \leq i \leq r} (j^2 - i^2) = \sum_{w=0}^{r} \eta_w j^{2w}$. We define

$$G_1(n) := \sum_{k=1}^{n} G_0(k) \qquad \text{and} \qquad G_2(n) := \sum_{k=1}^{n} G_1(k).$$

The polynomial $G(z)$ of real number z is defined by

$$G(z) := (-1)^r \frac{z^2 r!^2}{2} + \sum_{w=1}^{r} \eta_w \Big(\frac{P_{2w}(z-1)}{2} + \sum_{j=0}^{w} \zeta_j(2w) P_{2w-2j+1}(z-1) \Big). \tag{7.5}$$

Lemma 7.2. *The function $G(z)$ defined in* (7.5) *satisfies the following relations:*

$$G(0) = 0, \tag{7.6}$$

$$G(n) = (-1)^r \frac{n r!^2}{2} + G_2(n-1), \qquad (n \in \mathbb{N}) \tag{7.7}$$

$$G(n) = G(-n). \qquad (n \in \mathbb{N}) \tag{7.8}$$

Proof. It's obvious that $P_n(0) = 0$ and thus $P_n(-1) = 0$ by (7.3). So that $G(0) = 0$ follows from (7.5). By definitions of G_0, G_1 and G_2 we have

$$\begin{aligned}
G_2(n-1) &= \sum_{k=1}^{n-1} \sum_{j=1}^{k} \sum_{w=0}^{r} \eta_w j^{2w} \\
&= \sum_{k=1}^{n-1} \sum_{j=1}^{k} \eta_0 + \sum_{w=1}^{r} \eta_w \sum_{k=1}^{n-1} P_{2w}(k) \\
&= \eta_0 \binom{n}{2} + \sum_{w=1}^{r} \eta_w \sum_{k=1}^{n-1} \Big(\frac{k^{2w}}{2} + \sum_{j=0}^{w} \zeta_j(2w) k^{2w-2j+1} \Big) \\
&= (-1)^r r!^2 \binom{n}{2} + \sum_{w=1}^{r} \eta_w \Big(\frac{P_{2w}(n-1)}{2} + \sum_{j=0}^{w} \zeta_j(2w) P_{2w-2j+1}(n-1) \Big).
\end{aligned}$$

Hence (7.7) is true. By (7.3),

$$\begin{aligned}
G(n)-G(-n) &= \sum_{w=1}^{r}\eta_w\Bigl(\frac{P_{2w}(n-1)}{2}+\sum_{j=0}^{w}\zeta_j(2w)P_{2w-2j+1}(n-1)\Bigr)\\
&\qquad -\sum_{w=1}^{r}\eta_w\Bigl(-\frac{P_{2w}(n)}{2}+\sum_{j=0}^{w}\zeta_j(2w)P_{2w-2j+1}(n)\Bigr)\\
&=\sum_{w=1}^{r}\eta_w\Bigl(P_{2w}(n)-\frac{n^{2w}}{2}-\sum_{j=0}^{w}\zeta_j(2w)n^{2w-2j+1}\Bigr)\\
&=0. \qquad \square
\end{aligned}$$

The above lemma implies that $G(n)$ is an even polynomial of the integer n with degree $2r+2$, which means that there exist some rational numbers ξ_i such that

$$G(n)=\sum_{i=1}^{r+1}\xi_i n^{2i}. \tag{7.9}$$

Proof of Theorem 7.1. By (7.2) we obtain

$$\begin{aligned}
S(\lambda,r) &= \sum_{\square\in\lambda} G_0(h_\square)\\
&=\sum_{0\le j<i\le m}\bigl(G_2(x_i-x_j-1)+G_2(y_i-y_{j+1}-1)\\
&\qquad -G_2(x_i-y_{j+1}-1)-G_2(y_i-x_j-1)\bigr)\\
&=\sum_{0\le j<i\le m}\bigl(G(x_i-x_j)+G(y_i-y_{j+1})-G(x_i-y_{j+1})-G(y_i-x_j)\bigr).
\end{aligned}$$

The last equality is due to (7.7) and

$$(x_i-x_j)+(y_i-y_{j+1})-(x_i-y_{j+1})-(y_i-x_j)=0.$$

Thus by (7.9), we have

$$\begin{aligned}
S(\lambda,r) &= \sum_{1\le k\le r+1}\xi_k\sum_{0\le j<i\le m}\bigl((x_i-x_j)^{2k}+(y_i-y_{j+1})^{2k}\\
&\qquad -(x_i-y_{j+1})^{2k}-(y_i-x_j)^{2k}\bigr)\\
&=\sum_{1\le k\le r+1}\xi_k V(k),
\end{aligned}$$

where

$$V(k)=\sum_{0\le i\le j\le m}(x_i-x_j)^{2k}+\sum_{1\le i\le j\le m}(y_i-y_j)^{2k}-\sum_{0\le i\le m}\sum_{1\le j\le m}(x_i-y_j)^{2k}.$$

Notice that ξ_k is independent of λ since $G(n)$ is independent of λ. Comparing the coefficients of z^{2k} $(1\le k\le r+1)$ on both sides of the following trivial identity

$$\begin{aligned}
&\Bigl(\sum_{i=0}^{m}e^{x_iz}-\sum_{j=1}^{m}e^{y_jz}\Bigr)\Bigl(\sum_{i=0}^{m}e^{-x_iz}-\sum_{j=1}^{m}e^{-y_jz}\Bigr)\\
&=\sum_{i=0}^{m}\sum_{j=0}^{m}e^{(x_i-x_j)z}+\sum_{i=1}^{m}\sum_{j=1}^{m}e^{(y_i-y_j)z}-\sum_{i=0}^{m}\sum_{j=1}^{m}e^{(x_i-y_j)z}-\sum_{i=0}^{m}\sum_{j=1}^{m}e^{(y_j-x_i)z},
\end{aligned}$$

we obtain there exist some rational numbers b'_ν such that

$$V(k) = \sum_{|\nu| \le 2k} b'_\nu q_\nu(\lambda) \tag{7.10}$$

for every partition λ. This achieves the proof. $\square$

For each partition $\nu = (\nu_1, \nu_2, \cdots, \nu_\ell)$ we define

$$S_\nu(\lambda) := \prod_{1\le i\le \ell} S(\lambda, \nu_i).$$

Combining Theorems 7.1 and 6.1 we derive the following result.

Theorem 7.3. *Let $\nu = (\nu_1, \nu_2, \cdots, \nu_\ell)$ be a given partition. Then $S_\nu(\lambda)$ is a D-polynomial with degree at most $|\nu| + \ell$. Furthermore, there exist some $b_\delta \in \mathbb{Q}$ indexed by partitions δ such that*

$$D^k\Big(\frac{S_\nu(\lambda)}{H_\lambda}\Big) = \sum_{|\delta|\le 2|\nu|+2\ell-2k} b_\delta \frac{q_\delta(\lambda)}{H_\lambda} \tag{7.11}$$

for every partition λ.

Now we are ready to prove Theorem 1.3.

Proof of Theorem 1.3. It is easy to see that for any symmetric function $F(z_1, z_2, \ldots)$ of infinite variables, $F(h_\square^2 : \square \in \lambda)$ can be written as a linear combination of some $S(\lambda, \nu)$. Then by Theorem 7.3 we obtain Theorem 1.3. $\square$

By Theorem 1.3 and Theorem 1.4, we obtain

Theorem 7.4. *Let μ be a given partition and k a nonnegative integer. For each power sum symmetric function $p_\nu(z_1, z_2, \ldots)$ indexed by the integer partition $\nu = (\nu_1, \nu_2, \ldots, \nu_\ell)$ we have*

$$\sum_{|\lambda/\mu|=n} f_{\lambda/\mu} D^k\Big(\frac{p_\nu(h_\square^2 : \square \in \lambda)}{H_\lambda}\Big) = \sum_{0\le i\le |\nu|+\ell-k} d_{i+k}\binom{n}{i}$$

is a polynomial of n, where

$$d_i = D^i\Big(\frac{p_\nu(h_\square^2 : \square \in \mu)}{H_\mu}\Big).$$

In particular,

$$\frac{1}{(n+|\mu|)!}\sum_{|\lambda/\mu|=n} f_\lambda f_{\lambda/\mu} p_\nu(h_\square^2 : \square \in \lambda) \tag{7.12}$$

is a polynomial of n with degree at most $|\nu| + \ell$.

8. Okada-Panova hook length formula

Okada's conjecture on hook lengths (1.5) was first proved by Panova [21] by means of Theorem 1.1. In this section, we give another proof of Okada-Panova formula by using difference operators. In fact, the constants K_r arise directly from the computation for a single partition λ, without the summation ranging over all partitions of size n.

Proof of Theorem 1.5. By (6.3) and Theorem 7.3 there exist $a, b \in \mathbb{Q}$ such that for every λ,

$$H_\lambda D^r\Big(\frac{S(\lambda, r)}{H_\lambda}\Big) = a|\lambda| + b.$$

The explicit values of a and b are determined by taking two special partitions $\lambda = \emptyset$ and $\lambda = (1)$. Since $S(\lambda, r) = 0$ if λ does not have any hook length greater than r, we have

$$b = D^r\Big(\frac{S(\lambda, r)}{H_\lambda}\Big)\Big|_{\lambda=\emptyset} = 0$$

by (1.7). On the other hand, it's obvious that the only partitions of size $r+1$ who have hook lengths greater than r are $\{\lambda^{(k)} : 0 \le k \le r\}$ where

$$\lambda^{(k)} = (k+1, \underbrace{1, 1, \cdots, 1}_{r-k}).$$

Then

$$f_{\lambda^{(k)}} = \binom{r}{k} \qquad \text{and} \qquad S(\lambda^{(k)}, r) = \prod_{1\le i\le r}\big((r+1)^2 - i^2\big).$$

By (1.7) we have

$$a = D^r\Big(\frac{S(\lambda, r)}{H_\lambda}\Big)\Big|_{\lambda=(1)} = \sum_{|\lambda|=r+1} f_\lambda \frac{S(\lambda, r)}{H_\lambda} = \sum_{0\le k\le r} f_{\lambda^{(k)}} \frac{S(\lambda^{(k)}, r)}{H_{\lambda^{(k)}}},$$

so that

$$a = \frac{(2r+1)!}{r!(r+1)^2} \sum_{0\le k\le r} \binom{r}{k}^2 = \frac{(2r+1)!}{r!(r+1)^2}\binom{2r}{r} = K_r.$$

Hence (1.8) is true. Consequently, (1.9) and (1.10) are derived from (1.8) by applying the difference operator D. □

Proof of Theorem 1.2. Since $S(\lambda, r) = 0$ if λ does not have any hook length greater than r, we have

$$D^i\Big(\frac{S(\lambda, r)}{H_\lambda}\Big)\Big|_{\lambda=\emptyset} = 0 \tag{8.1}$$

for $0 \le i \le r$ by (1.7). Substituting $g(\lambda)$ by $S(\lambda, r)/H_\lambda$ and μ by $\emptyset$ in (1.6) we get

$$\sum_{|\lambda|=n} f_\lambda \frac{S(\lambda, r)}{H_\lambda} = \sum_{k=0}^{n} \binom{n}{k} D^k\Big(\frac{S(\mu, r)}{H_\mu}\Big)\Big|_{\mu=\emptyset} = K_r\binom{n}{r+1}$$

by (8.1), (1.9) and (1.10). □

9. Fujii-Kanno-Moriyama-Okada content formula

Recall $C(\lambda, r) = \sum_{\square\in\lambda}\prod_{i=0}^{r-1}(c_\square^2 - i^2)$.

Theorem 9.1. *There exist some $b_\nu \in \mathbb{Q}$ indexed by partitions ν such that*

$$H_\lambda D\Big(\frac{C(\lambda, r)}{H_\lambda}\Big) = \sum_{|\nu|\le 2r} b_\nu q_\nu(\lambda)$$

for every partition λ.

Proof. We have

$$\sum_{\square\in\lambda^{i+}} c_\square^{2r} - \sum_{\square\in\lambda} c_\square^{2r} = (\beta_i - \alpha_{i+1})^{2r} = x_i^{2r}.$$

Therefore

$$H_\lambda D\Big(\frac{\sum_{\square\in\lambda} c_\square^{2r}}{H_\lambda}\Big) = \sum_{\lambda^{i+}} \frac{H_\lambda}{H_{\lambda^{i+}}}\Big(\sum_{\square\in\lambda^{i+}} c_\square^{2r} - \sum_{\square\in\lambda} c_\square^{2r}\Big) = \sum_{\lambda^{i+}} \frac{H_\lambda}{H_{\lambda^{i+}}} x_i^{2r}.$$

The proof is achieved by Lemma 6.4 and linearity. □

Proof of Theorem 1.7. By (6.3), Theorems 9.1 and 6.1 there exist $a, b \in \mathbb{Q}$ such that for every λ,

$$H_\lambda D^r\Big(\frac{C(\lambda,r)}{H_\lambda}\Big) = a|\lambda| + b.$$

The explicit values of a and b are determined by taking two special partitions $\lambda = \emptyset$ and $\lambda = (1)$. Since $C(\lambda, r) = 0$ if λ does not have any content whose absolute value is greater than $r-1$, we have

$$b = D^r\Big(\frac{C(\lambda,r)}{H_\lambda}\Big)\Big|_{\lambda=\emptyset} = 0$$

by (1.7). On the other hand, it's obvious that the only partitions of size $r+1$ who have contents with absolute values greater than $r-1$ are (1^{r+1}) and $(r+1)$. By (1.7) we have

$$a = D^r\Big(\frac{C(\lambda,r)}{H_\lambda}\Big)\Big|_{\lambda=(1)} = \sum_{|\lambda|=r+1} f_\lambda \frac{C(\lambda,r)}{H_\lambda} = \frac{(2r)!}{(r+1)!}.$$

Hence (1.13) is true. Consequently, (1.14) and (1.15) are derived from (1.13) by applying the difference operator D. □

Proof of Theorem 1.8. Since $C(\lambda, r) = 0$ if λ does not have any content whose absolute value is greater than $r-1$, we have

$$D^i\Big(\frac{C(\lambda,r)}{H_\lambda}\Big)\Big|_{\lambda=\emptyset} = 0 \tag{9.1}$$

for $0 \le i \le r$ by (1.7). Substituting $g(\lambda)$ by $C(\lambda,r)/H_\lambda$ and μ by $\emptyset$ in (1.6) we get

$$\sum_{|\lambda|=n} f_\lambda \frac{C(\lambda,r)}{H_\lambda} = \sum_{k=0}^{n} \binom{n}{k} D^k\Big(\frac{C(\mu,r)}{H_\mu}\Big)\Big|_{\mu=\emptyset} = \Big(\frac{(2r)!}{(r+1)!}\Big)\binom{n}{r+1}$$

by (9.1), (1.14) and (1.15). □

Proof of Theorem 1.9. Directly by Theorem 1.4 and Lemma 1.7. □

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