REPRESENTATION OF POSITIVE INTEGERS BY THE FORM $x^3 + y^3 + z^3 - 3xyz$

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Abstract. We study a representation of positive integers by the form $x^3 + y^3 + z^3 - 3xyz$ in the conditions $0 \le x \le y \le z,\ \ z \ge x + 1$.

1. Introduction

Let $F(x, y, z) = x^3 + y^3 + z^3 - 3xyz$. For a positive integer n, denote by $\nu(n)$ the number of ways to write n in the form $F(x, y, z)$ in the conditions $0 \le x \le y \le z,\ \ z \ge x+1$. Indeed, the case $z = x$ is not interesting, since in this case $F(x, y, z) = F(x, x, x) = 0$. Below we proved the following results:
(i)for every positive n, except for $n \equiv \pm 3 \pmod 9$ (cf.A074232 [3]), $\nu(n) >= 1$;
(ii) for the exceptional n, $\nu(n) = 0$;
(iii) for every prime $p \ne 3$, $\nu(p) = \nu(2p) = 1$;
(iv) $\limsup(\nu(n)) = \infty$;
(v) for every positive n, there exists k such that $\nu(k) = n$.

2. Lower estimate of $F(x, y, z)$

Proposition 1. *If* $z \ge x + 1$, *then*

$$F(x, y, z) \ge 3z - 2. \tag{1}$$

Proof. Previously note that
a) $F((z-1), (z+1), (z+1)) = 12z + 4 > 3(z+1) - 2$;
b) $F(z, (z+1), (z+1)) \ = \ 3z + 2 > 3(z+1) - 2$;
c) F(z, z, (z+1)) = 3(z+1) - 2.
Now we use induction over $z \ge 1$. Evidently, for $z = 1$, when either $(x, y) = (0, 0)$ or $(x, y) = (0, 1)$, the inequality (1) holds. Suppose (1) holds for some $z \ge 1$. Now setting $z := z + 1$, in view of a), b), c), we can take $0 \le x \le z-1,\ \ y \le z$. Then $F(x, y, z+1) = F(x, y, z)+3z^2+3z+1-3xy$ and, according the supposition, $F(x, y, z+1) \ge (3z-2)+3z^2+3z+1-3(z-1)z = 9z - 1 > 3(z + 1) - 2$. $\square$

The second proof.

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Proof. We have

$$F'_z(x,y,z)=3z^2-3xy\geq 3z^2-3(z-1)z=3z\geq 3.$$

So, for any fixed x,y, $F(x,y,z)$ increases over z. Hence, $F(x,y,z)\geq F(x,y,z_{min})$.
1) In the case $y=x+1$, $z_{min}=x+1$; 2) If $y<x+1$, then $y=x$. Since $z>x$, then $z_{min}=x+1$; 3) If $y>x+1$, then $z_{min}=y$.
In case 1) $F(x,y,z)\geq F(x,x+1,x+1)=3x+2=3(z-1)+2=3z-1>3z-2$;
In case 2) $F(x,y,z)\geq F(x,x,x+1)=3x+1=3z-2$;
In case 3) $F(x,y,z)\geq F(x,y,y)=x^3+2y^3-3xy^2$. Note that $F(x,y,y)'_y=6y^2-6xy\geq 6y^2-6(y-2)y=12y\geq 24$. Since $y_{min}=x+2$, then we have $F(x,y,z)\geq F(x,y,y)\geq F(x,x+2,x+2)=12x+16=12(z-2)+16\geq 3z-2$. □

Proposition 2. *If $z\geq x+2$, then*

$$F(x,y,z)\geq 9z-10. \tag{2}$$

Here there exist also at least two possibilities of proof. We show the second way.

Proof. Again

$$F'_z(x,y,z)=3z^2-3xy\geq 3z^2-3(z-2)z=6z\geq 12.$$

1)-3) $y=x,x+1,x+2$ respectively, $z_{min}=x+2$;
4) $y>x+2, z_{min}=y$.
We have
in case 1) $F(x,x,x+2)=12x+8=12z-16\geq 9z-10,\ \ z\geq 2$;
in case 2) $F(x,x+1,x+2)=9x+9=9(z-1)>9z-10$;
in case 3) $F(x,x+2,x+2)=12x+16=12z-8>9z-10$;
in case 4) $F(x,y,y)=x^3+2y^3-3xy^2$. As in proof of Proposition 1, $F(x,y,y)'_y>0$. Since $y_{min}=x+3$, then we have $F(x,y,z)\geq F(x,y,y)\geq F(x,x+3,x+3)=27x+54=27(z-1)>10z-1,\ \ z\geq 3$. □

3. Results $(i),(ii)$

Proposition 3. 1) *For every positive n, except for $n\equiv\pm 3 \pmod 9$, $\nu(n)\geq 1$;* 2) *If $n\equiv\pm 3\pmod 9$, then $\nu(n)=0$.*

Proof. 1) The statement follows from the following three equalities:

$$F(k-1,k,k)=3k-1; \tag{3}$$

$$F(k-1,k-1,k)=3k-2; \tag{4}$$

$$F(k,k+1,k+2)=9(k+1). \tag{5}$$

2) Let, for $n \equiv \pm 3 \pmod 9$, we have

$$n = F(x,y,z)). \tag{6}$$

However, we show that, if $F(x,y,z)$ is divisible by 3, then it divisible by 9. Note that, since $x^3 \equiv x \mod 3$, then

$$F(x,y,z) \equiv x+y+z \pmod 3. \tag{7}$$

So, by (6)

$$x+y+z \equiv 0 \pmod 3. \tag{8}$$

By the symmetry, it is sufficient to consider the cases $(x,y,z) \equiv (i,i,i) \pmod 3$, $i = 0,1,2$, and $(x,y,z) \equiv (0,1,2) \pmod 3$. Furthermore, note that

$$F(x,y,z) = (x+y+z)(x^2+y^2+z^2-xy-xz-yz) \tag{9}$$

and it is easy to see that in the considered cases also

$$x^2+y^2+z^2-xy-xz-yz \equiv 0 \pmod 3. \tag{10}$$

So, by (8) - (10), $F(x,y,z) \equiv 0 \pmod 9$ which contradicts the representation (6). □

4. Result (*iii*)

Proposition 4. *For every prime $p \neq 3$, $\nu(p) = \nu(2p) = 1$.*

Proof. In view of (3)-(4), for every prime p other than 3, we have $\nu(p) \geq 1$. However, in (3)-(4) are used the only two possibilities, when $z = x+1$. In both these cases

$$x^2+y^2+z^2-xy-xz-yz = 1. \tag{11}$$

Let us show that, if $z \geq x+2$, a representation of prime p is impossible. In this case $x+y+z \geq 2$. In view of (9), if $p = F(x,y,z)$, then it should be $x+y+z = p$ such that (11) holds. However, using Proposition 2, we have

$$x^2+y^2+z^2-xy-xz-yz = \frac{F(x,y,z)}{x+y+z} \geq$$

$$\frac{9z-10}{(z-2)+2z} \geq 2, \quad z \geq 2, \tag{12}$$

and (11) is impossible. So, for $p \neq 3$, $\nu(p) = 1$. Finally, for the representation of $2p$ in case $z \geq x+2$, note that, since (11) does not hold, it should be $x+y+z = p$ and $x^2+y^2+z^2-xy-xz-yz = 2$. But, according to (12), it is possible only if $z = 2$. In this case $x = 0$, $y = 0, 1$ *or* 2 and $F(x,y,z) = 8, 9$ *or* 16. Thus, for $p \neq 3$, $\nu(2p) = 1$. □

For example, we have a unique representation

$$p = x^3 + y^3 + z^3 - 3xyz$$

with $x = y = z - 1 = \frac{p-1}{3}$ if prime $p \equiv 1 \pmod 3$ and with $x + 1 = y = z = \frac{p+1}{3}$ if prime $p \equiv 2 \pmod 3$.
Also we have a unique representation

$$2p = x^3 + y^3 + z^3 - 3xyz$$

with $x+1 = y = z = \frac{2p+1}{3}$ if prime $p \equiv 1 \pmod 3$ and with $x = y = z-1 = \frac{2p-1}{3}$ if prime $p \equiv 2 \pmod 3$.

5. Result (iv)

Lemma 1.

$$(F(x,y,z))^3 = F(u,v,w), \tag{13}$$

where

$$u = F(x,y,z) + 9xyz, \ \ v = 3(x^2y + y^2z + z^2x), \ \ w = 3(x^2z + z^2y + y^2x).$$

Proof. The identity is proved straightforward. □

Lemma 2. *If the numbers x, y, z in* (13) *form an arithmetic progression with the difference $d \geq 1$, then the numbers v, u, w form an arithmetic progression with the difference $3d^3$.*

Proof. Let for $x \geq 0, d \geq 1$, we have $y = x + d, \ \ z = x + 2d$. Then

$$v = 3(x^2y + y^2z + z^2x) = 9x^3 + 27x^2d + 27xd^2 + 6d^3,$$
$$u = x^3 + y^3 + z^3 + 6xyz = 9x^3 + 27x^2d + 27xd^2 + 9d^3,$$
$$w = 3(x^2z + z^2y + y^2x) = 9x^3 + 27x^2d + 27xd^2 + 12d^3.$$

Thus $u = v + d_1, \ \ w = v + 2d_1$, where $d_1 = 3d^3$. □

Remark 1. *Since here $v < u < w$, then* (13) *we can write in the form $(F(x,y,z))^3 = F(v,u,w)$; further $(F(v,u,w))^3 = F(\xi, \eta, \zeta)$, such that $\xi < \eta < \zeta$, etc.*

Proposition 5. $\limsup(\nu(n)) = \infty$.

Proof. Consider sequence $27, 27^3, 27^{3^2}, 27^{3^3}, ..., 27^{3^k}, ...$ Representation $27^{3^k} = F(0, 0, 27^{3^{k-1}})$ we call trivial. We are interested in non-trivial representations of $b_k = 27^{3^k}$. Note that $b_0 = 27$ has a unique non-trivial representation defined by (5): $b_0 = F(2,3,4)$. Thus, by Lemma 2, $b_1 = b_0^3$ has at least 2 distinct non-trivial representations: by (5) with $d_1 = 1$ and with $d_2 = 3$.

Further, again by Lemma 2, $b_2 = b_1^3$ has at least 3 distinct non-trivial representations: by (5) with $d_1 = 1$, $d_2 = 3$ and $3 \cdot 3^3 = 3^4$. Analogously $b_3 = b_2^3$ has at least 4 distinct non-trivial representations: by (5) with $d_1 = 1$, $d_2 = 3$, $d_3 = 3^4$ and $d_4 = 3 \cdot 81^3 = 3^{13}$; ..., $b_k = b_{k-1}^3$ has at least $k+1$ distinct non-trivial representations: $1, 3, 3^4, 3^{13}, ..., 3^{(3^k-1)/2}$. This completes the proof. □

We give also the second proof.

Proof. We use the homogeneity of $F(x, y, z)$ of degree 3. By induction, show that $\nu(8^k) \geq k+1$. It is evident for $k = 0$. Suppose that it is true for some value of k. Take $k+1$ triples (x_i, y_i, z_i) such that $8^k = F(x_i, y_i, z_i)$, $i = 1, ..., k+1$. Then for $k+1$ triples of even numbers $(2x_i, 2y_i, 2z_i)$, we have $8^{k+1} = F(2x_i, 2y_i, 2z_i)$. But, by (3)-(4), always there is a triple of not all even numbers $x = (n-1)/3$, $y = (n-1)/3$, $z = (n+2)/3)$ or $x = ((n-2)/3$, $y = (n+1)/3$, $z = (n+1)/3)$, where $n = 8^{k+1}$, for which $8^{k+1} = F(x, y, z)$. So $\nu(8^{k+1}) \geq k+2$. □

6. Result (v)

Lemma 3. *There is a unique representation of* 8^k *by the form* $F(x, y, z)$ *with not all even numbers* x, y, z.

Proof. In (3)-(4) we used the only two possibilities, when $z = x+1$ and in both these cases we have the equality (11). This gives one representation of 8^k, when $8^k \equiv 1 \pmod 3$ (even k) and one representation of 8^k, when $8^k \equiv 2 \pmod 3$.(odd k). Let now $z \geq x+2$. Since $Fx, y, z) = (x+y+z)(x^2+y^2+z^2-xy-xz-yz)$, and, in view of the symmetry, the case, when the numbers x, y, z in a representation of 8^k are not all even, reduces, say, for the case when x and y are odd, while z is even. But then $x^2+y^2+z^2-xy-xz-yz$ is odd. In Section 3 we saw that, in the condition $z \geq x+2$, $x^2+y^2+z^2-xy-xz-yz \geq 2$. So it is odd ≥ 3. This is impossible in representation of 8^k by $F(x, y, z)$. □

Theorem 1. *For every positive* n, *there exists* k *such that* $\nu(k) = n$.

Proof. In the second proof of Result *iv*, we showed that $\nu(8^k) \geq k+1$. To prove the theorem, it suffices to prove that really we have here the equality: $\nu(8^k) = k+1$. Again use induction. Suppose that it is true for some value of k. As in the second proof of Result *iv*, take $k+1$ triples (x_i, y_i, z_i) such that $8^k = F(x_i, y_i, z_i)$, $i = 1, ..., k+1$. By Lemma 3, among these triples there exists a unique triple, say, $(x_{k+1}, y_{k+1}, z_{k+1})$ with not all even numbers.

Then for $k+1$ triples of even numbers $(2x_i, 2y_i, 2z_i)$, we have $8^{k+1} = F(2x_i, 2y_i, 2z_i)$, and only one of them $(2x_{k+1}, 2y_{k+1}, 2z_{k+1})$ contains not all numbers divisible by 4. Besides, there is a unique triple with odd two numbers. Suppose now, that there is an additional $(k+3)$-th triple $(x*, y*, z*)$ such that $8^{k+1} = F(x*, y*, z*)$. All numbers $x*, y*, z*$ should be divisible by 4. But then a triple $(x*/2, y*/2, z*/2)$ is an additional $(k+2)$-th triple for representation of 8^k. This contradicts the inductional supposition. The theorem follows. □

In conclusion, note that the sequence of $\{\nu(n)\}$ is A261029 [3] (including also $n = 0$). Besides, the smallest numbers $k = k(n)$ from Theorem 1 are presented in our with Peter J. C. Moses sequence A260935 [3].

7. On a Carmichael paper

While browsing the Bulletin of the American Mathematical Society, Michel Marcus found a Carmichael paper [1] on the same topic (now it is available in the sequence A074232). The methods of [1] and the present paper are quite different. So comparing the results, we can consider proof of (i)-(iii) as a *short proof* of the main results of [1], while (iv)-(v) give new results.

The author is happy to unwittingly continue with a new approach a research of the outstanding mathematician Robert Daniel Carmichael in exact CENTENARY (Aug 1915 - Aug 2015) of his paper.

Note that we published almost at the same time also the paper [2] which was inspired by the sequences A072670 and A260803 [3] by R. Zumkeller and D. A. Corneth respectively. These sequences with its restriction conditions essentially inspired also the present paper, since the author always remembered the remarkable form $x^3+y^3+z^3-3xyz$ which is the determinant of the circulant matrix with the first row (x, y, z).

References

[1] R. D. Carmichael, *On the representation of numbers in the form $x^3+y^3+z^3-3xyz$*, Bull. Amer. Math. Soc. 22 (1915), 111-117.

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[3] N. J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences* http://oeis.org.

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