IMPARTIAL AVOIDANCE AND ACHIEVEMENT GAMES FOR GENERATING SYMMETRIC AND ALTERNATING GROUPS

BRET J. BENESH, DANA C. ERNST, AND NÁNDOR SIEBEN

Abstract. Anderson and Harary introduced two impartial games on finite groups. Both games are played by two players who alternately select previously-unselected elements of a finite group. The first player who builds a generating set from the jointly-selected elements wins the first game. The first player who cannot select an element without building a generating set loses the second game. We determine the nim-numbers, and therefore the outcomes, of these games for symmetric and alternating groups.

1. Introduction

Anderson and Harary [3] introduced two impartial games *Generate* and *Do Not Generate* in which two players alternately take turns selecting previously-unselected elements of a finite group G. The first player who builds a generating set for the group from the jointly-selected elements wins the achievement game $\mathsf{GEN}(G)$. The first player who cannot select an element without building a generating set loses the avoidance game $\mathsf{DNG}(G)$. The outcomes of both games were studied for some of the more familiar finite groups, including abelian, dihedral, symmetric, and alternating groups in [3, 4], although the analysis was incomplete for alternating groups.

Brandenburg studies a similar game in [6]. This variation of the game is played on a finitely generated abelian group G. Players alternate replacing G by the quotient $G/\langle g \rangle$ for a choice of a nontrivial element g. The player with the last possible move wins.

A fundamental problem in the theory of impartial combinatorial games is finding the nim-number of a game. The nim-number determines the outcome of the game and also allows for the easy calculation of the nim-numbers of game sums. The last two authors [11] developed tools for studying the nim-numbers of both games, which they applied in the context of certain finite groups including cyclic, abelian, and dihedral. In [5], we developed a complete set of criteria for determining the nim-numbers of avoidance games and calculated the corresponding nim-numbers for several families of groups, including symmetric groups.

The previous work in [5, 11] left open the determination of the nim-numbers of $\mathsf{DNG}(A_n)$, $\mathsf{GEN}(S_n)$, and $\mathsf{GEN}(A_n)$, except for a handful of cases. In this paper, we provide a complete classification of the nim-numbers of the games involving the symmetric and alternating groups. Since nim-numbers determine the outcome, our classification completes the analysis from [4]. In particular, under optimal play, the first player will win the achievement game and lose the avoidance game for most symmetric and alternating groups. Moreover, building on the work of [5], we lay the foundation for how one might ascertain the nim-numbers for finite simple groups.

The paper is organized as follows. In Section 2, we recall the basic terminology of impartial games, establish our notation, and review several necessary results from [11]. The following section describes how to compute $\mathsf{GEN}(G)$ for G in a family called Γ_1, which includes the nonabelian alternating and symmetric groups, as well as all other finite nonabelian simple groups. In Section 4, we review the known results for $\mathsf{DNG}(S_n)$ and compute the nim-numbers for $\mathsf{GEN}(S_n)$, and then we find the nim-numbers for $\mathsf{DNG}(A_n)$ and $\mathsf{GEN}(A_n)$ in Section 5. We close with some open questions.

The authors thank the referee for making suggestions that improved the quality of this paper.

2. Preliminaries

2.1. Impartial Games.
In this section, we briefly recall the basic terminology of impartial games and establish our notation. For a comprehensive treatment of impartial games, see [2, 19].

An *impartial game* is a finite set X of *positions* together with a starting position and a collection $\{\mathrm{Opt}(P) \subseteq X \mid P \in X\}$, where $\mathrm{Opt}(P)$ is the set of possible *options* for a position P. Two players take turns replacing the current position P with one of the available options in $\mathrm{Opt}(P)$. A position P with $\mathrm{Opt}(P) = \emptyset$ is called *terminal*. The player who moves into a terminal position *wins*. All games must come to an end in finitely many turns.

The *minimum excludant* $\mathrm{mex}(A)$ of a set A of ordinals is the smallest ordinal not contained in the set. The *nim-number* of a position P is the minimum excludant

$$\mathrm{nim}(P) := \mathrm{mex}\{\mathrm{nim}(Q) \mid Q \in \mathrm{Opt}(P)\}$$

of the set of nim-numbers of the options of P. Since the minimum excludant of the empty set is 0, the terminal positions of a game have nim-number 0. The *nim-number of a game* is the nim-number of its starting position. The nim-number of a game determines the outcome since the second player has a winning strategy if and only if the nim-number for a game is 0. The winning strategy is to always pick an option that has nim-number 0. This puts the opponent into a losing position at every turn.

The *sum* of the games P and R is the game $P + R$ whose set of options is

$$\mathrm{Opt}(P + R) := \{Q + R \mid Q \in \mathrm{Opt}(P)\} \cup \{P + S \mid S \in \mathrm{Opt}(R)\}.$$

We write $P = R$ if the second player of the game $P + R$ has a winning strategy.

The one-pile NIM game with n stones is denoted by the *nimber* $*n$. The set of options of $*n$ is $\mathrm{Opt}(*n) = \{*0, \ldots, *(n-1)\}$. The Sprague–Grundy Theorem [2, 19] states that $P = *\mathrm{nim}(P)$ for every impartial game P.

2.2. Achievement and Avoidance Games for Generating Groups.
We now provide a more precise description of the achievement and avoidance games for generating a finite group G. For additional details, see [3, 4, 11]. For both games, the starting position is the empty set. The players take turns choosing previously-unselected elements to create jointly-selected sets of elements, which are the positions of the game.

In the avoidance game $\mathsf{DNG}(G)$, the first player chooses $x_1 \in G$ such that $\langle x_1 \rangle \neq G$ and at the kth turn, the designated player selects $x_k \in G \setminus \{x_1, \ldots, x_{k-1}\}$ such that $\langle x_1, \ldots, x_k \rangle \neq G$ to create the position $\{x_1, \ldots, x_k\}$. Notice that the positions of $\mathsf{DNG}(G)$ are exactly the non-generating subsets of G. A position Q is an option of P if $Q = P \cup \{g\}$ for some $g \in G \setminus P$, where $\langle Q \rangle \neq G$. The player who cannot select an element without building a generating set is the loser. We note that there is no avoidance game for the trivial group since the empty set generates the whole group.

In the achievement game $\mathsf{GEN}(G)$, the first player chooses any $x_1 \in G$ and at the kth turn, the designated player selects $x_k \in G \setminus \{x_1, \ldots, x_{k-1}\}$ to create the position $\{x_1, \ldots, x_k\}$. A player wins on the nth turn if $\langle x_1, \ldots, x_n \rangle = G$. In this case, the positions of $\mathsf{GEN}(G)$ are subsets of terminal positions, which are certain generating sets of G. Note that the second player has a winning strategy if G is trivial since $\langle \emptyset \rangle = G$, so the first player has no legal opening move. In particular, $\mathsf{GEN}(S_1) = \mathsf{GEN}(A_1) = \mathsf{GEN}(A_2) = *0$.

Maximal subgroups play an important role because a subset S of a finite group is a generating set if and only if S is not contained in any maximal subgroup. Let $\mathcal{M}$ be the set of maximal subgroups of G. Following [11], the set of *intersection subgroups* is defined to be the set

$$\mathcal{I} := \{\cap \mathcal{N} \mid \emptyset \neq \mathcal{N} \subseteq \mathcal{M}\}$$

containing all possible intersections of maximal subgroups. Note that the elements of $\mathcal{I}$ are in fact subgroups of G. The smallest intersection subgroup is the Frattini subgroup $\Phi(G)$ of G, and we say that an intersection subgroup I is non-Frattini if $I \neq \Phi(G)$. Not every subgroup of G need be an intersection subgroup. For instance, no proper subgroup of a nontrivial Frattini subgroup is an intersection subgroup. Even a subgroup that contains the Frattini subgroup need not be intersection subgroup, as shown in [5, Example 1].

The set $\mathcal{I}$ of intersection subgroups is partially ordered by inclusion. For each $I \in \mathcal{I}$, we let

$$X_I := \mathcal{P}(I) \setminus \cup \{\mathcal{P}(J) \mid J \in \mathcal{I}, J < I\}$$

be the collection of those subsets of I that are not contained in any other intersection subgroup smaller than I. We define $\mathcal{X} := \{X_I \mid I \in \mathcal{I}\}$ and call an element of $\mathcal{X}$ a *structure class*. The starting position $\emptyset$ is in $X_{\Phi(G)}$ for both games.

The set $\mathcal{X}$ of structure classes is a partition of the set of game positions of $\mathsf{DNG}(G)$. The partition $\mathcal{X}$ is compatible with the option relationship between game positions [11, Corollary 3.11]: if $X_I, X_J \in \mathcal{X}$ and $P, Q \in X_I \neq X_J$, then $\mathrm{Opt}(P) \cap X_J \neq \emptyset$ if and only if $\mathrm{Opt}(Q) \cap X_J \neq \emptyset$. We say that X_J is an *option* of X_I and write $X_J \in \mathrm{Opt}(X_I)$ if $\mathrm{Opt}(I) \cap X_J \neq \emptyset$.

For the achievement game $\mathsf{GEN}(G)$, we must include an additional structure class X_G containing terminal positions. A subset $S \subseteq G$ belongs to X_G whenever S generates G while $S \setminus \{s\}$ does not for some $s \in S$. Note that we are abusing notation since X_G may not contain G. The positions of $\mathsf{GEN}(G)$ are the positions of $\mathsf{DNG}(G)$ together with the elements of X_G. The set $\mathcal{Y} := \mathcal{X} \cup \{X_G\}$ is a partition of the set of game positions of $\mathsf{GEN}(G)$. As in the avoidance case, the partition $\mathcal{Y}$ is compatible with the option relationship between positions [11, Corollary 4.3].

For any position $\{g\}$ of either $\mathsf{DNG}(G)$ or $\mathsf{GEN}(G)$, we define $\lceil g \rceil$ to be the unique element of $\mathcal{I} \cup \{G\}$ such that $\{g\} \in X_{\lceil g \rceil}$. For example, if e is the identity of G, then $\lceil e \rceil = \Phi(G)$.

We define $\mathrm{pty}(S)$ of a set S to be 0 if $|S|$ is even and 1 if $|S|$ is odd, and we say that S is *even* if $\mathrm{pty}(S) = 0$ and *odd* if $\mathrm{pty}(S) = 1$. The *type* of the structure class X_I is the triple

$$\mathrm{type}(X_I) := (\mathrm{pty}(I), \mathrm{nim}(P), \mathrm{nim}(Q)),$$

where $P, Q \in X_I$ with $\mathrm{pty}(P) = 0$ and $\mathrm{pty}(Q) = 1$; by [11, Proposition 3.15], this is well-defined. The type can be calculated as in the following example: if an odd X_I has options of type $(0, a, b)$ and $(1, c, d)$, then $\mathrm{type}(X_I) = (1, \mathrm{mex}\{b, d, x\}, x)$, where $x = \mathrm{mex}\{a, c\}$. In the achievement case, the type of the structure class X_G is defined to be $\mathrm{type}(X_G) = (\mathrm{pty}(G), 0, 0)$. The *option type* of X_I is the set

$$\mathrm{otype}(X_I) := \{\mathrm{type}(X_K) \mid X_K \in \mathrm{Opt}(X_I)\}.$$

For both the avoidance and achievement games, the nim-number of the game is the same as the nim-number of the initial position $\emptyset$, which is an even subset of $\Phi(G)$. Because of this, the nim-number of the game is the second component of $\text{type}(X_{\Phi(G)})$.

2.3. **Achievement and Avoidance Game Strategy and Intuition.** Let G be a nontrivial finite group. In general, describing strategies in terms of nim-numbers is difficult, however we are able to provide a full description of the relationship between the strategy and the nim-number of $\text{DNG}(G)$. A complete characterization is elusive for $\text{GEN}(G)$.

After playing the avoidance game, the two players can notice that they spent the entire game choosing elements from a single maximal subgroup M. The first player will have won if M is odd, and the second player will have won if M is even. These facts immediately yield the winning strategies for the two players. The second player should attempt to choose any legal element of even order since that will ensure a final maximal subgroup of even order, while the first player should try to select an element $g \in G$ such that $\langle g, t \rangle = G$ for any element $t \in G$ of even order. Of course, only one of these strategies can be successful. Note that a winning strategy for the first player is equivalent to choosing an element $g \in G$ that is not contained in any even maximal subgroup. The nim-numbers of the avoidance game were characterized in the following theorem.

Theorem 2.1. [5, Theorem 6.3] Let G be a nontrivial finite group.
  (1) If $|G| = 2$ or G is odd, then $\text{DNG}(G) = *1$.
  (2) If G is cyclic with $|G| \equiv_4 0$ or the set of even maximal subgroups covers G, then $\text{DNG}(G) = *0$.
  (3) Otherwise, $\text{DNG}(G) = *3$.

It follows that a nim-number of 0 indicates that there is always an element of even order for the second player to choose, regardless of the first player's initial choice. Notice that G only has odd maximal subgroups if $|G| = 2$ or G is odd, so a nim-number of 1 indicates that the first player cannot lose, independent of strategy. A nim-number of 3 indicates that there is an element not contained in any even maximal subgroup, and choosing this element on the first move is sufficient to guarantee victory for the first player. However, the first player may lose by failing to choose this element. Thus, the difference between $\text{DNG}(G) = *1$ and $\text{DNG}(G) = *3$ is that the first player needs to make a wise first move to ensure a win only in the latter case.

Now let G be S_n or A_n for $n \geq 5$. We provide a description of the strategy for $\text{GEN}(G)$, but different positive nim-numbers for $\text{GEN}(G)$ do not seem to translate to any obvious group-theoretic description. The group G has the property that for every nontrivial $g \in G$, there is an $x \in G$ such that $\langle g, x \rangle = G$. The first player can win by choosing the identity with the first selection and an x such that $\langle g, x \rangle = G$ for the third move after the second player chooses a nonidentity element g.

3. Achievement Games for Γ_1 Groups

In this section we develop some general results about $\text{GEN}(G)$ for a special class of groups containing all of the nonabelian alternating groups and all but one of the nonabelian symmetric groups.

Following [13, Definition 1.1], two elements x and y of a group G are called *mates* if $\langle x, y \rangle = G$. We say that a finite nonabelian group is in Γ_1 if every nontrivial element of the group has a mate (see [7, Definition 1.01] for a general definition of Γ_r). It is known that $S_n \in \Gamma_1$ for $n \notin \{1, 2, 4\}$ [16] and $A_n \in \Gamma_1$ for $n \notin \{1, 2, 3\}$ [8]. Moreover, Guralnick and Kantor [14] proved that every finite nonabelian simple group is in Γ_1.

For the remainder of this paper, we will rely on Γ_1 groups having trivial Frattini subgroups.

Proposition 3.1. If $G \in \Gamma_1$, then $\Phi(G) = \{e\}$.

Proof. For sake of a contradiction suppose that x is a nontrivial element of $\Phi(G)$. Since $G \in \Gamma_1$, x has a mate y. Then $G = \langle x, y \rangle \leq \langle \Phi(G), y \rangle = \langle y \rangle$, where the last equality holds because $\Phi(G)$ is the set of all nongenerators of G (see [15, Problem 2.7], for instance). This implies G is cyclic and hence abelian, which contradicts the assumption that $G \in \Gamma_1$. □

The following general results about Γ_1 groups will be used to determine the nim-numbers of $\mathsf{GEN}(S_n)$ and $\mathsf{GEN}(A_n)$. We define $T := \{t_0, \dots, t_4\}$, where the types t_i are defined in Table 1.

	$\mathrm{otype}(X_I)$	$\mathrm{type}(X_I)$	$\mathrm{otype}(X_I)$	$\mathrm{type}(X_I)$
	$\{t_0\}$	t_2 or t_4	$\{t_0, t_4\}$	t_1 or t_4
$t_0 := (\mathrm{pty}(G), 0, 0)$	$\{t_0, t_1\}$	t_1	$\{t_0, t_1, t_4\}$	t_1
$t_1 := (1, 1, 2)$	$\{t_0, t_2\}$	t_2	$\{t_0, t_2, t_4\}$	t_3
$t_2 := (1, 2, 1)$	$\{t_0, t_3\}$	t_2	$\{t_0, t_3, t_4\}$	t_1
$t_3 := (1, 4, 3)$	$\{t_0, t_1, t_2\}$	t_3	$\{t_0, t_1, t_2, t_4\}$	t_3
$t_4 := (0, 1, 2)$	$\{t_0, t_1, t_3\}$	t_1	$\{t_0, t_1, t_3, t_4\}$	t_1
	$\{t_0, t_2, t_3\}$	t_2	$\{t_0, t_2, t_3, t_4\}$	t_3
	$\{t_0, t_1, t_2, t_3\}$	t_3	$\{t_0, t_1, t_2, t_3, t_4\}$	t_3

TABLE 1. Complete list of possible option types for a nonterminal structure class X_I in $\mathsf{GEN}(G)$ with $I \neq \Phi(G)$ when G is a Γ_1 group. Note that $\mathrm{pty}(I)$ and $\mathrm{otype}(X_I)$ determine $\mathrm{type}(X_I)$.

Proposition 3.2. If I is a non-Frattini intersection subgroup of a Γ_1 group G, then $\mathrm{type}(X_I)$ in $\mathsf{GEN}(G)$ satisfies

$$\mathrm{type}(X_I) = \begin{cases} t_1, & I \text{ is odd, } \mathrm{otype}(X_I) \cap \{t_1, t_4\} \neq \emptyset, \ t_2 \notin \mathrm{otype}(X_I) \\ t_2, & I \text{ is odd, } \mathrm{otype}(X_I) \cap \{t_1, t_4\} = \emptyset \\ t_3, & I \text{ is odd, } \mathrm{otype}(X_I) \cap \{t_1, t_4\} \neq \emptyset, \ t_2 \in \mathrm{otype}(X_I) \\ t_4, & I \text{ is even.} \end{cases}$$

Proof. The type of a terminal structure class must be t_0. The option type of any nonterminal structure class X_I of a Γ_1 group with $I \neq \Phi(G)$ contains t_0. Structural induction and the calculations in Table 1 shows that $\mathrm{otype}(X_I) \subseteq T$ implies $\mathrm{type}(X_I) \in T$. That is, no types outside of T arise. Note that a structure class of type t_4 can never have an odd option by Lagrange's Theorem, so t_4 only appears if $\mathrm{otype}(X_I)$ is a subset of $\{t_0, t_4\}$. □

For odd G, only some of the calculations from Table 1 are needed since the only possible option types are $\{t_0\}$ and $\{t_0, t_2\}$. This observation is sufficient to determine $\mathsf{GEN}(G)$ for odd G.

Corollary 3.3. If I is a non-Frattini intersection subgroup of an odd Γ_1 group G, then $\mathrm{type}(X_I)$ in $\mathsf{GEN}(G)$ is t_2.

Proposition 3.4. If G is an odd Γ_1 group, then $\mathsf{GEN}(G) = *2$.

Proof. Every option of $\Phi(G)$ has type t_2 by Corollary 3.3, so $\mathrm{otype}(X_{\Phi(G)}) = \{t_2\}$. Then

$$\mathrm{type}(X_{\Phi(G)}) = (1, \mathrm{mex}\{0, 1\}, \mathrm{mex}\{2\}) = (1, 2, 0),$$

so $\mathsf{GEN}(G) = *2$. $\square$

Example 3.5. Since $H := \mathbb{Z}_7 \rtimes \mathbb{Z}_3$ is in Γ_1, $\mathsf{GEN}(H) = *2$ by Proposition 3.4. Note that H is the smallest odd Γ_1 group. The smallest odd nonabelian group that is not Γ_1 is $K := (\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3$, yet $\mathsf{GEN}(K) = *2$, too. It is possible to get nim-numbers other than $*2$ for odd groups; in fact, $\mathsf{GEN}(\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3) = *1$. These three examples agree with [11, Corollary 4.8], which states that $\mathsf{GEN}(G) \in \{*1, *2\}$ for odd nontrivial G.

Proposition 3.6. Assume $G \in \Gamma_1$ and I is an odd non-Frattini intersection subgroup of G. If I is only contained in odd (respectively, even) maximal subgroups, then $\mathrm{type}(X_I)$ is t_2 (respectively, t_1).

Proof. Since $I \neq \Phi(G)$ and $G \in \Gamma_1$, $t_0 \in \mathrm{otype}(X_I)$. In both cases of the proof, we use structural induction on the structure classes.

First, we assume that I is only contained in odd maximal subgroups. If X_J is a nonterminal option of X_I, then J is also only contained in odd maximal subgroups. So $\mathrm{type}(X_J) = t_2$ by induction. Hence $\{t_0\} \subseteq \mathrm{otype}(X_I) \subseteq \{t_0, t_2\}$, which implies $\mathrm{type}(X_I) = t_2$ using Proposition 3.2.

Next, we assume that I is only contained in even maximal subgroups. Since I is odd, there is an involution $t \notin I$ such that $I \cup \{t\}$ is contained in an even maximal subgroup. The structure class X_J containing $I \cup \{t\}$ is a type t_4 option of X_I. So we may conclude that $\{t_0, t_4\} \subseteq \mathrm{otype}(X_I)$. Let X_J be a nonterminal option X_I. If X_I is even, then $\mathrm{type}(X_J) = t_4$ by Table 1. If X_J is odd, then $\mathrm{type}(X_J) = t_1$ by induction, since J is contained only in even maximal subgroups. Hence $\{t_0, t_4\} \subseteq \mathrm{otype}(X_I) \subseteq \{t_0, t_1, t_4\}$, which implies $\mathrm{type}(X_I) = t_1$ using Proposition 3.2. $\square$

For the rest of the paper we will only consider even groups.

Proposition 3.7. If G is an even Γ_1 group, then

$$\mathsf{GEN}(G) = \begin{cases} *1, & t_2 \notin \mathrm{otype}(X_{\Phi(G)}) \\ *3, & t_2 \in \mathrm{otype}(X_{\Phi(G)}), t_3 \notin \mathrm{otype}(X_{\Phi(G)}) \\ *4, & t_2, t_3 \in \mathrm{otype}(X_{\Phi(G)}). \end{cases}$$

Proof. Since $G \in \Gamma_1$, $X_{\Phi(G)}$ has no option of type t_0. Let t be an involution of G and X_J be the structure class containing $\Phi(G) \cup \{t\}$. Then X_J is a type t_4 option of $X_{\Phi(G)}$. Hence $\{t_4\} \subseteq \mathrm{otype}(X_{\Phi(G)}) \subseteq \{t_1, \ldots, t_4\}$. We compute $\mathrm{type}(X_{\Phi(G)})$ for every possibility for $\mathrm{otype}(X_{\Phi(G)})$ in Table 2. The result follows from this calculation and the fact that $\mathsf{GEN}(G)$ is equal to the second component of $\mathrm{type}(X_{\Phi(G)})$. $\square$

Recall that a subset $\mathcal{C}$ of the power set of a group G is a *covering* of G if $\bigcup \mathcal{C} = G$; in this case, we also say that $\mathcal{C}$ *covers* G or G *is covered by* $\mathcal{C}$.

Corollary 3.8. If a Γ_1 group G is covered by the set of even maximal subgroups of G, then $\mathsf{GEN}(G) = *1$.

Proof. We will demonstrate that $t_2 \notin \mathrm{otype}(X_{\Phi(G)})$, which will suffice by Proposition 3.7. Let X_I be an option of $X_{\Phi(G)}$. Then $I = \lceil g \rceil$ for some $g \in G$. If $\lceil g \rceil$ is even, then $\mathrm{type}(X_{\lceil g \rceil}) = t_4 \neq t_2$ by Proposition 3.2. Now assume that $\lceil g \rceil$ is odd. Since G is covered by the set of even maximal subgroups, g is contained in an even maximal subgroup M. By Cauchy's Theorem, there is

otype($X_{\Phi(G)}$)	otype($X_{\Phi(G)}$)	type($X_{\Phi(G)}$)
$\{t_4\}$	$\{t_1, t_4\}$	$(1,1,0)$
$\{t_2, t_4\}$	$\{t_1, t_2, t_4\}$	$(1,3,0)$
$\{t_3, t_4\}$	$\{t_1, t_3, t_4\}$	$(1,1,0)$
$\{t_2, t_3, t_4\}$	$\{t_1, t_2, t_3, t_4\}$	$(1,4,0)$

TABLE 2. Spectrum of type($X_{\Phi(G)}$) for $G \in \Gamma_1$. Note that $\Phi(G)$ is the trivial group in this case.

an involution t in M. The structure class X_J containing $\{g, t\}$ is a type t_4 option of $X_{\lceil g \rceil}$ by Proposition 3.2. So type($X_{\lceil g \rceil}$) cannot be t_2 again by Proposition 3.2. $\qquad \square$

If a noncyclic group G has only even maximal subgroups, then the set of even maximal subgroups covers G. The next corollary then follows immediately from Corollary 3.8.

Corollary 3.9. If G is a Γ_1 group with only even maximal subgroups, then $\mathsf{GEN}(G) = *1$.

4. SYMMETRIC GROUPS

In light of Corollary 3.9, we need only a simple proposition to completely determine the nim-numbers for the achievement and avoidance games for symmetric groups.

Proposition 4.1. If $n \geq 4$, then every maximal subgroup of S_n has even order.

Proof. An odd subgroup cannot be maximal since it is contained in the even subgroup A_n. $\qquad \square$

The nim-numbers for the avoidance game for generating symmetric groups were computed in [11] for $n \in \{2, 3, 4\}$ and in [5] for $n \geq 5$. We include the statement of the result here for completeness. Note that S_1 is trivial, so $\mathsf{DNG}(S_1)$ does not exist.

Theorem 4.2. The values of $\mathsf{DNG}(S_n)$ are

$$\mathsf{DNG}(S_n) = \begin{cases} *1, & n = 2 \\ *3, & n = 3 \\ *0, & n \geq 4. \end{cases}$$

We are now ready to determine the nim-numbers for $\mathsf{GEN}(S_n)$. This result verifies the portion of [11, Conjecture 9.1] on symmetric groups.

Theorem 4.3. The values of $\mathsf{GEN}(S_n)$ are

$$\mathsf{GEN}(S_n) = \begin{cases} *0, & n \in \{1, 4\} \\ *2, & n = 2 \\ *3, & n = 3 \\ *1, & n \geq 5. \end{cases}$$

Proof. The empty set is a generating set for the trivial group, so $\mathsf{GEN}(S_1) = *0$. The cases where $n \in \{2, 3, 4\}$ were done in [11], so assume $n \geq 5$. By [16], $S_n \in \Gamma_1$. By Proposition 4.1, every maximal subgroup of S_n has even order. Hence $\mathsf{GEN}(S_n) = *1$ by Corollary 3.9. $\qquad \square$

Theorems 4.2 and 4.3 immediately yield the following result.

Corollary 4.4. The first player has a winning strategy for

- DNG(S_n) if and only if $n \in \{2, 3\}$;
- GEN(S_n) if and only if $n \notin \{1, 4\}$.

5. ALTERNATING GROUPS

Determining the nim-numbers of DNG(A_n) and GEN(A_n) requires much more background. The following well-known proposition follows from Feit and Thompson's Odd Order Theorem [12] and the fact that every group of order $2m$ for some odd m has a normal subgroup of order m (see [15, Corollary 6.12] for example).

Proposition 5.1. If U is a nonabelian simple group, then 4 divides $|U|$.

Below, we will make use of a special case of the O'Nan–Scott Theorem (see [17] for instance). Recall that the *general affine group* of degree n over a field of size p^k for a prime p is defined to be the semidirect product $\mathrm{AGL}(n, p^k) := V \rtimes \mathrm{GL}(n, p^k)$ of a vector space V of size p^{nk} (i.e., dimension n) and the general linear group, where the latter acts on V by linear transformations.

Theorem 5.2 (O'Nan–Scott). Let A_n act on a set Ω of size n. If M is a maximal subgroup of A_n, then M must be one of the following:

(1) (Intransitive Case) $M = (S_m \times S_k) \cap A_n$ with $n = m + k$;
(2) (Imprimitive Case) $M = (S_m \wr S_k) \cap A_n$ with $n = mk$, $m, k > 1$;
(3) (Affine Case) $M = \mathrm{AGL}(k, p) \cap A_n$ with $n = p^k$ and p prime;
(4) (Diagonal Case) $M = (U^k.(\mathrm{Out}(U) \times S_k)) \cap A_n$ with U a nonabelian simple group, $k \geq 2$, and $n = |U|^{k-1}$;
(5) (Wreath Case) $M = (S_m \wr S_k) \cap A_n$ with $n = m^k$, $m \geq 5$, $k > 1$, and either $k \neq 2$ or $m \not\equiv_4 2$;
(6) (Almost Simple Case) $U \lhd M \leq \mathrm{Aut}(U)$, where U is a nonabelian simple group and M acts primitively on Ω.

Remark 5.3. We will see that $\mathrm{AGL}(1, p)$ plays an important role in determining the nim-numbers for alternating groups. Note that $\mathrm{GL}(1, p) \cong \mathbb{Z}_p^\times$, where $\mathbb{Z}_p^\times$ is the group of units of $\mathbb{Z}_p$, which is cyclic of order $p - 1$. Then $\mathrm{AGL}(1, p) \cong \mathbb{Z}_p \rtimes \mathbb{Z}_p^\times$, where the action is field multiplication. It is also easy to check that every element of $\mathrm{AGL}(1, p)$ either has order p or has order dividing $p - 1$.

Corollary 5.4. Let $n \geq 5$. If A_n has an odd maximal subgroup, then n is prime, $n \equiv_4 3$, and every odd maximal subgroup of A_n is isomorphic to $\mathrm{AGL}(1, n) \cap A_n$, and hence has order $\frac{1}{2}n(n-1)$.

Proof. If H is a subgroup of S_n that is not contained in A_n, then $|H \cap A_n| = \frac{1}{2}|H|$. Then $H \cap A_n$ is even if 4 divides $|H|$. By inspection, $S_m \times S_k$ and $S_m \wr S_k$ have orders that are divisible by 4 under the conditions specified in the Intransitive, Imprimitive, and Wreath Cases of Theorem 5.2, so A_n cannot have an odd maximal subgroup in those cases. Similarly, any $U^k.(\mathrm{Out}(U)) \times S_k)$ and any almost simple M are divisible by 4 by Proposition 5.1, so A_n cannot have an odd maximal subgroup in the Diagonal or Almost Simple Cases of Theorem 5.2.

This leaves only the Affine Case to be considered. Assume that $n = p^k$ for some prime p, and let M be a maximal subgroup of A_n in the Affine case of Theorem 5.2. The order of $\mathrm{AGL}(k, p)$ is $p^k(p^k - 1)(p^k - p)(p^k - p^2) \cdots (p^k - p^{k-1})$, which is divisible by 4 if $k \geq 2$. Then we may assume that $k = 1$, so $p = n \geq 5$ and we conclude that p is odd. Then $M \cong \mathrm{AGL}(1, p) \cap A_p \cong (\mathbb{Z}_p \rtimes \mathbb{Z}_p^\times) \cap A_p$ by Remark 5.3. Since A_p does not contain a $(p-1)$-cycle, we conclude that $|M| = \frac{1}{2}p(p-1)$, which is odd if and only if $p \equiv_4 3$. $\qquad\square$

The next result follows directly from Remark 5.3 and Corollary 5.4.

Corollary 5.5. Let $n \geq 5$. If A_n has an odd maximal subgroup M, then every nontrivial element of M is either an n-cycle or a power of a product of two $\frac{1}{2}(n-1)$-cycles.

Proposition 5.6. Let $n \geq 5$. Then A_n has an odd maximal subgroup if and only if n is prime, $n \equiv_4 3$, and $n \notin \{7, 11, 23\}$.

Proof. Suppose that A_n has an odd maximal subgroup M. Then by Corollary 5.4, n is prime, $n \equiv_4 3$, and $M \cong \mathrm{AGL}(1, n) \cap A_n$. The subgroup $M \cong \mathrm{AGL}(1, n) \cap A_n$ is not maximal if $n \in \{7, 11, 23\}$ by the main theorem from [17].

Thus, we may assume that n is prime, $n \equiv_4 3$, and $n \notin \{7, 11, 23\}$. Again by the main theorem from [17], we have that $\mathrm{AGL}(1, n) \cap A_n$ is maximal in A_n. Its order is $\frac{1}{2}n(n-1)$ by Corollary 5.4, which is odd because $n \equiv_4 3$. $\qquad\square$

Recall [10] that a prime p is a *generalized repunit prime* if there is a prime-power q and integer n such that $p = (q^n - 1)/(q - 1)$. The next definition will simplify the statement of the proposition that follows.

Definition 5.7. A prime p is said to be a ζ-*prime* if all of the following conditions hold:

(1) $p \equiv_4 3$;
(2) $p \notin \{11, 23\}$;
(3) p is not a generalized repunit prime.

The ζ-primes that are less than 100 are $19, 43, 47, 59, 67, 71, 79,$ and 83 [1]. Note that $7 = 111_2$ is a generalized repunit prime, so we did not have to explicitly exclude the prime 7 from Condition (2) to match the set of exceptions from Proposition 5.6.

Proposition 5.8. If $n \geq 5$, then the following are equivalent.

(1) The even maximal subgroups of n fail to cover A_n.
(2) There exists an n-cycle of A_n that is not in any even maximal subgroup.
(3) n is a ζ-prime.

Proof. First, we show that Items (1) and (2) are equivalent. If there exists an n-cycle of A_n that is not in any even maximal subgroup, then the even maximal subgroups of A_n fail to cover A_n by definition. Now suppose that the even maximal subgroups of n fail to cover A_n. Then there must be an odd maximal subgroup of A_n, so n is equal to some prime p with $p \equiv_4 3$ and $p \notin \{7, 11, 23\}$ by Proposition 5.6. Let r be the integer such that $p = 2r + 1$. If M is an odd maximal subgroup of A_p, then each element of M is either trivial, a p-cycle, or a power of two disjoint r-cycles by Corollary 5.5. The identity and every power of two disjoint r-cycles is contained in an even subgroup isomorphic to $(S_r \times S_r) \cap A_p$, which is contained in some even maximal subgroup. Therefore, it must be a p-cycle that is not contained an even maximal subgroup.

To finish, we show that Items (2) and (3) are equivalent. Let g be an n-cycle of A_n that is not in any even maximal subgroup. Then g must be contained in an odd maximal subgroup because A_n is not cyclic. Proposition 5.6 implies that n is a prime such that $n \equiv_4 3$ with $n \notin \{7, 11, 23\}$. Theorem 5.2 and the fact that n is prime imply that any maximal subgroup containing g must be isomorphic to $\mathrm{AGL}(1, n) \cap A_n$ or an almost simple group H that acts primitively on a set of size n. The former has odd order, and [18, Table 3] lists all the possibilities for H. Therefore, g will be contained in an even maximal subgroup H if and only if H appears in [18, Table 3]. The only rows in this table where the second column (labeled by n) is equal to the fourth column

(labeled by p, which is equal to n in our case) correspond to the first $\mathrm{PSL}(d, q)$, $\mathrm{Sz}(q)$, M_{23}, and M_{11}. But the row for $\mathrm{PSL}(d, q)$ implies that n is a generalized repunit prime, the row for $\mathrm{Sz}(q)$ implies that $n \equiv_4 1$, and the other two imply that $n \in \{7, 11, 23\}$. So the ζ-primes were defined exactly to exclude the entries in this table. Therefore, if g is not contained in any even maximal subgroup, then g is not in any H listed in [18, Table 3] and hence p is a ζ-prime.

Conversely, assume that n is a ζ-prime. Then n is a prime such that $n \equiv_4 3$ and $n \notin \{7, 11, 23\}$. So A_n has no subgroup H from [18, Table 3], and hence g is not contained in any even maximal subgroup, proving Item (2). $\qquad\square$

5.1. **Avoidance Games for Generating Alternating Groups.** We will use the following result to determine the nim-numbers of $\mathsf{DNG}(A_n)$.

Proposition 5.9. [5, Corollary 6.4] Let G be a nontrivial finite group.
  (1) If all maximal subgroups of G are odd, then $\mathsf{DNG}(G) = *1$.
  (2) If all maximal subgroups of G are even, then $\mathsf{DNG}(G) = *0$.
  (3) Assume G has both even and odd maximal subgroups.
   (a) If the set of even maximal subgroups covers G, then $\mathsf{DNG}(G) = *0$.
   (b) If the set of even maximal subgroups does not cover G, then $\mathsf{DNG}(G) = *3$.

Theorem 5.10. The values of $\mathsf{DNG}(A_n)$ are
$$\mathsf{DNG}(A_n) = \begin{cases} *3, & n \in \{3, 4\} \text{ or } n \text{ is a } \zeta\text{-prime} \\ *0, & \text{otherwise.} \end{cases}$$

Proof. The cases where $n \in \{3, 4\}$ were done in [11], so assume $n \geq 5$. If n is not a ζ-prime, then the set of even maximal subgroups covers A_n by Proposition 5.8; in this case, $\mathsf{DNG}(A_n) = *0$ by Proposition 5.9. If n is a ζ-prime, then the set of even maximal subgroups fails to cover A_n by Proposition 5.8. This implies that A_n has an odd maximal subgroup. The group A_n contains the proper subgroup $\langle (1, 2)(3, 4) \rangle$ of order 2, and so A_n must also contain an even maximal subgroup. We may conclude that $\mathsf{DNG}(A_n) = *3$ if n is a ζ-prime by Proposition 5.9. $\qquad\square$

Just like for S_1, A_1 and A_2 are trivial, so $\mathsf{DNG}(A_1)$ and $\mathsf{DNG}(A_2)$ do not exist.

5.2. **Achievement Games for Generating Alternating Groups.** We will see that ζ-primes play an important role in determining the nim-numbers of $\mathsf{GEN}(A_n)$ as they did for $\mathsf{DNG}(A_n)$. The following theorem refutes the portion of [11, Conjecture 9.1] on alternating groups.

Theorem 5.11. The values of $\mathsf{GEN}(A_n)$ are
$$\mathsf{GEN}(A_n) = \begin{cases} *0, & n \in \{1, 2\} \\ *2, & n = 3 \\ *3, & n = 4 \\ *4, & n \text{ is a } \zeta\text{-prime} \\ *1, & \text{otherwise.} \end{cases}$$

Proof. The empty set is a generating set for the trivial group, so $\mathsf{GEN}(A_1) = *0 = \mathsf{GEN}(A_2)$. The cases where $n \in \{3, 4\}$ were done in [11], so assume $n \geq 5$. By [8], $A_n \in \Gamma_1$. If every maximal subgroup of A_n has even order, then $\mathsf{GEN}(A_n) = *1$ by Corollary 3.9. We only need to determine what happens in the case that A_n has an odd maximal subgroup. Then $n = p$ for some prime $p \notin \{7, 11, 23\}$ such that $p \equiv_4 3$ by Proposition 5.6. We may write $p = 2r + 1$ for some odd r. If p is not a ζ-prime, then the even maximal subgroups cover A_p by Proposition 5.8, so $\mathsf{GEN}(A_p) = *1$ by Corollary 3.8.

So assume that p is a ζ-prime. Then there is an odd maximal subgroup M of A_p; we know that M must be isomorphic to $\mathrm{AGL}(1, p) \cap A_p$ by Corollary 5.4. Then $M = \langle g, x \rangle$ where g is a p-cycle and x is the product of two r-cycles and each element of M is either trivial, a p-cycle, or a power of two disjoint r-cycles by Corollary 5.5. By Proposition 3.6, $\mathrm{type}(M) = t_2$.

The element x is contained in an even subgroup isomorphic to $(S_r \times S_r) \cap A_p$, so $X_{\lceil x \rceil}$ has an even option of type t_4. The structure class $X_{\lceil x \rceil}$ has the type t_2 option X_M, since $\{x, g\} \in X_M$. Proposition 3.2 implies $\mathrm{type}(X_{\lceil x \rceil}) = t_3$.

Since p is a ζ-prime, there is a p-cycle y that is not contained in any even maximal subgroup by Proposition 5.8. Since all p-cycles are conjugate in S_p, we conclude that g is also only contained in odd maximal subgroups, so $\mathrm{type}(X_{\lceil g \rceil}) = t_2$ by Proposition 3.6. Then $t_2, t_3 \in \mathrm{otype}(X_{\Phi(A_p)})$, so $\mathsf{GEN}(A_p) = *4$ by Proposition 3.7. $\qquad\square$

5.3. Outcomes for Alternating Groups.
Theorems 5.10 and 5.11 immediately yield the following result.

Corollary 5.12. The first player has a winning strategy for
- $\mathsf{DNG}(A_n)$ if and only if $n \in \{3, 4\}$ or n is a ζ-prime;
- $\mathsf{GEN}(A_n)$ if and only if $n \notin \{1, 2\}$.

6. FURTHER QUESTIONS

We conclude with a few open problems.

(1) Recall that every finite simple group is in Γ_1 by [14]. It is well-known that many finite simple groups have the property that every maximal subgroup has even order (see [5], [9], and [20]). For such G, $\mathsf{DNG}(G) = *0$ by the main results from [5] and $\mathsf{GEN}(G) = *1$ by Corollary 3.9. Can we determine the nim-numbers for $\mathsf{DNG}(G)$ and $\mathsf{GEN}(G)$ for every finite simple group G?

(2) Can we determine the nim-numbers for $\mathsf{DNG}(G)$ and $\mathsf{GEN}(G)$ if G is almost simple?

(3) Can the conditions from Proposition 3.7 be translated into group-theoretic conditions? For instance, can one describe when $t_2 \in \mathrm{otype}(X_{\Phi(G)})$ based on the subgroup structure of G?

REFERENCES

1. *The On-Line Encyclopedia of Integer Sequences*, published electronically at `https://oeis.org` (2010), Sequence A270123.
2. M.H. Albert, R.J. Nowakowski, and D. Wolfe, *Lessons in play: an introduction to combinatorial game theory*, AMC **10** (2007), 12.
3. M. Anderson and F. Harary, *Achievement and avoidance games for generating abelian groups*, Internat. J. Game Theory **16** (1987), no. 4, 321–325.
4. F.W. Barnes, *Some games of F. Harary, based on finite groups*, Ars Combin. **25** (1988), no. A, 21–30, Eleventh British Combinatorial Conference (London, 1987).
5. B.J. Benesh, D.C. Ernst, and N. Sieben, *Impartial avoidance games for generating finite groups*, North-Western European Journal of Mathematics **2** (2016), 83–102.
6. M. Brandenburg, *Algebraic games*, `arXiv:1205.2884` (2012).
7. J.L. Brenner and J. Wiegold, *Two-generator groups. I*, Michigan Math. J. **22** (1975), 53–64.
8. N. Chigira et al., *Generating alternating groups*, Hokkaido Mathematical Journal **26** (1997), 435–438.
9. J.H. Conway, R.T. Curtis, S.P. Norton, R.A. Parker, and R.A. Wilson, *Atlas of finite groups: maximal subgroups and ordinary characters for simple groups*, Oxford University Press Eynsham, 1985.
10. H. Dubner, *Generalized repunit primes*, Math. Comp. **61** (1993), no. 204, 927–930.
11. D.C. Ernst and N. Sieben, *Impartial achievement and avoidance games for generating finite groups*, `arXiv:1407.0784` (2014).

12. W. Feit and J.G. Thompson, *Solvability of groups of odd order*, Pacific J. Math. **13** (1963), 775–1029.
13. T. Foguel, *Finite groups with a special 2-generator property*, Pacific J. Math. **170** (1995), no. 2, 483–495.
14. R.M. Guralnick and W.M. Kantor, *Probabilistic generation of finite simple groups*, J. Algebra **234** (2000), no. 2, 743–792, Special issue in honor of Helmut Wielandt.
15. I.M. Isaacs, *Algebra: a graduate course*, Graduate Studies in Mathematics, vol. 100, American Mathematical Society, Providence, RI, 2009, Reprint of the 1994 original.
16. I.M. Isaacs and T. Zieschang, *Generating symmetric groups*, American Mathematical Monthly (1995), 734–739.
17. M.W. Liebeck, C.E. Praeger, and J. Saxl, *A classification of the maximal subgroups of the finite alternating and symmetric groups*, J. Algebra **111** (1987), no. 2, 365–383.
18. M.W. Liebeck and J. Saxl, *Primitive permutation groups containing an element of large prime order*, J. Lond. Math. Soc. **2** (1985), no. 2, 237–249.
19. A.N. Siegel, *Combinatorial game theory*, Graduate Studies in Mathematics, vol. 146, American Mathematical Society, Providence, RI, 2013.
20. R.A. Wilson, *The finite simple groups*, Graduate Texts in Mathematics, vol. 251, Springer-Verlag London, Ltd., London, 2009.

BRET BENESH, DEPARTMENT OF MATHEMATICS, COLLEGE OF SAINT BENEDICT AND SAINT JOHN'S UNIVERSITY, 37 COLLEGE AVENUE SOUTH, SAINT JOSEPH, MN 56374-5011, USA
  E-mail address: bbenesh@csbsju.edu

DANA ERNST AND NÁNDOR SIEBEN, DEPARTMENT OF MATHEMATICS AND STATISTICS, NORTHERN ARIZONA UNIVERSITY, PO BOX 5717, FLAGSTAFF, AZ 86011-5717, USA
  E-mail address: Dana.Ernst@nau.edu, Nandor.Sieben@nau.edu