i-Mark: A NEW SUBTRACTION DIVISION GAME

ÉRIC SOPENA

Abstract. Given two finite sets of integers $S \subseteq \mathbb{N} \setminus \{0\}$ and $D \subseteq \mathbb{N} \setminus \{0, 1\}$, the impartial combinatorial game i-Mark(S, D) is played on a heap of tokens. From a heap of n tokens, each player can move either to a heap of $n - s$ tokens for some $s \in S$, or to a heap of n/d tokens for some $d \in D$ if d divides n. Such games can be considered as an integral variant of Mark-type games, introduced by Elwyn Berlekamp and Joe Buhler and studied by Aviezri Fraenkel and Alan Guo, for which it is allowed to move from a heap of n tokens to a heap of $\lfloor n/d \rfloor$ tokens for any $d \in D$.

Under normal convention, it is observed that the Sprague-Grundy sequence of the game i-Mark(S, D) is aperiodic for any sets S and D. However, we prove that, in many cases, this sequence is almost periodic and that the set of winning positions is periodic. Moreover, in all these cases, the Sprague-Grundy value of a heap of n tokens can be computed in time $O(\log n)$.

We also prove that, under misère convention, the outcome sequence of these games is purely periodic.

Keywords: Combinatorial games; Subtraction games; Subtraction division games; Sprague-Grundy sequence; Aperiodicity.

Mathematics Subject Classification (MSC 2010): 91A46.

1. Introduction

The impartial combinatorial game Mark, due to Mark Krusemeyer according to Elwyn Berlekamp and Joe Buhler [2], is played on a heap of tokens. On their turn, each player can move from a heap of n tokens, $n \geq 1$, either to a heap of $n - 1$ tokens or to a heap of $\lfloor n/2 \rfloor$ tokens. Under *normal convention*, the first player unable to move (when the heap is empty) loses the game. This game is a particular case of a more general family of games, that we call *subtraction division games* following [8], defined as follows[1]. Let us denote by $[a, b]$, $a \leq b$, the set of integers $\{i \mid a \leq i \leq b\}$, by $\mathbb{N}_{\geq i}$, $i \in \mathbb{N}$, the set $\mathbb{N} \setminus [0, i-1]$ and let $S \subseteq \mathbb{N}_{\geq 1}$ and $D \subseteq \mathbb{N}_{\geq 2}$ be two finite sets of integers. The subtraction division game $SD(S, D)$ is played on a heap of n tokens, $n \geq 0$. From a heap of n tokens, each player can move either to a heap of $n - s$ tokens for some $s \in S$, or to a heap of $\lfloor n/d \rfloor$ tokens for some $d \in D$. The Mark game is therefore the game $SD(\{1\}, \{2\})$.

Univ. Bordeaux, LaBRI, UMR5800, F-33400 Talence, France.
CNRS, LaBRI, UMR5800, F-33400 Talence, France.
E-mail address: eric.sopena@labri.fr.
Date: November 10, 2015.
This work has been supported by the ANR-14-CE25-0006 project of the French National Research Agency.

[1] In [8], Elizabeth Kupin used a slightly different definition for subtraction division games, where division-type moves leave a heap of $\lceil n/d \rceil$ tokens, instead of $\lfloor n/d \rfloor$ tokens, and the game stops as soon as the heap contains a unique token, instead of no token at all.

In this paper, we study a variant of such subtraction division games, that we propose to name i-MARK(S, D) for *integral* MARK, obtained by restricting division-type moves, allowing a move from a heap of n tokens to a heap of $\lfloor n/d \rfloor$ tokens, $d \in D$, only when d divides n, so that $\lfloor n/d \rfloor = n/d$.

1.1. **Combinatorial Game Theory: basic notions and terminology.** We first briefly recall the main notions of Combinatorial Game Theory that we will need in this paper. More details can be found on any of the reference books [1], [3], [4] or [9].

Let us call a *heap-game* any impartial combinatorial game played on a heap of tokens. We denote by $\mathbf{n}$ a heap of n tokens, or $\mathbf{n}_{\text{GAME}}$ if we want to specify that the considered game is GAME. Those heaps $\mathbf{n}$ for which the **N**ext player to move has a winning strategy are N-positions, whereas those for which the **P**revious player has a winning strategy are P-positions. We denote by $\mathcal{N}$ the set of P-positions and by $\mathcal{P}$ the set of P-positions. Note that a position $\mathbf{n}$ is in $\mathcal{P}$ if and only if every move from $\mathbf{n}$ leads to a position in $\mathcal{N}$, whereas $\mathbf{n}$ is in $\mathcal{N}$ if and only if there exists at least one move from $\mathbf{n}$ to a position in $\mathcal{P}$. We also say that the *outcome* of a position $\mathbf{n}$ is N if $\mathbf{n} \in \mathcal{N}$ and P otherwise. It is not difficult to prove that any position of an impartial combinatorial game is either an N-position or a P-position, which implies that the two sets $\mathcal{N}$ and $\mathcal{P}$ are *complementary* – that is, $\mathcal{N} \cap \mathcal{P} = \emptyset$ and $\mathcal{N} \cup \mathcal{P} = \mathbb{N}$.

Heap-games can be played on any finite number of heaps, by means of *sums of games*. In that case, a player move consists in first selecting one of the heaps and then making a legal move on that heap. Knowing whether each heap is an N- or a P-position is not sufficient to determine whether the whole game is an N- or a P-position. Under normal convention, we use the *Sprague-Grundy function* which assigns to every position G of an impartial game its *g-value* $g(G)$ (sometimes called *nim-value* or *nimber*), defined as the unique positive integer k such that G is equivalent to the NIM position $\mathbf{k}_{\text{NIM}}$. (Equivalence here means that the sum $G + \mathbf{k}_{\text{NIM}}$ is a P-position.) Therefore, a position is in $\mathcal{P}$ if and only if its g-value is 0. The g-value of any position G of an impartial game can be inductively computed using the *mex function*, defined by

$$\text{mex}(S) = \min(\mathbb{N} \setminus S)$$

for every finite set $S \subseteq \mathbb{N}$. We then have

$$g(G) = \text{mex}(\text{opt}(G)),$$

where $\text{opt}(G)$ denotes the set of *options* of G – that is, the set of positions that can be reached from G by making a legal move. Note here that if a position has k options then its g-value is at most k.

Let $\oplus$ denote the *nim-sum function*, defined by

$$n_1 \oplus n_2 = R^{-1}(R(n_1) \text{ XOR } R(n_2))$$

for any two positive integers n_1 and n_2, where R denotes the function that associates with each integer its binary representation. For a position $G = G_1 + \ldots + G_p$ of a sum of games, the g-value of G is then given by

$$g(G) = g(G_1) \oplus \ldots \oplus g(G_p).$$

Computing the g-value of a game recursively using the mex function can be exponential in the 'size' of G and thus inefficient. For a heap-game GAME we define the *Sprague-Grundy sequence* (sometimes called the *nim-sequence*) of GAME as the sequence

$$\mathcal{G}(\text{GAME}) = (g(\mathbf{1}_{\text{GAME}}), g(\mathbf{2}_{\text{GAME}}), \ldots).$$

Knowing this sequence clearly allows to compute the g-value of any position of GAME on a finite number of heaps. This can be efficient whenever the Sprague-Grundy sequence has "nice properties", in particular if it is periodic. We say that an integer sequence $(a_i)_{i\in\mathbb{N}}$ is *periodic*, with preperiod of length $q \geq 0$ and period of length $p \geq 1$, if for every $i \geq q$, $a_i = a_{i+p}$. In such a case, any value a_i of the sequence is determined by the value of $a_i \mod p$, which can be computed in time $O(\log a_i)$ for a fixed p. This is namely the case for all *subtraction games* – that is, games of the form $SD(S, \emptyset)$ – whenever the subtraction set S is finite [3].

Similarly, we say that a set of integers $S \subseteq \mathbb{N}$ is periodic if there exist q, p, with $q \geq 0$ and $p \geq 1$, such that for every $i \geq q$, $i \in S$ if and only if $i + p \in S$. Note that since the sets $\mathcal{N}$ and $\mathcal{P}$ of any heap-game are complementary, either both these sets are periodic or none of them is periodic.

We define the *outcome sequence* of a game GAME as the sequence $(o_i)_{i\in\mathbb{N}}$ given by $o_i = N$ if $\mathbf{i}$ is an N-position for GAME and $o_i = P$ otherwise. We will say that such an outcome sequence is periodic whenever the sets $\mathcal{N}$ and $\mathcal{P}$ are periodic.

1.2. **The game** MARK**.** In [5, 6], Aviezri Fraenkel developped a study of the MARK game $SD(\{1\}, \{2\})$ and gave a characterization of the corresponding sets $\mathcal{N}$ and $\mathcal{P}$. Although aperiodic, these two sets reveals a nice structure: a MARK position $\mathbf{n}$ is in $\mathcal{N}$ if and only if the binary representation $R(n)$ of n has an even number of trailing 0's[2]. This allows to compute the outcome of any MARK position $\mathbf{n}$ in time $O(\log n)$. It is also proved that the Sprague-Grundy sequence $\mathcal{G}(\text{MARK})$ has the following property (note that since any MARK position has at most two options, $g(\mathbf{n}) \leq 2$ for any MARK position $\mathbf{n}$):

- $g(\mathbf{n}) = 0$ is and only if $R(n)$ has an odd number of trailing 0's,
- $g(\mathbf{n}) = 1$ if and only if $R(n)$ has an even number of trailing 0's and an odd number of 1's,
- $g(\mathbf{n}) = 2$ if and only if $R(n)$ has an even number of trailing 0's and an even number of 1's.

Again, this allows to compute the g-value $g(\mathbf{n})$ of any MARK position $\mathbf{n}$ in time $O(\log n)$.

Aviezri Fraenkel also studied the MARK game under *misère convention* (the first player unable to move wins the game), the game UPMARK (allowing moves from $\mathbf{n}$ to $\lceil \mathbf{n/2} \rceil$ instead of $\lfloor \mathbf{n/2} \rfloor$) and introduced the general game MARK-t = $SD([1, t-1], \{t\})$, for any given $t \geq 1$, whose sets $\mathcal{N}$ and $\mathcal{P}$ again are aperiodic for every t.

This latter game has been studied by Alan Guo in [7]. He proved that the g-value of any MARK-t position can be computed in quadratic time. More precisely, he proved that $g(\mathbf{n}) = k$, $k \leq t-2$, if and only if $R_t(n)$ has an odd number of trailing k's, where $R_t(n)$ denotes the representation of n written in base t, and that deciding whether $g(\mathbf{n}) = t-1$ or $g(\mathbf{n}) = t$ can be done in quadratic time.

[2] Aviezri Fraenkel called such numbers *vile* numbers, and *dopey* numbers those numbers whose binary representation has an odd number of trailing 0's.

1.3. **The game i-Mark: an integral subtraction division game.** We study in this paper an integral variant of Mark-type games. For any two sets $S \subseteq \mathbb{N}_{\geq 1}$ and $D \subseteq \mathbb{N}_{\geq 2}$, the allowed moves in the game i-Mark(S, D) are those leading from a position $\mathbf{n}$ to any of the positions $\mathbf{n} - \mathbf{s}$, with $s \in S$, or $\mathbf{n}/\mathbf{d}$, with $d \in D$ and $d|n$.

As for the game Mark, the Sprague-Grundy sequence of the game i-Mark(S, D) is aperiodic whenever the set D is non-empty:

Theorem 1. *For every finite set $S \subseteq \mathbb{N}_{\geq 1}$ and every non-empty finite set $D \subseteq \mathbb{N}_{\geq 2}$, the Sprague-Grundy sequence of the game i-Mark(S, D) is aperiodic.*

Proof. Assume to the contrary that $\mathcal{G}(i\text{-Mark}(S, D))$ is periodic, with preperiod of length q and period of length p, and let $d \in D$. Let $\mathbf{n}$ be any position with $n = kp \geq q$ for some $k \geq 1$. Since $d \in D$, there is a move from $\mathbf{dn}$ to $\mathbf{n}$ and, therefore $g(\mathbf{dn}) \neq g(\mathbf{n})$. This contradicts our assumption since $dn - n = (d-1)n = (d-1)kp$. □

However, and in contrast to the game Mark, we will prove that the outcome sequence of such integral games is periodic in many cases. Moreover, the Sprague-Grundy sequence appears to be "almost periodic" in most of these cases, in the following sense:

Definition 1. An integer sequence $(a_i)_{i \in \mathbb{N}}$ is *ℓ-almost periodic* for some $\ell \geq 0$, with preperiod of length $q \geq 0$, period of length $p > \ell$ and *exception set* $\mathcal{E} = \{j_1, \dots, j_\ell\} \subseteq [0, p-1]$ if, for every $i \geq q$ with $(i \mod p) \notin \mathcal{E}$, $a_i = a_{i+p}$.

Intuitively speaking, a sequence is ℓ-almost periodic if it is "periodic, except on ℓ columns of the period". A 0-almost periodic sequence is thus a periodic sequence.

1.4. **Organisation of the paper.** In the next section, we provide more details on the results proposed by Aviezri Fraenkel in [5] on the game Mark and consider its integral version i-Mark$(\{1\}, \{2\})$ as an introductory example.

We then study games of the form i-Mark$([1, t-1], \{d\})$, with $t, d \geq 2$, in Section 3, and of the form i-Mark$(\{a, 2a\}, \{2\})$, with $a \geq 2$, in Section 4.

We finally consider the misère version of the game i-Mark in Section 5 and propose some open questions in Section 6.

2. A first sample game: i-Mark$(\{1\}, \{2\})$

We consider in this section the simple game i-Mark$(\{1\}, \{2\})$ since it corresponds to our variant of the original game Mark. We first briefly recall some results presented by Aviezri Fraenkel in [5].

2.1. **The game Mark.** Let $A = (a_n)_{n \geq 1}$ and $B = (b_n)_{n \geq 0}$ be the two sequences recursively defined by:

$$a_n = \text{mex}\{a_i, b_i \mid 0 \leq i < n\},$$

$$b_0 = 0 \text{ and } b_n = 2a_n,\ n \geq 1.$$

These two sequences are clearly complementary and respectively correspond to the sequences A003159 and A036554 of the "On-Line Encyclopedia of Integer Sequences" of Neil Sloane [10].

A position $\mathbf{n}$ is then an N-position for MARK if and only if $n \in A$, and thus a P-position for MARK if and only if $n \in B$. The first elements of these two sequences are given in the following table:

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
a_n		1	3	4	5	7	9	11	12	13	15	16	17	19	20	21	23
b_n	0	2	6	8	10	14	18	22	24	26	30	32	34	38	40	42	46

Moreover, it can be observed that both these sets are aperiodic and, therefore, the Sprague-Grundy sequence $\mathcal{G}(\text{MARK})$ is aperiodic. The first elements of this sequence are the following:

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$g(\mathbf{n})$	0	1	0	2	1	2	0	1	0	2	0	1	2	1	0	2	1

2.2. **The game i-MARK$(\{1\},\{2\})$.** Let us now consider the integral version of the game MARK, denoted i-MARK$(\{1\},\{2\})$. By Theorem 1, we know that the Sprague-Grundy sequence of the game i-MARK$(\{1\},\{2\})$ is aperiodic. We first prove that its outcome sequence is periodic.

Theorem 2. *The outcome sequence of the game i-MARK$(\{1\},\{2\})$ is periodic, with preperiod of length 4 and period of length 2. More precisely, we have*

(i) $\mathcal{N} = \{\mathbf{1},\mathbf{3}\} \cup \{\mathbf{2k} \mid k \geq 2\}$,
(ii) $\mathcal{P} = \{\mathbf{0},\mathbf{2}\} \cup \{\mathbf{2k+1} \mid k \geq 2\}$.

Proof. Clearly, $\mathbf{0} \in \mathcal{P}$. Since $\text{opt}(\mathbf{1}) = \{\mathbf{0}\}$, $\text{opt}(\mathbf{2}) = \{\mathbf{1}\}$, $\text{opt}(\mathbf{3}) = \{\mathbf{2}\}$ and $\text{opt}(\mathbf{4}) = \{\mathbf{2},\mathbf{3}\}$, we get $\mathbf{1},\mathbf{3},\mathbf{4} \in \mathcal{N}$ and $\mathbf{2} \in \mathcal{P}$. The result then follows by induction since (i) $\text{opt}(\mathbf{n}) = \{\mathbf{n-1}\}$ for every odd n, so that $\mathbf{n} \in \mathcal{P}$, and (ii) for every P-position $\mathbf{n}$, $\mathbf{n} \in \text{opt}(\mathbf{n+1})$ and thus $\mathbf{n+1}$ is an N-position. □

The first elements of the Sprague-Grundy sequence $\mathcal{G}(i\text{-MARK}(\{1\},\{2\}))$ are the following:

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$g(\mathbf{n})$	0	1	0	1	2	0	2	0	1	0	1	0	1	0	1	0
n	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31
$g(\mathbf{n})$	2	0	1	0	2	0	1	0	2	0	1	0	2	0	1	0

By Theorem 1, we know that this sequence is aperiodic. By Theorem 2, we know that $g(\mathbf{n}) = 0$ for every odd n, $n \geq 5$, and $g(\mathbf{n}) \in \{1,2\}$ for every even n, $n \geq 4$. Hence, this sequence is 1-almost periodic. We now prove that deciding whether $g(\mathbf{n}) = 2$ for any position n is easy.

For every integer $n > 0$, we denote by $R^1(n)$ the binary number obtained from the binary representation $R(n)$ of n by deleting all the trailing 0's. Then we have:

Theorem 3. *Let $\mathbf{n}$ be any position of the game i-MARK$(\{1\},\{2\})$. Then $g(\mathbf{n}) = 2$ if and only if n is even and either:*

(i) *$R^1(n) = 11_2$ and $R(n)$ has an odd number of trailing 0's, or*
(ii) *$R^1(n) \neq 11_2$ and $R(n)$ has an even number of trailing 0's.*

Proof. The theorem clearly holds for $n \leq 6$. Suppose now that the theorem holds up to position $\mathbf{n-1}$, $n \geq 7$, and consider the position $\mathbf{n}$.

If n is odd, we know by Theorem 2 that $n \in \mathcal{P}$ and thus $g(\mathbf{n}) = 0$.

Assume thus that n is even, so that $\text{opt}(\mathbf{n}) = \{\mathbf{n/2}, \mathbf{n-1}\}$. Since $g(\mathbf{n-1}) = 0$, we get $g(\mathbf{n}) = 2$ if and only if $g(\mathbf{n/2}) = 1$ and the result directly follows from the induction hypothesis. □

Let $\mathcal{N}_2$ denote the set of integers n such that $g(\mathbf{n}) = 2$. The set $\mathcal{N}_2$ is as follows:

$$\mathcal{N}_2 = \{4, 6, 16, 20, 24, 28, 36, 44, 52, 60, 64, 68, 76, 80, 84, 92, 96, 100, 108, 112, \dots\}.$$

Note that from Theorems 2 and 3, we get that computing the g-value of any position $\mathbf{n}$ for the game i-Mark$(\{1\},\{2\})$ can be done in time $O(\log n)$.

3. The game i-Mark$([1,t-1],\{d\})$

We consider in this section the game i-Mark$([1,t-1],\{d\})$, with $t \geq 2$ and $d \geq 2$. If $t = d = 2$, we get the game i-Mark$(\{1\},\{2\})$ considered in the previous section. More generally, if $t = d$, we get the integral version of the game Mark-t introduced by Aviezri Fraenkel in [6] and studied by Alan Guo in [7].

We will prove that for every $t \geq 2$ and $d \geq 2$, $d \not\equiv 1 \pmod t$, the outcome sequence of the game i-Mark$([1,t-1],\{d\})$ is periodic, while its Sprague-Grundy sequence is 1-almost periodic.

We first consider the outcome sequence of the game i-Mark$([1,t-1],\{d\})$ when $d \not\equiv 1 \pmod t$, and prove the following:

Theorem 4. *For every integers $t \geq 2$ and $d \geq 2$, $d \not\equiv 1 \pmod t$, the outcome sequence of the game i-Mark$([1,t-1],\{d\})$ is periodic, with preperiod of length $d(t-1)+2$ and period of length t. More precisely, the set $\mathcal{P}$ of P-positions is given by*

$$\mathcal{P} = \{\mathbf{qt} \mid 0 \leq q < d\} \cup \{\mathbf{qt+1} \mid q \geq d\}.$$

Proof. We clearly have $\mathbf{0} \in \mathcal{P}$, which implies $\{\mathbf{1}, \dots, \mathbf{t-1}\} \subseteq \mathcal{N}$ since $\mathbf{0}$ is an option of all these positions.

Suppose now that the theorem holds up to position $\mathbf{n-1}$, $n \geq t$, and consider the position $\mathbf{n}$. Let $n = qt + r$, with $q > 0$ and $r \in [0, t-1]$.

If $q < d$ and $r \neq 0$ then $\mathbf{qt} \in \text{opt}(\mathbf{n})$ and thus, since $\mathbf{qt} \in \mathcal{P}$ by induction hypothesis, $\mathbf{n} \in \mathcal{N}$.

If $q < d$ and $r = 0$, $\text{opt}(\mathbf{n}) = \{(\mathbf{qt-t+1}, \dots, \mathbf{qt-1}\}$ and thus, since all these options are in $\mathcal{N}$ by induction hypothesis, $\mathbf{n} \in \mathcal{P}$.

If $q = d$ and $r = 0$ – that is, $n = dt$ – then, since $\mathbf{t} \in \text{opt}(\mathbf{n})$ and $\mathbf{t} \in \mathcal{P}$ by induction hypothesis, $\mathbf{n} \in \mathcal{N}$.

If $q > d$ and $r = 0$ then, since $(\mathbf{q-1})\mathbf{t+1} \in \text{opt}(\mathbf{n})$ and $(\mathbf{q-1})\mathbf{t+1} \in \mathcal{P}$ by induction hypothesis, $\mathbf{n} \in \mathcal{N}$.

If $q \geq d$ and $r = 1$ we consider two cases. If $d \not| n$ then $\text{opt}(\mathbf{n}) = \{(\mathbf{q-1})\mathbf{t+2}, \dots, \mathbf{qt}\}$ and thus, since all these options are in $\mathcal{N}$ by induction hypothesis, $\mathbf{n} \in \mathcal{P}$. If $d | n$ then $\text{opt}(\mathbf{n}) = \{\mathbf{n/d}\} \cup \{(\mathbf{q-1})\mathbf{t+2}, \dots, \mathbf{qt}\}$. Since $d \not\equiv 1 \pmod t$ and $n \equiv 1 \pmod t$ we have $n/d \not\equiv 1 \pmod t$. Hence, all the options of $\mathbf{n}$ are in $\mathcal{N}$ by induction hypothesis, and thus $\mathbf{n} \in \mathcal{P}$.

Finally, if $q \geq d$ and $r > 1$ then, since $\mathbf{qt+1} \in \text{opt}(\mathbf{n})$ and $\mathbf{qt+1} \in \mathcal{P}$ by induction hypothesis, $\mathbf{n} \in \mathcal{N}$.

This gives that the outcome sequence of the game i-MARK$([1, t-1], \{d\})$ is periodic, with preperiod of length $dt - 1$ and period of length t. □

Using computer check, it seems that, for every integers $t \geq 2$ and $d \geq 2$, $d \not\equiv 1 \pmod t$, the Sprague-Grundy sequence of the game i-MARK$([1, t-1], \{d\})$ is 1-almost periodic with period of length t. However, the length of the preperiod seems to be more "erratic".

We thus prove this property only for two particular cases, namely $d = t$ and $d = 2$. Moreover, we will also prove that, in these two cases, the non-periodic column has a nice structure, so that the g-value of any position $\mathbf{n}$ can be computed in time $O(\log n)$.

3.1. **The game i-MARK$([1, t-1], \{t\})$.** We now prove that the Sprague-Grundy sequence of the game i-MARK$([1, t-1], \{t\})$ is 1-almost periodic for every $t \geq 2$ and that the g-value of any position $\mathbf{n}$ can be computed in time $O(\log n)$.

Theorem 5. *For every $t \geq 2$, the Sprague-Grundy sequence of the game i-MARK$([1, t-1], \{t\})$ is 1-almost periodic, with preperiod of length $t^2 - 1$, period of length t and exception set $\mathcal{E} = \{0\}$. More precisely, for every integer $n = qt + r$ with $q \geq 0$ and $r \in [0, t-1]$, we have*

(i) *if $q < t$ and $r = 0$, or $q \geq t$ and $r = 1$, then $g(\mathbf{n}) = 0$,*
(ii) *if $q < t$ and $r \neq 0$, then $g(\mathbf{n}) = r$,*
(iii) *if $q \geq t$ and $r > 1$, then $g(\mathbf{n}) = r - 1$,*
(iv) *if $q \geq t$ and $r = 0$, then $g(\mathbf{n}) \in \{t-1, t\}$.*

Proof. Claim (i) direclty follows from Theorem 4. For the remaining claims, the proof easily follows by induction. Suppose that the theorem holds up to position $\mathbf{n-1}$, $n \geq 1$, and consider the position $\mathbf{n}$. Let $n = qt + r > 1$, with $q \geq 0$ and $r \in [0, t-1]$.

If $q = 0$ and $r \neq 0$ then $\text{opt}(\mathbf{n}) = \{\mathbf{0}, \ldots, \mathbf{r-1}\}$ and the result follows thanks to the induction hypothesis.

Similarly, if $q > 0$ and $r \neq 0$ then $\text{opt}(\mathbf{n}) = \{\mathbf{qt+r-t+1}, \ldots, \mathbf{qt+r-1}\}$ and the result follows thanks to the induction hypothesis.

Finally, if $q \geq t$ and $r = 0$ then $\text{opt}(\mathbf{n}) = \{\mathbf{q}\} \cup \{\mathbf{qt-t+1}, \ldots, \mathbf{qt-1}\}$ and, again, the result follows thanks to the induction hypothesis.

Hence, the Sprague-Grundy function $\mathcal{G}$(i-MARK-t) is 1-almost periodic, with preperiod of length $t^2 - 1$, period of length t and exception set $\mathcal{E} = \{0\}$. □

As quoted in Section 1, Alan Guo obtained in [7] a result with a similar flavour for the game MARK-t. He also proved that the g-value of any position $\mathbf{n}$ can be computed in time $O(n^2)$, using the representation of n in base t.

Our next result, combined with Theorem 5, shows that the situation for the game i-MARK$([1, t-1], \{t\})$ is easier since the g-value of any position $\mathbf{n}$ can be computed in time $O(\log n)$.

For every integer $n \geq 0$, let us denote by $R_t(n)$ the representation of n in base t and by $R_t^1(n)$ the number in base t obtained from $R_t(n)$ by deleting all the trailing 0's. We then have:

Theorem 6. *Let $t \geq 2$ be an integer. Let* $\mathbf{n}$ *be any position of the game i-*Mark$([1, t-1], \{t\})$ *with $n = qt$, $q \geq t$. Then $g(\mathbf{n}) = t$ if and only if either:*

(i) $R_t^1(n) \in \{R_t^1(k) \mid k < t^2,\ k \equiv t-1 \pmod t\}$ *and $R_t(q)$ has an odd number of trailing 0's, or*

(ii) $R_t^1(n) \notin \{R_t^1(k) \mid k < t^2,\ k \equiv t-1 \pmod t\}$ *and $R_t(q)$ has an even number of trailing 0's.*

Proof. From Theorem 5, we know that for every position $\mathbf{n}$, with $n = qt$ and $q \geq t$, $g(\mathbf{n}) = t$ if and only if $g(\mathbf{q}) = t - 1$, which happens only if either $q = q't - 1$ with $q' \leq t$, or $q = q't$ with $q' \geq t$ and $g(\mathbf{q'}) = t$.

The result directly follows from this observation. □

3.2. **The game** ***i*****-Mark$([1, t-1], \{2\})$.** For the game i-Mark$([1, t-1], \{2\})$ we will prove that the Sprague-Grundy sequence is 1-almost periodic for every $t \geq 3$ and that the g-value of any position $\mathbf{n}$ can be computed in time $O(\log n)$.

Note that the case $t = 2$ corresponds to the game i-Mark$(\{1\}, \{2\})$ considered in Subsection 2.2. We will consider separately the case $t = 3$, and then the general case $t \geq 4$.

3.2.1. *The game i-*Mark$([1, 2], \{2\})$. We first consider the case $t = 3$ and prove that the Sprague-Grundy sequence of the game i-Mark$([1, 2], \{2\})$ is 1-almost periodic.

Theorem 7. *The Sprague-Grundy sequence of the game i-*Mark$([1, 2], \{2\})$ *is 1-almost periodic, with preperiod of length 18, period of length 3, and exception set $\mathcal{E} = \{0\}$. More precisely, for every integer $n = 3q + r$ with $q \geq 6$ and $r \in [0, 2]$, we have*

(i) *if $r = 1$ then $g(\mathbf{n}) = 0$,*

(ii) *if $r = 2$ then $g(\mathbf{n}) = 1$,*

(iii) *if $r = 0$ then $g(\mathbf{n}) \in \{2, 3\}$.*

Proof. We will determine the value of $g(\mathbf{n})$ for every position n, $n \geq 0$, from which the theorem will follow.

An easy calculation gives the following first values of $\mathcal{G}(i$-Mark([1,2],{2})):

n	0	1	2	3	4	5	6	7	8	9	10
$g(\mathbf{n})$	0	1	2	0	1	2	3	0	2	1	0
n	11	12	13	14	15	16	17	18	19	20	21
$g(\mathbf{n})$	2	1	0	2	1	0	2	3	0	1	2

In particular, $g(\mathbf{18}) = 3$, $g(\mathbf{19}) = 0$ and $g(\mathbf{20}) = 1$, so that the claimed property holds for $q = 6$ and $r \in [0, 2]$. Suppose now that the theorem holds up to $n - 1$, $n \geq 21$, and consider the position $\mathbf{n}$. Let $n = 3q + r$, with $q \geq 7$ and $r \in [0, 2]$.

If $r = 1$, the options of $\mathbf{n}$ are $\mathbf{3(q-1)+2}$, $\mathbf{3q}$, and $\mathbf{(3q+1)/2}$ if q is odd. Since $(3q+1)/2 \equiv 2 \pmod 3$ if q is odd, these three potential options are all N-positions thanks to the induction hypothesis, which gives $g(\mathbf{n}) = 0$.

If $r = 2$, the options of $\mathbf{n}$ are $\mathbf{3q}$, $\mathbf{3q+1}$, and $\mathbf{(3q+2)/2}$ if q is even. Since $(3q+2)/2 \equiv 1 \pmod 3$ if q is even, the g-value of any of these three potential options is either 0, 2 or 3, thanks to the induction hypothesis, which gives $g(\mathbf{n}) = 1$.

Finally, if $r = 0$, the options of $\mathbf{n}$ are $\mathbf{3(q-1)+1}$, $\mathbf{3(q-1)+2}$, and $\mathbf{3q/2}$ if q is even. Thanks to the induction hypothesis, we have $g(\mathbf{3(q-1)+1}) = 0$

and $g(\mathbf{3(q-1)+2}) = 1$. Moreover, since $3q/2 \equiv 0 \pmod 3$ if q is even, we have $g(\mathbf{3q/2}) \in \{2, 3\}$ and, therefore, $g(\mathbf{n}) = 2$ if q is odd, or q is even and $g(\mathbf{3q/2}) = 3$, and $g(\mathbf{n}) = 3$ otherwise.

Hence, the Sprague-Grundy sequence of the game i-MARK$([1, 2], \{2\})$ is 1-almost periodic, with preperiod of length 18, period of length 3, and exception set $\mathcal{E} = \{0\}$. □

From the proof of Theorem 7, we can see that the set of integers n, $n \geq 18$, for which $g(\mathbf{n}) = 3$ has somehow a nice structure. This fact is stated in a more explicit way in the following theorem:

Theorem 8. *Let* $\mathbf{n}$ *be any position of the game* i-MARK$([1, 2], \{2\})$ *with* $n = 3q$, $q \geq 6$. *Then* $g(\mathbf{n}) = 3$ *if and only if either:*

(i) *q is even, $R^1(q) = 1_2$ or $R^1(q) = 101_2$, and $R(q)$ has an even number of trailing 0's, or*
(ii) *q is even, $R^1(q) \neq 1_2$, $R^1(q) \neq 101_2$, and $R(q)$ has an odd number of trailing 0's.*

Proof. The proof directly follows from the last part of the proof of Theorem 7, by observing that $g(\mathbf{24}) = 2$ (which corresponds to the case $q = 8$, $8 = 1000_2$), $g(\mathbf{30}) = 2$ (which corresponds to the case $q = 10$, $10 = 1010_2$), while $g(\mathbf{18}) = 3$ (which corresponds to the case $q = 6$, $6 = 110_2$), and $g(\mathbf{3q}) = 2$ for every odd q, $q \geq 6$. □

Let $\mathcal{N}_3$ denote the set of integers n such that $g(\mathbf{n}) = 3$. The set $\mathcal{N}_3$ is as follows:

$$\mathcal{N}_2 = \{6, 18, 42, 48, 54, 60, 66, 72, 78, 90, 102, 114, 126, 138, 150, 162, 168, \dots\}.$$

From Theorems 7 and 8, we get that the g-value of any position $\mathbf{n}$ of the game i-MARK$([1, 2], \{2\})$ can be computed in time $O(\log n)$.

3.2.2. *The game* i-MARK$([1, t-1], \{2\})$, $t \geq 4$. We now turn to the general case and prove that the Sprague-Grundy sequence $\mathcal{G}(i\text{-MARK}([1, t-1], \{2\}))$ is 1-almost periodic for every $t \geq 4$, as for the game considered in the previous subsection, except that the g-values 1 and 2 are "switched".

Theorem 9. *For every $t \geq 4$, the Sprague-Grundy sequence of the game i-MARK$([1, t-1], \{2\})$ is 1-almost periodic, with preperiod of length $2t-1$, period of length t, and exception set $\mathcal{E} = \{0\}$. More precisely, for every integer $n = qt + r$ with $q \geq 0$ and $r \in [0, t-1]$, we have*

(i) *if $q < 2$ and $r = 0$, or $q \geq 2$ and $r = 1$, then $g(\mathbf{n}) = 0$,*
(ii) *if $q < 2$ and $r \neq 0$, then $g(\mathbf{n}) = r$,*
(iii) *if $q \geq 2$ and $r = 2$, then $g(\mathbf{n}) = 2$,*
(iv) *if $q \geq 2$ and $r = 3$, then $g(\mathbf{n}) = 1$,*
(v) *if $q \geq 2$ and $r > 3$, then $g(\mathbf{n}) = r - 1$,*
(vi) *if $q \geq 2$ and $r = 0$, then $g(\mathbf{n}) \in \{t-1, t\}$.*

Proof. The proof is similar to the proof of Theorem 7. Claim (i) direclty follows from Theorem 4 since, in that case, $\mathbf{n}$ has an option in $\mathcal{P}$ and is thus an N-position. For the remaining claims, the proof easily follows by induction. Suppose the theorem holds up to position $\mathbf{n-1}$, $n \geq 1$, and consider the position $\mathbf{n}$. Let $n = qt + r > 1$, with $q \geq 0$ and $r \in [0, t-1]$.

If $q = 0$ and $r \neq 0$ then $\mathrm{opt}(\mathbf{n}) = \{\mathbf{0}, \dots, \mathbf{r-1}\}$ and the result follows thanks to the induction hypothesis.

Similarly, if $q > 0$ and $r \neq 0$ then $\mathrm{opt}(\mathbf{n}) = \{\mathbf{qt+r-t+1}, \dots, \mathbf{qt+r-1}\}$ and the result follows thanks to the induction hypothesis.

Finally, if $q > 0$ and $r = 0$ then $\mathrm{opt}(\mathbf{n}) = \{\mathbf{qt-t+1}, \dots, \mathbf{qt-1}\}$ if q and t are odd, and $\mathrm{opt}(\mathbf{n}) = \{\mathbf{qt/2}\} \cup \{\mathbf{qt-t+1}, \dots, \mathbf{qt-1}\}$ otherwise. Hence, thanks to the induction hypothesis, we get $g(\mathbf{n}) = t-1$ if q and t are odd, or at least one of them is even and $g(\mathbf{n/2}) = t$, and $g(\mathbf{n}) = t$ otherwise.

Hence, the Sprague-Grundy sequence of the game i-Mark$([1,t-1],\{2\})$ is 1-almost periodic, with preperiod of length $2t-1$, period of length t, and exception set $\mathcal{E} = \{0\}$. □

As for the game i-Mark$([1,2],\{2\})$, the g-value of any position $\mathbf{n}$ of the game i-Mark$([1,t-1],\{2\})$ can be computed in time $O(\log n)$, thanks to Theorem 9 and the following characterization, the proof of which is similar to the proof of Theorem 8 and is thus omitted:

Theorem 10. *Let $t \geq 2$ be an integer. Let $\mathbf{n}$ be any position of the game i-*Mark$([1,t-1],\{2\})$ *with $n = qt$, $q \geq 2$, m be the smallest integer such that $m \geq 2t$ and $R^1(n) = R^1(m)$, and z_n (resp. z_m) denote the number of trailing 0's of $R(n)$ (resp. of $R(m)$). Then $g(\mathbf{n}) = t$ if and only if either:*

(i) *qt is even, $g(\mathbf{m}) = t-1$ and $z_n - z_m$ is odd, or*
(ii) *qt is even, $g(\mathbf{m}) = t$ and $z_n - z_m$ is even.*

4. The game i-Mark$(\{a,2a\},\{2\})$

We consider in this section the game i-Mark$(\{a,2a\},\{2\})$, $a \geq 1$. Note that when $a = 1$, this game is the game i-Mark$([1,2],\{2\})$ considered in Subsection 3.2.1. We will prove that, in some cases, the outcome sequence of the game i-Mark$(\{a,2a\},\{2\})$ is 1-almost periodic. We will also prove that, in those cases, the g-value of any position $\mathbf{n}$ can be computed in time $O(\log n)$.

For any even a, the g-value of any position $\mathbf{n}$, n odd, is easy to determine. For every integer sequence $A = (a_i)_{i\in\mathbb{N}}$, we define the *odd subsequence* of A as the subsequence $A_{odd} = (a_{2q+1})_{q\in\mathbb{N}}$. We then have:

Theorem 11. *For every integer $a \geq 2$, a even, the odd subsequence of the Sprague-Grundy sequence of the game i-*Mark$(\{a,2a\},\{2\})$ *is purely periodic, with period of length $3a$. More precisely, for every integer n with $n = 3qa + r$, $q \geq 0$ and $r \in [1, 3a-1]$, r odd, we have*

(i) *If $r < a$ then $g(\mathbf{n}) = 0$,*
(ii) *If $a < r < 2a$ then $g(\mathbf{n}) = 1$,*
(iii) *If $r > 2a$ then $g(\mathbf{n}) = 2$.*

Proof. If $n < a$ then $\mathbf{n}$ has no option and, therefore, $g(\mathbf{n}) = 0$. If $a < n < 2a$ then $\mathrm{opt}(\mathbf{n}) = \{\mathbf{n-a}\}$ and, therefore, $g(\mathbf{n}) = 1$. If $2a < n < 3a$ then $\mathrm{opt}(\mathbf{n}) = \{\mathbf{n-2a}, \mathbf{n-a}\}$ and, therefore, $g(\mathbf{n}) = 2$. Hence, the theorem holds for every position $\mathbf{n}$ with n odd, $n < 3a$. The result then easily follows by induction since, for every odd n, $\mathrm{opt}(\mathbf{n}) = \{\mathbf{n-2a}, \mathbf{n-a}\}$. □

However, the whole Sprague-Grundy sequence of the game i-MARK$(\{a, 2a\}, \{2\})$ is not always 1-almost periodic with period of length $3a$. The smallest length of the period for which this sequence is 1-almost periodic is for instance 18 if $a = 3$, 60 if $a = 5$ and 36 if $a = 6$.

The Sprague-Grundy sequence of the game i-MARK$(\{a, 2a\}, \{2\})$ has been proved to be 1-almost periodic with period of length $3a$ when $a = 1$ in Theorem 7. We will consider the cases $a = 2$ and $a = 4$ in the two following subsections.

4.1. **The game i-MARK$(\{2, 4\}, \{2\})$.** We first consider the case $a = 2$ and prove that the Sprague-Grundy sequence of the game i-MARK$(\{2, 4\}, \{2\})$ is 1-almost periodic.

Theorem 12. *The Sprague-Grundy sequence of the game i-MARK$(\{2, 4\}, \{2\})$ is 1-almost periodic, with preperiod of length 11, period of length 6, and exception set $\mathcal{E} = \{0\}$. More precisely, for any integer $n \geq 11$, $n = 6q + r$ with $r \in [0, 5]$, we have*

(i) *if $r = 1$ or $r = 4$ then $g(\mathbf{n}) = 0$,*
(ii) *if $r = 2$ or $r = 3$ then $g(\mathbf{n}) = 1$,*
(iii) *if $r = 5$ then $g(\mathbf{n}) = 2$,*
(iii) *if $r = 0$ then $g(\mathbf{n}) \in \{2, 3\}$.*

Proof. We will determine the value of $g(\mathbf{n})$ for every position n, $n \geq 0$, from which the theorem will follow.

An easy calculation gives the following first values of $\mathcal{G}(i$-MARK$(\{2,4\},\{2\}))$:

n	0	1	2	3	4	5	6	7	8	9	10	11
$g(\mathbf{n})$	0	0	1	1	2	2	0	0	1	1	3	2
n	12	13	14	15	16	17	18	19	20	21	22	23
$g(\mathbf{n})$	2	0	1	1	0	2	2	0	1	1	0	2

Hence, the theorem holds up to $n = 23$, that is $q = 3$ and $r = 5$. Suppose now that the theorem holds up to $n - 1$, $n \geq 23$, and consider the position $\mathbf{n}$. Let $n = 6q + r$, with $q \geq 4$ and $r \in [0, 5]$.

If r is odd, the options of $\mathbf{n}$ are $\mathbf{n-2}$ and $\mathbf{n-4}$ and the claim follows thanks to the induction hypothesis.

If r is even, the options of $\mathbf{n}$ are $\mathbf{n-2}$, $\mathbf{n-4}$ and $\mathbf{n/2}$. Again, the claim follows thanks to the induction hypothesis.

Hence, the Sprague-Grundy sequence of the game i-MARK$(\{2, 4\}, \{2\})$ is 1-almost periodic, with preperiod of length 11, period of length 6, and exception set $\mathcal{E} = \{0\}$. □

From the proof of Theorem 12, we can see that the set of integers n, $n \geq 11$, for which $g(\mathbf{n}) = 3$ is not difficult to characterize:

Theorem 13. *Let $\mathbf{n}$ be any position of the game i-MARK$(\{2, 4\}, \{2\})$ with $n = 6q$, $q \geq 2$. Then $g(\mathbf{n}) = 3$ if and only if either:*

(i) *q is even, $q \geq 4$, $R^1(q) = 1_2$ and $R(q)$ has an even number of trailing 0's, or*
(ii) *q is even, $q \geq 4$, $R^1(q) \neq 1_2$, and $R(q)$ has an odd number of trailing 0's.*

Proof. The theorem clearly holds for $q = 2$ and $q = 3$ since $g(\mathbf{12}) = g(\mathbf{18}) = 2$. Assume the theorem holds up to $q-1$, $q \geq 4$ and consider the position $\mathbf{n}$, $n = 6q$. We have $\mathrm{opt}(\mathbf{n}) = \{\mathbf{n-2}, \mathbf{n-4}, \mathbf{n/2}\}$ with, by Theorem 12, $g(\mathbf{n-2}) = 0$ and $g(\mathbf{n-4}) = 1$. Therefore, $g(\mathbf{n}) = 3$ if and only if $g(\mathbf{n/2}) = 2$. This happens only if q is even – since, if q is odd then $n/2 \equiv 3 \pmod 6$ and $g(\mathbf{n/2}) = 1$ by Theorem 12 – and, thanks to the induction hypothesis, if either $R^1(q) = 1_2$ and $R(q)$ has an even number of trailing 0's, or $R^1(q) \neq 1_2$ and $R(q)$ has an odd number of trailing 0's. □

From Theorems 12 and 13, we get that the g-value of any position $\mathbf{n}$ of the game i-Mark$(\{2,4\},\{2\})$ can be computed in time $O(\log n)$.

4.2. **The game i-Mark$(\{4,8\},\{2\})$.** We now consider the case $a = 4$. The results are similar to the previous ones and we omit the proofs, which use the same technique.

Theorem 14. *The Sprague-Grundy sequence of the game i-Mark$(\{4,8\},\{2\})$ is 1-almost periodic, with preperiod of length 17, period of length 12, and exception set $\mathcal{E} = \{0\}$. More precisely, for any integer $n \geq 17$, $n = 12q + r$ with $r \in [0,11]$, we have*

(i) *if $r \in \{1,3,4,10\}$ then $g(\mathbf{n}) = 0$,*
(ii) *if $5 \leq r \leq 8$ then $g(\mathbf{n}) = 1$,*
(iii) *if $r \in \{2,9,11\}$ then $g(\mathbf{n}) = 2$,*
(iii) *if $r = 0$ then $g(\mathbf{n}) \in \{2,3\}$.*

We just give the first values of $\mathcal{G}(i$-Mark$(\{4,8\},\{2\}))$, including the preperiod and the period:

n	0	1	2	3	4	5	6	7	8	9	10	11
$g(\mathbf{n})$	0	0	1	0	2	1	2	1	1	2	0	2
n	12	13	14	15	16	17	18	19	20	21	22	23
$g(\mathbf{n})$	0	0	3	0	2	1	1	1	1	2	0	2
n	24	25	26	27	28	29	30	31	32	33	34	35
$g(\mathbf{n})$	3	0	2	0	0	1	1	1	1	2	0	2
n	36	37	38	39	40	41	42	43	44	45	46	47
$g(\mathbf{n})$	3	0	2	0	0	1	1	1	1	2	0	2

The set of integers n, $n \geq 17$, for which $g(\mathbf{n}) = 3$ is characterized as follows:

Theorem 15. *Let $\mathbf{n}$ be any position of the game i-Mark$(\{4,8\},\{2\})$ with $n = 12q$, $q \geq 2$. Then $g(\mathbf{n}) = 3$ if and only if $R(q)$ has an odd number of trailing 0's.*

From Theorems 14 and 15, we get that the g-value of any position $\mathbf{n}$ of the game i-Mark$(\{4,8\},\{2\})$ can be computed in time $O(\log n)$.

5. i-MiMark: i-Mark under misère convention

We consider in this section the game i-Mark under misère convention – that is, when the first player unable to move wins the game –, called i-MiMark. The position $\mathbf{0}$ is thus a P-position for the game i-MiMark.

In [6], Aviezri Fraenkel characterized the sets of N- and P-positions of the game MiMark – Mark under misère convention – and, as for the normal convention, these two sets appeared to be aperiodic.

We will show that the outcome sequence of the game i-MiMark(S, D) is *purely* periodic – that is, with no preperiod – in many cases, in particular for most of the cases considered in the previous sections.

5.1. **The game i-MiMark$([1, t-1], D)$.** We first consider the case $S = [1, t-1]$ with $t \geq 2$. We first prove the following general result:

Theorem 16. *For every integer $t \geq 2$ and every set $D \in \mathbb{N}_{\geq 2}$ such that $d \not\equiv 1 \pmod t$ for every $d \in D$, the outcome sequence of the game i-MiMark$([1, t-1], D)$ is purely periodic with period of length t. More precisely, the set $\mathcal{P}$ of P-positions is given by*

$$\mathcal{P} = \{\mathbf{qt+1} \mid q \geq 0\}.$$

Proof. We clearly have $\mathbf{0} \in \mathcal{N}$ and $\mathbf{1} \in \mathcal{P}$. Hence, for every $n \in [2, t]$, $\mathbf{n} \in \mathcal{N}$ since, in that case, $\mathbf{1} \in \text{opt}(\mathbf{n})$. Consider now the position $\mathbf{n}$, $n = qt + r$, $q \geq 1$ and $r \in [0, t-1]$, and assume that the theorem is true up to position $\mathbf{n-1}$.

If $r = 0$, then $\mathbf{(q-1)t+1} \in \text{opt}(\mathbf{n})$ and thus $\mathbf{n} \in \mathcal{N}$ since $\mathbf{(q-1)t+1} \in \mathcal{P}$ by induction hypothesis. Similarly, if $r > 1$ then $\mathbf{qt+1} \in \text{opt}(\mathbf{n})$ and thus $\mathbf{n} \in \mathcal{N}$ since $\mathbf{qt+1} \in \mathcal{P}$ by induction hypothesis.

Finally, if $r = 1$ then

$$\text{opt}(\mathbf{n}) = \{\mathbf{qt-t+2}, \dots, \mathbf{qt}\} \cup \{\mathbf{(qt+1)/d} \mid d \in D,\ d|n\}.$$

Since $d \not\equiv 1 \pmod t$ for every $d \in D$, $(qt+1)/d \not\equiv 1 \pmod t$ and, therefore, $\mathbf{n} \in \mathcal{P}$ thanks to induction hypothesis.

Hence, the outcome sequence of the game i-MiMark$([1, t-1], D)$ is purely periodic with period of length t. □

Note that for every integer $t \geq 3$ the game i-MiMark$([1, t-1], \{t\})$ – the misère version of the game considered in Subection 3.1, or in Subsection 3.2 if $t = 2$ – satisfies the hypothesis of Theorem 16.

5.2. **The game i-MiMark$(\{a, 2a\}, \{2\})$.** We now consider the case $S = \{a, 2a\}$, $a \geq 1$, and $D = \{2\}$. We will prove that the sets $\mathcal{N}$ and $\mathcal{P}$ are purely periodic, with period of length $3a$, whenever $a = 2$ or a is odd.

This claim is easy to prove when $a = 2$:

Lemma 17. *Let $\mathbf{n}$ be any position of the game i-MiMark$(\{2, 4\}, \{2\})$. We then have $\mathbf{n} \in \mathcal{P}$ if and only if $n \equiv 2 \pmod 6$ or $n \equiv 3 \pmod 6$.*

Proof. We prove this result by induction. We clearly have $\mathbf{0}, \mathbf{1} \in \mathcal{N}$ and $\mathbf{2} \in \mathcal{P}$. Suppose now that the theorem holds up to $n-1$, $n \geq 4$, and consider the position $\mathbf{n}$. If $n \equiv 0, 1, 4$ or $5 \pmod 3$ then $\mathbf{n} \in \mathcal{N}$ since $\mathbf{n-4}$, $\mathbf{n-4}$, $\mathbf{n-2}$ or $\mathbf{n-2}$, respectively, are P-positions thanks to the induction hypothesis. If $n \equiv 3 \pmod 6$ then $\mathbf{n} \in \mathcal{P}$ since, thanks to the induction hypothesis, $\mathbf{n-2} \in \mathcal{N}$ and $\mathbf{n-4} \in \mathcal{N}$. Finally, if $n \equiv 2 \pmod 6$ then $\mathbf{n} \in \mathcal{P}$ since, thanks to the induction hypothesis, $\mathbf{n-2} \in \mathcal{N}$, $\mathbf{n-4} \in \mathcal{N}$ and $\mathbf{n/2} \in \mathcal{N}$. □

We now consider the case of a odd. The following lemma gives the outcome of any position $\mathbf{n}$ with $n < a$.

Lemma 18. *Let* $\mathbf{n}$ *be any position of the game* i-MIMARK$(\{a,2a\},\{2\})$*, a odd, with* $n \in [0,a-1]$*. We then have* $\mathbf{n} \in \mathcal{N}$ *if and only if* $n = 0$ *or* $R(n)$ *has an even number of trailing 0's.*

Proof. Clearly, the result holds for $n = 0$. Assume now that $n > 0$. We use induction on the number $z(n)$ of trailing 0's of $R(n)$. If $z(n) = 0$ then n is odd, so that $\mathbf{n}$ has no option, which implies $\mathbf{n} \in \mathcal{N}$. Suppose that the result holds up to $z(n) = k \geq 0$ and let n be such that $z(n) = k+1$. Since n is even, $\text{opt}(\mathbf{n}) = \{\mathbf{n/2}\}$, so that $\mathbf{n}$ is an N-position if and only if $\mathbf{n/2}$ is a P-position and the result follows thanks to the induction hypothesis. □

Incidently, note that Lemma 18 also holds when a is even. We are now able to prove the main result of this subsection:

Theorem 19. *Let* a *be an integer,* $a \geq 1$*. If* $a = 2$ *or* a *is odd then the outcome sequence of the game* i-MIMARK$(\{a,2a\},\{2\})$ *is purely periodic with period of length* $3a$*.*

Proof. The case $a = 2$ directly follows from Lemma 17. Suppose now that a is odd. We will prove by induction that, for every $n \geq 0$, positions $\mathbf{n}$ and $\mathbf{n+3a}$ have the same outcome.

Since $\mathbf{0},\mathbf{1} \in \mathcal{N}$ and $\text{opt}(\mathbf{a}) = \{\mathbf{0},\mathbf{1}\}$, we have $\mathbf{a} \in \mathcal{P}$. Therefore, $\mathbf{3a} \in \mathcal{N}$ since $\mathbf{a} \in \text{opt}(\mathbf{3a})$, and thus the property holds for $n = 0$.

Suppose now that the theorem holds up to $n-1$, $n \geq 1$, and consider the position $\mathbf{n}$. We consider two cases, according to the outcome of $\mathbf{n}$.

(1) $\mathbf{n} \in \mathcal{P}$.
In that case, we have $\mathbf{n+a}, \mathbf{n+2a} \in \mathcal{N}$. If n is even then $n+3a$ is odd and thus $\text{opt}(\mathbf{n+3a}) = \{\mathbf{n+a}, \mathbf{n+2a}\}$. Hence, $\mathbf{n+3a} \in \mathcal{P}$ and we are done.

Suppose now that n is odd and, to the contrary, that $\mathbf{n+3a} \in \mathcal{N}$. Note first that since $\mathbf{n} \in \mathcal{P}$ we necessarily have $n \geq a$ by Lemma 18. Since $\text{opt}(\mathbf{n+3a}) = \{\mathbf{n+a}, \mathbf{n+2a}, (\mathbf{n+3a})/\mathbf{2}\}$ and $\mathbf{n+a}, \mathbf{n+2a} \in \mathcal{N}$, we necessarily have $(\mathbf{n+3a})/\mathbf{2} \in \mathcal{P}$, which implies $(\mathbf{n+a})/\mathbf{2} \in \mathcal{N}$ and $(\mathbf{n-a})/\mathbf{2} \in \mathcal{N}$. Since $\mathbf{n} \in \mathcal{P}$, we also have $\mathbf{n-a} \in \mathcal{N}$ which implies that at least one option of $\mathbf{n-a}$ is in $\mathcal{P}$. The possible options of $\mathbf{n-a}$ are (i) $(\mathbf{n-a})/\mathbf{2}$, but $(\mathbf{n-a})/\mathbf{2} \in \mathcal{N}$ as observed above, (ii) $\mathbf{n-2a}$ if $n \geq 2a$, but in that case $\mathbf{n-2a} \in \mathcal{N}$ since $\mathbf{n} \in \mathcal{P}$, and (iii) $\mathbf{n-3a}$ if $n \geq 3a$, which therefore necessarily exists and must be in $\mathcal{P}$. This implies $(\mathbf{n-3a})/\mathbf{2} \in \mathcal{N}$ and we finally get

$$(\mathbf{n-3a})/\mathbf{2} \in \mathcal{N},\ (\mathbf{n-a})/\mathbf{2} \in \mathcal{N} \text{ and } (\mathbf{n+a})/\mathbf{2} \in \mathcal{N},$$

which contradicts the induction hypothesis.

(2) $\mathbf{n} \in \mathcal{N}$.
If $\mathbf{n+a} \in \mathcal{P}$ or $\mathbf{n+2a} \in \mathcal{P}$ then $\mathbf{n+3a} \in \mathcal{N}$ and we are done.

Suppose therefore that $\mathbf{n}, \mathbf{n+a}, \mathbf{n+2a} \in \mathcal{N}$. By induction hypothesis, this implies in particular $\mathbf{n-a} \in \mathcal{N}$ if $n \geq a$. If n is even then $n+a$ is odd, so that $\text{opt}(\mathbf{n+a}) = \{\mathbf{n-a}, \mathbf{n}\}$ (resp. $\text{opt}(\mathbf{n+a}) = \{\mathbf{n}\}$) if $n \geq a$ (resp. $n < a$) and thus $\mathbf{n+a} \in \mathcal{P}$, contradicting our assumption. Similarly, if n is odd then $n+2a$ is odd, so that $\text{opt}(\mathbf{n+2a}) = \{\mathbf{n}, \mathbf{n+a}\}$ and thus $\mathbf{n+2a} \in \mathcal{P}$, again contradicting our assumption.

This concludes the proof. □

a	$3a$	period
1	3	NPN
2	6	$NNPPNN$
3	9	$NNPPNNNPN$
5	15	$NNPNNPPNPP\ NNNNN$
7	21	$NNPNNNPPPN\ PPNNNNNNNP\ N$
9	27	$NNPNNNPNPP\ PNNPPNNNNN\ NPNNNPN$
11	33	$NNPNNNPNPN\ PPNNPPNNPN\ NNNPNNNPNN\ NPN$

Table 1. Period of the outcome sequence of the game i-MiMark($\{a, 2a\}, \{2\}$)

Table 1 shows the period, of length $3a$, of the outcome sequence of the game i-MiMark($\{a, 2a\}, \{2\}$) for $a = 2$ or a odd, $a \leq 11$.

The period of the outcome sequence of the game i-MiMark($\{2, 4\}, \{2\}$) is given by Lemma 17. The next two propositions will determine the outcome of any position $\mathbf{n}$, with $a \leq n \leq 3a - 1$, for the game i-MiMark($\{a, 2a\}, \{2\}$), a odd. These two propositions, together with Lemma 18, thus determine the corresponding outcome sequences.

Proposition 20. *Let* $\mathbf{n}$ *be any position of the game* i-MiMark($\{a, 2a\}, \{2\}$)*,* a *odd, with* $n \in [a, 2a - 1]$*. We then have:*

(i) *if* n *is even then* $\mathbf{n} \in \mathcal{P}$*,*
(ii) *if* n *is odd then* $\mathbf{n} \in \mathcal{P}$ *if and only if* $a = n$ *or* $R(n - a)$ *has an even number of trailing 0's.*

Proof. Suppose first that n is even. This implies that $n-a$ is odd, so that $\mathbf{n} - \mathbf{a} \in \mathcal{N}$ by Lemma 18. Since $\mathbf{n} - \mathbf{a} \in \text{opt}(\mathbf{n})$, we then get $\mathbf{n} \in \mathcal{P}$.

Suppose now that n is odd. If $n = a$ then $\text{opt}(\mathbf{n}) = \{\mathbf{0}, \mathbf{1}\}$ which gives $\mathbf{n} \in \mathcal{P}$ since $\mathbf{0}, \mathbf{1} \in \mathcal{N}$ by Lemma 18. If $n \neq a$, the result again follows from Lemma 18 since $\text{opt}(\mathbf{n}) = \{\mathbf{n} - \mathbf{a}\}$ and $\mathbf{n} - \mathbf{a} \in \mathcal{N}$ if and only if $R(n - a)$ has an even number of trailing 0's. □

Proposition 21. *Let* $\mathbf{n}$ *be any position of the game* i-MiMark($\{a, 2a\}, \{2\}$)*,* a *odd, with* $n \in [2a, 3a - 1]$*. We then have:*

(i) *if* $n = 2a$ *or* n *is odd then* $\mathbf{n} \in \mathcal{N}$*,*
(ii) *if* n *is even,* $n \neq 2a$*, then* $\mathbf{n} \in \mathcal{N}$ *if and only if one of the following conditions holds:*
   (a) $R(n - 2a)$ *has an odd number of trailing 0's,*
   (b) $R(n - a)$ *has an even number of trailing 0's,*
   (c) $n \equiv 0 \pmod 4$*,*
   (d) $n \equiv 2 \pmod 4$ *and* $R(n/2 - a)$ *has an even number of trailing 0's.*

Proof. Since $\mathbf{a} \in \mathcal{P}$ by Proposition 20, we have $\mathbf{2a} \in \mathcal{N}$. If n is odd then $n - a$ is even, which implies $\mathbf{n} \in \mathcal{N}$ since $\mathbf{n} - \mathbf{a} \in \text{opt}(\mathbf{n})$ and $\mathbf{n} - \mathbf{a} \in \mathcal{P}$ by Proposition 20 (case (i)).

Suppose now that n is even, $n \neq 2a$. We then have $\mathbf{n} \in \mathcal{N}$ if and only if at least one of the positions $\mathbf{n} - \mathbf{2a}$, $\mathbf{n} - \mathbf{a}$, $\mathbf{n/2}$ is in $\mathcal{P}$ (note that $n/2 \in [a+1, a+(a-1)/2]$). By Lemma 18, $\mathbf{n} - \mathbf{2a} \in \mathcal{P}$ if and only if $R(n - 2a)$ has an odd number of trailing 0's (case (ii.a)). Since $n - a$ is odd, by Proposition 20, $\mathbf{n} - \mathbf{a} \in \mathcal{P}$ if and only if

a	4	6	8	10	12	14	16
length of the preperiod	7	9	61	193	105	105	313
number of exceptions	2	2	6	6	10	8	14
a	18	20	22	24	26	28	30
length of the preperiod	345	397	425	497	1129	1217	585
number of exceptions	10	14	12	26	16	28	18

Table 2. Preperiod of the outcome sequence of the game i-MiMark($\{a, 2a\}, \{2\}$), a even

$R(n-a)$ has an even number of trailing 0's (case (ii.b)). Finally, by Proposition 20, $\mathbf{n/2} \in \mathcal{P}$ if and only if either $n/2$ is even, which means $n \equiv 0 \pmod 4$ (case (ii.c)), or $n/2$ is odd, which means $n \equiv 2 \pmod 4$, and $R(n/2 - a)$ has an even number of trailing 0's (case (ii.d)). □

When a is even, $a \geq 4$, the outcome sequence of the game i-MiMark($\{a, 2a\}, \{2\}$) seems to be more "erratic". Using computer check, we observed that the outcome sequence seems to be always periodic, also with period of length $3a$, but with a preperiod of, at that time, undetermined length.

We show in Table 2, for the first even values of a, the observed length of the preperiod and the number of *exceptions* contained in the preperiod – by exception we mean here a position whose outcome is different from the outcome to which it would correspond in the period. The last exception is thus considered as the last element of the preperiod.

6. Discussion

In this paper, we initiated the study of a new family of impartial combinatorial games – the integral subtraction division games – obtained by restricting in a natural way the availability of division-type moves in subtraction division games. We proved that in many cases these games have a "nice behaviour" since, under normal convention, their Sprague-Grundy sequence is almost periodic and the g-value of any heap of n tokens can be computed in time $O(\log n)$. Moreover, we proved that, under misère convention and again in many cases, the outcome sequence is purely periodic.

We finally list below a few open questions related to integral subtraction division games that could be of interest.

1. Do there exist sets S and D for which the outcome sequence of the game i-Mark(S, D) under normal convention is not periodic?
2. Is it true that, for every $d \not\equiv 1 \pmod t$, the Sprague-Grundy sequence of the game i-Mark($[1, t-1], \{d\}$) is 1-almost periodic with period of length t?
3. What can be said about games of the form i-Mark($[1, t-1], \{d\}$), when $d \equiv 1 \pmod t$, under normal convention? under misère convention?
4. Is it true that, for every $a \geq 1$, the Sprague-Grundy sequence of the game i-Mark($\{a, 2a\}, \{2\}$) is 1-almost periodic?
5. What can be said about games of the form i-Mark($\{a, 2a, \ldots, ka\}, \{2\}$), $k \geq 3$, under normal convention? under misère convention?

(6) What can be said about games of the form i-MARK(S, D), with $|D| > 1$, under normal convention? under misère convention?

(7) Is it true that for every even integer $a \geq 4$ the outcome sequence of the game i-MIMARK$(\{a, 2a\}, \{2\})$ is periodic with period of length $3a$?

(8) What can be said, when a is odd, about the outcome sequence of the game i-MIMARK$(\{a, 2a\}, \{2\})$?

REFERENCES

[1] Michael H. Albert, Richard J. Nowakowski and David Wolfe. *Lessons in Play: An Introduction to Combinatorial Game Theory*. A K Peters, Wellesley, MA (2007).

[2] Elwyn Berlekamp and Joe P. Buhler. Puzzles Column. EMISSARY – MSRI Gazette (Fall 2009), p. 6.

[3] Elwyn R. Berlekamp, John H. Conway and Richard K. Guy. *Winning Ways for your Mathematical Plays*, Vol. 1–4, A K Peters, Wellesley, MA, 2nd edition: vol. 1 (2001), vols. 2, 3 (2003), vol. 4 (2004).

[4] John H. Conway. *On Numbers and Games*. CRC Press, 2nd edition (2000).

[5] Aviezri S. Fraenkel. Aperiodic subtraction games. *The Electronic Journal of Combinatorics* 18(2) (2011), #P19.

[6] Aviezri S. Fraenkel. The vile, dopey, evil and odious game players. *Discrete Mathematics* 312 (2012), 42–46.

[7] Alan Guo. Winning strategies for aperiodic subtraction games. *Theoretical Computer Science* 421 (2012), 70–73.

[8] Elizabeth J. Kupin. Subtraction division games. Available at `http://arxiv.org/abs/1201.0171`.

[9] Aaron N. Siegel. *Combinatorial Game Theory*. American Mathematical Society, Graduate Studies in Mathematics, Vol. 146 (2013).

[10] Neil J. A. Sloane. *The On-Line Encyclopedia of Integer Sequences*. Available at `https://oeis.org/`.