Permutations r_j such that $\sum_{i=1}^{n}\prod_{j=1}^{k} r_j(i)$ is maximized or minimized

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Abstract

Tabulation of the set of permutations r_j of $\{1, \cdots, n\}$ such that $\sum_{i=1}^{n}\prod_{j=1}^{k} r_j(i)$ is maximized or minimized.

1 Introduction

Consider k permutations of the integers $\{1, \cdots, n\}$ denoted as $\{r_1, \cdots, r_k\}$ and the value $v(n,k) = \sum_{i=1}^{n}\prod_{j=1}^{k} r_j(i)$. The maximal value of v among all k-sets of permutations, denoted as $v_{\max}(n,k)$, is $\sum_{i=1}^{n} i^k$ and is achieved when all the k permutations are the same. This is a consequence of the following result in [1]:

Lemma 1. *Consider a set of nonnegative numbers* $\{a_{ij}\}$, $i = 1, \cdots, m$, $j = 1, \cdots, n$. *Let* $a'_{i1}, a'_{i2}, \cdots, a'_{in}$ *be the numbers* $a_{i1}, a_{i2}, \cdots, a_{in}$ *reordered such that* $a'_{i1} \geq a'_{i2} \geq \cdots \geq a'_{in}$. *Then*

$$\sum_{j=1}^{n}\prod_{i=1}^{m} a_{ij} \leq \sum_{j=1}^{n}\prod_{i=1}^{m} a'_{ij}$$

$$\prod_{j=1}^{n}\sum_{i=1}^{m} a_{ij} \geq \prod_{j=1}^{n}\sum_{i=1}^{m} a'_{ij}$$

Finding the minimal value of $v(n,k)$ among all k-sets of permutations, denoted as $v_{\min}(n,k)$, appears to be more complicated for $k > 2$ and $n > 2$. Clearly $v(1,k) = 1$ and $v(n,1) = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$. Since $v(n,k)$ is invariant under simultaneous reordering of the permutations, we can use this to define equivalence classes among k-sets of permutations.

2 The case $n = 2$

There are only two permutations on the integers $\{1, 2\}$. In this case $v_{\max}(2, k) = 1 + 2^k$. If k is even, $v_{\min}(2, k) = 2^k$ is achieved with $k/2$ of the permutations of one kind and the other half the other kind. If $k = 2m + 1$ is odd, $v_{\min}(2, k) = 3 \cdot 2^m$ is achieved with m of the permutations of one kind and $m + 1$ of them the other kind.

3 The case $k = 2$

Lemma 2 (Rearrangement inequality)**.**

$$x_n y_1 + \cdots + x_1 y_n \le x_{\sigma(1)} y_1 + \cdots + x_{\sigma(n)} y_n \le x_1 y_1 + \cdots x_n y_n$$

for real numbers x_i, y_i such that $x_1 \le \cdots \le x_n$ and $y_1 \le \cdots \le y_n$ and all permutations σ.

A proof of this can be found in [2].

In this case $v_{\max}(n, 2) = \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$. Consider the two permutations (1,2,$\cdots$, n) and (n,$n-1$,$\cdots$, 2, 1). The value of $v(n, 2)$ is equal to $\sum_{i=1}^{n} i(n-i+1) = (n+1)\sum_{i=1}^{n} i - \sum_{i=1}^{n} i^2 = \frac{n(n+1)^2}{2} - \frac{n(n+1)(2n+1)}{6} = n(n+1)\left(\frac{n+1}{2} - \frac{2n+1}{6}\right) = \frac{n(n+1)(n+2)}{6}$ and is in fact equal to $v_{\min}(n, 2)$ by Lemma 2.

4 $v_{\min}(n, k)$ for $k = 3, 4, \cdots$

The values of $v_{\min}(n, k)$ for different k's are listed in OEIS sequences [3] A070735 ($k = 3$, `https://oeis.org/A070735`), A070736 ($k = 4$, `https://oeis.org/A070736`), A260356 ($k = 5$, `https://oeis.org/A260356`), A260357 ($k = 6$, `https://oeis.org/A260357`), A260358 ($k = 7$, `https://oeis.org/A260358`), and sequence A260359 (for the case $k = 8$, `https://oeis.org/A260359`).

Partial list of values of $v_{\min}(n, k)$ (with some data taken from OEIS) are listed in Table 1. The antidiagonals of Table 1 can be found in `https://oeis.org/A260355` (OEIS sequence A260355).

Table 2 is a partial table of the number of nonequivalent k-sets of permutations achieving $v_{\min}(n, k)$ where equivalence is described in Section 1. We will denote these numbers as $N_{\min}(n, k)$. For $n \le 2$ or $k \le 2$, $N_{\min}(n, k) = 1$. On the other hand, $N_{\min}(n, k)$ can be larger than 1 if $n > 2$ or $k > 2$. For example, the 2 sets of nonequivalent permutations that achieves $v_{\min}(3, 6) = 108$ are $(123, 123, 231, 231, 312, 312)$ and $(123, 132, 213, 231, 312, 321)$. The 3 sets of nonequivalent permutations that achieves $v_{\min}(5, 3) = 89$ are $(12345, 34251, 52314)$, $(12345, 35214, 52341)$ and $(12345, 35241, 52314)$.

Note that $N_{\max}(n, k)$ corresponding to v_{max} satisfies $N_{\max}(n, k) = 1$ for all n and k.

Table 3 lists $N_{\min}(3, k)$ for various values of k.

	$k=1$	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$n=1$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	3	4	6	8	12	16	24	32	48	64	96	128	192	256	384
3	6	10	18	33	60	108	198	360	648	1188	2145	3888	7083	12844	23328
4	10	20	44	96	214	472	1043	2304	5136	11328	24993	55296	122624	271040	599832
5	15	35	89	231	600	1564	4074	10618							
6	21	56	162	484	1443	4320									
7	28	84	271	915	3089										
8	36	120	428	1608											
9	45	165	642	2664											
10	55	220	930	4208											
11	66	286	1304												
12	78	364	1781												
13	91	455	2377												
14	105	560	3111												
15	120	680	4002												

Table 1: Partial list of $v_{\min}(n,k)$.

	$k=1$	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$n=1$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
3	1	1	1	1	1	2	1	2	2	2	1	3	1	1	3
4	1	1	2	4	11	10	10	81	791	533	24	1461	3634	192	2404
5	1	1	3	12	16	188	211	2685							
6	1	1	10	110	16										
7	1	1	6												
8	1	1	16												
9	1	1	4												
10	1	1	12												
11	1	1													
12	1	1													
13	1	1													
14	1	1													
15	1	1													

Table 2: Partial list of $N_{\min}(n,k)$, the number of nonequivalent k-sets of permutations that achieve $v_{\min}(n,k)$.
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k	$N_{\min}(3,k)$
1	1
2	1
3	1
4	1
5	1
6	2
7	1
8	2
9	2
10	2
11	1
12	3
13	1
14	1
15	3
16	2
17	1
18	4
19	3
20	2
21	4
22	1
23	2
24	5
25	1
26	3
27	5
28	2
29	3
30	6
31	2
32	4
33	6
34	3
35	1

Table 3: $N_{\min}(3,k)$ for various values of k.

5 Computing $v_{\min}(n,k)$

Since we can always pick the member in the equivalent class such that $r_1 = \{1, \cdots, n\}$, we can fix r_1. By Lemma 2, after $r_1, \cdots r_{k-1}$ are chosen, the permutation r_k that minimize $v(n,k)$ among all choices of r_k is the permutation of $\{1, \cdots, n\}$ that is in reverse order from the order of the sequence of numbers $\left\{\prod_{j=1}^{k-1} r_j(1), \prod_{j=1}^{k-1} r_j(2), \cdots, \prod_{j=1}^{k-1} r_j(n)\right\}$. This means we only need to test $v(n,k)$ by choosing only $r_2, \cdots, r_{k-1}$, as r_1 is fixed and r_k is determined by the choices for the other permutations. The number of combinations we need to check (for $k > 2$) is equal to the number of combinations of $n!$ objects chosen $k-2$ times with replacement, which is equal to

$$\binom{n! + k - 3}{k - 2}$$

The following Python function computes $v_{\min}(n,k)$ (for $k \geq 2$):

```
from itertools import permutations, combinations_with_replacement
def vmin(n,k):
    ntuple, count = tuple(range(1,n+1)), n**(k+1)
    for s in combinations_with_replacement(permutations(ntuple,n),k-2):
        t = list(ntuple)
        for d in s:
            for i in range(n):
                t[i] *= d[i]
        t.sort()
        v = 0
        for i in range(n):
            v += (n-i)*t[i]
        if v < count:
            count = v
    return count
```

6 Permutation sets achieving $v_{\min}(n,k)$

The next sections list for each n and k one k-set of permutations (there may be many) that achieves $v_{\min}(n,k)$. Concatenating the permutations and considering them as a number in base $n+1$, the listed k-set of permutations is the k-set with the smallest such number that achieves $v_{\min}(n,k)$. We use the letters a, b, etc. to represent the numbers 10, 11, $\cdots$. We omit the $n \leq 2$ or $k \leq 2$ cases as they were discussed above.

6.1 $k = 3$

- $n = 3$: $(123, 231, 312)$

- $n = 4$: $(1234, 2341, 4213)$
- $n = 5$: $(12345, 34251, 52314)$
- $n = 6$: $(123456, 435261, 642315)$
- $n = 7$: $(1234567, 5463271, 7523416)$
- $n = 8$: $(12345678, 64572381, 86425317)$
- $n = 9$: $(123456789, 854673291, 976324518)$
- $n = 10$: $(123456789a, 96485372a1, a783452619)$
- $n = 11$: $(123456789ab, a65847932b1, b984632571a)$

6.2 $k = 4$

- $n = 3$: $(123, 132, 312, 321)$
- $n = 4$: $(1234, 2143, 3412, 4321)$
- $n = 5$: $(12345, 23145, 42531, 54312)$
- $n = 6$: $(123456, 235146, 632541, 653412)$
- $n = 7$: $(1234567, 3264571, 6724513, 7542136)$
- $n = 8$: $(12345678, 32457168, 87423651, 87452613)$

6.3 $k = 5$

- $n = 3$: $(123, 123, 231, 312, 321)$
- $n = 4$: $(1234, 1234, 3214, 4231, 4321)$
- $n = 5$: $(12345, 21453, 34512, 45231, 53124)$
- $n = 6$: $(123456, 213564, 453612, 563241, 643125)$
- $n = 7$: $(1234567, 2317564, 5371624, 6574312, 7534162)$

6.4 $k = 6$

- $n = 3$: $(123, 123, 231, 231, 312, 312)$
- $n = 4$: $(1234, 1234, 2134, 4312, 4321, 4321)$
- $n = 5$: $(12345, 12345, 31254, 45213, 54321, 54321)$
- $n = 6$: $(123456, 123465, 421536, 564312, 635142, 654321)$

6.5 $k = 7$

- $n = 3$: $(123, 123, 132, 231, 312, 312, 321)$
- $n = 4$: $(1234, 1234, 1234, 4231, 4231, 4312, 4312)$
- $n = 5$: $(12345, 12345, 21534, 45132, 45231, 52314, 54321)$

6.6 $k = 8$

- $n = 3$: $(123, 123, 123, 231, 231, 312, 312, 321)$
- $n = 4$: $(1234, 1234, 1243, 3124, 3421, 4213, 4321, 4321)$
- $n = 5$: $(12345, 12345, 12345, 42513, 45123, 53142, 53421, 53421)$

6.7 $k = 9$

- $n = 3$: $(123, 123, 123, 231, 231, 231, 312, 312, 312)$
- $n = 4$: $(1234, 1234, 1234, 2134, 3241, 3412, 4213, 4321, 4321)$

6.8 $k = 10$

- $n = 3$: $(123, 123, 123, 132, 231, 231, 312, 312, 312, 321)$
- $n = 4$: $(1234, 1234, 1234, 1234, 3124, 4213, 4321, 4321, 4321, 4321)$

6.9 $k = 11$

- $n = 3$: $(123, 123, 123, 123, 132, 312, 312, 321, 321, 321, 321)$
- $n = 4$: $(1234, 1234, 1234, 1234, 2314, 3412, 4213, 4231, 4231, 4231, 4231)$

6.10 $k = 12$

- $n = 3$: $(123, 123, 123, 123, 231, 231, 231, 231, 312, 312, 312, 312)$
- $n = 4$: $(1234, 1234, 1234, 1243, 2143, 3124, 3412, 3421, 4213, 4321, 4321, 4321)$

6.11 $k = 13$

- $n = 3$: $(123, 123, 123, 123, 132, 132, 312, 312, 312, 321, 321, 321, 321)$
- $n = 4$: $(1234, 1234, 1234, 1234, 1234, 2143, 4213, 4213, 4321, 4321, 4321, 4321, 4321)$

6.12 $k = 14$

- $n = 3$: $(123, 123, 123, 123, 123, 123, 213, 312, 321, 321, 321, 321, 321, 321)$
- $n = 4$: $(1234, 1234, 1234, 1234, 1234, 1324, 4132, 4132, 4312, 4312, 4321, 4321, 4321, 4321)$

6.13 $k = 15$

- $n = 3$: $(123, 123, 123, 123, 123, 231, 231, 231, 231, 231, 312, 312, 312, 312, 312)$
- $n = 4$: (1234, 1234, 1234, 1234, 1234, 1234, 3214, 3421, 4213, 4213, 4231, 4231, 4231, 4321, 4321)

6.14 $k = 16$

- $n = 3$: (123, 123, 123, 123, 123, 132, 132, 231, 312, 312, 312, 312, 321, 321, 321, 321)
- $n = 4$: (1234, 1234, 1234, 1234, 1243, 1243, 3124, 3124, 3421, 3421, 4213, 4213, 4321, 4321, 4321, 4321)

7 Version history

- August 12, 2015: initial version

References

[1] H. D. Ruderman, Two New Inequalities, The American Mathematical Monthly, Vol. 59, No. 1 (Jan., 1952), pp. 29-32.

[2] Hardy, G.H., Littlewood, J.E. and Pólya, G. (1952), Inequalities, Cambridge Mathematical Library (2nd ed.), Cambridge University Press.

[3] The on-line encyclopedia of integer sequences, founded in 1964 by N. J. A. Sloane.