On Eigenvalue Distribution of Random Matrices of Ihara Zeta Function of Large Random Graphs*

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Abstract

We consider the ensemble of real symmetric random matrices $H^{(n,\rho)}$ obtained from the determinant form of the Ihara zeta function of random graphs that have n vertices with the edge probability ρ/n. We prove that the normalized eigenvalue counting function of $H^{(n,\rho)}$ weakly converges in average as $n, \rho \to \infty$ and $\rho = o(n^\alpha)$ for any $\alpha > 0$ to a shift of the Wigner semi-circle distribution. Our results support a conjecture that the large Erdős-Rényi random graphs satisfy in average the weak graph theory Riemann Hypothesis.

1 Ihara zeta function of graphs and random matrices

The Ihara zeta function (IZF) associated to a finite connected graph $\Gamma = (V, E)$ is defined at $u \in \mathbb{C}$, for $|u|$ sufficiently small, by

$$Z_\Gamma(u) = \prod_{[C]} (1 - u^{\nu(C)})^{-1}, \tag{1.1}$$

where the product runs over the equivalence classes of primitive closed backtrackless, tail-less cycles $C = (\alpha_1, \alpha_2, \ldots, \alpha_l = \alpha_1)$ of positive length l in Γ, $\alpha_i \in V$ and $\nu(C) = l - 1$ is the number of edges in C [24]. Being introduced by Y. Ihara [13] in the algebraic context, IZF represents now an intensively developing subject of combinatorial graph theory with applications in the number theory and the spectral theory (see e.g. [12, 24] and references therein); it has also been studied in various

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other aspects, in particular in relations with the heat kernels on graphs [3], quantum walks on graphs [22], certain theoretical physics models [27].

Ihara's theorem [13] proved first for $(q+1)$-regular graphs Γ says that the IZF (1.1) is the reciprocal of a polynomial and that for sufficiently small $|u|$

$$Z_\Gamma(u)^{-1} = (1-u^2)^{r-1}\det(I + u^2(B-I) - uA), \tag{1.2}$$

where $A = (a_{ij})_{i,j=1,\dots,n}$ is the adjacency matrix of $\Gamma = \Gamma_n$, n is the number of vertices of Γ_n, $B = diag(\sum_{j=1}^n a_{ij})_{i=1,\dots,n}$ and $r-1 = \mathrm{Tr}(B-2I)/2$. This statement has been proven also for possibly irregular finite graphs (see [1, 23] for the combinatorial proofs of (1.2) and references related). The right-hand side of (1.2) represents an entire function; this means that $Z_\Gamma(u)$ has a meromorphic continuation to the whole complex u-plane. Note also that $r-1$ can be expressed in terms of the Euler characteristic of the graph because $\mathrm{Tr}(B-2I)/2 = |E| - n$, where $|E|$ is the total number of edges of Γ.

While the Ihara's determinant formula (1.2) gives a powerful tool in the studies of the Ihara zeta function, the explicit form of $Z_\Gamma(u)$ can be computed for relatively narrow families of finite graphs. Regarding the case of infinite graphs, the definition of the Ihara zeta function represents an important problem that requires a number of additional restrictions and assumptions (see, in particular, [4, 10, 11, 17]). In the most cases, the graphs under consideration have a bounded vertex degree (in particular, regular or essentially regular graphs).

A complementary approach is represented by a stochastic point of view, when the graphs are chosen at random from a set of all possible graphs on n vertices. This description "in the whole" naturally leads to the limiting transition of infinitely increasing dimension of the graphs, $n \to \infty$. Certain aspects of large random d-regular graphs have been studied in [8]. The present note is related with the Ihara zeta function (1.1) of random graphs whose average vertex degree ρ infinitely increases in the limit $n \to \infty$.

Let us consider an ensemble of $n \times n$ real symmetric matrices $A^{(n,\rho)}$ whose entries are determined by a collection of jointly independent Bernoulli random variables $\mathcal{A}_n^{(\rho)} = \{a_{ij}^{(n,\rho)}, 1 \le i \le j \le n\}$ such that

$$\left(A^{(n,\rho)}\right)_{ij} = a_{ij}^{(n,\rho)} = \begin{cases} 1-\delta_{ij}, & \text{with probability } \dfrac{\rho}{n}\,, \\ 0, & \text{with probability } 1-\dfrac{\rho}{n}\,, \end{cases} \tag{1.3}$$

where δ_{ij} is the Kronecker δ-symbol and $0 < \rho < n$, $\rho \in \mathbb{N}$. The adjacency matrices $\{A^{(n,\rho)}\}$ represent the ensemble of random graphs $\{\Gamma^{(n,\rho)}\}$ that can be referred to as the Erdős-Rényi random graphs [2]. With this definition in hands, one can determine the random Ihara zeta function $Z_{\Gamma^{(n,\rho)}}$ and to study it in the limit of infinite graph dimension, $n \to \infty$. In the present note, we consider the asymptotic regime of sparse random graphs, when $1 \ll \rho \ll n^\alpha$ for any $\alpha > 0$.

Passing to the normalized logarithm of (1.2), we get the following relation,

$$-\frac{1}{n}\log Z_{\Gamma^{(n,\rho)}}(u) = \frac{1}{2n}\ \mathrm{Tr}\left(B^{(n,\rho)} - 2I\right)\ \log(1-u^2)$$

$$+\frac{1}{n}\log\det\left((1-u^2)I + u^2 B^{(n,\rho)} - uA^{(n,\rho)}\right), \tag{1.4}$$

where

$$\left(B^{(n,\rho)}\right)_{ij} = \delta_{ij}\sum_{l=1}^{n} a_{il}^{(n,\rho)}.$$

Regarding the first term of the right-hand side of (1.4)

$$\Theta^{(n,\rho)}(u) = \frac{1}{2n}\ \mathrm{Tr}\left(B^{(n,\rho)} - 2I\right)\ \log(1-u^2),$$

it is easy to compute its mathematical expectation $\mathbf{E}\Theta^{(n,\rho)}(u)$ with respect to the measure generated by the family $\mathcal{A}_n^{(\rho)}$ (1.3),

$$\mathbf{E}\Theta^{(n,\rho)}(u) = \left(\frac{n-1}{2n}\rho - 1\right)\ln(1-u^2) \to \left(\frac{\rho}{2} - 1\right)\ln(1-u^2), \quad n\to\infty. \tag{1.5}$$

The last expression shows that to obtain a finite value of $\mathbf{E}\Theta^{(n,\rho)}(u)$ in the limit $n,\rho\to\infty$, one has to rescale the parameter $u = u_\rho$ as follows,

$$u_\rho^2 = \frac{v^2}{\rho}. \tag{1.6}$$

Then the last term of (1.4) takes the form

$$\Xi^{(n,\rho)}(v) = \frac{1}{n}\log\det\left((1-v^2/\rho)I + H^{(n,\rho)}(v)\right), \tag{1.7}$$

where

$$H^{(n,\rho)}(v) = \frac{v^2}{\rho}B^{(n,\rho)} - \frac{v}{\sqrt{\rho}}A^{(n,\rho)}. \tag{1.8}$$

The presence of the factor $\rho^{-1/2}$ in front of $A^{(n,\rho)}$ is fairly natural from the point of view of the spectral theory of large random matrices. The normalization of $B^{(n,\rho)}$ by v^2/ρ is less common and arises because of the condition (1.5). Therefore it would be natural to say that $\{H^{(n,\rho)}(v)\}$ is the random matrix ensemble of the Ihara zeta function of random graphs. It is interesting to note that similar rescaling of the spectral parameter u^2 (1.6) by the vertex degree $q+1$ is needed when instead of $\{\Gamma^{(n,\rho)}\}$ the ensemble of $(q+1)$-regular random graphs is considered in the limit of infinite q [18].

Let us note that in the case of $u = 1$, the last term of the right-hand side of (1.4) gives the discrete version of the Laplace operator $\Delta_\Gamma^{(n,\rho)} = B^{(n,\rho)} - A^{(n,\rho)}$ determined on the graph $\Gamma^{(n,\rho)}$. The spectral properties of $\Delta_\Gamma^{(n,\rho)}$ of large random graphs have been studied in a number of works (see e. g. [5, 15]). The eigenvalue distribution of a version of the graph's Laplacian given by $\rho^{-1/2}\Delta_\Gamma^{(n,\rho)}$ for finite and infinite values of $0 < \rho < n$ was studied in the limit $n \to \infty$ in [16]. Let us stress that this matrix $\rho^{-1/2}\Delta_\Gamma^{(n,\rho)}$ is again different from (1.8). The "zeta-function" normalization induced by (1.5) essentially changes the properties of the ensemble with respect to the graph's Laplacian. In particular, in contrast to $\Delta_{\Gamma^{(n,\rho)}}$ the matrix $H^{(n,\rho)}(v)$ with given $v > 0$ (1.8) is no more positively determined for large values of ρ, even the finite ones.

Summing up, we can say that the random matrix ensemble $H^{(n,\rho)}(v)$ (1.8) is the new one that up to our knowledge has not been studied. In the present note we consider (1.8) with $v \in \mathbb{R}$ and study the eigenvalue distribution of real symmetric random matrices $H^{(n,\rho)}(v)$ in the limit

$$n, \rho \to \infty, \quad \rho = o(n^\alpha), \text{ for any } \alpha > 0 \tag{1.9}$$

that we denote by $(n, \rho)_\alpha \to \infty$. Our main proposition is that the normalized eigenvalue counting function of $H^{(n,\rho)}(v)$ weakly converges in average to the well-known Wigner semicircle distribution shifted by v^2. This statement is proved in Section 2. In Section 3, we discuss our results in relation with the limiting values of the correspondingly re-normalized Ihara zeta function for complex $v \in \mathbb{C}$.

2 Limiting eigenvalue distribution of $H^{(n,\rho)}(v)$

Let us rewrite definition (1.8) in the form

$$H^{(n,\rho)}(v) = v^2 \tilde{B}^{(n,\rho)} - v\tilde{A}^{(n,\rho)}, \quad v \in \mathbb{R}. \tag{2.1}$$

Denoting the eigenvalues of $H^{(n,\rho)}(v)$ by $\lambda_1^{(n,\rho)}(v) \le \cdots \le \lambda_n^{(n,\rho)}(v)$, one introduces the normalized eigenvalue counting function,

$$\sigma_v^{(n,\rho)}(\lambda) = \frac{1}{n} \# \left\{ j : \lambda_j^{(n,\rho)}(v) \le \lambda \right\}, \quad \lambda \in \mathbb{R}. \tag{2.2}$$

The moments of this measure satisfy the following relation,

$$M_k^{(n,\rho)}(v) = \mathbf{E}\left(\frac{1}{n}\operatorname{Tr}\left(H^{(n,\rho)}(v)\right)^k\right) = \int_{-\infty}^{+\infty} \lambda^k d\bar{\sigma}_v^{(n,\rho)}(\lambda), \quad k = 0, 1, 2, \ldots, \tag{2.3}$$

where $\bar{\sigma}_v^{(n,\rho)}$ represents the averaged eigenvalue counting function, $\bar{\sigma}_v^{(n,\rho)} = \mathbf{E}\sigma_v^{(n,\rho)}$. The main result of the present note is as follows.

Theorem 1. *For any given $k \in \mathbb{N}$, the averaged moment (2.3) converges in the limit (1.9),*

$$\lim_{(n,\rho)_\alpha \to \infty} M_k^{(n,\rho)}(v) = \tilde{m}_k(v) = \begin{cases} v^{4l+2} \sum_{p=0}^{l} \frac{1}{v^{2p}} \binom{2l+1}{2p} \mathrm{t}_p, & \text{if } k = 2l+1, \\ v^{4l} \sum_{p=0}^{l} \frac{1}{v^{2p}} \binom{2l}{2p} \mathrm{t}_p, & \text{if } k = 2l, \end{cases} \tag{2.4}$$

where $l = 0, 1, 2, \ldots$ and

$$\mathrm{t}_p = \frac{(2p)!}{p!\,(p+1)!}, \quad p = 0, 1, 2, \ldots \tag{2.5}$$

are the Catalan numbers.

Proof. To study the moment $M_k^{(n,\rho)}$, we consider the product

$$L_k^{(n,\rho)}(\bar{P}, \bar{Q}) = \frac{1}{n} \sum_{i_0=1}^{n} \mathbf{E} \left(\tilde{A}^{p_1} \tilde{B}^{q_1} \cdots \tilde{A}^{p_s} \tilde{B}^{q_s} \right)_{i_0 i_0}, \tag{2.6}$$

where $\sum_{i=1}^{s} p_i = P$, $\sum_{i=1}^{s} q_i = Q$, $P+Q = k$, and where we denoted $\bar{P} = (p_1, \ldots, p_s)$ and $\bar{Q} = (q_1, \ldots, q_s)$. In (2.6), we assume $p_2 \geq 1, \ldots, p_s \geq 1$, $q_1 \geq 1, \ldots, q_{s-1} \geq 1$ and $p_1 \geq 0, q_s \geq 0$. In what follows, we omit the bars over P and Q when no confusion can arise.

We study the limiting behavior of variables $L_k^{(n,\rho)}(\bar{P}, \bar{Q})$ with the help of the diagram technique close to that developed in [16]. We consider the product

$$\left(\tilde{A}^{p_1} \cdots \tilde{B}^{q_s} \right)_{i_0 i_0} = \sum_{i_0, i_1, \ldots i_{p_s} = 1}^{n} (\tilde{A}^{p_1})_{i_0 i_{p_1}} (\tilde{B}^{(q_1)})_{i_{p_1} i_{p_1}} \cdots (\tilde{A}^{p_s})_{i_{p_1+\cdots+p_{s-1}}+1, i_0} (\tilde{B}^{(q_s)})_{i_0 i_0}$$

as a sum over the generalized trajectories $\left(\mathcal{I}_{p_1}^{(1)}, \mathcal{J}_{q_1}^{(1)}, \ldots, \mathcal{I}_{p_s}^{(s)}, \mathcal{J}_{q_s}^{(s)}\right)$, where the closed trajectory of P steps is given by

$$\mathcal{I}_P = (\mathcal{I}_{p_1}^{(1)}, \mathcal{I}_{p_2}^{(2)}, \ldots, \mathcal{I}_{p_s}^{(s)}) = (i_0, i_1, \ldots, i_{p_1}, i_{p_1+1}, \ldots, i_{p_1+p_2}, \ldots, i_{P-1}, i_0) \tag{2.7}$$

and $\mathcal{J}_{q_k}^{(k)} = (j_1^{(k)}, \ldots, j_{q_k}^{(k)})$, $k = 1, \ldots, s$.

Regarding (2.7), we associate to i_0 a root vertex α and draw the new vertex $\beta, \gamma, \delta, \ldots$ each time when we see a value of i_l that is not equal to one of the values previously seen. We draw the blue edges that correspond to the steps of $\mathcal{I}_P$ and get a graphical representation of the trajectory $\mathcal{I}_P$ by a closed chain G_P with P blue edges. Clearly, G_P is a connected multi-graph with the root vertex α. We denote by $\beta_1, \dots, \beta_h$ the vertices of G_P that correspond to different values of variables $i_0, i_{p_1}, i_{p_1+p_2}, \dots, i_{p_1+p_2+\dots+p_{s-1}}$, $1 \le h \le P$.

The q_1-plet $\mathcal{J}_{q_1}^{(1)}$ can be represented by a set of q_1 oriented red edges (β_1, γ_l) with $1 \le l \le q_1$. We can put a flesh at the head vertex γ_l. If the value of $j_r^{(1)}$ does not coincide with any element of $\mathcal{I}_P$ (2.7), then we say that the corresponding vertex γ_r is the red one. In the opposite case we have $\gamma_r \in V(G_P)$. Then we construct, by the same procedure, a representation of the remaining parts $\mathcal{J}_{q_2}^{(2)}, \dots, \mathcal{J}_{q_s}^{(s)}$ and get a multi-graph that we denote by $\mathcal{H}(\bar{P}, \bar{Q}) = G_{\bar{P}} \uplus F_{\bar{Q}}$. We say that $G_{\bar{P}}$ and $F_{\bar{Q}}$ represent the blue and the red sub-graphs of the diagram $\mathcal{H}(\bar{P}, \bar{Q})$, respectively.

Let denote by $\bar{G} = \bar{G}_{\bar{P}}$ and $\bar{F} = \bar{F}_{\bar{Q}}$ the simple graphs associated with $G = G_{\bar{P}}$ and $F = F_{\bar{Q}}$, respectively and consider a part $\bar{F}'_{\bar{Q}}$ of the red sub-graph of $\bar{\mathcal{H}}(\bar{P}, \bar{Q})$ such that there is no edge of $E(\bar{F}'_{\bar{Q}})$ that coincide with the elements of $E(\bar{G}_{\bar{P}})$. We denote by $V_r(\bar{F}')$ the set of red vertices with fleshes of $\bar{F}'_{\bar{Q}}$ and write that $\bar{F}''_{\bar{Q}} = \bar{F}_{\bar{Q}} \setminus \bar{F}'_{\bar{Q}}$. It is clear that

$$\Pi_{A,B}(\mathcal{I}_P, \mathcal{J}_Q) = \left(\frac{\rho}{n}\right)^{|E(\bar{G})|+|E(\bar{F}')|}$$

where $\bar{F}' = \bar{F}'_{\bar{Q}}$. Also it is easy to see that

$$|\mathcal{C}(G_{\bar{P}} \uplus F_{\bar{Q}})| = n(n-1)\cdots(n-|V(\bar{G})|-|V_r(\bar{F}')|+1) = n^{|V(\bar{G})|+|V_r(\bar{F}')|}(1+o(1)),\ n \to \infty.$$

Then

$$\frac{1}{n\rho^{P/2+Q}} \sum_{\{\mathcal{I}_P, \mathcal{J}_Q\} \in \mathcal{C}(G_{\bar{P}} \uplus F_{\bar{Q}})} \Pi_{A,B}(\mathcal{I}_P, \mathcal{J}_Q)$$
$$= \frac{\rho^{|E(\bar{G})|-P/2}}{n^{|E(\bar{G})|-|V(\bar{G})|+1}} \cdot \frac{\rho^{|E(\bar{F}')|-|E(F')|-|E(F'')|}}{n^{|E(\bar{F}')|-|V_r(\bar{F}')|}} (1+o(1)),\ n \to \infty, \tag{2.8}$$

where $F'' = F''_{\bar{Q}}$.

By using (2.8), it is not hard to prove that the terms of (2.6) that do not vanish in the limit $(n, \rho)_\alpha \to \infty$ (1.9) have the diagrams that satisfy inequality

$$|E(\bar{G})| - |V(\bar{G})| + |E(\bar{F}')| - |V_r(\bar{F}')| + 1 \le 0. \tag{2.9}$$

Indeed, if (2.9) is not satisfied, then the right-hand side of (2.8) gets a factor n^{-k}, $k \ge 1$ that suppress any power of ρ. The following two conditions are also necessary to have a non-zero limit: $|E(\bar{G})| \ge P/2$ and

$$|E(\bar{F}')| - |E(F')| - |E(F'')| \ge 0. \tag{2.10}$$

Let us denote a diagram that verifies these three conditions by $\tilde{\mathcal{H}}(\bar{P}, \bar{Q}) = \tilde{G}_{\bar{P}} \uplus \tilde{F}_{\bar{Q}}$. Due to the Euler relation for the planar embedding of connected graphs, the only equality is possible in (2.9) and this means that the simple graph $\tilde{G} \uplus \tilde{F}'$ is given by a tree. This tree is a plane rooted tree such that its blue sub-graph $\tilde{G}$ is also a tree with $P/2 = p$ edges. Indeed, let us denote by (δ_1, ϵ_1) the first leaf of the tree $\tilde{G}$. Then in the trajectory $\mathcal{I}_P$ (2.7) there is a couple of steps $(i_{l-1}, i_l), (i_l, i_{l+1})$ such that $i_{l-1} = i_{l+1}$. Removing this couple, we get the reduced trajectory $\mathcal{I}'_{p-2}$ such that its representation $\tilde{G}$ is again a tree. Proceeding by recurrence, we see that P is pair, $P = 2p$ and that $|E(\tilde{G})| \le P/2$. Thus, $p = |E(\tilde{G})|$.

Let us note that given a graph $\tilde{G}_{\bar{P}}$ that is a rooted tree $\mathcal{T}_p$, one can easily restore the original multi-graph $\Gamma_{\bar{P}}$ and corresponding sequence of vertices $\mathcal{W}_{2p}$ by considering the chronological run over the tree $\mathcal{T}_p$. In this case the walk $\mathcal{W}_{2p}$ is such that $\Gamma_{\bar{P}}$ is a multi-graph, where each couple of vertices $\{\gamma, \delta\}$ is joined by either zero edges or exactly two edges - in there and back directions, (γ, δ) and (δ, γ).

Regarding the red sub-graph $F = F' \sqcup F''$ of $\tilde{\mathcal{H}}(\bar{P}, \bar{Q})$, one can easily see that $|E(\bar{F}')| - |E(F')| \in \{0, -1, -2, \dots\}$ and that $|E(F'')| \in \{0, 1, 2, \dots\}$. Then (2.10) is possible only when $\tilde{F}''$ is empty and when $\bar{F}' = \tilde{F}'$. The last relation means that the red part $\tilde{F}' = \tilde{F}$ has no multiple edges.

It is known that the number of rooted trees of p edges is given by the Catalan number t_p. The positions and numbers of red edges being determined by $(\bar{P}, \bar{Q})$, we deduce from (2.8) that the following relation is true,

$$L_k^{(n,\rho)}(\bar{P}, \bar{Q}) = \mathrm{t}_p v^{2Q+p}(1 + o(1)), \quad (n, \rho)_\alpha \to \infty. \tag{2.11}$$

Returning to the moments (2.3) and regarding the trace

$$\frac{1}{n}\mathbf{E}\left(\mathrm{Tr} H_n^k\right) = \frac{1}{n}\sum_{i_0=1}^{n} \mathbf{E}\big(\underbrace{H_n\, H_n \cdots H_n}_{k\ times}\big)_{i_0 i_0},$$

we have to choose $2p$ elements H_n of the last product that will be represented by $-\tilde{A}$. This can be done in $\binom{k}{2p}$ ways and this choice determines uniquely the s-plets $\bar{P}$, $\bar{Q}$. Combining this observation with (2.11), we get relation (2.4). Theorem 1 is proved. $\square$

It is known that the Catalan numbers t_k verify the recurrence

$$\mathrm{t}_{k+1} = \sum_{j=0}^{k} \mathrm{t}_{k-j}\mathrm{t}_j, \; k \ge 0 \quad \text{and} \quad \mathrm{t}_0 = 1, \tag{2.12}$$

and that the family of moments

$$v^{2p}\mathrm{t}_p = \int_{\mathbb{R}} \lambda^{2p} d\mu_v(\lambda), \quad p = 0, 1, 2, \dots$$

uniquely determines an even measure μ_v with the density

$$\frac{d\mu_v}{d\lambda} = \mu_v'(\lambda) = \frac{1}{2\pi v^2}\begin{cases}\sqrt{4v^2-\lambda^2}, & \text{if } \lambda\in[-2v,2v],\\ 0, & \text{otherwise,}\end{cases} \tag{2.13}$$

known in the spectral theory of random matrices as the semi-circle or the Wigner distribution [26]. It follows from (2.4) that the limiting moments $\tilde m_k(v)$ can be represented as follows,

$$\tilde m_k(v) = \int_{\mathbb{R}} (v^2+\lambda)^k \, d\mu_v(\lambda), \quad k=0,1,2,\dots \tag{2.14}$$

and therefore the corresponding measure $\tilde\sigma_v$ such that $\tilde m_k(v) = \int \lambda^k d\tilde\sigma_v(\lambda)$ is given by a shift of the semi-circle distribution (2.13),

$$\tilde\sigma_v'(\lambda) = \frac{1}{2\pi v^2}\sqrt{4v^2-(\lambda-v^2)^2}, \quad |\lambda - v^2| \le 2|v|. \tag{2.15}$$

It follows from (2.15) that

$$\tilde m_k(v) \le (2v+v^2)^k, \quad k\ge 0.$$

The family of moments $\{\tilde m_k(v)\}$ satisfies the Carleman condition and therefore the measure $\tilde\sigma_v$ is uniquely determined. Thus, Theorem 1 implies the weak convergence in average of measures $\sigma_v^{(n,\rho)}$ to $\tilde\sigma_v$; this means that for any continuous bounded function $f:\mathbb{R}\mapsto\mathbb{R}$ the following is true,

$$\lim_{n,\rho\to\infty}\int_{\mathbb{R}} f(\lambda)\, d\bar\sigma_v^{(n,\rho)}(\lambda) = \int_{\mathbb{R}} f(\lambda)\, d\tilde\sigma_v(\lambda). \tag{2.16}$$

In this connection it should be noted that the generating function

$$f(\xi) = -\sum_{k=0}^{\infty}\frac{1}{\xi^{k+1}}v^{2k}t_k = \int_{-2v}^{2v}\frac{d\mu_v(\lambda)}{\lambda-\xi}$$

verifies the following equation that can be easily deduced from (2.12),

$$f(\xi) = \frac{1}{-\xi - v^2 f(\xi)}. \tag{2.17}$$

Regarding the generation function $g(\xi)$ of $\tilde m_k(v)$ and using (2.14), we get equality

$$g(\xi) = -\sum_{k=0}^{\infty}\frac{1}{\xi^{k+1}}\tilde m_k(v) = \int_{-\infty}^{\infty}\frac{d\sigma_v(\lambda)}{\lambda-\xi} = \int_{-\infty}^{\infty}\frac{d\mu_v(\lambda-v^2)}{\lambda-\xi} = f(\xi - v^2).$$

This means that $g(\xi)$ verifies the deformed version of (2.17),

$$g(\xi) = \frac{1}{v^2 - \xi - v^2 g(\xi)}. \tag{2.18}$$

Relation (2.18) shows that the numbers $\tilde{m}_k(v), k \ge 0$ are determined by the recurrence (cf. (2.12))

$$\tilde{m}_{k+1}(v) = v^2 \tilde{m}_k(v) + v^2 \sum_{j=0}^{k-1} \tilde{m}_{k-1-j}(v)\, \tilde{m}_j(v), \quad k \ge 1 \tag{2.19}$$

with the initial conditions $\tilde{m}_0(v) = 1$ and $\tilde{m}_1(v) = v^2$. In the particular case of $v^2 = 1$ the first ten values of $\tilde{m}_k(1), k \ge 0$ (2.19) are given by 1, 1, 2, 4, 9, 21, 51, 127, 323, 835. One can compute this also with the help of (2.4). These numbers are the Motzkin numbers [20].

Using the diagram technique developed above, one can get the following improvement of Theorem 1.

Theorem 2. *Given k, the following relation holds,*

$$\lim_{(n,\rho)_\alpha \to \infty} \rho \left(M_k^{(n,\rho)} - \tilde{m}_k(v) \right) = R_k^{(1)}(v), \tag{2.20}$$

where

$$R_k^{(1)}(v) = v^{2k-2p}\, \mathrm{t}_p$$
$$\times \left(\sum_{p=0}^{\lfloor k/2 \rfloor} \binom{k}{2p} \frac{p(p-1)}{p+2} + 4 \sum_{p=0}^{\lfloor (k-1)/2 \rfloor} \binom{k}{2p+1} p + \sum_{p=0}^{\lfloor (k-2)/2 \rfloor} \binom{k}{2p+2} \frac{4p+2}{p+2} \right). \tag{2.21}$$

Proof. Let us consider a diagram $\hat{\mathcal{H}}(\bar{P}, \bar{Q})$ such that corresponding sum over $\mathcal{C}(\hat{\mathcal{H}}(\bar{P}, \bar{Q}))$ represents the term of the order $O(\rho^{-1})$ of $M_k^{(n,\rho)}$ in the limit $n, \rho \to \infty$, $\rho = o(n)$. It follows from (2.8) that the diagram $\hat{\mathcal{H}}(\bar{P}, \bar{Q})$ verifies condition (2.9) and therefore its blue part $\hat{G}$ together with its red part $\hat{F}'$ represents a plane rooted tree $\hat{G} \uplus \hat{F}' = \mathcal{T}_r$. In these relations, we denoted by $\hat{G}$ and $\hat{F}$ simple graphs obtained from corresponding multi-graphs G and F. The red part $\hat{F}'$ contains such red edges that do not coincide with edges of $\hat{G}$.

The next consequence of (2.8) is that the following condition is verified by $\hat{\mathcal{H}}$,

$$|E(\hat{G})| - P/2 + |E(\hat{F}')| - |E(F')| - |E(F'')| = -1, \tag{2.22}$$

where $F'' = F \setminus F'$. It is clear that (2.22) is possible in one of the three following situations:

- either $|E(\hat{G})| - P/2 = -1$, $|E(\hat{F}')| - |E(F')| = 0$ and $|E(F'')| = 0$,
- or $|E(\hat{G})| - P/2 = 0$, $|E(\hat{F}')| - |E(F')| = 0$ and $|E(F'')| = -1$,
- or $|E(\hat{G})| - P/2 = 0$, $|E(\hat{F}')| - |E(F')| = -1$ and $|E(F'')| = 0$.

In the first case the chain of vertices of $\hat{G}_P$ is such that the corresponding walk $\hat{\mathcal{W}}_{2p}$ generates a multi-graph of the tree-type such that there exists one edge passed four times and that remaining $p-2$ edges are passed two times in there and back directions. The number of such walks is given by the formula

$$\# \left\{ \hat{\mathcal{W}}_{2p} \right\} = \frac{(2p)!}{(p-2)!\,(p+2)!} = \mathrm{t}_p \, \frac{p(p-1)}{(p+2)} \tag{2.23}$$

(see [14] for the proof). Then the remaining $k-2p$ red edges are to be distributed over $2p+1$ instants of time. This gives the factor $\binom{k}{2p}$ in the right-hand side of (2.21).

The second case describes a graph such that the chain of vertices such that the blue diagram $\hat{G}_P$ is a rooted tree of $p = P/2$ edges passed two times and there exists one red edge of the form (β, γ) such that $\gamma \in V(\hat{G}_P)$. It is not hard to see that obtain the corresponding chain, we have to consider a tree $\mathcal{T}_p$, to choose an edge e from it, to join one red edge η' to one or another side of e, to choose the orientation of the flesh of e' and to distribute $k-2p-1$ red edges over $2p+2$ instants of time. This gives the second term of the right-hand side of (2.21).

Finally, the third case determines such diagrams $\hat{\mathcal{H}}$ that $\hat{G}_P$ is a tree of p edges and there exist two red edges that make a multi-edge and the remaining red edges are simple. Let us describe how to construct corresponding diagrams and chains of vertices (walks). We consider two red edges of the form $e' = (\beta, \gamma)$ and $e'' = (\beta, \gamma)$. Then we attach to the vertex β a tree $\mathcal{T}_a$ that have a edges and point out an instant of time that will represent the starting and ending point of the corresponding walk. This can be done in $2a+1$ ways. Finally, we attach to β a tree $\mathcal{T}'_b$ of b edges, $b = p - a$ in the way that it will represent the part of the walk performed between the fist and the second passages of (β, γ) given by e' and e'', respectively. Now it remains to distribute $r - 2 - 2p$ red edges among $2p+1+2$ instant of times. This gives the factor $\binom{k}{2p+2}$. Taking into account elementary equality

$$\sum_{a,b \geq 0, a+b=p} (2a+1)\mathrm{t}_a \mathrm{t}_b = (p+1)\mathrm{t}_{p+1}, \tag{2.24}$$

we get the third term of the right-hand side of (2.21). Relation (2.24) can be proved with the help of the generating function $f(\xi)$ [14]. Also one can observe that the right-hand side of (2.24) represents the number of Catalan trees of $p+1$ edges with one marked edge; this gives the right-hand side of (2.24).

Theorem 2 is proved. □

3 Applications to IZF

Variable (1.7) can be rewritten with the help of (2.2) in the following form,

$$\Xi^{(n,\rho)}(v) = \frac{1}{n} \log\det\left(1_\rho + v^2 \tilde{B}^{(n,\rho)} - v\tilde{A}^{(n,\rho)}\right) = \int_{-\infty}^{\infty} \log\left(1_\rho + \lambda\right) d\sigma_v^{(n,\rho)}(\lambda), \quad (3.1)$$

where we denoted $1_\rho = 1 - v^2/\rho$. Then the convergence of IZF for a sequence of graphs can be reduced to the question of the convergence of the corresponding spectral measures. This approach has been used for the first time in the studies of IZF of infinite regular graphs in paper [11].

It is known that the normalized adjacency matrix $\tilde{A}^{(n,\rho)} = \rho^{-1/2} A^{(n,\rho)}$ of the Erdős-Rényi random graphs has all eigenvalues, excepting the maximal one, concentrated with probability 1 in the limit $n, \rho \to \infty$ on the interval $[-2, 2]$, while this maximal eigenvalue is of the order $\sqrt{\rho}$ [6, 9]. Regarding (3.1) for the negative values of v, one can therefore expect that the limit

$$\lim_{(n,\rho)_\alpha \to \infty} \Xi^{(n,\rho)}(v) = \Xi(v) \quad (3.2)$$

exists with probability 1 for all v such that $-1/2 < v < 0$. Moreover, observing that $\tilde{B}^{(n,\rho)}$ is asymptotically close to the unit matrix in the limit (1.9), it would be natural to assume that convergence in average

$$\lim_{(n,\rho)_\alpha \to \infty} \frac{1}{n} \mathbf{E}\big(\log Z_{\Gamma^{(n,\rho)}}(v/\sqrt{\rho})\big) = \frac{v^2}{2} - \Xi(v), \quad (3.3)$$

where

$$\Xi(v) = \frac{1}{2\pi v^2} \int_{-2|v|}^{2|v|} \log(1+\lambda)\, \sqrt{4v^2 - (\lambda - v^2)^2}\, d\lambda \quad (3.4)$$

holds for all $v \in (-1, 1)$. While (3.3) is close to the weak convergence result $\bar{\sigma}_v^{(n,\rho)} \to \tilde{\sigma}_v$ (2.16) established by Theorem 1 above, one cannot use it directly because the function $\log(1+\lambda)$ is not bounded in the vicinity of -1. To justify (3.3), the detailed analysis of fine properties of eigenvalues of $\tilde{H}^{(n,\rho)}$ is needed. This goes beyond the scope of the present note.

Expression (3.4) can be rewritten in the form

$$\Xi(v) = \frac{2}{\pi} \int_{-1}^{1} \log\left(1 + v^2 + 2v\nu\right) \sqrt{1-\nu^2}\, d\nu, \quad (3.5)$$

where the last integral converges for any real v. Moreover, the right-hand side of (3.5) can be continued to a function holomorphic in any domain

$$C_\epsilon = \{v : \ v \in \mathbb{C}, |v| \le 1 - \epsilon\}, \quad 0 < \epsilon < 1.$$

Thus, assuming that (3.2) and (3.3) hold and remembering the relation between parameters u and v (1.6), one can say that our results support the conjecture that the family of random graphs $\{\Gamma^{(n,\rho)}\}$ satisfies in the limit $(n,\rho)_\alpha \to \infty$ (1.9) a version of the weak graph theory Riemann Hypothesis [12, 25] that says that the Ihara zeta function $Z_\Gamma(u)$ is pole free for $|u| < 1/\sqrt{q}$, where $u \in \mathbb{C}$ and $q+1$ is the maximum degree of all the vertices of Γ. In our setting, the maximum degree normalization is naturally replaced by the averaged vertex degree ρ. Let us say that to prove this conjecture, one has to establish convergence (3.2) for complex v and this problem goes also far beyond the frameworks of our results.

In the present note we follow a stochastic approach to the studies of the Ihara zeta function of graphs. We assume the graph's edges to be present at random and consider the ensemble of such graphs in the limit of their infinite dimensions. Such a reasoning is completely in the spirit of the spectral theory of large random matrices used for the first time in the spectral theory of heavy atomic nuclei [21, 26], see also [19]. In the frameworks of the stochastic approach, the collective properties of complex systems remain valid while the difficulties are mostly related with special cases that are relatively rare. This can be regarded as a kind of simplified description that nevertheless catches the principal features of the system under consideration.

It should be noted that even in the frameworks of the stochastic approach to IZF of graphs, the problem of the limiting transition of the infinitely increasing dimension of graphs, $n \to \infty$, still persists. The main difficulty in the establishing of convergence of the normalized logarithm of IZF (1.4) is that because of possible presence of negative eigenvalues of $H^{(n,\rho)}$, one can guarantee the existence of the function $\Xi^{(n,\rho)}(v)$ (1.7) for small values of $|v|$ only, and this smallness can converge to 0 as $n \to \infty$. However, the proportion of the graphs that exhibit such anomalously small area of convergence can be sufficiently weak. The fact that the integral of the right-hand side of (3.5) converges confirms this conjecture. We also see that due to (3.3), a kind of the averaged version of the the graph theory Riemann Hypothesis can be true for the large irregular random graphs.

Let us finally point out that statements similar to our results could be obtained for the family of d-regular random graphs $\hat\Gamma^{(n,d)}$. Regarding $\log Z_{\hat\Gamma^{(n,d)}}(u)$ (1.4) with the spectral parameter $u = v/\sqrt{d-1}$, we get the following version of (3.1),

$$\hat\Xi^{(n,d)}(v) = \frac{1}{n} \log\det\left(I(1+v^2) - \frac{v}{\sqrt{d-1}} \hat A^{(n,d)} \right), \tag{3.6}$$

where $\hat A^{(n,d)}$ is the adjacency matrix of $\hat\Gamma^{(n,d)}$. With the help of the results of [18], one can show that the right-hand side of (3.6) converges as $n, d \to \infty$ for all $-1 < v < 0$ to the corresponding integral over the shifted semicircle distribution (2.15). Here again, regarding convergence of $\hat\Xi^{(n,d)}(v)$ in average, one can extend the above domain up to $v \in (-1, 1)$. To justify this, one would need to know the fine properties of the graph's spectrum, including statements similar to the well-known Alon's second eigenvalue conjecture (see e. g. [7]) as well as estimates of the lowest eigenvalue of $\hat{A}^{(n,d)}/\sqrt{d-1}$ in the limit $n, d \to \infty$.

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