

BINARY CODES AND PERIOD-2 ORBITS OF SEQUENTIAL DYNAMICAL SYSTEMS

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ABSTRACT. Let $[K_n, f, \pi]$ be the (global) SDS map of a sequential dynamical system (SDS) defined over the complete graph K_n using the update order $\pi \in S_n$ in which all vertex functions are equal to the same function $f: \mathbb{F}_2^n \rightarrow \mathbb{F}_2^n$. Let η_n denote the maximum number of periodic orbits of period 2 that an SDS map of the form $[K_n, f, \pi]$ can have. We show that η_n is equal to the maximum number of codewords in a binary code of length $n - 1$ with minimum distance at least 3. This result is significant because it represents the first interpretation of this fascinating coding-theoretic sequence other than its original definition.

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1. INTRODUCTION

Suppose we wish to model a finite system in which objects have various states and update their states in discrete time steps. Moreover, assume that the state to which an object updates depends only on the current state of that object along with the states of other nearby or connected objects. We can capture such a system's behavior with a *graph dynamical system*. A graph dynamical system contains a graph representing the connections between objects, a set of states that the objects can adopt, a collection of functions that model how each individual object updates its state in reaction to its neighbor's states, and a rule determining the scheme by which the objects update their states.

In a series of papers published between 1999 and 2001, Barrett, Mortveit, and Reidys introduced the notion of a sequential dynamical system (SDS), a graph dynamical system in which vertices update their states sequentially [9, 6, 7]. Subsequently, several researchers have worked to develop a general theory of SDS (see, for example, [3, 4, 5, 10, 11, 12, 13, 19]). The article [10] is interesting because it shows how SDS, originally proposed as models of computer simulation, are now being studied in relation with Hecke-Kiselman monoids in algebraic combinatorics. We draw most of our terminology and background information concerning SDS from [16], a valuable reference for anyone interested in exploring this field.

In the theory of SDS, the primary focus of many research articles is to count or otherwise characterize periodic orbits in the phase spaces of sequential dynamical systems [1, 2, 4, 8, 21, 22, 23]. For example, the recent paper [1] studies which periodic orbits can coexist in certain SDS and when certain SDS must necessarily have unique fixed points. In particular, that article shows that analogues of Sharkovsky's theorem from continuous dynamics completely fail to hold for many families of SDS. It is common to analyze the dynamics of sequential dynamical systems defined

using classical Boolean functions such as OR, AND, NOR, and NAND (see, for example, [?, ?]). The article [23] focuses more generally on SDS defined using so-called “bi-threshold” functions. By contrast, we will consider SDS defined using a completely arbitrary update function f . As we describe later, this function will be the vertex function for every vertex in the graph (this is possible because we will only consider base graphs that are complete). We now proceed to clarify some of these remarks by establishing some notation and definitions.

If v is a vertex of a graph Y , we let $d(v)$ denote the degree of v . We often work in the finite field $\mathbb{F}_2 = \{0, 1\}$. In doing so, we let $\bar{x} = 1 + x$ for any $x \in \mathbb{F}_2$. For any vector $\vec{x} = (x_1, x_2, \dots, x_k) \in \mathbb{F}_2^k$, let $\text{inv}(\vec{x}) = (\bar{x}_1, \bar{x}_2, \dots, \bar{x}_k)$. Furthermore, id will denote the identity permutation $123 \cdots n$ (the length n of the permutation id will always be clear from context).

An SDS is built from the following parts:

- An undirected simple graph Y with vertices v_1, v_2, \dots, v_n .
- A set of states A . We will typically use the set of states $A = \mathbb{F}_2 = \{0, 1\}$.
- A collection of *vertex functions* $\{f_{v_i}\}_{i=1}^n$. Each vertex v_i of Y is endowed with its own vertex function $f_{v_i}: A^{d(v_i)+1} \rightarrow A$.
- A permutation $\pi \in S_n$. The permutation π is known as the *update order*.

Let $q(v)$ denote the state of a vertex v . Suppose a vertex v_i has neighbors $v_{j_1}, v_{j_2}, \dots, v_{j_{d(v_i)}}$, where $j_1 < j_2 < \dots < j_s < i < j_{s+1} < j_{s+2} < \dots < j_{d(v_i)}$. We let

$$X(v_i) = (q(v_{j_1}), q(v_{j_2}), \dots, q(v_{j_s}), q(v_i), q(v_{j_{s+1}}), q(v_{j_{s+2}}), \dots, q(v_{j_{d(v_i)}})).$$

For example, if the vertex v_3 has neighbors v_1, v_4 , and v_6 , we let $X(v_3) = (q(v_1), q(v_3), q(v_4), q(v_6))$. The vector $(q(v_1), q(v_2), \dots, q(v_n))$, which lists all of the states of the vertices of Y in the order corresponding to the order of the vertex indices, is known as the *system state* of the SDS. Note that if Y is a complete graph and v_i is any vertex of Y , then $X(v_i)$ is equal to the system state of the SDS.

From each vertex function f_{v_i} , define the *local update function* $L_{v_i}: A^n \rightarrow A^n$ by

$$L_{v_i}(x_1, x_2, \dots, x_n) = (x_1, x_2, \dots, x_{i-1}, f_{v_i}(X(v_i)), x_{i+1}, \dots, x_n).$$

Combining these local update functions with the update order $\pi = \pi(1)\pi(2) \cdots \pi(n)$ (we have written the permutation π as a word), we obtain the SDS map $F: A^n \rightarrow A^n$ given by

$$F = L_{v_{\pi(n)}} \circ L_{v_{\pi(n-1)}} \circ \cdots \circ L_{v_{\pi(1)}}.$$

We will find it useful to introduce an “intermediate” SDS map $G_i: A^n \rightarrow A^n$ for each $i \in \{1, 2, \dots, n\}$, which we define by

$$G_i = L_{v_{\pi(i)}} \circ L_{v_{\pi(i-1)}} \circ \cdots \circ L_{v_{\pi(1)}}.$$

Thus, $F = G_n$. We use the convention that G_0 denotes the identity map from A^n to A^n . The vector $G_i(\vec{x})$ represents the system state of the SDS obtained by starting with a system state \vec{x} and updating only the first i vertices in the update order π . Once the system updates all n vertices (known as a system update), the new system state is $F(\vec{x})$.

Given any SDS on a graph Y with vertex functions $\{f_{v_i}\}_{i=1}^n$ and update order π , we denote its SDS map F by the triple $[Y, \{f_{v_i}\}_{i=1}^n, \pi]$. If all of the vertex functions f_{v_i} are equal to the same function f , we will simply write $[Y, f, \pi]$ for the corresponding SDS map (this situation can only occur if the base graph is regular).

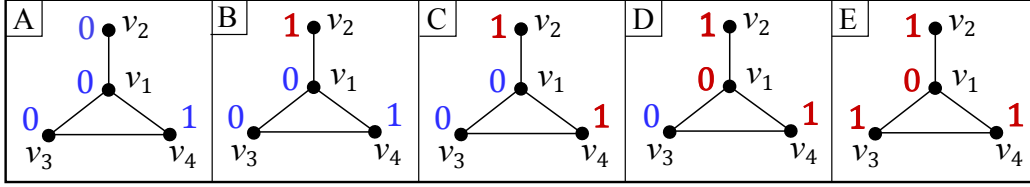


FIGURE 1. A system update of the SDS of Example 1.1. Block A shows the initial state of the SDS. Blocks B, C, and D show the intermediate steps of the system update. Block E shows the system state obtained after completing the system update.

Example 1.1. Consider the graph Y shown in block A of Figure 1. We define an SDS over Y using the update order $\pi = 2413$ and the vertex functions f_{v_i} given by

$$\begin{aligned} f_{v_1}(x_1, x_2, x_3, x_4) &= x_1x_3 + x_2 + x_4, \\ f_{v_2}(x_1, x_2) &= x_1x_2 + 1, \\ f_{v_3}(x_1, x_2, x_3) &= x_1 + x_2 + x_3, \end{aligned}$$

and

$$f_{v_4}(x_1, x_2, x_3) = x_1x_2 + x_3.$$

The initial system state of this SDS is $(0, 0, 0, 1)$, as shown by the blue labels in block A of Figure 1. Because $\pi(1) = 2$, we first update the vertex v_2 using the vertex function f_{v_2} . We have

$$f_{v_2}(X(v_2)) = f_{v_2}(q(v_1), q(v_2)) = f_{v_2}(0, 0) = 1,$$

so the vertex v_2 updates to the new state 1. This intermediate update is shown in block B of Figure 1. Another way to understand the transition from block A to block B in the figure is to see that we have changed the system state of the SDS by applying the local update function L_{v_2} . Indeed, $L_{v_2}(0, 0, 0, 1) = (0, 1, 0, 1)$. In a similar fashion, we next update vertices v_4 , v_1 , and v_3 . Letting $F = [Y, \{f_{v_i}\}_{i=1}^4, \pi]$, we find that $F(0, 0, 0, 1) = (0, 1, 1, 1)$, as shown in block E of the figure. In other words, through a sequence of local updates, the system update transformed the system state $(0, 0, 0, 1)$ into the new system state $(0, 1, 1, 1)$.

The SDS map F tells us how the states of the vertices of the graph Y change when we update the graph in a sequential manner. A useful tool for visualizing how F acts on the system's states is the *phase space* of the SDS. The phase space, denoted $\Gamma(F)$, is the directed graph with vertex set $V(\Gamma(F)) = A^n$ and edge set

$$E(\Gamma(F)) = \{(\vec{x}, \vec{y}) \in A^n \times A^n : \vec{y} = F(\vec{x})\}.$$

In other words, we draw a directed edge from \vec{x} to $F(\vec{x})$ for each $\vec{x} \in A^n$. As an example, the phase space of the SDS given in Example 1.1 is shown in Figure 2. Notice that the phase space shown in Figure 2 has a single 2-cycle (formed from the vertices 0111 and 0101). In general, a phase space of an SDS can have many cycles of various lengths. This leads us to some interesting questions. For example, when is it possible to have a phase space composed entirely of 2-cycles? Can we find an upper bound on the number of 2-cycles that can appear in the phase spaces of certain SDS? If we can show, for instance, that certain SDS defined over a graph with n vertices cannot have phase spaces consisting entirely of 2-cycles, then it will follow that the very natural function $\text{inv}: \mathbb{F}_2^n \rightarrow \mathbb{F}_2^n$ cannot be the SDS map of any of those SDS.

In this paper, we study the number of 2-cycles that can appear in the phase spaces of SDS defined over a complete graph K_n in which all vertex functions are the same. More formally, we give the following definition.

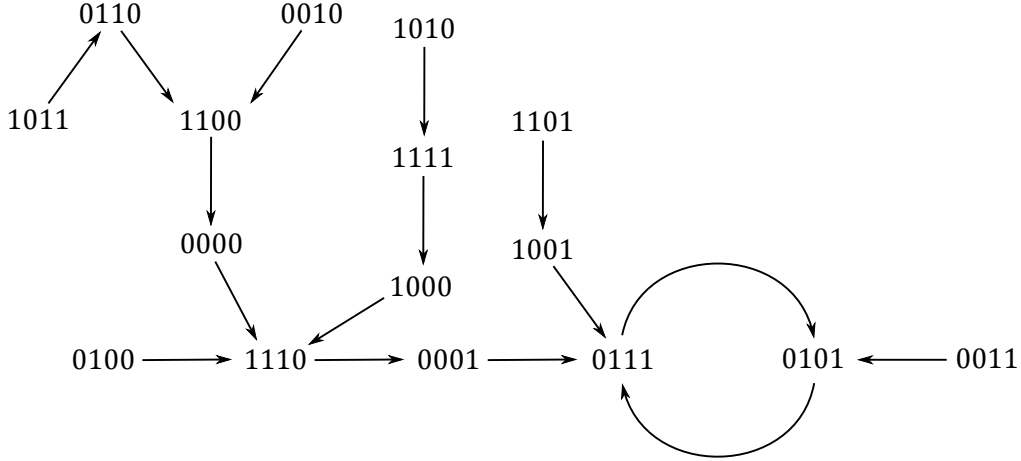


FIGURE 2. The phase space of the SDS described in Example 1.1. Note that we have omitted parentheses and commas from the vectors in order to improve the aesthetics of the image. For example, the vector $(1, 0, 1, 1)$ is written as 1011.

Definition 1.1. For a positive integer n , let $(\mathbb{F}_2)^{\mathbb{F}_2^n}$ be the set of all functions $g: \mathbb{F}_2^n \rightarrow \mathbb{F}_2$. For each $g \in (\mathbb{F}_2)^{\mathbb{F}_2^n}$ and $\pi \in S_n$, let $\eta(g, \pi)$ denote the number of 2-cycles in the phase space $\Gamma([K_n, g, \pi])$. Define

$$\eta_n = \max_{g \in (\mathbb{F}_2)^{\mathbb{F}_2^n}} \eta(g, \text{id}).$$

Remark 1.1. Our decision to use the identity update order id in the definition of η_n in Definition 1.1 stems from a desire for convenience, but we lose no generality in making such a decision because we are working over complete graphs. In other words,

$$\eta_n = \max_{g \in (\mathbb{F}_2)^{\mathbb{F}_2^n}} \eta(g, \text{id}) = \max_{\substack{g \in (\mathbb{F}_2)^{\mathbb{F}_2^n} \\ \pi \in S_n}} \eta(g, \pi).$$

In the next section, we reformulate the problem of determining η_n in terms of finding the clique number of a certain graph. We then show that $\eta_{n+1} = A(n, 3)$, where $A(n, 3)$ denotes the maximum number of codewords in a binary code of length n with minimum distance at least 3. The sequence $A(n, 3)$, which is sequence A005864 in Sloane's Online Encyclopedia of Integer Sequences [17], has been a fascinating and mysterious subject of inquiry in coding theory [14, 15, 18, 20]. Our result is noteworthy because, to the best of our knowledge, it provides the only known interpretation of this sequence other than its original definition.

2. SEARCHING FOR η_n

If X is a set and $F: X \rightarrow X$ is a function, we say an element \vec{x} of X is a periodic point of period 2 of F if $F^2(\vec{x}) = \vec{x}$ and $F(\vec{x}) \neq \vec{x}$. Typically, we use the field \mathbb{F}_2 as our set of states. However, in the following lemma, we may use any set of states A so long as $|A| \geq 2$.

Lemma 2.1. *Let $n \geq 2$ be an integer. Let $f: A^n \rightarrow A$ be a function, and let $\pi = \pi(1)\pi(2) \cdots \pi(n) \in S_n$ be a permutation. Suppose $\vec{x} = (x_1, x_2, \dots, x_n)$ is a periodic point of period 2 of the SDS map $F = [K_n, f, \pi]$. Write $F(\vec{x}) = \vec{z} = (z_1, z_2, \dots, z_n)$.*

For each $k \in \{1, 2, \dots, n\}$, we have $x_k \neq z_k$. In particular, if $A = \mathbb{F}_2$, then $F(\vec{x}) = \text{inv}(\vec{x})$.

Proof. For any $j \in \{1, 2, \dots, n\}$, recall that $G_{j-1}(\vec{x})$ is the system state that results from starting with the initial system state \vec{x} and then updating the vertices $v_{\pi(1)}, v_{\pi(2)}, \dots, v_{\pi(j-1)}$ (in this order). When we update the state of the vertex $v_{\pi(j)}$, the state to which $v_{\pi(j)}$ updates is given by $f(G_{j-1}(\vec{x}))$. On the other hand, the state to which $v_{\pi(j)}$ updates must be $z_{\pi(j)}$, the state of $v_{\pi(j)}$ in the system state \vec{z} . It follows that $f(G_{j-1}(\vec{x})) = z_{\pi(j)}$. Note that

$$f(G_n(\vec{x})) = f(F(\vec{x})) = f(\vec{z}) = f(G_0(\vec{z})) = x_{\pi(1)}.$$

These same arguments show that $f(G_{j-1}(\vec{z})) = x_{\pi(j)}$ for any $j \in \{1, 2, \dots, n\}$ and that $f(G_n(\vec{z})) = z_{\pi(1)}$.

If $i \in \{1, 2, \dots, n-1\}$, then the preceding paragraph tells us that

$$(1) \quad \begin{aligned} f(G_{i-1}(\vec{x})) &= z_{\pi(i)}, & f(G_{i-1}(\vec{z})) &= x_{\pi(i)}, \\ f(G_i(\vec{x})) &= z_{\pi(i+1)}, & f(G_i(\vec{z})) &= x_{\pi(i+1)}. \end{aligned}$$

We also have

$$(2) \quad \begin{aligned} f(G_{n-1}(\vec{x})) &= z_{\pi(n)}, & f(G_{n-1}(\vec{z})) &= x_{\pi(n)}, \\ f(G_n(\vec{x})) &= x_1, & f(G_n(\vec{z})) &= z_1. \end{aligned}$$

Notice that if $x_{\pi(i)} = z_{\pi(i)}$, then updating the vertex $v_{\pi(i)}$ doesn't "change" anything; that is, $G_{i-1}(\vec{x}) = G_i(\vec{x})$ and $G_{i-1}(\vec{z}) = G_i(\vec{z})$. With the help of the equations in (1) and (2), this shows that if $x_{\pi(i)} = z_{\pi(i)}$ for some $i \in \{1, 2, \dots, n\}$, then $x_{\pi(i+1 \pmod n)} = z_{\pi(i+1 \pmod n)}$. As a consequence, we see that if $x_{\pi(i)} = z_{\pi(i)}$ for some $i \in \{1, 2, \dots, n\}$, then $\vec{x} = \vec{z} = F(\vec{x})$. However, we are assuming that \vec{x} is a periodic point of F of period 2, so $F(\vec{x}) \neq \vec{x}$ by definition. Thus, $x_{\pi(i)} \neq z_{\pi(i)}$ for all $i \in \{1, 2, \dots, n\}$, which proves the lemma. \square

Let $w = w_1 w_2 \dots w_k$ be a finite word over the alphabet \mathbb{F}_2 . We say that a vector $(x_1, x_2, \dots, x_n) \in \mathbb{F}_2^n$ contains the subsequence w if there exist $i_1, i_2, \dots, i_k \in \{1, 2, \dots, n\}$ such that $i_1 < i_2 < \dots < i_k$ and $x_{i_j} = w_j$ for all $j \in \{1, 2, \dots, k\}$ (this is sometimes expressed by saying that w is a "scattered subword" of (x_1, x_2, \dots, x_n)). For example, the vector $(x_1, x_2, x_3, x_4, x_5) = (1, 0, 1, 1, 0)$ contains the subsequence 100 because $x_1 = 1$, $x_2 = 0$, and $x_5 = 0$. However, the vector $(1, 0, 1, 1, 0)$ does not contain the subsequence 001. This leads us to the following definition.

Definition 2.1. Let w be a finite word over the alphabet \mathbb{F}_2 . Define $D_n(w)$ to be the set of vectors $\vec{x} \in \mathbb{F}_2^n$ such that \vec{x} contains the subsequence w .

Definition 2.2. Let $n \geq 2$ be an integer. Let

$$\widehat{\mathbb{F}}_2^n = \{(x_1, x_2, \dots, x_n) \in \mathbb{F}_2^n : x_1 = 0\}.$$

Define \widehat{H}_n to be the undirected simple graph with vertex set

$$V(\widehat{H}_n) = \widehat{\mathbb{F}}_2^n$$

and edge set

$$E(\widehat{H}_n) = \{\{\vec{x}, \vec{y}\} \subseteq \widehat{\mathbb{F}}_2^n : \vec{x} + \vec{y} \in D_n(101)\}.$$

The following two lemmas link the graphs \widehat{H}_n to our study of 2-cycles in the phase spaces of SDS defined over complete graphs.

Lemma 2.2. Let $n \geq 2$ be an integer, and let $C = \{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_k\}$ be a clique of order k of \widehat{H}_n . There exists a map $f: \mathbb{F}_2^n \rightarrow \mathbb{F}_2$ such that each $\vec{x}_i \in C$ is in a 2-cycle of $\Gamma([K_n, f, \text{id}])$. Moreover, no two distinct \vec{x}_i, \vec{x}_j are contained in the same 2-cycle of $\Gamma([K_n, f, \text{id}])$.

Proof. For each $i \in \{1, 2, \dots, k\}$, let $\vec{x}_i = (a_{i1}, a_{i2}, \dots, a_{in})$, where $a_{i1} = 0$ by the definition of \widehat{H}_n . Define the map $f: \mathbb{F}_2^n \rightarrow \mathbb{F}_2$ as follows:

- If there are some i, ℓ such that

$$\vec{\alpha} = (\overline{a_{i1}}, \overline{a_{i2}}, \dots, \overline{a_{i\ell}}, a_{i(\ell+1)}, a_{i(\ell+2)}, \dots, a_{in}),$$

then let $f(\vec{\alpha}) = \overline{a_{i(\ell+1)}}$.

- If there are some i, ℓ such that

$$\vec{\alpha} = (a_{i1}, a_{i2}, \dots, a_{i\ell}, \overline{a_{i(\ell+1)}}, \overline{a_{i(\ell+2)}}, \dots, \overline{a_{in}}),$$

then let $f(\vec{\alpha}) = a_{i(\ell+1)}$.

- Otherwise, let $f(\vec{\alpha}) = 0$.

We first need to show that f is well-defined. To do so, we show that for any $i, j \in \{1, 2, \dots, k\}$ and any $\ell, m \in \{0, 1, \dots, n-1\}$ with $i \neq j$ or $\ell \neq m$, we have

$$(3) \quad (\overline{a_{i1}}, \dots, \overline{a_{i\ell}}, a_{i(\ell+1)}, \dots, a_{in}) \neq (\overline{a_{j1}}, \dots, \overline{a_{jm}}, a_{j(m+1)}, \dots, a_{jn}),$$

$$(4) \quad (\overline{a_{i1}}, \dots, \overline{a_{i\ell}}, a_{i(\ell+1)}, \dots, a_{in}) \neq (a_{j1}, \dots, a_{jm}, \overline{a_{j(m+1)}}, \dots, \overline{a_{jn}}),$$

and

$$(5) \quad (a_{i\ell}, \dots, a_{i\ell}, \overline{a_{i(\ell+1)}}, \dots, \overline{a_{in}}) \neq (a_{j1}, \dots, a_{jm}, \overline{a_{j(m+1)}}, \dots, \overline{a_{jn}}).$$

This will show that we have not accidentally defined $f(\vec{\alpha}) = 0$ and $f(\vec{\alpha}) = 1$ for the same vector $\vec{\alpha}$. Applying the function inv to each side of (5) yields (3), and applying inv to each side of (3) yields (5) (we are using the injectivity of inv). This shows that (3) and (5) are equivalent, so we only need to prove (3) and (4) to show that f is well-defined.

Choose such i, j, ℓ, m , and assume without loss of generality that $\ell \leq m$. First, suppose $i = j$. Because we are assuming either $i \neq j$ or $\ell \neq m$, this implies that $\ell < m$. The two vectors in (3) cannot be the same because their m^{th} coordinates are different. Similarly, the two vectors in (4) cannot be the same because their ℓ^{th} coordinates are different. This proves (3) and (4) in the case in which $i = j$.

We now assume $i \neq j$. Because \vec{x}_i and \vec{x}_j are adjacent in \widehat{H}_n , the vector $\vec{x}_i + \vec{x}_j$ contains the subsequence 101. This implies that there exist $r_1, r_2, r_3 \in \{2, 3, \dots, n\}$ such that $r_1 < r_2 < r_3$, $a_{ir_1} \neq a_{jr_1}$, $a_{ir_2} = a_{jr_2}$, and $a_{ir_3} \neq a_{jr_3}$. If $r_1 \leq \ell$, then the two vectors in (3) differ in their r_1^{th} coordinates. If $r_3 \geq m+1$, then the two vectors in (3) differ in their r_3^{th} coordinates. If $\ell < r_1$ and $r_3 \leq m$, then $\ell+1 \leq r_2 \leq m$, so the two vectors in (3) differ in their r_2^{th} coordinates. This proves (3) in all cases. We next prove (4). Note that $a_{i1} = a_{j1} = 0$. Since $m > \ell \geq 0$, the first coordinate of the vector on the right-hand side of (4) is 0. If $\ell \geq 1$, then the first coordinate of the vector on the left-hand side of (4) is 1. This shows that (4) holds whenever $\ell \geq 1$, so assume $\ell = 0$. If $r_2 \geq m+1$, then the two vectors in (4) differ in their r_2^{th} coordinates. Hence, we may also assume $r_2 \leq m$. Since $\ell = 0$, the r_1^{th} coordinate of the vector on the left-hand side of (4) is a_{ir_1} . Since $r_1 < r_2 \leq m$, the r_1^{th} coordinate of the vector on the right-hand side of (4) is a_{jr_1} . Because $a_{ir_1} \neq a_{jr_1}$, the two vectors in (4) differ in their r_1^{th} coordinates. This proves (4), showing that f is indeed well-defined.

Let $F = [K_n, f, \text{id}]$. We now show that for each $i \in \{1, 2, \dots, k\}$, the vector \vec{x}_i is in a 2-cycle of $\Gamma(F)$. Choose some $i \in \{1, 2, \dots, k\}$. It follows from the definition of f that for each $\ell \in \{1, 2, \dots, n\}$

$$G_\ell(\vec{x}_i) = (\overline{a_{i1}}, \dots, \overline{a_{i\ell}}, a_{i(\ell+1)}, \dots, a_{in})$$

and

$$G_\ell(\text{inv}(\vec{x}_i)) = G_\ell(\overline{a_{i1}}, \overline{a_{i2}}, \dots, \overline{a_{in}}) = (a_{i1}, \dots, a_{i\ell}, \overline{a_{i(\ell+1)}}, \dots, \overline{a_{in}}).$$

In particular, $F(\vec{x}_i) = \text{inv}(\vec{x}_i) \neq \vec{x}_i$ and $F^2(\vec{x}_i) = F(\text{inv}(\vec{x}_i)) = \vec{x}_i$. In other words, \vec{x}_i is in a 2-cycle of $\Gamma(F)$.

Finally, choose some distinct $i, j \in \{1, 2, \dots, k\}$. It is easy to see that the vectors \vec{x}_i and \vec{x}_j are in distinct 2-cycles of $\Gamma(F)$. Since $\vec{x}_i \neq \vec{x}_j$, the only way the vectors \vec{x}_i and \vec{x}_j could be in the same 2-cycle of $\Gamma(F)$ is if $F(\vec{x}_i) = \vec{x}_j$. We have just shown that $F(\vec{x}_i) = \text{inv}(\vec{x}_i)$, so \vec{x}_i could only be in the same 2-cycle as \vec{x}_j if $\vec{x}_j = \text{inv}(\vec{x}_i)$. However, this is impossible because \vec{x}_i and \vec{x}_j have the same first coordinate (namely, 0). \square

Lemma 2.3. *Let $n \geq 2$ be an integer, and let \vec{x} and \vec{y} be distinct nonadjacent vertices of \widehat{H}_n . Let $f: \mathbb{F}_2^n \rightarrow \mathbb{F}_2$ be a function. If \vec{x} is in a 2-cycle of the phase space $\Gamma([K_n, f, \text{id}])$, then \vec{y} is not in a 2-cycle of $\Gamma([K_n, f, \text{id}])$.*

Proof. Suppose, by way of contradiction, that \vec{x} and \vec{y} are both in 2-cycles of $\Gamma([K_n, f, \text{id}])$. Let $F = [K_n, f, \text{id}]$. Let $\vec{x} = (a_1, a_2, \dots, a_n)$ and $\vec{y} = (b_1, b_2, \dots, b_n)$. Because $\vec{x} \neq \vec{y}$, we may let r be the smallest element of $\{1, \dots, n\}$ such that $a_r \neq b_r$. Similarly, we may let s be the largest element of $\{1, \dots, n\}$ such that $a_s \neq b_s$. Note that $r \geq 2$ because $a_1 = b_1 = 0$. Because \vec{x} and \vec{y} are not adjacent in \widehat{H}_n , the vector $\vec{x} + \vec{y} = (a_1 + b_1, \dots, a_n + b_n)$ does not contain the subsequence 101. This implies that

$$a_i + b_i = \begin{cases} 1, & \text{if } r \leq i \leq s; \\ 0, & \text{otherwise.} \end{cases}$$

In other words, $\vec{y} = (a_1, \dots, a_{r-1}, \overline{a_r}, \dots, \overline{a_s}, a_{s+1}, \dots, a_n)$. For the sake of convenience, let $a_{n+1} = \overline{a_1}$.

Because we are assuming that each of the vectors \vec{x} and \vec{y} is in a 2-cycle of $\Gamma(F)$, we know from Lemma 2.1 that $F(\vec{x}) = \text{inv}(\vec{x})$ and $F(\vec{y}) = \text{inv}(\vec{y})$. Hence, for each $\ell \in \{1, \dots, n\}$,

$$G_\ell(\vec{x}) = (\overline{a_1}, \dots, \overline{a_\ell}, a_{\ell+1}, \dots, a_n)$$

and

$$G_\ell(\vec{y}) = (\overline{b_1}, \dots, \overline{b_\ell}, b_{\ell+1}, \dots, b_n).$$

It follows that

$$f(\overline{a_1}, \dots, \overline{a_s}, a_{s+1}, \dots, a_n) = f(G_s(\vec{x})) = \overline{a_{s+1}},$$

but also

$$f(\overline{a_1}, \dots, \overline{a_s}, a_{s+1}, \dots, a_n) = f(\overline{b_1}, \dots, \overline{b_{r-1}}, b_r, \dots, b_n) = f(G_{r-1}(\vec{y})) = \overline{b_r} = a_r.$$

Therefore, $a_r = \overline{a_{s+1}}$. Similarly, we have

$$f(\overline{a_1}, \dots, \overline{a_{r-1}}, a_r, \dots, a_n) = f(G_{r-1}(\vec{x})) = \overline{a_r}$$

and

$$f(\overline{a_1}, \dots, \overline{a_{r-1}}, a_r, \dots, a_n) = f(\overline{b_1}, \dots, \overline{b_s}, b_{s+1}, \dots, b_n) = f(G_s(\vec{y})) = \overline{b_{s+1}} = \overline{a_{s+1}}.$$

This implies that $\overline{a_r} = \overline{a_{s+1}} = a_r$, which is a contradiction. \square

We are now in a position to prove one of our crucial theorems. Let $\omega(G)$ denote the clique number of a graph G . That is, G contains a clique of order $\omega(G)$ but does not contain a clique of order $\omega(G) + 1$.

Theorem 2.1. *For any integer $n \geq 2$, we have*

$$\eta_n = \omega(\widehat{H}_n).$$

Proof. Choose an integer $n \geq 2$, and let $k = \omega(\widehat{H}_n)$. Let $C = \{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_k\}$ be a clique of order k of \widehat{H}_n . By Lemma 2.2, there exists a map $f: \mathbb{F}_2^n \rightarrow \mathbb{F}_2$ such that any distinct vectors $\vec{x}_i, \vec{x}_j \in C$ are in distinct 2-cycles of $\Gamma([K_n, f, \text{id}])$. In particular, $\Gamma([K_n, f, \text{id}])$ contains at least k 2-cycles. In the notation of Definition 1.1, $f \in (\mathbb{F}_2)^{\mathbb{F}_2^n}$, and $\eta(f, \text{id}) \geq k$. Thus,

$$\eta_n = \max_{g \in (\mathbb{F}_2)^{\mathbb{F}_2^n}} \eta(g, \text{id}) \geq \eta(f, \text{id}) \geq k.$$

We now show that $\eta_n \leq k$. By Definition 1.1, there exists a function $g \in (\mathbb{F}_2)^{\mathbb{F}_2^n}$ such that $\eta(g, \text{id}) = \eta_n$. In other words, there are η_n 2-cycles in the phase space $\Gamma([K_n, g, \text{id}])$. It follows from Lemma 2.1 that each 2-cycle of $\Gamma([K_n, g, \text{id}])$ contains exactly one vector whose first coordinate is 0. In the notation of Definition 2.2, each of the η_n 2-cycles of $\Gamma([K_n, g, \text{id}])$ contains exactly one vector that is a vertex of \widehat{H}_n . Let $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_{\eta_n}$ be these vertices of \widehat{H}_n . Choose some distinct $i, j \in \{1, 2, \dots, \eta_n\}$. Because each of the vectors \vec{x}_i, \vec{x}_j is in a 2-cycle of $\Gamma([K_n, g, \text{id}])$, it follows from Lemma 2.3 that \vec{x}_i and \vec{x}_j must be adjacent in \widehat{H}_n . Because i and j were arbitrary, this shows that $\{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_{\eta_n}\}$ is a clique of \widehat{H}_n . Consequently, $\eta_n \leq k$. \square

Given an integer $n \geq 2$, we may define a map $\theta_n: \widehat{\mathbb{F}}_2^n \rightarrow \mathbb{F}_2^{n-1}$ by

$$\theta_n(x_1, x_2, \dots, x_n) = (x_2, x_3, \dots, x_n).$$

Because the first coordinate of each vector in $\widehat{\mathbb{F}}_2^n$ is 0, it should be clear that θ_n is a vector space isomorphism. Furthermore, if $\vec{x}, \vec{y} \in \widehat{\mathbb{F}}_2^n$, then $\vec{x} + \vec{y} \in D_n(101)$ if and only if $\theta_n(\vec{x}) + \theta_n(\vec{y}) \in D_{n-1}(101)$. This motivates the following definition.

Definition 2.3. Given an integer $m \geq 2$, define H_m to be the undirected simple graph with vertex set

$$V(H_m) = \mathbb{F}_2^m$$

and edge set

$$E(H_m) = \{\{\vec{x}, \vec{y}\} \subseteq \mathbb{F}_2^m : \vec{x} + \vec{y} \in D_m(101)\}.$$

It follows from the preceding paragraph that θ_{n+1} is a graph isomorphism from the graph \widehat{H}_{n+1} to the graph H_n . Therefore, $\eta_{n+1} = \omega(\widehat{H}_{n+1}) = \omega(H_n)$. In the following section, we relate the graphs H_n to error-correcting binary codes.

3. BINARY CODES

In coding theory, a binary code of length n is a subset of \mathbb{F}_2^n . The elements of a code are known as *codewords*. The *Hamming distance* between two codewords $\vec{x} = (x_1, x_2, \dots, x_n)$ and $\vec{y} = (y_1, y_2, \dots, y_n)$, which we shall denote by $\delta(\vec{x}, \vec{y})$, is simply the number of positions in which the vectors \vec{x} and \vec{y} have different coordinates. That is,

$$\delta(\vec{x}, \vec{y}) = |\{i \in \{1, 2, \dots, n\} : x_i \neq y_i\}|.$$

If C is a nonempty binary code of length n , then the *minimum distance* of C , denoted $\Delta(C)$, is the quantity

$$\Delta(C) = \min_{\substack{\vec{x}, \vec{y} \in C \\ \vec{x} \neq \vec{y}}} \delta(\vec{x}, \vec{y}).$$

If a code C consists of a single codeword, we convene to let $\Delta(C) = \infty$.

Of particular interest in coding theory are binary codes with minimum distance at least 3. Such codes are known as one-bit error-correcting codes because any error formed by flipping a single bit (that is, changing a single coordinate) in a codeword can be detected and corrected (this is not the case for binary codes with minimum distance less than 3). Let $A(n, 3)$ denote the maximum number of codewords that can appear in a binary code of length n that has minimum distance at least 3. An important problem in coding theory is the determination of the values of $A(n, 3)$. It is known that $A(n, 3) = 2^{n - \log_2(n+1)}$ if $n + 1$ is a power of 2, but most values of $A(n, 3)$ are not known.

Given an integer $m \geq 2$, define J_m to be the undirected simple graph with vertex set

$$V(J_m) = \mathbb{F}_2^m$$

and edge set

$$E(J_m) = \{\{\vec{x}, \vec{y}\} \subseteq \mathbb{F}_2^m : \vec{x} + \vec{y} \in D_m(111)\}.$$

Observe that two vectors $\vec{x}, \vec{y} \in \mathbb{F}_2^m$ are adjacent in J_m if and only if the Hamming distance $\delta(\vec{x}, \vec{y})$ between \vec{x} and \vec{y} is at least 3. Therefore, $A(n, 3)$ is equal to the clique number $\omega(J_n)$ of the graph J_n . From this, we obtain the following theorem.

Theorem 3.1. *For any positive integer n ,*

$$\eta_{n+1} = A(n, 3).$$

Proof. We saw at the end of the preceding section that $\eta_{n+1} = \omega(H_n)$. Therefore, in light of the preceding paragraph, we see that it suffices to show that $\omega(H_n) = \omega(J_n)$. Consider the linear transformation $T: \mathbb{F}_2^n \rightarrow \mathbb{F}_2^n$ given by

$$T(x_1, x_2, \dots, x_n) = (x_1, x_1 + x_2, x_1 + x_2 + x_3, \dots, x_1 + x_2 + \dots + x_n).$$

The linear transformation T is an endomorphism with a trivial kernel, so it is an isomorphism. Furthermore, a vector $\vec{z} \in \mathbb{F}_2^n$ contains the subsequence 111 if and only if $T(\vec{z})$ contains the subsequence 101. This shows that T is in fact a graph isomorphism from J_n to H_n , so $\omega(H_n) = \omega(J_n)$. \square

As mentioned in the introduction, Theorem 3.1 provides the first known interpretation of the numbers $A(n, 3)$ outside of coding theory.

4. CONCLUDING REMARKS

We have defined η_n to be the maximum number of 2-cycles that can appear in a phase space of the form $\Gamma([K_n, f, \text{id}])$. One could easily generalize these numbers by defining $\eta_n(m)$ to be the maximum number of m -cycles that can appear in such a phase space. One may show that $\eta_n(1) = 2$ for all positive integers n (the only possible fixed points of an SDS map $[K_n, f, \text{id}]$ are the all-0's and all-1's vectors of length n). Is it possible to obtain general bounds for the numbers $\eta_n(m)$? Could we perhaps relate the numbers $\eta_n(m)$ to codes as we have done for the numbers $\eta_n(2)$?

There are, of course, many other natural ways to generalize the problems considered here. One might wish to replace complete graphs with graphs that are, in some sense, “almost” complete (such as complements of cycle graphs). We could also choose to ask similar questions about SDS maps of the form $[K_n, f, \text{id}]$ defined using a set of states A with $|A| \geq 3$.

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