REPRESENTATION OF POSITIVE INTEGERS BY THE FORM $x_1...x_k + x_1 + ... + x_k$

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Abstract. For an arbitrary given $k \geq 3$, we consider a possibility of representation of a positive number n by the form $x_1...x_k + x_1 + ... + x_k,\ \ 1 \leq x_1 \leq ... \leq x_k$. We also study a question on the smallest value of $k \geq 3$ in such a representation.

1. Introduction

In 2002, R. Zumkeller published in OEIS the sequence A072670: "Number of ways to write n as $ij + i + j,\ \ 0 < i <= j$". This sequence possesses a remarkable property.

Proposition 1. *Positive integer n is not represented by the form $ij + i + j,\ \ 0 < i <= j$, if and only if $n = p - 1$, where p is prime.*

Proof. Condition $n = p-1$ is sufficient, since if $n = ij+i+j$, then $n+1 = (i+1)(j+1)$ cannot be prime. Thus n of the form $p-1$ is not represented by the form $ij + i + j,\ \ 0 < i <= j$. Suppose that, conversely, n is not represented by this form. Show that $n + 1$ is prime. If $n + 1 \geq 4$ is composite, then $n + 1 = rs,\ \ s \geq r \geq 2$. Set $i = r - 1,\ \ j = s - 1$. We have

$$ij + i + j = (r - 1)(s - 1) + (r - 1) + (s - 1) = n + 1 - 1 = n.$$

This contradicts the supposition. So $n + 1$ is prime. □

In this note, for an arbitrary given $k \geq 3$, we consider a more general form $x_1...x_k + x_1 + ... + x_k,\ \ 1 \leq x_1 \leq ... \leq x_k$. In particular, we study a question on the smallest value of $k \geq 3$ in a a possible representation of n.

2. Necessary condition for non-representation of n

Denote by $\nu_k(n)$ the number of ways to write n by the

$$F_k = F(x_1, ..., x_k) =$$

$$x_1...x_k + x_1 + ... + x_k,\ \ 1 \leq x_1 \leq ... \leq x_k,\ \ k \geq 3. \tag{1}$$

Proposition 2. *If, for a given $k \geq 3$, for $n \geq k - 1$ we have $\nu_k(n) = 0$, then $n - k + 3$ is prime.*

1991 *Mathematics Subject Classification.* 11N32.

Proof. If $n - k + 3 \geq 4$ is composite, then $n - k + 3 = rs,\ \ s \geq r \geq 2$. Set $x_i = 1\ \ for\ \ i = 1, ..., k - 2$ and $x_{k-1} = r - 1,\ \ x_k = s - 1$. We have

$F_k = (r-1)(s-1)+(k-2)+(r-1)+(s-1) = (n-k+3)+(k-2)-1 = n.$

This contradicts the condition $\nu_k(n) = 0$. So $n - k + 3$ is prime. □

Proposition 3. *If* $k_1 < k_2$ *and* $\nu_{k_1}(n) > 0$, *then* $\nu_{k_2}(n + k_2 - k_1) > 0$.

Proof. By the condition, there exist $x_1, ..., x_{k_1}$ such that

$$n = x_1...x_{k_1} + x_1 + ... + x_{k_1},\ \ 1 \leq x_1 \leq ... \leq x_{k_1},\ \ k_1 \geq 3.$$

Set $y_i = 1,\ \ i = 1, ..., k_2 - k_1$, and $y_{k_2-k_1+1} = x_1, ..., y_{k_2} = x_{k_1}$. Then we have

$y_1...y_{k_2} + y_1 + ... + y_{k_2} = x_1...x_{k_1} + k_2 - k_1 + x_1 + ... + x_{k_1} = n + x_{k_2} - x_{k_1}.$

□

Corollary 1. *If* $k_1 < k_2$ *and* $\nu_{k_1}(n + k_1 - 3) > 0$, *then* $\nu_{k_2}(n + k_2 - 3) > 0$.

Corollary 2. *If* $k_1 < k_2$ *and* $\nu_{k_2}(n + k_2 - 3) = 0$, *then* $\nu_{k_1}(n + k_1 - 3) = 0$.

Note that, by Proposition 2, in Corollary 2 the number n is prime.

3. Cases $k = 3$ and $k = 4$

Consider more detail the case $k = 3$, when

$$F_3 = x_1x_2x_3 + x_1 + x_2 + x_3,\ \ 1 \leq x_1 \leq x_2 \leq x_3.$$

The numbers of ways to write the positive numbers by the form F_3 are given in the sequence A260803 by D. A. Corneth. Note that, by Proposition 2, a number $n \geq 2$, could be not represented by F_3 only in case when n is prime. However, note that sequence of primes p not represented by F_3 should grow fast enough. Indeed, p should not be a prime of the form

$$(2t + 1)m + (t + 2),\ \ t, m \geq 2, \tag{2}$$

where $t \equiv 0\ \ or\ \ 2 \pmod 3$. Indeed, in this case $p = x_1x_2x_3 + x_1 + x_2 + x_3$ for $x_1 = 2,\ \ x_2 = t,\ \ x_3 = m$, if $t \leq m$, and for $x_1 = 2,\ \ x_2 = m,\ \ x_3 = t$ otherwise. Since $\gcd(2t + 1, t + 2) = \gcd(2(t + 2) - 3, t + 2) = 1$, then, by Dirichlet's theorem, for any admissible $t \geq 2$, the progression (2) contains infinitely many primes p. For all these primes, $\nu_3(p) > 0$.

Question 1. *Is the sequence of primes* $\{p \mid \nu_3(p) = 0\}$ *infinite*?

However, in case of $k = 4$, in view of Corollary 1, to the set of progressions (2) one can add, for example, the following set of progressions

$$(4t + 1)m + (t + 3),\ \ t, m \geq 2. \tag{3}$$

Here $\gcd(4t+1, t+3) = \gcd(4(t+3)-11, t+3) = 1$, except for $t \equiv -3 \pmod{11}$. Hence, for any admissible $t \geq 2$ the progression (3) contains infinitely many primes p. For such p we have

$$p + k - 3 = p + 1 = 2 \cdot 2tm + 2 + 2 + t + m = F_4$$

with $x_1 = x_2 = 2,\ x_3 = t,\ x_4 = m$, if $t \leq m$, and $x_1 = x_2 = 2,\ x_3 = m,\ x_4 = t$, if $t > m$. So for such p, $\nu_4(p+1) > 0$. Therefore, and, by the observations in table in Corneth's sequence A260804 for $k = 4$, the following question has another tint.

Question 2. *Is the sequence of primes* $\{p \mid \nu_4(p+1) = 0\}$ *only finite?*

4. Smallest k for representation of *prime* $+ k - 3$

According to Proposition 2, if m is not represented in the form F_k, then $m - k + 3$ is prime. Denote by p_n the n-th prime. Let $m - k + 3 = p_n$. Then, for every n, it is interesting a question, for either smallest $k \geq 3$ the number $p_n + k - 3$ is represented by F_k? Denote by $s(n),\ n \geq 1$, this smallest k and let us write $s(n) = 0$, if $p_n + k - 3$ is not represented by F_k for any $k \geq 3$. The sequence $\{s(n)\}$ starts with the following terms (A260965):

$$0, 0, 0, 0, 0, 0, 0, 3, 4, 3, 0, 0, 4, 0, 3, 0, 3, 3, 0, 4, 3, 3, 4, 3,$$

$$4, 0, 3, 5, 3, 4, 3, ... \tag{4}$$

Conjecture 1. *The sequence* (4) *contains only a finite number of zero terms.*

For example, a solution in affirmative of Question 2, immediately proofs Conjecture 1. Here we will concern only a question on estimates of $s(n)$.

Proposition 4.

$$s(n) \leq \lfloor (\log_2(p_n)) \rfloor. \tag{5}$$

Proof. Suppose, for a given p_n, there exists k such that $p_n + k - 3$ is represented by the form F_k. Then for the smallest possible k such a representation we call an *optimal representation* with a given p_n. Let us show that in an optimal representation all $x_i >= 2$. Indeed, let $x_1 = ... = x_u = 1$ and $x_i >= 2$ for $u + 1 <= i <= k$, such that $p_n + k - 3 = x_{u+1}...x_k + u + x_{u+1} + ... + x_k$ be an optimal representation. Note that $u < k$, otherwise $F_k = 1 + k$ which is not $k - 3+$ prime. Set $k_1 = k - u;\ y_j = x_{u+j}$. Then $p_n + k_1 - 3 = y_1...y_{k_1} + y_1 + ... + y_{k_1}$. Since $k_1 < k$, it contradicts the optimality of the form F_k. The contradiction shows that all x_i in an optimal representation are indeed more than or equal 2. So for an optimal representation, $p_n + k - 3 = F_k >= 2^k + 2k$ and $2^k + k + 3 <= p_n$. Hence $s(n) = k_{min} < \log_2(p_n)$ and the statement follows. □

Now we need a criterion for $s(n) > 0$.

Proposition 5. *$s(n) > 0$ if and only if either there exists $t_2 \geq$ such that*

$$B(t_2) = 2^{t_2} + t_2 + 3 = p_n$$

or there exist $t_2 \geq 0, t_3 \geq 1$ such that

$$B(t_2, t_3) = 2^{t_2}3^{t_3} + t_2 + 2t_3 + 3 = p_n$$

or there exist $t_2 \geq 0, t_3 \geq 0, t_4 \geq 1$ such that

$$B(t_2, t_3, t_4) = 2^{t_2}3^{t_3}4^{t_4} + t_2 + 2t_3 + 3t_4 + 3 = p_n,$$

etc.

Proof. Distinguish the following cases for $x_i \geq 2, i = 1, ..., k$, and $F_k = x_1...x_k + x_1 + ... + x_k$:

(i) All $x_i = 2, i = 1, ..., t_2$. Here $k = t_2$ and $F_k = 2^{t_2} + 2t_2$. If this is $t_2 - 3 + p_n$, then $p_n = 2^{t_2} + t_2 + 3 = B(t_2)$.
(ii) The first t_2 consecutive $x_i = 2$ and t_3 consecutive $x_i = 3$. Note that $t_3 \geq 1$ (otherwise, we have case (i)). Here $k = t_2 + t_3$ and $F_k = 2^{t_2}3^{t_3} + 2t_2 + 3t_3$. If this is $k-3+p_n = t_2+t_3-3+p_n$, then $p_n = 2^{t_2}3^{t_3}+t_2+2t_3+3 = B(t_2, t_3)$, etc. □

Note that in the expressions $B(t_2), B(t_2, t_3), ...$ defined in Proposition 5, we can consider only the case when the last variable is positive. Indeed, in $B(t_2)$, $t_2 \geq 1$ and if $t_{j+1} = 0$, then, evidently, $B(t_2, ..., t_j, 0) = B(t_2, ..., t_j)$.

Corollary 3. *If $v < j$ is the smallest number such that, for some $t_2, ..., t_v, t_j$ $B(t_2, ..., t_v, t_j) = p_n$, then $s(n) = t_2 + ... + t_v + t_j$. If, for a given n, for any j there is no such v, then $s(n) = 0$.*

Practically, using this algorithm for different j (cf. Section 5), we rather quickly reduce the number of variables t_i for the evaluation of $s(n)$.

5. Cases of $p_n = 97$ and $p_n = 101$

Here we show that, for $p_{25} = 97, p_{26} = 101$, we have $s(25) = 4$ and $s(26) = 0$. Note that $B(0, 0, ..., 0, t_j) = 2(j + 1)$ and, for $j \geq 3$, $B(t_2, 0, ..., 0, t_j) = (2^{t_2}+1)j+t_2+2$. For $t_2 = 1, , ..., 5$, we have $3j+3, 5j+4, 9j+5, 17j+6, 33j+7$ respectively. None of these expressions is equal to 97 or 101.

Further, for $j \geq 4$, $B(t_2, t_3, 0, ..., 0, t_j) = (2^{t_2}3^{t_3} + 1)j + t_2 + 2t_3 + 2$. Here $t_2 > 0$, otherwise we have even values. For $(t_2, t_3) = (1, 1), (2, 1), (3, 1)$, we have $7j + 5, 13j + 6, 25j + 7$ respectively. None of these expressions is equal to 97 or 101, expect for $13j + 6 = 97$ for $j = 7$ which corresponds to $t_2 = 2, t_3 = 1, t_7 = 1$. Hence, by Corollary 3, $s(25) = 2 + 1 + 1 = 4$. Continuing the research for $p = 101$, note that, for $j \geq 5$, $B(t_2, t_3, t_4, 0, ..., 0, t_j) = (2^{t_2}3^{t_3}4^{t_4} + 1)j + t_2 + 2t_3 + 3t_4 + 2$. Here already for: $(t_2, t_3, t_4) = (1, 1, 1)$ we have $25j + 8 > 101$. It completes the case $t_j = 1$. In case $t_j = 2$ we have $B(t_2, 0, ..., 0, t_j) = 2^{t_2}j^2 + t_2 + 2(j - 1) + 3, \;\; j \geq 3$. Here t_2 should be even (otherwise $B(t_2, 0, ..., 0, t_j)$ is even). For $t_2 = 2, 4$, we have $4j^2 + 2j + 3, 16j^2 + 2j + 5$. respectively. None of these expressions is equal 101. For $j \geq 4$, $B(t_2, t_3, 0, ..., 0, t_j) = 2^{t_2}3^{t_3}j^2 + t_2 + 2t_3 + 2(j - 1) + 3$ is ≥ 108 already for $t_2 = t_3 = 1$. Finally, in case $t_j \geq 3, \;\; j \geq 3$ we have $B(t_2, 0, ..., 0, t_j) = 64$ for $t_2 = 1, j = 3, t_j = 3$ and > 101 otherwise. So, $s(26) = 0$.

References

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