

COMBINATORIAL AUSLANDER-REITEN QUIVERS AND REDUCED EXPRESSIONS

SE-JIN OH[†], UHIRINN SUH[‡]

ABSTRACT. In this paper, we introduce the notion of combinatorial Auslander-Reiten(AR) quiver for commutation classes $[\tilde{w}]$ of w in finite Weyl group. This combinatorial object visualizes the convex partial order $\prec_{[\tilde{w}]}$ on the subset $\Phi(w)$ of positive roots. By analyzing properties of the combinatorial AR-quivers with labelings and reflection maps, we can apply their properties to the representation theory of KLR algebras and multiplication structure of dual PBW generators associated to any commutation class $[\tilde{w}_0]$ of the longest element w_0 .

INTRODUCTION

For a Dynkin quiver Q of finite type ADE, the Auslander-Reiten quiver Γ_Q encodes the representation theory of the path algebra $\mathbb{C}Q$ in the following sense: (i) the set of vertices corresponds to the set $\text{Ind } Q$ of isomorphism classes of indecomposable $\mathbb{C}Q$ -modules, (ii) the set of arrows corresponds to the set of irreducible morphisms for $M, N \in \text{Ind } Q$. On the other hand, by reading the residues of vertices of Γ_Q in a *compatible way* ([3]), one can obtain reduced expressions \tilde{w}_0 of the longest element w_0 in the Weyl group W . Such reduced expressions can be grouped into one class $[Q]$ via commutation equivalence \sim :

$\tilde{w}_0 \sim \tilde{w}'_0$ if and only if \tilde{w}'_0 can be obtained by applying the short braid relations $s_i s_j = s_j s_i$ properly.

A reduced expression in $[Q]$ is called *adapted to* Q . Reduced expressions in $[Q]$ have been used in representation theory intensively. For example, [9, 14, 17] to name a few. However, there are many reduced expressions of w_0 which are not adapted to *any* Dynkin quiver Q .

Another important role of Γ_Q in Lie theory is a realization of the convex partial order \prec_Q on Φ^+ , which is defined as follows: For a reduced expression $\tilde{w}_0 = s_{i_1} s_{i_2} \cdots s_{i_N} \in [Q]$, we label a positive root $s_{i_1} s_{i_2} \cdots s_{i_{k-1}} \alpha_k \in \Phi^+$ as $\beta_k^{\tilde{w}_0}$ and assign the *residue* i_k to $\beta_k^{\tilde{w}_0}$. Then each reduced expression $\tilde{w}_0 \in [Q]$ induces the total order $\prec_{\tilde{w}_0}$ on Φ^+ , $\beta_k^{\tilde{w}_0} \prec_{\tilde{w}_0} \beta_l^{\tilde{w}_0} \iff k < l$. Using the total orders $\prec_{\tilde{w}'_0}$ for $\tilde{w}'_0 \in [Q]$, we obtain \prec_Q on Φ^+ as follows:

- $\alpha \prec_Q \beta$ if and only if $\alpha \prec_{\tilde{w}'_0} \beta$ for all $\tilde{w}'_0 \in [Q]$,
- $\alpha \prec_Q \beta$ and $\gamma = \alpha + \beta \in \Phi^+$ imply $\alpha \prec_Q \gamma \prec_Q \beta$ (the convexity).

Note that it is proved in [23, 31] that any convex total order \prec on Φ^+ is $\prec_{\tilde{w}_0}$ for some \tilde{w}_0 of w_0 . As the definition itself, \prec_Q is quite complicated since there are lots of reduced expressions in each $[Q]$. Interestingly, Γ_Q realizes \prec_Q in the sense that

$\alpha \prec_Q \beta$ if and only if there exists a path from β to α in Γ_Q .

For non-adapted reduced expressions \tilde{w}_0 and their commutation classes $[\tilde{w}_0]$, the definition $\prec_{[\tilde{w}_0]}$ is still applicable. However, there was no study on the order $\prec_{[\tilde{w}_0]}$ on Φ^+ for non-adapted $[\tilde{w}_0]$ and apply them to the representation theory, to the best knowledge of authors.

Date: April 28, 2017.

Key words and phrases. combinatorial AR-quiver, reduced expressions.

[†]This work was supported by NRF Grant #2016R1C1B2013135.

[‡]This work was supported by NRF Grant #2016R1C1B1010721.

In this paper, we introduce new quiver $\Upsilon_{[\tilde{w}]}$, named as the *combinatorial AR-quiver*, for each reduced expression \tilde{w} of $w \in W$. It realizes the convex partial order $\prec_{[\tilde{w}]}$ on $\Phi(w)$ (Theorem 2.29); that is,

$$\alpha \prec_{[\tilde{w}]} \beta \text{ if and only if there exists a path from } \beta \text{ to } \alpha \text{ in } \Upsilon_{[\tilde{w}]}.$$

and hence it can be understood as the Hasse quiver associated to the order $\prec_{[\tilde{w}]}$ ([4]). We first suggest an algorithm for constructing $\Upsilon_{[\tilde{w}]}$ of $\tilde{w} = s_{i_1} \cdots s_{i_\ell}$ (Algorithm 2.1). If we use *residues* as labels of the vertices $\Upsilon_{[\tilde{w}]}^0$ in $\Upsilon_{[\tilde{w}]}$ instead of $\Phi(w) \subset \Phi^+$, one can construct $\Upsilon_{[\tilde{w}]}$ instantly. Then we can prove that $\Upsilon_{[\tilde{w}]} = \Upsilon_{[\tilde{w}']}$ if and only if $[\tilde{w}] = [\tilde{w}']$, and hence any reduced expressions in $[\tilde{w}]$ can be obtained by reading residues of $\Upsilon_{[\tilde{w}]}$ in a *compatible way* (Theorem 2.23).

As one can expect, $\Upsilon_{[Q]}$ is isomorphic to Γ_Q so that $\Upsilon_{[\tilde{w}]}$ is understood as a generalizations of Γ_Q (Theorem 2.28). Hence combinatorial AR quivers have analogous properties to original AR quivers. For instance, we prove that (i) an arrow $\alpha \rightarrow \beta$ in $\Upsilon_{[\tilde{w}]}$ implies $-(\alpha, \beta) = (\alpha_i, \alpha_j) > 0$ where i and j are residues of α and β , (ii) positive roots in a *sectional path* of $\Upsilon_{[\tilde{w}]}$ (Definition 2.12) satisfy distinguished property with respect to the non-degenerate pairing $(\ , \)$ on the set of root lattice.

There are enormous number of reduced expressions for w_0 and, by grouping into commutation equivalence classes, we reduce the efforts to understand all reduced expressions. However, there are still too many commutation classes of reduced expressions so that we consider another equivalence relation called *reflection equivalence relations* between commutation equivalence classes. A family of equivalence classes induced from the reflection equivalences is called an r -cluster point $[[\tilde{w}_0]]$. We would like to point out that there are lots of similarities between representation theories related to commutation classes $[Q]$ and $[Q']$ in the r -cluster point $[[Q]]$ (for example, [9, 14, 20, 21, 22], see also Corollary 5.16). In addition, we introduce the notion of Coxeter composition (Definition 3.12) with respect to a non-trivial Dynkin diagram automorphism σ . Our conjecture is that Coxeter compositions classify r -cluster points (Conjecture 1).

The most useful for applications but complicate part in combinatorial AR quivers is computing labels in terms of positive roots. One can see in Algorithm 2.1 that the labeling of $\Upsilon_{[\tilde{w}]}^0$ with $\Phi(w)$ needs lots of computations. In Section 4, we suggest an efficient way to reduce large amount of computations in general. Roughly speaking, every positive root in each sectional path shares a *component* (Definition 4.5). Hence the labeling for a given vertex follows from joining information of sectional paths passing through it.

In Section 5, we apply our observations in previous sections to the representation theory of KLR-algebras ([6, 9, 10, 11, 18]) which categorifies each dual PBW-basis $\{P_{\tilde{w}_0}(\beta) \mid \beta \in \Phi^+\}$ ([15, 28]) associated to the reduced expression \tilde{w}_0 of w_0 . Using the $\prec_{[\tilde{w}_0]}$ realized by $\Upsilon_{[\tilde{w}_0]}$, we can prove that *the proper standard module* $S_{\tilde{w}_0}(\beta)$ ([6, 18]) over KLR-algebra of each finite type *depends only on* its commutation class $[\tilde{w}_0]$ (up to $q^{\mathbb{Z}}$) and hence so does the dual PBW-monomial associated to \tilde{w}_0 (Theorem 5.9). Note that such property was observed in [22] (see also [11] for ADE cases). Here we give an alternative proof. Furthermore, we prove that the proper standard modules $S_{[\tilde{w}_0]}(\beta)$'s for β 's lying on the same sectional path q -*commute* to each other and hence so does the dual PBW-generators $P_{[\tilde{w}_0]}(\beta)$'s (Proposition 5.13). Using the reflection maps on each $[[\tilde{w}_0]]$, we also observe similarities among $\{S_{[\tilde{w}_0]}(\alpha)\}$ and $\{S_{[\tilde{w}'_0]}(\alpha')\}$ for $[\tilde{w}_0]$ and $[\tilde{w}'_0]$ in the same r -cluster point $[[\tilde{w}_0]]$ (Corollary 5.16).

Acknowledgements. The first author would like to express his sincere gratitude to Euiyong Park for many fruitful discussions.

1. AUSLANDER-REITEN QUIVERS ARISING FROM DYNKIN QUIVERS AND ORDERS ON THE SET OF POSITIVE ROOTS

In this section, we recall combinatorial constructions of Auslander-Reiten quivers from Dynkin quivers. We refer to [1, 2, 8, 14] for the basic theories on quiver representations and Auslander-Reiten quivers. For the combinatorial properties, we refer to [3, 20].

1.1. Auslander-Reiten quivers. Let $A = (a_{ij})_{i,j \in I}$ for $I = \{1, \dots, n\}$ be a Cartan matrix of a finite-dimensional simple Lie algebra \mathfrak{g} . Let Δ be the Dynkin diagram associated to A .

Definition 1.1. For vertices $i, j \in I$ in Δ , the number of edges between i and j is called the *distance* between i and j is denoted by $d_\Delta(i, j)$.

We denote by $\Pi_n = \{\alpha_i \mid i \in I\}$ the set of simple roots, Φ the set of roots, Φ^+ (resp. Φ^-) the set of positive roots (resp. negative roots). Let $\{\epsilon_i \mid 1 \leq i \leq m\}$ be the set of orthonormal basis of \mathbb{C}^m . The free abelian group $\mathbf{Q} := \bigoplus_{i \in I} \mathbb{Z}\alpha_i$ is called the *root lattice*. Set $\mathbf{Q}^+ = \sum_{i \in I} \mathbb{Z}_{\geq 0}\alpha_i \subset \mathbf{Q}$ and $\mathbf{Q}^- = \sum_{i \in I} \mathbb{Z}_{\leq 0}\alpha_i \subset \mathbf{Q}$. For $\beta = \sum_{i \in I} m_i \alpha_i \in \mathbf{Q}^+$, we set $\text{ht}(\beta) = \sum_{i \in I} m_i$. Let (\cdot, \cdot) be the symmetric bilinear form on $\mathbf{Q} \times \mathbf{Q}$ (we refer [5, Plate I~IX]).

A Dynkin quiver Q is obtained by adding an orientation to each edge in the Dynkin diagram Δ . In other words, $Q = (Q^0, Q^1)$ where Q^0 is the set of vertices indexed by I and Q^1 is the set of oriented edges with the underlying graph Δ . We say the vertex i is a sink (resp. source) if every edge between i and j is oriented as follows: $j \rightarrow i$ (resp. $i \rightarrow j$).

Let $\text{Mod}(\mathbb{C}Q)$ be the category of finite dimensional modules over the path algebra $\mathbb{C}Q$. An object $M \in \text{Mod } \mathbb{C}Q$ consists of the following data:

- (1) a finite dimensional module M_i for each $i \in Q^0$,
- (2) a linear map $\psi_{i \rightarrow j} : M_i \rightarrow M_j$ for each oriented edge $i \rightarrow j$.

The *dimension vector* of the module M is $\underline{\dim} M = \sum_{i \in I} (\dim M_i) \alpha_i$ and a simple object in $\text{Mod } \mathbb{C}Q$ is $S(i)$ for some $i \in I$ where $\underline{\dim} S(i) = \alpha_i$. In $\text{Mod } \mathbb{C}Q$, the set of isomorphism classes $[M]$ of indecomposable modules is denoted by $\text{Ind } Q$.

Theorem 1.2 (Gabriel's theorem). *Let Q and Φ^+ be a Dynkin quiver and the set of positive roots of type A_n, D_n or E_n . Then there is a bijection between $\text{Ind } Q$ and Φ^+ :*

$$[M] \mapsto \underline{\dim} M.$$

The Weyl group W of a finite type with rank n is generated by simple reflections $s_i \in \text{Aut}(\mathbf{Q})$, $i \in I$. Note that

$$(w(\alpha), w(\beta)) = (\alpha, \beta)$$

for any $w \in W$ and $\alpha, \beta \in \mathbf{Q}$.

For $w \in W$, the length of w is

$$\ell(w) = \min\{l \in \mathbb{Z}_{\geq 0} \mid s_{i_1} \cdots s_{i_l} = w, s_{i_k} \text{ are simple reflections}\}.$$

If $w = s_{i_1} s_{i_2} \cdots s_{i_{\ell(w)}}$ then the sequence of simple reflections $\tilde{w} = (s_{i_1}, \dots, s_{i_{\ell(w)}})$ is called a reduced expression associated to w .

We denote by w_0 the longest element in W and by $*$ the involution on I induced by w_0 ; i.e.,

$$(1.1) \quad w_0(\alpha_i) := -\alpha_{i^*} \quad \text{for all } i \in I.$$

For $w \in W$, consider the subset ([5])

$$(1.2) \quad \Phi(w) = \{\alpha \in \Phi^+ \mid w^{-1}(\alpha) \in \Phi^-\} = \{s_{i_1} s_{i_2} \cdots s_{i_{k-1}}(\alpha_{i_k}) \mid k = 1, \dots, l\} \text{ such that } |\Phi(w)| = \ell(w).$$

Let us consider the action of a simple reflection s_i , $i \in I$, on the set

$$[[Q]] = \{Q \mid Q \text{ is a Dynkin quiver with } \Delta \text{ as the underlying diagram}\},$$

defined by $s_i(Q) = Q'$, where $s_i(j) = j$ for all $j \in Q_0$ and

$$(j \rightarrow k) \mapsto \begin{cases} (j \leftarrow k) & \text{if } j = i \text{ and } i \text{ is a source in } Q, \\ (j \leftarrow k) & \text{if } k = i \text{ and } i \text{ is a sink in } Q, \\ (j \rightarrow k) & \text{otherwise,} \end{cases} \quad \text{for all } j \rightarrow k \in Q_1.$$

Definition 1.3. A reduced expression $\tilde{w} = (s_{i_1}, \dots, s_{i_l})$ of w is said to be *adapted* to a Dynkin quiver Q if

$$i_k \text{ is a sink of } Q_{k-1} = s_{i_{k-1}} \cdots s_{i_1}(Q).$$

The followings are well-known:

- (1.3) (i) A reduced expression \tilde{w}_0 of w_0 is adapted to at most one Dynkin quiver Q .
(ii) For each Dynkin quiver Q , there is a reduced expression \tilde{w}_0 of w_0 adapted to Q .

Note that the converse of (1.3) (i) is not true; that is, two different reduced expressions of w_0 can be adapted to the same Dynkin quiver Q . Actually, we can assign a *class* of reduced expressions of w_0 to each Dynkin quiver Q .

Definition 1.4. [3, 14] Let $\tilde{w} = (s_{i_1}, s_{i_2}, \dots, s_{i_k})$ and $\tilde{w}' = (s_{i'_1}, s_{i'_2}, \dots, s_{i'_k})$ be reduced expressions of $w \in W$. If \tilde{w}' can be obtained by a sequence of commutation relations, $s_i s_j = s_j s_i$ for $d_\Delta(i, j) > 1$, from \tilde{w} then we say \tilde{w} and \tilde{w}' are *commutation equivalent* and write $\tilde{w} \sim \tilde{w}'$. The *equivalence class* of \tilde{w} is denoted by $[\tilde{w}]$.

Proposition 1.5. [3, 14] *Reduced expressions $\tilde{w}_0 = (s_{i_1}, s_{i_2}, \dots, s_{i_l})$ and $\tilde{w}'_0 = (s_{i'_1}, s_{i'_2}, \dots, s_{i'_l})$ of w_0 are adapted to a quiver Q if and only if $\tilde{w}_0 \sim \tilde{w}'_0$ and \tilde{w}_0 is adapted to Q .*

Thus we can denote by $[Q]$ the equivalence class of w_0 consisting of all reduced expressions adapted to the Dynkin quiver Q .

For each Dynkin quiver Q , there is a unique *Coxeter element* ϕ_Q in W such that all of reduced expressions of ϕ_Q are adapted to Q : The element $\phi_Q \in W$ has the form

$$\phi_Q = s_{i_1} s_{i_2} \cdots s_{i_n} \quad \text{where } \{i_1, \dots, i_n\} = I.$$

Now we recall the Auslander-Reiten (AR) quiver Γ_Q associated to a Dynkin quiver Q of type A_n , D_n , or E_n . For the rest of this section, we assume that Q is a Dynkin quiver of type A_n , D_n , or E_n . Let us denote by $\text{Ind } Q$ the set of isomorphism classes $[M]$ of indecomposable modules in $\text{Mod } \mathbb{C}Q$, where $\text{Mod } \mathbb{C}Q$ is the category of finite dimensional modules over the path algebra $\mathbb{C}Q$.

Definition 1.6. Let \tilde{w}_0 be a reduced expression of w_0 adapted to a Dynkin quiver Q . The quiver $\Gamma_Q = (\Gamma_Q^0, \Gamma_Q^1)$ is called the Auslander-Reiten quiver (AR quiver) if

- (1) each vertex V_M in Γ_Q^0 corresponds to an isomorphism class $[M]$ in $\text{Ind } Q$,
- (2) an arrow $V_M \rightarrow V_{M'}$ in Γ_Q^1 implies that there exists an irreducible morphism $M \rightarrow M'$.

Gabriel's theorem (Theorem 1.2) tells that there is a natural one-to-one correspondence between the set Γ_Q^0 of vertices in Γ_Q and the set Φ^+ of positive roots. Hence we use Φ^+ as the index set Γ_Q^0 .

The quiver Γ_Q of type A_n , D_n and E_n can be obtained by a purely combinatorial method. In order to show this, we introduce another quiver $A(Q)$ below. Denote by h the Coxeter number and \mathbf{a}_i (resp. \mathbf{b}_i) the number of arrows in Q directed to the vertex i (resp. i^*) between the vertices indexed by i and i^* (see [3, 8, 24]).

- (1) Consider the quiver $\mathbb{N}Q$ whose vertices are indexed by $I \times \mathbb{N}$ and the set of arrows is $\{(i, m) \rightarrow (j, m), (j, m) \rightarrow (i, m-1) \mid i \rightarrow j \in Q_1\}$.
 - (2) The subquiver $A(Q)$ of $\mathbb{N}Q$ consists of the vertices $\{(i, m) \mid 1 \leq m \leq r_i\}$, where
- $$(1.4) \quad r_i = (h + \mathbf{a}_i - \mathbf{b}_i)/2.$$

The following proposition shows the relation between two quivers $A(Q)$ and Γ_Q .

Proposition 1.7. [3, 24] *As quivers, $A(Q)$ is isomorphic to Γ_Q . More precisely, let $\tilde{w}_0 = (s_{i_1}, s_{i_2}, \dots, s_{i_l}) \in [Q]$. The isomorphism $\iota_Q : A(Q) \rightarrow \Gamma_Q$ is given by the following correspondence between their vertices:*

$$(1.5) \quad (i, m) \longleftrightarrow \beta = s_{i_1} s_{i_1} \cdots s_{i_{k-1}} \alpha_{i_k} \in \Phi^+ \quad \text{for } m = \#\{i_t = i \mid 1 \leq t \leq k-1\} + 1 \text{ and } i = i_k.$$

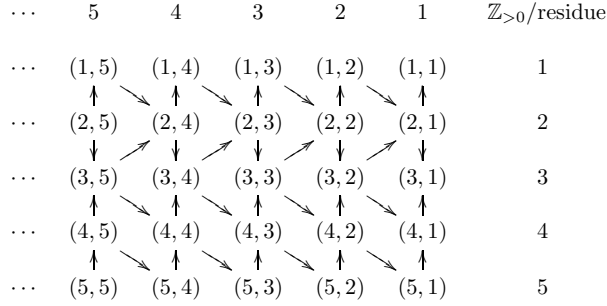
Here the value (i, m) corresponding to β does not depend on the choice of reduced expression \tilde{w}_0 of w_0 .

We call the i of β in (1.5) the *residue* of β (with respect to Q). By the above proposition, the quiver Γ_Q does *not* depend on the choice of reduced expression \tilde{w}_0 in $[Q]$.

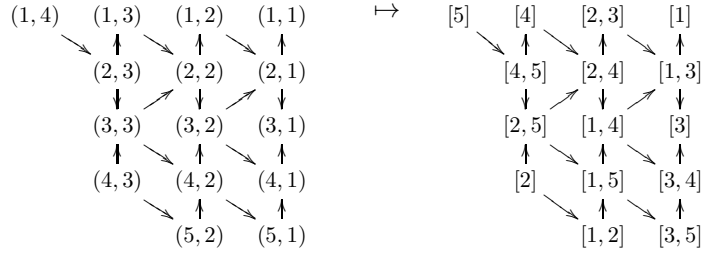
Remark 1.8. We sometimes denote by $\Gamma_{[\tilde{w}_0]}$ for $\tilde{w}_0 \in [Q]$ instead of Γ_Q to emphasize that it *does* depend *only* on the equivalent class $[\tilde{w}_0]$.

Example 1.9. 1) Let $\tilde{w}_0 = (s_1, s_3, s_2, s_4, s_1, s_3, s_5, s_2, s_4, s_1, s_3, s_5, s_2, s_4, s_1)$ of A_5 , which is adapted to the Dynkin quiver $Q = \circ_1 \leftarrow \circ_2 \rightarrow \circ_3 \leftarrow \circ_4 \rightarrow \circ_5$.

2) The quiver $\mathbb{N}Q$ is



3) Compute $r_i = (\mathbf{h} + \mathbf{a}_i - \mathbf{b}_i)/2$. In this case $(r_i | i \in I) = (4, 3, 3, 3, 2)$ since $\mathbf{h} = 6$. Take the finite subquiver $A(Q)$ of $\mathbb{N}Q$ consisting of the vertices $\{(i, j) | 1 \leq j \leq r_i\}$. Then $A(Q)$ is isomorphic to Γ_Q by the map $\iota_Q : A(Q) \rightarrow \Gamma_Q$:



where $[i, j] = \sum_{k=i}^j \alpha_k \in \Phi^+$ and $[i, i] = [i] = \alpha_i$.

Now we introduce the height function $\xi : I \rightarrow \mathbb{Z}$ associated to Q and describe another combinatorial description of Γ_Q using the height function and the Coxeter element ϕ_Q (see [9]).

Definition 1.10. The *height function* ξ associated to the quiver Q is a map on Q satisfying $\xi(j) = \xi(i) - 1 \in \mathbb{Z}$ if there exists an arrow $i \rightarrow j$.

The *repetition quiver* $\mathbb{Z}Q$ of Q has the set of vertices

$$(\mathbb{Z}Q)^0 = \{(i, p) \in I \times \mathbb{Z} \mid p - \xi(i) \in 2\mathbb{Z}\}$$

and arrows $(\mathbb{Z}Q)^1 = \{(j, p+1) \rightarrow (i, p), (i, p) \rightarrow (j, p-1) \mid i \text{ and } j \text{ as connected in } Q\}$. For $i \in I$, we define positive roots γ_i and θ_i in the following way:

$$(1.6) \quad \gamma_i = \sum_{j \in \overleftarrow{i}} \alpha_j \quad \text{and} \quad \theta_i = \sum_{j \in \overrightarrow{i}} \alpha_j \quad \text{where}$$

- \overleftarrow{i} is the set of vertices j in Q^0 such that there exists a path from i to j ,
- \overrightarrow{i} is the set of vertices j in Q^0 such that there exists a path from j to i .

Note that $\{\gamma_i \mid i \in I\} = \Phi(\phi_Q)$ and $\{\theta_i \mid i \in I\} = \Phi(\phi_Q^{-1})$.

Consider the map $\pi : \Phi^+ \rightarrow (\mathbb{Z}Q)_0$ such that

$$(1.7) \quad \gamma_i \mapsto (i, \xi(i)), \quad \phi_Q(\alpha) \mapsto (i, p-2) \quad \text{if } \pi(\alpha) = (i, p) \text{ and } \phi_Q(\alpha), \alpha \in \Phi^+.$$

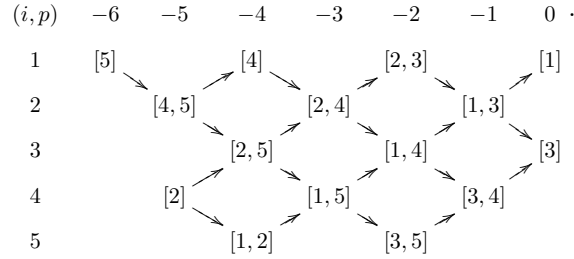
Then we have the following:

- (1) The subquiver $B(Q)$ of $\mathbb{Z}Q$ consisting of $\pi(\Phi^+)$ is the same as the quiver Γ_Q by identifying their vertices as Φ^+ .
- (2) Recall that $A(Q)$ in Proposition 1.7 is isomorphic to Γ_Q . The map $A(Q) \rightarrow B(Q)$ is given by $(i, m) \mapsto (i, \xi(i) - 2(m - 1))$ by considering coordinates of all $\beta \in \Phi^+$.

Since $A(Q)$ and $B(Q)$ are isomorphic quivers to Γ_Q , indices of $A(Q)$ and $B(Q)$ give coordinates to positive roots in Γ_Q . The coordinate induced by $B(Q)$ has meanings in the description of reflection map related to $[\tilde{w}_0]$ for \tilde{w}_0 which is adapted to some Dynkin quiver Q (see (3.2) below).

Definition 1.11. A path $\beta_0 \rightarrow \beta_1 \rightarrow \cdots \rightarrow \beta_s$ in Γ_Q is called a *sectional* path if $\phi_Q(\beta_{k+1}) = \beta_{k-1}$ for all $1 \leq k \leq s - 1$.

Example 1.12. The AR quiver Γ_Q associated to Q in Example 1.9 is



Here the coordinate (i, p) is induced from that of $B(Q)$.

Combinatorially, a path is sectional if the path is *upwards* or *downwards* in Γ_Q under our convention.

1.2. Partial orders on $\Phi(w)$. Let w be an element in W of finite type. An order \preceq on the set $\Phi(w)$ is said to be *convex* if

$$\alpha, \beta, \alpha + \beta \in \Phi(w) \text{ and } \alpha \preceq \beta \text{ implies } \alpha \preceq \alpha + \beta \preceq \beta.$$

Each reduced expression \tilde{w} of $w \in W$ induces a total order on $\Phi(w)$ using the position function defined as follows:

Definition 1.13. Let $\tilde{w} = (s_{i_1}, s_{i_2}, \dots, s_{i_l})$ be a reduced expression of w . The *position function* $\pi_{\tilde{w}} : \Phi(w) \rightarrow \mathbb{N}$ associated to the reduced expression \tilde{w} is defined by $\beta_k^{\tilde{w}} = s_{i_1} s_{i_2} \cdots s_{i_{k-1}}(\alpha_{i_k}) \mapsto k$.

Definition 1.14. The total order $<_{\tilde{w}}$ on $\Phi(w)$ associated to \tilde{w} is defined by

$$\beta_j^{\tilde{w}} <_{\tilde{w}} \beta_k^{\tilde{w}} \quad \text{if and only if} \quad j = \pi_{\tilde{w}}(\beta_j^{\tilde{w}}) < \pi_{\tilde{w}}(\beta_k^{\tilde{w}}) = k.$$

Definition 1.15. Let $\alpha, \beta \in \Phi(w) \subset \Phi^+$. We define an order $\prec_{[\tilde{w}]}$ on $\Phi(w)$ as follows:

$$\alpha \prec_{[\tilde{w}]} \beta \quad \text{if and only if} \quad \alpha <_{\tilde{w}'} \beta \quad \text{for any } \tilde{w}' \in [\tilde{w}].$$

Proposition 1.16. [23] *The total order $<_{\tilde{w}}$ and the partial order $\prec_{[\tilde{w}]}$ are convex orders on $\Phi(w)$.*

The AR quiver Γ_Q visualizes the convex partial order $\prec_{[Q]}$ when Q is a Dynkin quiver Q of type ADE in the following sense:

Proposition 1.17. [25] *For $\tilde{w}_0 \in [Q]$ and $\alpha, \beta \in \Phi^+$, we have $\alpha \prec_{[\tilde{w}_0]} \beta$ if and only if there is a path from β to α in $\Gamma_{[\tilde{w}_0]}$.*

Recall that the order of $\Phi(w)$ via $<_{\tilde{w}}$ is determined by the value of the position function $\pi_{\tilde{w}}$. Now, we introduce a function called level function $\lambda_{\tilde{w}}$, which is closely related to the partial order $\prec_{[\tilde{w}]}$.

Definition 1.18. [3] Let $\tilde{w} = (s_{i_1}, s_{i_2}, \dots, s_{i_l})$ be a reduced expression of w . Given $\alpha \in \Phi(w)$, let

$$(1.8) \quad \beta_1, \beta_2, \dots, \beta_k = \alpha$$

be a sequence of distinct elements of $\Phi(w)$ ending with α such that $\pi_{\tilde{w}}(\beta_{i-1}) < \pi_{\tilde{w}}(\beta_i)$ and $(\beta_i, \beta_{i-1}) \neq 0$. The function $\lambda_{\tilde{w}} : \Phi(w) \rightarrow \mathbb{N}$ associated to the reduced expression \tilde{w} is defined as follows:

$$\lambda_{\tilde{w}}(\alpha) = \max \{k \geq 1 \mid \beta_1, \beta_2, \dots, \beta_k = \alpha \text{ is the sequence in (1.8)}\}.$$

We call it *the level function* associated to \tilde{w} .

Proposition 1.19. [3] *Two reduced expressions \tilde{w} and \tilde{w}' of w are commutation equivalent if and only if $\lambda_{\tilde{w}} = \lambda_{\tilde{w}'}$.*

The above proposition tells that the level function $\lambda_{\tilde{w}}$ does depend only on the equivalent class of \tilde{w} and hence we shall write $\lambda_{[\tilde{w}]}$ instead of $\lambda_{\tilde{w}}$. In particular, the level function $\lambda_{[\tilde{w}_0]}$ for $\tilde{w}_0 \in [Q]$ is closely related to the AR quiver Γ_Q .

Proposition 1.20. [3] *For Q and $\tilde{w}_0 \in [Q]$, define a function $\lambda_{\Gamma_Q} : \Phi^+ \rightarrow \mathbb{N}$ in an inductive way:*

$$\alpha \mapsto \begin{cases} 1, & \text{if } \alpha \text{ is a sink in } \Gamma_Q, \\ \max\{\lambda_{\Gamma_Q}(\beta) \mid \alpha \rightarrow \beta \text{ in } \Gamma_Q\} + 1, & \text{otherwise.} \end{cases}$$

Then we have $\lambda_{\Gamma_Q} = \lambda_{[\tilde{w}_0]}$.

Another closely related notion to the AR quiver Γ_Q is compatible readings of positive roots. To see the relation, we introduce a *compatible reading* of Γ_Q . A sequence s_{i_1}, \dots, s_{i_N} (resp. i_1, \dots, i_N) of simple reflections (resp. indices) is called a *compatible reading* of the AR quiver Γ_Q if

whenever there is an arrow from (i_q, n_q) to (i_r, n_r) in $A(Q) \simeq \Gamma_Q$, read s_{i_r} before s_{i_q} .

According to Proposition 1.17, a compatible reading of Γ_Q gives a compatible reading of positive roots, in the sense that α is read before β if $\alpha \prec_{[Q]} \beta$ for $\alpha, \beta \in \Phi^+$.

Theorem 1.21. [3] *Let Q be a Dynkin quiver of finite type A_n, D_n, E_n . Then any reduced expression of $w_0 \in W$ adapted to the quiver Q can be obtained by a compatible reading of the AR quiver Γ_Q .*

2. COMBINATORIAL AR-QUIVERS AND RELATED CONVEX PARTIAL ORDERS

In this section, we shall construct a quiver which visualizes the convex partial order $\prec_{[\tilde{w}]}$ for *any* w of all finite Weyl group W and its reduced expression \tilde{w} .

2.1. Combinatorial AR-quivers. Let us take

$$\tilde{w} = (s_{i_1}, s_{i_2}, s_{i_3}, \dots, s_{i_{\ell(w)}})$$

a reduced expression of an element $w \in W$. Then we can label $\Phi(w)$ as follows:

$$\beta_k^{\tilde{w}} = s_{i_1} \cdots s_{i_{k-1}}(\alpha_k) \quad \text{for } 1 \leq k \leq \ell(w).$$

Recall that the *residue* of $\beta_k^{\tilde{w}}$ is defined by $\text{res}(\beta_k^{\tilde{w}}) = i_k$.

Algorithm 2.1. *The quiver $\Upsilon_{\tilde{w}} = (\Upsilon_{\tilde{w}}^0, \Upsilon_{\tilde{w}}^1)$ associated to \tilde{w} is constructed in the following algorithm:*

(Q1) $\Upsilon_{\tilde{w}}^0$ consists of $\ell(w)$ vertices labeled by $\beta_1^{\tilde{w}}, \dots, \beta_{\ell(w)}^{\tilde{w}}$.

(Q2) There is an arrow from $\beta_k^{\tilde{w}}$ to $\beta_j^{\tilde{w}}$ if

(i) $k > j$, (ii) $d_{\Delta}(i_k, i_j) = 1$ and (iii) $\{t \mid j < t < k, i_t = i_j \text{ or } i_k\} = \emptyset$.

(Q3) Assign the color $m_{jk} = -(\alpha_{i_j}, \alpha_{i_k})$ to each arrow $\beta_k^{\tilde{w}} \rightarrow \beta_j^{\tilde{w}}$ in (Q2); that is, $\beta_k^{\tilde{w}} \xrightarrow{m_{jk}} \beta_j^{\tilde{w}}$. Replace $\xrightarrow{1}$ by \rightarrow , $\xrightarrow{2}$ by \Rightarrow and $\xrightarrow{3}$ by \Rightarrow .

We call the quiver $\Upsilon_{\tilde{w}}$ the *combinatorial AR-quiver* associated to \tilde{w} .

Remark 2.2.

- (1) By replacing labels $\beta_k^{\tilde{w}}$'s with i_k 's, one can obtain the usual Hasse quiver. To compute $\beta_k^{\tilde{w}}$, we need lots of computations in general.

- (2) In Proposition 2.15 and Proposition 2.16, we will show that if two roots α and β are connected by a sequence of arrows with the *same direction*, then the product of colors of the arrows corresponds to (α, β) (see the propositions for rigorous statements).

The following proposition follows from the construction of the quiver $\Upsilon_{\tilde{w}}$:

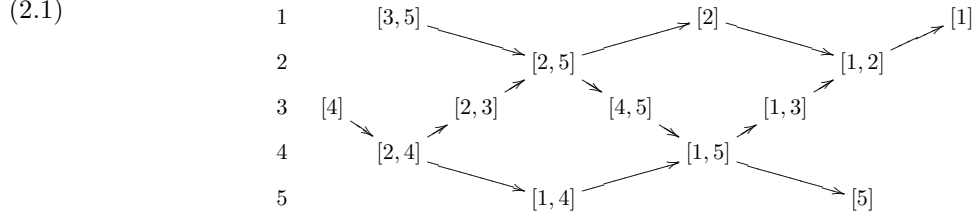
Proposition 2.3. *If two reduced expressions \tilde{w} and \tilde{w}' are commutation equivalent then $\Upsilon_{\tilde{w}} = \Upsilon_{\tilde{w}'}$.*

By Proposition 2.3, we can define $\Upsilon_{[\tilde{w}]}$ for an equivalence class $[\tilde{w}]$ of $w \in W$.

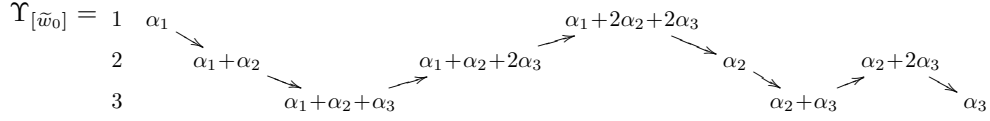
Example 2.4. Let $\tilde{w} = (s_1, s_2, s_3, s_5, s_4, s_3, s_1, s_2, s_3, s_5, s_4, s_3, s_1)$ of A_5 . Then one can easily check that \tilde{w} is *not* adapted to *any* Dynkin quiver Q of type A_5 . According to Algorithm 2.1, labels of vertices of the combinatorial AR quiver $\Upsilon_{[\tilde{w}]}$ are

$$\begin{aligned} & (\beta_k^{\tilde{w}} \mid 1 \leq k \leq \ell(w) = 13) \\ & = ([1], [1, 2], [1, 3], [5], [1, 5], [4, 5], [2], [2, 5], [2, 3], [1, 4], [2, 4], [4], [3, 5]). \end{aligned}$$

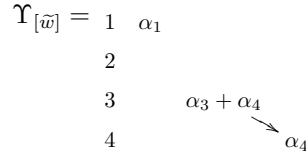
The quiver $\Upsilon_{[\tilde{w}]}$ is drawn as follows:



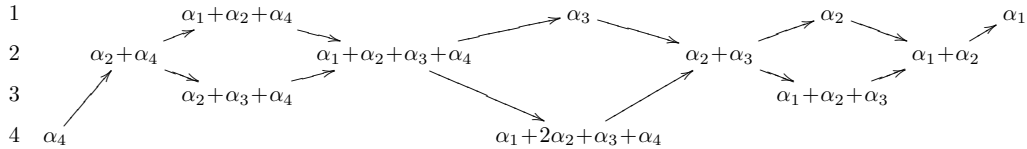
Example 2.5. Let $\tilde{w}_0 = (s_3, s_2, s_3, s_2, s_1, s_2, s_3, s_2, s_1)$ of B_3 . The combinatorial AR quiver of \tilde{w}_0 is



Example 2.6. A combinatorial AR quiver is not necessarily connected. For example, let $\tilde{w} = (s_4, s_3, s_1)$ of A_4 . Then



Example 2.7. Let $\tilde{w}_0 = (s_1, s_2, s_3, s_1, s_2, s_4, s_1, s_2, s_3, s_1, s_2, s_4)$ of D_4 . Note that \tilde{w}_0 is not adapted to any Dynkin quiver of type D_4 . We can draw the combinatorial AR quiver $\Upsilon_{[\tilde{w}_0]}$ as follows:



Example 2.8. Let $\tilde{w} = (s_1, s_2, s_1, s_2, s_1)$ of G_2 . Then

$$\begin{array}{ccccccc} \Upsilon_{[\tilde{w}]} = 1 & \alpha_1 + 3\alpha_2 & \rightleftharpoons & 2\alpha_1 + 3\alpha_2 & \rightleftharpoons & \alpha_1 \\ 2 & \alpha_1 + 2\alpha_2 & \rightleftharpoons & \alpha_1 + \alpha_2 & & \end{array}$$

We showed that a combinatorial AR quiver is not necessarily connected. The following proposition shows a property about connectedness of a combinatorial AR quiver.

Proposition 2.9. *For $w \in W$ and its reduced expression \tilde{w} of w , let $\beta_{k_1}^{\tilde{w}}$ and $\beta_{k_2}^{\tilde{w}}$ be two vertices in $\Upsilon_{[\tilde{w}]}$. Suppose both of the two vertices have the same residue i and $k_1 < k_2$. Then there is a path from $\beta_{k_2}^{\tilde{w}}$ to $\beta_{k_1}^{\tilde{w}}$ in $\Upsilon_{[\tilde{w}]}$.*

Proof. It is enough to show when there is no vertex $\beta_k^{\tilde{w}}$ with the residue i such that $k_1 < k < k_2$.

Suppose there is no index $i' \in I$ which satisfies the following property:

$$(2.2) \quad \text{there is a vertex } \beta_{k_3}^{\tilde{w}} \in \Upsilon_{[\tilde{w}]}^0 \text{ with the residue } i' \text{ for some } k_1 < k_3 < k_2 \text{ and } d_{\Delta}(i, i') = 1.$$

Then it is a contradiction to the fact that \tilde{w} is a reduced expression. Hence there is $i' \in I$ satisfying (2.2).

Take $i' \in I$ which satisfies (1) and (2) above.

- (i) If there is a unique vertex $\beta_{k_3}^{\tilde{w}} \in \Upsilon_{[\tilde{w}]}^0$ with the residue i' -th such that $k_1 < k_3 < k_2$ then there is a path from $\beta_{k_2}^{\tilde{w}}$ to $\beta_{k_1}^{\tilde{w}}$ via $\beta_{k_3}^{\tilde{w}}$.
- (ii) If there are more than one vertices k_3, k_4, \dots with the residue i' such that $k_1 < k_3, k_4, \dots < k_2$. Without loss of generality, let us assume there are arrows from $\beta_{k_2}^{\tilde{w}}$ to $\beta_{k_4}^{\tilde{w}}$ and from $\beta_{k_3}^{\tilde{w}}$ to $\beta_{k_1}^{\tilde{w}}$ in $\Upsilon_{[\tilde{w}]}$. Then it is enough show there is a path from $\beta_{k_4}^{\tilde{w}}$ to $\beta_{k_3}^{\tilde{w}}$ in $\Upsilon_{[\tilde{w}]}$. Again, we can assume that there is no vertex $\beta_l^{\tilde{w}}$ with the residue i' such that $k_3 < l < k_4$. Inductively, we can reduce the situation to the case (i).

Hence we proved that there is a path from $\beta_{k_2}^{\tilde{w}}$ to $\beta_{k_1}^{\tilde{w}}$. □

Proposition 2.10. *Let \tilde{w} be a reduced expression consisting of simple reflections $\{s_{i_1}, \dots, s_{i_k}\}$. The subdiagram of Δ consisting of the set of indices $\{i_1, \dots, i_k\}$ is connected if and only if $\Upsilon_{[\tilde{w}]}$ is connected.*

Recall that the level function $\lambda_{[\tilde{w}]}$ is defined for any reduced expression $\tilde{w} \in W$ of any finite type. In the adapted cases (Definition 1.18 and Proposition 1.20), the level function $\lambda_{[Q]}$ is visualized by Γ_Q . More precisely,

- the existence of an arrow between α and β in Γ_Q implies $(\alpha, \beta) \neq 0$,
- $\lambda_{[Q]}(\alpha)$ is the length of the longest path in Γ_Q starting from α .

Now we shall prove that $\Upsilon_{[\tilde{w}]}$ plays the roles of Γ_Q for *any* \tilde{w} of w in *any* finite Weyl group of W .

Now we introduce paths in combinatorial AR quivers which have a particular property.

Definition 2.11. For a combinatorial AR quiver $\Upsilon_{[\tilde{w}]}$, let us choose positive roots $\alpha, \beta \in \Phi(w)$ which are connected by a path. The smallest number of arrows consisting a path between α and β is called the *distance* between α and β in $\Upsilon_{[\tilde{w}]}$ and denote it by $d_{[\tilde{w}]}(\alpha, \beta)$.

Definition 2.12. Consider a path P in a combinatorial AR quiver $\Upsilon_{[\tilde{w}]}$. If a pair of roots α, β in P whose residues are i and j satisfies

$$d_{[\tilde{w}]}(\alpha, \beta) = d_{\Delta}(i, j)$$

then the path is called a *sectional path*.

When $[\tilde{w}] = [Q]$ for some Dynkin quiver Q of type ADE, the above definition coincides with Definition 1.11.

Example 2.13. In Example 2.4, $[2, 4]$ and $[2]$ whose residues are 4 and 1 lie in the sectional path:

$$[2, 4]_4 \rightarrow [2, 4]_3 \rightarrow [2, 5]_2 \rightarrow [2]_1$$

Here each subindex i denotes the residue for its vertex.

Remark 2.14. [29, Section 3] In the theory of AR quivers for the path algebra $\mathbb{C}Q$, sectional paths provide information on $\dim(M, N)$ and $\text{Ext}^1(M, N)$ for $M, N \in \text{Ind } Q$.

Proposition 2.15. *Let α and β have residues i and j in the combinatorial Auslander-Reiten quiver $\Upsilon_{[\tilde{w}]}$. If $d_{[\tilde{w}]}(\alpha, \beta) = 1$ then we have $(\alpha, \beta) = -(\alpha_i, \alpha_j) > 0$.*

Proof. Take a reduced expression $\tilde{w} = (s_{i_1}, \dots, s_{i_{\ell(w)}}) \in [\tilde{w}]$ and denote $\alpha = \beta_k^{\tilde{w}}$ and $\beta = \beta_l^{\tilde{w}}$ for $1 \leq k < l \leq \ell(w)$. Then the arrow is directed from β to α . If $l = k + 1$, then our assertion follows from the formula below:

$$(\alpha, \beta) = (s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k}), s_{i_1} \cdots s_{i_k}(\alpha_{i_l})) = (-\alpha_{i_k}, \alpha_{i_l}).$$

Assume that $l > k + 1$ and set $\tilde{w}_{k \leq \cdot \leq l} := (s_{i_k}, \dots, s_{i_l})$. It is enough to show that there exists a reduced expression $\tilde{w}' \in [\tilde{w}]$ such that $\beta_{k'}^{\tilde{w}'} = \alpha$ and $\beta_{k'+1}^{\tilde{w}'} = \beta$ for some $k' \in \{1, \dots, \ell(w) - 1\}$.

Observe that the following property is followed by the algorithm of combinatorial AR quivers

$$\{i_t \mid k < t < l, i_t = i\} = \{i_t \mid k < t < l, i_t = j\} = \emptyset.$$

Hence the direct path from β to α is the only path starting from β to α . Otherwise, we get a loop in Δ as a consequence, which is a contradiction.

Now let

$$P = \left\{ a_t \in \mathbb{N} \mid \begin{array}{l} \text{(i) } k < a_t < l, t = 1, \dots, m, \text{ (ii) } a_1 < a_2 < \dots < a_m, \\ \text{(iii) each } \beta_{a_t}^{\tilde{w}} \text{ is on a path to } \alpha \text{ in } \Upsilon_{[\tilde{w}]} \end{array} \right\}$$

and let $P^c = \{k, k+1, \dots, l\} \setminus P = \{b_1, b_2, \dots, b_{(l-k+1)-m}\}$ where $b_1 < b_2 < \dots < b_{(l-k+1)-m}$. By our observation, there is no path from $\beta_{b_{t'}}^{\tilde{w}}$ to $\beta_{a_t}^{\tilde{w}}$ for any $t = 1, \dots, m$ and $t' = 1, \dots, (l-k+1)-m$.

Hence $\beta_{b_1}^{\tilde{w}}$ is not connected with any of vertices in $\{\beta_i^{\tilde{w}} \mid k \leq i < b_1\} \subset \{\beta_{a_t}^{\tilde{w}} \mid t = 1, \dots, m\}$ and

$$\tilde{w}_{k \leq \cdot \leq l} = (s_{i_k}, \dots, s_{i_{b_1-1}}, s_{i_{b_1}}, s_{i_{b_1+1}}, \dots, s_{i_l}) \sim (s_{i_{b_1}}, s_{i_k}, \dots, s_{i_{b_1-1}}, s_{i_{b_1+1}}, \dots, s_{i_l}).$$

Inductively, we can do the same thing with $b_2, \dots, b_{(l-k+1)-m}$ and finally get the following equivalent reduced expression to $\tilde{w}_{k \leq \cdot \leq l}$:

$$\tilde{w}_{k \leq \cdot \leq l} \sim (s_{i_{b_1}}, \dots, s_{i_{b_{(l-k+1)-m}}}, s_{i_k}, s_{i_{a_1}}, \dots, s_{i_{a_m}}, s_{i_l}).$$

Since $\beta_{a_m}^{\tilde{w}}$ is not connected to β , we have $s_{i_{a_m}} s_{i_l} = s_{i_l} s_{i_{a_m}}$. Hence

$$\tilde{w}_{k \leq \cdot \leq l} \sim (s_{i_{b_1}}, \dots, s_{i_{b_{(l-k+1)-m}}}, s_{i_k}, s_{i_{a_1}}, s_{i_{a_{m-1}}}, \dots, s_{i_l}, s_{i_{a_m}}).$$

Inductively, we get

$$\tilde{w}_{k \leq \cdot \leq l} \sim (s_{i_{b_1}}, \dots, s_{i_{b_{(l-k+1)-m}}}, s_{i_k}, s_{i_l}, s_{i_{a_1}}, \dots, s_{i_{a_m}}).$$

Let $\tilde{w}' = (s_{i'_1}, \dots, s_{i'_{\ell(w)}})$ have the form

$$\tilde{w}' = (s_{i_1}, \dots, s_{i_{k-1}}, s_{i_{b_1}}, \dots, s_{i_{b_{(l-k+1)-m}}}, s_{i_k}, s_{i_l}, s_{i_{a_1}}, \dots, s_{i_{a_m}}, s_{i_{l+1}}, \dots, s_{i_{\ell(w)}}).$$

Then $s_{i'_{l-m+1}} = s_{i_k}$ (resp. $s_{i'_{l-m+2}} = s_{i_l}$) and $\beta_{l-m+1}^{\tilde{w}'} = \beta_k^{\tilde{w}} = \alpha$ (resp. $\beta_{l-m+2}^{\tilde{w}'} = \beta_l^{\tilde{w}} = \beta$). Hence we proved our assertion by setting $k' = l - m + 1$. \square

Proposition 2.16. *Let α and β have residues $i = i_0$ and $j = i_k$ in $\Upsilon_{[\tilde{w}]}$. If α and β are in a sectional path*

$$\beta = \gamma_k \xrightarrow{m_{i_{k-1}, i_k}} \gamma_{k-1} \xrightarrow{m_{i_{k-2}, i_{k-1}}} \dots \longrightarrow \gamma_1 \xrightarrow{m_{i_0, i_1}} \gamma_0 = \alpha$$

in $\Upsilon_{[\tilde{w}]}$, then we have

$$(2.3) \quad (\alpha, \beta) = \begin{cases} \prod_{t=1}^{k-1} 2^{\delta_{3, i_t}} \prod_{t=0}^{k-1} m_{i_t, i_{t+1}} & \text{for Type } F_4, \\ \prod_{t=0}^{k-1} m_{i_t, i_{t+1}} & \text{otherwise,} \end{cases}$$

where i_t is the residue of γ_t and $m_{a,b} := -(\alpha_a, \alpha_b)$ for $a, b \in I$ (Algorithm 2.1). Hence

$$(\alpha, \beta) > 0.$$

Proof. Note that, by induction on k , we can see that

$$s_{i_0} s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k}) = \alpha_{i_k} + \sum_{p=1}^k (-2)^p \frac{\prod_{t=0}^{p-1} (\alpha_{i_{k-t-1}}, \alpha_{i_{k-t}})}{\prod_{t=0}^{p-1} (\alpha_{i_{k-t-1}}, \alpha_{i_{k-t-1}})} \alpha_{i_{k-p}}.$$

Proposition 2.15 shows that there exists $w \in W$ such that

$$\alpha = w(\alpha_i) \text{ and } \beta = w s_i s_{i_1} s_{i_2} \cdots s_{i_{k-1}}(\alpha_j).$$

We have

$$(2.4) \quad \begin{aligned} (w(\alpha_i), w s_i s_{i_1} s_{i_2} \cdots s_{i_{k-1}}(\alpha_j)) &= (\alpha_{i_0}, s_{i_0} s_{i_1} s_{i_2} \cdots s_{i_{k-1}}(\alpha_j)) \\ &= \left(\alpha_{i_0}, \alpha_{i_k} + \sum_{p=1}^k (-2)^p \frac{\prod_{t=0}^{p-1} (\alpha_{i_{k-t-1}}, \alpha_{i_{k-t}})}{\prod_{t=0}^{p-1} (\alpha_{i_{k-t-1}}, \alpha_{i_{k-t-1}})} \alpha_{i_{k-p}} \right). \end{aligned}$$

Since $(\alpha_{i_0}, \alpha_{i_p}) = 0$ for $p \neq 0, 1$,

$$(2.5) \quad \begin{aligned} &(\alpha_{i_0}, s_{i_0} s_{i_1} s_{i_2} \cdots s_{i_{k-1}}(\alpha_j)) \\ &= \left(\alpha_{i_0}, (-2)^{k-1} \frac{\prod_{t=1}^{k-1} (\alpha_{i_t}, \alpha_{i_{t+1}})}{\prod_{t=1}^{k-1} (\alpha_{i_t}, \alpha_{i_t})} \alpha_{i_1} + (-2)^k \frac{\prod_{t=0}^{k-1} (\alpha_{i_t}, \alpha_{i_{t+1}})}{\prod_{t=0}^{k-1} (\alpha_{i_t}, \alpha_{i_t})} \alpha_{i_0} \right) \\ &= -(-2)^{k-1} \frac{\prod_{t=1}^{k-1} (\alpha_{i_t}, \alpha_{i_{t+1}})}{\prod_{t=1}^{k-1} (\alpha_{i_t}, \alpha_{i_t})} (\alpha_{i_0}, \alpha_{i_1}) = \prod_{t=1}^{k-1} \frac{2}{(\alpha_{i_t}, \alpha_{i_t})} \prod_{t=0}^{k-1} -(\alpha_{i_t}, \alpha_{i_{t+1}}). \end{aligned}$$

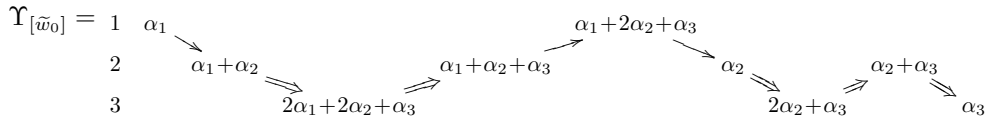
Here we note that the residue i_t for $t = 1, 2, \dots, k-1$ cannot be 1 or the rank n . (Only i_0 and i_k can be 1 or n .) According to [5], in the case of except F_4 , we can check that $(\alpha_{i_t}, \alpha_{i_t}) = 2$ for all $t = 1, 2, \dots, k-1$. In the case of type F_4 , we have $(\alpha_2, \alpha_2) = 2$ and $(\alpha_3, \alpha_3) = 1$. Hence we get the formula (2.3). \square

Remark 2.17. For any finite type other than F_4 , we have

$$(\alpha, \beta) = \prod_{t=0}^{k-1} (\gamma_t, \gamma_{t+1}) = \prod_{t=0}^{k-1} -(\alpha_{i_t}, \alpha_{i_{t+1}}) = \prod_{t=0}^{k-1} m_{i_t, i_{t+1}} > 0.$$

Here we use notations in Proposition 2.16.

Example 2.18. Let us consider $\tilde{w}_0 = (s_3, s_2, s_3, s_2, s_1, s_2, s_3, s_2, s_1)$ of type C_3 . Then



One can check that Proposition 2.16 holds in the above quiver. For instance,

$$\begin{aligned} 2 &= (\alpha_1 + 2\alpha_2 + \alpha_3, 2\alpha_1 + 2\alpha_2 + \alpha_3) \\ &= (\alpha_1 + 2\alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3)(\alpha_1 + \alpha_2 + \alpha_3, 2\alpha_1 + 2\alpha_2 + \alpha_3) \\ &= (\alpha_1, \alpha_2)(\alpha_2, \alpha_3). \end{aligned}$$

Proposition 2.19. Let $\alpha, \beta \in \Phi(w)$ and \tilde{w} be a reduced expression of $w \in W$. Suppose there is no path between α and β in $\Upsilon_{[\tilde{w}]}$. Then we have $(\alpha, \beta) = 0$.

Proof. Since $\prec_{\tilde{w}}$ is a total order, we can assume that $\beta_k^{\tilde{w}} = \alpha$ and $\beta_l^{\tilde{w}} = \beta$ for $k < l$ without loss of generality. If $l - k = 1$, then

$$(\alpha, \beta) = (s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k}), s_{i_1} \cdots s_{i_{k-1}} s_{i_k}(\alpha_{i_l})) = (\alpha_{i_k}, s_{i_k}(\alpha_{i_l})) = (\alpha_{i_k}, \alpha_{i_l}) = 0.$$

Now suppose $l - k \geq 2$. It is enough to find $\tilde{w}' \in [\tilde{w}]$ such that $\beta_{k'}^{\tilde{w}'} = \alpha$ and $\beta_{k'+1}^{\tilde{w}'} = \beta$ for some $k' \in \{1, \dots, \ell(w) - 1\}$. Take the set

$$P = \left\{ a_t \in \mathbb{N} \mid \begin{array}{l} k < a_t < l, t = 1, \dots, m, a_1 < a_2 < \dots < a_m, \\ \text{each } \beta_{a_t}^{\tilde{w}} \text{ is on a path to } \alpha \text{ in } \Upsilon_{[\tilde{w}]} \end{array} \right\}$$

and let $P^c = \{k, k+1, \dots, l\} \setminus P = \{b_1, b_2, \dots, b_{(l-k+1)-m}\}$ where $b_1 < b_2 < \dots < b_{(l-k+1)-m}$. Then $\beta_{b_1}^{\tilde{w}}$ is not connected with any of vertices in $\{\beta_i^{\tilde{w}} \mid k \leq i < b_1\}$. Hence

$$\tilde{w}_{k \leq \cdot \leq l} = (s_{i_k}, \dots, s_{i_{b_1-1}}, s_{i_{b_1}}, s_{i_{b_1+1}}, \dots, s_{i_l}) \sim (s_{i_{b_1}}, s_{i_k}, \dots, s_{i_{b_1-1}}, s_{i_{b_1+1}}, \dots, s_{i_l}).$$

Inductively, we can do the same thing with $b_2, \dots, b_{(l-k+1)-m}$ and finally get the following equivalent reduced expression to $\tilde{w}_{k \leq \cdot \leq l}$:

$$\tilde{w}_{k \leq \cdot \leq l} \sim (s_{i_{b_1}}, \dots, s_{i_{b_{(l-k+1)-m}}}, s_{i_k}, s_{i_{a_1}}, \dots, s_{i_{a_m}}, s_{i_l}).$$

Since $\beta_{a_m}^{\tilde{w}}$ is not connected to β , we have $s_{i_{a_m}} s_{i_l} = s_{i_l} s_{i_{a_m}}$. Hence

$$\tilde{w}_{k \leq \cdot \leq l} \sim (s_{i_{b_1}}, \dots, s_{i_{b_{(l-k+1)-m}}}, s_{i_k}, s_{i_{a_1}}, s_{i_{a_{m-1}}}, \dots, s_{i_l}, s_{i_{a_m}}).$$

Inductively, we get

$$\tilde{w}_{k \leq \cdot \leq l} \sim (s_{i_{b_1}}, \dots, s_{i_{b_{(l-k+1)-m}}}, s_{i_k}, s_{i_l}, s_{i_{a_1}}, \dots, s_{i_{a_m}}).$$

Then

$$\tilde{w}' = (s_{i_1}, \dots, s_{i_{k-1}}, s_{i_{b_1}}, \dots, s_{i_{b_{(l-k+1)-m}}}, s_{i_k}, s_{i_l}, s_{i_{a_1}}, \dots, s_{i_{a_m}}, s_{i_{l+1}}, \dots, s_{i_{\ell(w)}})$$

is the reduced expression in $[\tilde{w}]$ we desired. \square

We record the following corollary, appeared in the proof of the above proposition, for the later use.

Corollary 2.20. *Let $\alpha, \beta \in \Phi(w)$ and \tilde{w} be a reduced expression of $w \in W$. If there is no path between α and β in $\Upsilon_{[\tilde{w}]}$, then there are two distinct reduced expressions \tilde{w}' and \tilde{w}'' in $[\tilde{w}]$ and two integers $k, l \in \mathbb{N}$ such that $\beta_{k+1}^{\tilde{w}'} = \alpha$, $\beta_k^{\tilde{w}'} = \beta$ and $\beta_l^{\tilde{w}''} = \alpha$, $\beta_{l+1}^{\tilde{w}''} = \beta$.*

Proposition 2.21. *For a reduced expression \tilde{w} of $w \in W$ of any finite type, define a function $\lambda_{\Upsilon_{[\tilde{w}]}} : \Phi^+(w) \rightarrow \mathbb{N}$ in an inductive way:*

$$\alpha \mapsto \begin{cases} 1, & \text{if } \alpha \text{ is a sink in } \Upsilon_{[\tilde{w}]} \\ \max\{\lambda_{\Upsilon_{[\tilde{w}]}}(\beta) \mid \alpha \rightarrow \beta \text{ in } \Upsilon_{[\tilde{w}]}\} + 1, & \text{otherwise.} \end{cases}$$

Then we have $\lambda_{\Upsilon_{[\tilde{w}]}} = \lambda_{[\tilde{w}]}$.

Proof. Our assertion directly follows from Proposition 2.15 and Proposition 2.19. \square

By Proposition 2.21 and properties of the level function $\lambda_{[\tilde{w}]}$, we have the following theorem.

Theorem 2.22. *Two reduced expressions \tilde{w} and \tilde{w}' are commutation equivalent if and only if $\Upsilon_{[\tilde{w}]} = \Upsilon_{[\tilde{w}']}$.*

Proof. It is enough to show that if $\Upsilon_{[\tilde{w}]} = \Upsilon_{[\tilde{w}']}$ then $[\tilde{w}] = [\tilde{w}']$. However, since we know that $\lambda_{[\tilde{w}]} = \lambda_{\Upsilon_{[\tilde{w}]}} = \lambda_{\Upsilon_{[\tilde{w}']}} = \lambda_{[\tilde{w}']}$ and $\lambda_{[\tilde{w}]} = \lambda_{[\tilde{w}']}$ implies $[\tilde{w}] = [\tilde{w}']$, our assertion follows. \square

Theorem 2.22 implies that we can get every equivalent reduced expression \tilde{w}' to \tilde{w} by observing $\Upsilon_{[\tilde{w}]}$:

Theorem 2.23. *Every reduced expression of w in $[\tilde{w}]$ can be obtained by a compatible reading of $\Upsilon_{[\tilde{w}]}$.*

Remark 2.24. Throughout this section, we actually use residues as labels of $\Upsilon_{[\tilde{w}]}^0$, which need not compute. In Section 4, we will suggest an efficient algorithm for labeling of $\Upsilon_{[\tilde{w}]}^0$ with positive roots.

Now we will show that combinatorial AR quivers $\Upsilon_{[\tilde{w}_0]}$ can be considered as a generalization of AR quivers Γ_Q by providing isomorphism of quivers

$$\Upsilon_{[\tilde{w}_0]} \simeq \Gamma_Q \quad \text{when } [\tilde{w}_0] = [Q].$$

Lemma 2.25. *For a Dynkin quiver Q of type ADE, let $\tilde{w}_0 = (s_{i_1}, \dots, s_{i_N}) \in [Q]$ and $(i, j) \in I^2$ be a pair of indices with $d_\Delta(i, j) = 1$. If $i_k = i_{k'} = i$ for $k < k'$ then there exists t such that $k < t < k'$ and $i_t = j$.*

Proof. Let us denote

$$(2.6) \quad Q_m = s_{i_m} \cdots s_{i_1} Q \quad \text{for } m = 1, \dots, N.$$

Then i_k is a sink of Q_{k-1} . Suppose $k' = \min\{k_1 \mid k_1 > k, i_{k_1} = i\}$. If $\{t \mid k < t < k', i_k = j\} = \emptyset$ then the arrow between i and j in $Q_{k'-1}$ has the direction $i \rightarrow j$. Hence the vertex i cannot be a sink of $Q_{k'-1}$, which is a contradiction. \square

Example 2.26. In Example 2.4, we can obtain the following reduced expression in $[\tilde{w}_0]$ by compatible reading:

$$(s_1, s_2, s_5, s_3, s_4, s_3, s_1, s_2, s_5, s_1, s_3, s_4, s_3)$$

In the above reduced expression, one can check that Lemma 2.25 do *not* hold, since it is not adapted to any Q .

The following corollary directly follows from Lemma 2.25.

Corollary 2.27. *Suppose $\tilde{w} = (s_{i_1}, \dots, s_{i_{\ell(\tilde{w})}})$ is a reduced expression of $w \in W$ and let $i, j \in I$ be indices with $d_\Delta(i, j) = 1$. If there are $k, k' \in \{1, \dots, \ell(\tilde{w})\}$ with $k < k'$ such that*

- (1) $i_k = i_{k'} = i$,
- (2) $\{j' \mid k < j' < k', i_{j'} = j\} = \emptyset$.

then \tilde{w} is not adapted to any Dynkin quiver.

Theorem 2.28. *For a Dynkin quiver of type A_n, D_n or E_n , the combinatorial AR quiver $\Upsilon_{[Q]}$ is isomorphic to the AR quiver Γ_Q .*

Proof. Let us first show that $A(Q) \simeq \Gamma_Q$ is isomorphic to the combinatorial AR quiver $\Upsilon_{[Q]}$. Suppose the Dynkin quiver Q has an arrow from j to i . Then we have the following properties.

- (i) By the construction of $A(Q)$, there is an arrow from $(j, 1)$ to $(i, 1)$ in $A(Q)$,
- (ii) Recall the definition of the quiver Q_m in (2.6). Then i is a sink of Q_{k_i-1} and j is a sink of Q_{k_j-1} , where

$$k_i = \min\{k \mid s_{i_k} = s_i\} < k_j = \min\{k \mid s_{i_k} = s_j\}.$$

- (iii) $\{k \mid k_i < k < k_j, i_k = i\} = \emptyset$.

Here, the reason for (ii) is that, in order to make the vertex j a sink in the quiver Q_{k_j-1} , we need the reflection s_i which reverses the arrow from j to i . Also, Lemma 2.25 implies (iii). Using (ii), (iii) and the construction of the combinatorial AR quiver, we conclude that there is an arrow from $\beta_{k_j}^{\tilde{w}_0}$ to $\beta_{k_i}^{\tilde{w}_0}$ in $\Upsilon_{[\tilde{w}_0]}$ for $\tilde{w}_0 \in [Q]$.

Moreover, in the quiver $A(Q)$, there is an arrow from (i, m) to $(j, m-1)$ if

$$(2.7) \quad \#\{s_{i_k} = s_i \mid 1 \leq k \leq N\} \geq m, \quad \#\{s_{i_k} = s_j \mid 1 \leq k \leq N\} \geq m-1.$$

Let

$$k_{i,m} = \min \left\{ k' \mid \begin{array}{l} \text{there exists a sequence } k_1 < k_2 < \dots < k_m = k' \\ \text{such that } i_{k_1} = \dots = i_{k_m} = i \end{array} \right\}.$$

Then by Lemma 2.25, we have $k_i = k_{i,1} < k_j = k_{j,1} < k_{i,2} < k_{j,2} < k_{i,3} < \dots$. Hence, by the construction of $\Upsilon_{[\tilde{w}_0]}$, we have an arrow from $\beta_{k_{i,m}}^{\tilde{w}_0}$ to $\beta_{k_{j,m-1}}^{\tilde{w}_0}$ if (2.7) holds.

Similarly, if we have

$$\#\{s_{i_k} = s_i \mid 1 \leq k \leq N\} \geq m, \quad \#\{s_{i_k} = s_j \mid 1 \leq k \leq N\} \geq m,$$

then we have the arrow from (j, m) to (i, m) in $A(Q)$ and the arrow from $\beta_{k_{j,m}}^{\tilde{w}_0}$ to $\beta_{k_{i,m}}^{\tilde{w}_0}$ in $\Upsilon_{[\tilde{w}_0]}$.

As a conclusion, two quivers $A(Q)$ and $\Upsilon_{[Q]}$ are isomorphic by the map $\psi : A(Q) \rightarrow \Upsilon_{[Q]}$ such that $(i, m) \mapsto \beta_{k_{i,m}}^{\tilde{w}_0}$. Recall that the quiver isomorphism $\iota_Q : A(Q) \rightarrow \Gamma_Q$ was defined by $(i, m) \mapsto \beta_{k_{i,m}}^{\tilde{w}_0}$. Hence the ordinary AR quiver Γ_Q and the combinatorial AR quiver $\Upsilon_{[\tilde{w}_0]}$ are isomorphic to each other. \square

Now we can prove that, for any \tilde{w} of $w \in W$ of any finite type, $\Upsilon_{[\tilde{w}]}$ visualizes the convex partial order $\prec_{[\tilde{w}]}$ on $\Phi(w)$:

Theorem 2.29. *The combinatorial AR quiver $\Upsilon_{[\tilde{w}]}$ visualizes the convex partial order $\preceq_{[\tilde{w}]}$. That is $\alpha \preceq_{[\tilde{w}]} \beta$ if and only if there is a path from β to α in $\Upsilon_{[\tilde{w}]}$.*

Proof. It is obvious that then there is a path from β to α in $\Upsilon_{[\tilde{w}]}$ then we have $\alpha \prec_{[\tilde{w}]} \beta$.

Conversely, suppose that there is no path between β and α and $\pi_{\tilde{w}}(\beta) < \pi_{\tilde{w}}(\alpha)$. In the proof of Proposition 2.19, we showed that there is a reduced expression $\tilde{w}' = (s_{i_1}, \dots, s_{i_{\ell(w)}}) \in [\tilde{w}]$ such that $\alpha = \beta_{k+1}^{\tilde{w}'}$ and $\beta = \beta_k^{\tilde{w}'}$ and $(\alpha_{i_k}, \alpha_{i_{k+1}}) = 0$. Hence by exchanging s_{i_k} and $s_{i_{k+1}}$ in \tilde{w}' , we get $\tilde{w}'' = (s_{i_1}, \dots, s_{i_{k-1}}, s_{i_{k+1}}, s_{i_k}, s_{i_{k+2}}, \dots, s_{i_{\ell(w)}}) \in [\tilde{w}]$. Since $\pi_{\tilde{w}''}(\beta) = k + 1 > \pi_{\tilde{w}''}(\alpha) = k$, we conclude that β and α are not comparable via $\preceq_{[\tilde{w}]}$. \square

3. COMBINATORIAL REFLECTION MAPS

The following theorem is a well-known fact about sinks and sources of a Dynkin quiver Q and an AR quiver Γ_Q .

Theorem 3.1. *Let Q be a Dynkin quiver of type A_n , D_n , or E_n and Γ_Q be the associated AR quiver. The followings are equivalent.*

- (a) $i \in I$ is a sink (resp. source) of Q .
- (b) There are reduced expressions \tilde{w}_0 adapted to Q such that \tilde{w}_0 starts (resp. ends) with s_i (resp. s_{i^*}).
- (c) α_i is a sink (resp. source) of Γ_Q .

Let X_n be a simply laced type, i.e., X is A, D or E , and n is a proper integer depending of X . On the set of AR quiver $\Gamma_{[Q]} = \{\Gamma_Q \mid Q \text{ is a Dynkin quiver of type } X_n\}$, for $i \in I$, define *right (resp. left) reflection map*

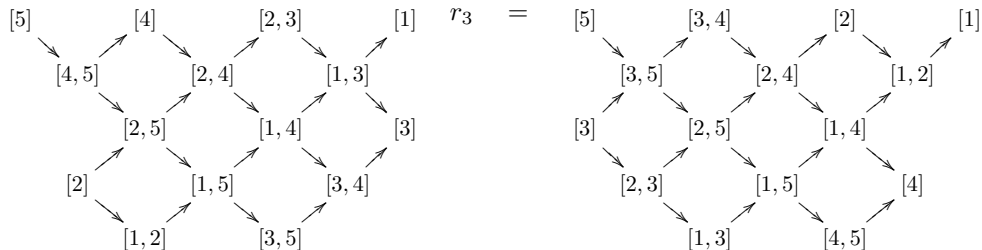
$$r_i : \Gamma_{[Q]} \rightarrow \Gamma_{[Q]}$$

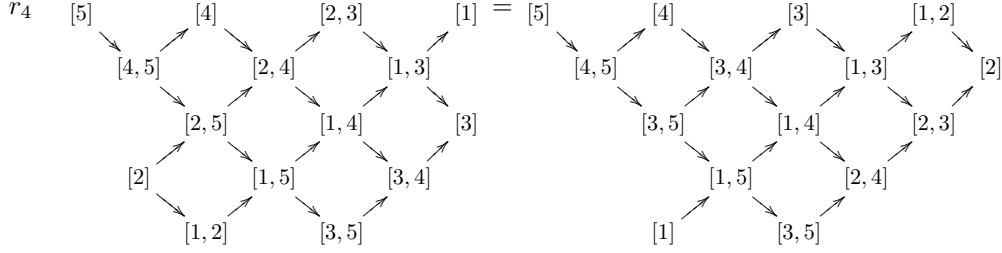
by $\Gamma_Q \mapsto \Gamma_Q r_i$ (resp. $\Gamma_Q \mapsto \Gamma_Q r_i$), where

$$(3.1) \quad \Gamma_Q r_i = \begin{cases} \Gamma_{s_i(Q)} & \text{if } i \text{ is a sink in } Q, \\ \Gamma_Q & \text{otherwise, and} \end{cases} \quad r_i \Gamma_Q = \begin{cases} \Gamma_{s_{i^*}(Q)} & \text{if } i^* \text{ is a source in } Q, \\ \Gamma_Q & \text{otherwise.} \end{cases}$$

Remark 3.2. The reflection maps in this section can be understood as a combinatorial version of the reflection functors in the representation theory over the path algebra $\mathbb{C}Q$ of a quiver Q (see [13, Section 3] for reference).

Example 3.3. Let $\tilde{w}_0 = (s_3, s_1, s_2, s_4, s_1, s_3, s_5, s_2, s_4, s_1, s_3, s_5, s_2, s_1, s_4)$ of A_5 . Note that \tilde{w}_0 is adapted one. Then α_3 is a sink of $\Gamma_{[\tilde{w}_0]}$ and α_2 is a source of $\Gamma_{[\tilde{w}_0]}$.





Let i be a sink (resp. source) in Q . The right (resp. left) reflection map r_i on Γ_Q can be described as follows:

- (3.2) (i) Delete the sink (resp. source) α_i (resp. α_{i^*}) in Γ_Q .
 (ii) Put a new vertex α_i (resp. α_{i^*}) with residue i^* at the beginning (resp. end) of Γ_Q and arrows starting from α_i (resp. ending at α_{i^*}) and ending at the first vertices (resp. starting from the last vertices) with residues j such that $d_\Delta(i^*, j) = 1$.
 (iii) Change each label β in $\Phi^+ \setminus \{\alpha_i\}$ (resp. $\Phi^+ \setminus \{\alpha_{i^*}\}$) with $s_i\beta$ (resp. $s_{i^*}\beta$).

Analogously, we can define reflection maps on combinatorial AR quivers. In order to do this, we need notions of source and sink of commutation classes $[\tilde{w}]$ of \mathcal{W} .

Definition 3.4. For a commutation equivalence class $[\tilde{w}]$, we say that $i \in I$ is a *sink* (resp. *source*) if there is a reduced expression $\tilde{w}' \in [\tilde{w}]$ of w starting with s_i (resp. ending with s_i).

The following proposition follows from the construction of the combinatorial AR quiver $\Upsilon_{[\tilde{w}]}$ and (1.2):

Proposition 3.5.

- (a) i is a sink of $[\tilde{w}]$ if and only if α_i is a sink in the quiver $\Upsilon_{[\tilde{w}]}$.
 (b) i is a source of $[\tilde{w}]$ if and only if $-\alpha_i$ is a source in the quiver $\Upsilon_{[\tilde{w}]}$.

Using sources and sinks of a commutation equivalence class, we shall define a reflection map on the set of combinatorial AR quivers

$$\Upsilon_{w_0} := \{ \Upsilon_{[\tilde{w}_0]} \mid \tilde{w}_0 \text{ is a reduced expression of } w_0 \}$$

and divide the set Υ_{w_0} into the orbits $\Upsilon_{[\tilde{w}_0]}$ of reflection maps (see also Definition 3.12 below):

$$\Upsilon_{w_0} = \bigsqcup_{[\tilde{w}_0]} \Upsilon_{[\tilde{w}_0]}$$

Definition 3.6. The right reflection map r_i on $[\tilde{w}_0]$ is defined by

$$[\tilde{w}_0] r_i = \begin{cases} [(s_{i_2}, \dots, s_{i_N}, s_{i^*})] & \text{if } i \text{ is a sink and } \tilde{w}'_0 = (s_i, s_{i_2}, \dots, s_{i_N}) \in [\tilde{w}_0], \\ [\tilde{w}_0] & \text{if } i \text{ is not a sink of } [\tilde{w}_0]. \end{cases}$$

On the other hand, the left reflection map r_i on $[\tilde{w}_0]$ is defined by

$$r_i [\tilde{w}_0] = \begin{cases} [(s_{i^*}, s_{i_1}, \dots, s_{i_{N-1}})] & \text{if } i \text{ is a source and } \tilde{w}'_0 = (s_{i_1}, \dots, s_{i_{N-1}}, s_i) \in [\tilde{w}_0], \\ [\tilde{w}_0] & \text{if } i \text{ is not a source of } [\tilde{w}_0]. \end{cases}$$

Lemma 3.7. For a reduced expression $\tilde{w}_0 = (s_{i_1}, \dots, s_{i_N})$ of w_0 , define $\tilde{w}_{<k} = s_{i_1} \cdots s_{i_{k-1}}$ for $1 \leq k \leq N$. Then

$$\tilde{w}_{<N}(\alpha_{i_N}) = \alpha_{i_N^*}.$$

Proof. $\tilde{w}_{<N}(\alpha_{i_N}) = w_0 \cdot s_{i_N}(\alpha_{i_N}) = w_0(-\alpha_{i_N}) = \alpha_{i_N^*}$. \square

The following propositions show the reflection map is well-defined on

$$\{ [\tilde{w}_0] \mid \tilde{w}_0 \text{ is a reduced expression of } w_0 \}.$$

Proposition 3.8. *Let $\tilde{w}_0 = (s_{i_1}, \dots, s_{i_{N-1}}, s_{i_N})$ be a reduced expression of w_0 .*

- (a) $\tilde{w}'_0 = (s_{i_N^*}, s_{i_1}, \dots, s_{i_{N-1}})$ is a reduced expression of w_0 which is not in $[\tilde{w}_0]$.
- (b) $\tilde{w}''_0 = (s_{i_2}, \dots, s_{i_{N-1}}, s_{i_N}, s_{i_1^*})$ is a reduced expression of w_0 which is not in $[\tilde{w}_0]$.

Proof. (a) Suppose \tilde{w}'_0 is not a reduced expression. Then \tilde{w}'_0 represents $w \in \mathbb{W}$ whose length is $N - 2$, that is

$$w = s_{i_N^*} s_{i_1} \cdots s_{i_{N-1}} \in \mathbb{W} \quad \text{and} \quad \ell(w) = N - 2.$$

Denote $\beta_k^{\tilde{w}_0} = s_{i_1} \cdots s_{i_{k-1}} \alpha_k$ for $k = 1, \dots, N$. Recall that $w_0^{-1} \beta_k^{\tilde{w}_0} \in \Phi^-$ and $|\{\beta_k^{\tilde{w}_0} \mid 1 \leq k \leq N\}| = |\Phi^+| = N$. Since $\beta_N^{\tilde{w}_0} = \alpha_{i_N^*}$ and $|\Phi(w)| = |\{\alpha \in \Phi^+ \mid w^{-1}(\alpha) \in \Phi^-\}| = N - 2$, there exists m such that $1 \leq m \leq N - 1$ and $w^{-1} \beta_m^{\tilde{w}_0} \in \Phi^+ \setminus \{\alpha_{i_N^*}\}$. Observe that $w^{-1} \beta_m^{\tilde{w}_0} = s_{i_N} w_0^{-1} s_{i_N^*} \beta_m^{\tilde{w}_0} \in \Phi^+$ and $w_0^{-1}(s_{i_N^*} \beta_m^{\tilde{w}_0}) = w_0(s_{i_N^*} \beta_m^{\tilde{w}_0}) \in \Phi^-$. Hence $w_0^{-1} s_{i_N^*} \beta_m^{\tilde{w}_0} = s_{i_N} w^{-1} \beta_m^{\tilde{w}_0} = -\alpha_{i_N}$ and $w^{-1} \beta_m^{\tilde{w}_0} = \alpha_{i_N}$. Now we get

$$\beta_m^{\tilde{w}_0} = w \alpha_{i_N} = s_{i_N^*} s_{i_1} \cdots s_{i_{N-1}} \alpha_{i_N} = s_{i_N^*} (\alpha_{i_N^*}) = -\alpha_{i_N^*},$$

which is a contradiction. Hence the length of w is N . In other words, $w = w_0$ and \tilde{w}'_0 is a reduced expression of w_0 .

Moreover, we have $[\tilde{w}_0] \neq [\tilde{w}'_0]$ since $\lambda_{\tilde{w}_0}(\alpha_{i_N^*}) > 1$ and $\lambda_{\tilde{w}'_0}(\alpha_{i_N^*}) = 1$.

(b) The analogous proof of (a) works to (b). □

Remark 3.9. To the experts, the fact that \tilde{w}'_0 and \tilde{w}''_0 are also reduced expressions of w_0 may be well-known (for example, [7, page 7] and [11, page 650]). However, we have had a difficulty finding its proof. Thus we provide a proof by using the system of positive roots.

Proposition 3.10. *Let $\tilde{w}_0 = (s_{i_1}, \dots, s_{i_N})$ and $\tilde{w}'_0 = (s_{i'_1}, \dots, s_{i'_N})$ be reduced expressions in $[\tilde{w}_0]$.*

- (a) If $i_1 = i'_1$ then $\tilde{w}_0^1 = (s_{i_2}, \dots, s_{i_N}, s_{i_1^*})$ and $\tilde{w}'_0^2 = (s_{i'_2}, \dots, s_{i'_N}, s_{i'_1^*})$ are in the same commutation equivalence class.
- (b) If $i_N = i'_N$ then $\tilde{w}_0^3 = (s_{i_N^*}, s_{i_1}, \dots, s_{i_{N-1}})$ and $\tilde{w}'_0^4 = (s_{i_N^*}, s_{i'_1}, \dots, s_{i'_{N-1}})$ are in the same commutation equivalence class.

Proof. Since we proved \tilde{w}_0^p , $p = 1, 2, 3, 4$, are all reduced expressions of w_0 , it is enough to show that $\Upsilon_{[\tilde{w}_0^1]} = \Upsilon_{[\tilde{w}'_0^2]}$ and $\Upsilon_{[\tilde{w}_0^3]} = \Upsilon_{[\tilde{w}'_0^4]}$. If $i_1 = i'_1$ then $\Upsilon_{[\tilde{w}]}$ and $\Upsilon_{[\tilde{w}']}$ are the same subquiver of $\Upsilon_{[\tilde{w}_0]}$ where $\tilde{w} = (s_{i_2}, \dots, s_{i_N})$ and $\tilde{w}' = (s_{i'_2}, \dots, s_{i'_N})$. By the algorithm of combinatorial AR quivers, there is a unique way to get another combinatorial AR quiver by putting the last vertex on the i_1^* -th residue of $\Upsilon_{[\tilde{w}]}$. Since the resulting AR quiver is $\Upsilon_{[\tilde{w}_0^1]} = \Upsilon_{[\tilde{w}'_0^2]}$, we have $[\tilde{w}_0^1] = [\tilde{w}'_0^2]$. Similarly, we can show that $[\tilde{w}_0^3] = [\tilde{w}'_0^4]$. □

The reflecting map on $[\tilde{w}_0]$ induces the right (resp. left) *reflection map* r_i for $i \in I$ on Υ_{w_0} as follows:

$$(3.3) \quad \Upsilon_{[\tilde{w}_0]} r_i = \Upsilon_{[\tilde{w}_0] r_i} \quad (\text{resp. } r_i \Upsilon_{[\tilde{w}_0]} = \Upsilon_{r_i[\tilde{w}_0]}).$$

If s_{i_1} is a sink in $[\tilde{w}_0]$, there is a reduced expression $\tilde{w}_0 = (s_{i_1}, s_{i_2}, \dots, s_{i_N}) \in [\tilde{w}_0]$. We know that $[\tilde{w}_0] s_{i_1} = [\tilde{w}'_0]$ where $\tilde{w}'_0 = (s_{i'_1}, s_{i'_2}, \dots, s_{i'_N}) = (s_{i_2}, \dots, s_{i_N}, s_{i_1^*})$. Hence for $k = 1, \dots, l - 1$, we have $\beta_k^{\tilde{w}'_0} = s_{i_1} \beta_{k+1}^{\tilde{w}_0}$.

When s_{i_N} is a source in $[\tilde{w}_0]$, there is a reduced expression $\tilde{w}_0 = (s_{i_1}, s_{i_2}, \dots, s_{i_N}) \in [\tilde{w}_0]$. We know that $s_{i_N}[\tilde{w}_0] = [\tilde{w}'_0]$ where $\tilde{w}'_0 = (s_{i'_1}, s_{i'_2}, \dots, s_{i'_N}) = (s_{i_N^*}, s_{i_1}, \dots, s_{i_{N-1}})$. Hence for $k = 2, \dots, l$, we have $\beta_k^{\tilde{w}'_0} = s_{i_N^*} \beta_{k-1}^{\tilde{w}_0}$.

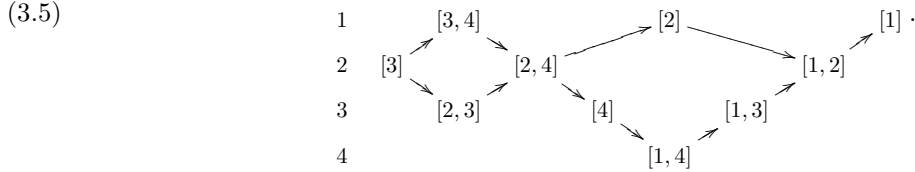
For $\tilde{w} = (s_{i_1}, \dots, s_{i_k})$, the right (resp. left) action of the reflection map $r_{\tilde{w}}$ is defined by

$$[\tilde{w}_0] r_{\tilde{w}} = [\tilde{w}_0] r_{i_1} \cdots r_{i_k} \quad (\text{resp. } r_{\tilde{w}}[\tilde{w}_0] = r_{i_k} \cdots r_{i_1}[\tilde{w}_0]).$$

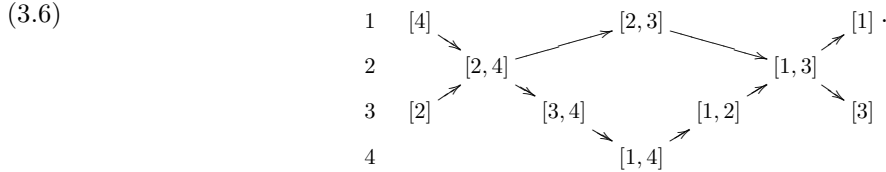
Then the right (resp. left) reflection map on $\Upsilon_{[\tilde{w}_0]}$ can be described as an analogue of (3.2):

- (3.4) (i) Delete the sink (resp. source) α_i (resp. α_{i^*}) with residue i (resp. i^*) and arrows incident with α_i (resp. α_{i^*}) in $\Upsilon_{[\tilde{w}_0]}$.
 (ii) Put a new vertex α_i (resp. α_{i^*}) in the end (resp. beginning) of $\Upsilon_{[\tilde{w}_0]}$ and arrows the conditions in Algorithm 2.1.
 (iii) Change each label β in $\Phi^+ \setminus \{\alpha_i\}$ (resp. $\Phi^+ \setminus \{\alpha_{i^*}\}$) with $s_i\beta$ (resp. $s_{i^*}\beta$).

Example 3.11. Let us consider reduced expression $\tilde{w}_0 = (s_1, s_2, s_1, s_3, s_4, s_3, s_2, s_3, s_1, s_2)$ of A_4 which is not adapted to any Dynkin quiver Q . Then we have



Since s_2 is a source of \tilde{w}_0 , we have $r_2[\tilde{w}_0] = (s_3, s_1, s_2, s_1, s_3, s_4, s_3, s_2, s_3, s_1)$ and $r_2\Upsilon_{[\tilde{w}_0]}$ is



Definition 3.12.

- (1) Let $[\tilde{w}_0]$ and $[\tilde{w}'_0]$ be two commutation equivalence classes. We say $[\tilde{w}_0]$ and $[\tilde{w}'_0]$ are *reflection equivalent* and write $[\tilde{w}_0] \stackrel{r}{\sim} [\tilde{w}'_0]$ if $[\tilde{w}'_0]$ can be obtained from $[\tilde{w}_0]$ by a sequence of reflection maps. The family of commutation equivalence classes $[[\tilde{w}_0]] := \{[\tilde{w}_0] \mid [\tilde{w}_0] \stackrel{r}{\sim} [\tilde{w}'_0]\}$ is called an *r-cluster point*.
 (2) If $[\tilde{w}_0] \stackrel{r}{\sim} [\tilde{w}'_0]$ then we say $\Upsilon_{[\tilde{w}_0]}$ and $\Upsilon_{[\tilde{w}'_0]}$ are *reflection equivalent* and write $\Upsilon_{[\tilde{w}_0]} \stackrel{r}{\sim} \Upsilon_{[\tilde{w}'_0]}$. Also, $\Upsilon_{[[\tilde{w}_0]]} := \{\Upsilon_{[\tilde{w}_0]} \mid [\tilde{w}_0] \stackrel{r}{\sim} [\tilde{w}'_0]\}$ is called an *r-cluster point*.

Now we shall observe what the equivalent classes in the same r-cluster point share:

Definition 3.13. Let σ be a Dynkin diagram automorphism and k be the number of σ -orbits of the index set I . Take a sequence of σ -orbits $\mathcal{O} = (o_1, o_2, \dots, o_k)$ where $o_i \neq o_j$ for $1 \leq i < j \leq k$. For a reduced expression $\tilde{w}_0 = (s_{i_1}, \dots, s_{i_N})$ of w_0 , the σ -composition of $[\tilde{w}_0]$ associated to \mathcal{O} is

$$(c_1, c_2, \dots, c_k) \in \mathbb{N}^k \quad \text{where } c_j = |\{s_{i_t} \mid i_t \in o_j \text{ for some } t \in \mathbb{Z}\}|.$$

Remark 3.14. Depending on the sequence of orbits \mathcal{O} , we get different σ -composition. However, every σ -composition is same up to order of components. Hence, we fix $\mathcal{O} = (o_1, o_2, \dots, o_k)$ satisfying

$$[\text{smallest element in } o_i] < [\text{smallest element in } o_{i+1}]$$

for all $i = 1 \dots, k-1$.

The well definedness of σ -composition follows by the fact that if $\tilde{w}_0 = (s_{i_1}, \dots, s_{i_N})$ and $\tilde{w}'_0 = (s_{i'_1}, \dots, s_{i'_N})$ are in the same commutation class then

$$\#\{i_k \mid i_k = i\} = \#\{i'_k \mid i'_k = i\} \text{ for any } i \in I.$$

Example 3.15. (1) In Example 3.11, the $*$ -composition of $[\tilde{w}_0]$ in (3.5) is

$$(4, 6)$$

since there are 4-many s_i for $i = 1$ or 4 in \tilde{w}_0 and 6-many s_j for $j = 2$ or 3 in \tilde{w}_0 .

(2) Let us take a Dynkin diagram involution σ of D_4 as $\sigma(i) = i$ for $1 \leq i \leq 2$ and $\sigma(3) = 4$. Then σ -composition of $[\tilde{w}_0]$ in Example 2.7 is

$$(4, 4, 4).$$

(3) Let us take a Dynkin diagram automorphism σ of D_4 as $\sigma(2) = 2$, $\sigma(1) = 3$, $\sigma(3) = 4$ and $\sigma(4) = 1$. Then σ -composition of $[\tilde{w}_0]$ for $\tilde{w}_0 = (s_1, s_2, s_3, s_2, s_1, s_2, s_4, s_2, s_1, s_2, s_3, s_2)$ is

$$(6, 6).$$

Proposition 3.16. *If two commutation equivalence classes $[\tilde{w}_0]$ and $[\tilde{w}'_0]$ of w_0 are in the same r -cluster point then σ -compositions of $[\tilde{w}_0]$ and $[\tilde{w}'_0]$ are same.*

Proof. Note that any Dynkin diagram automorphism σ is compatible with $*$, it is enough to show when $\sigma = *$. Let $\tilde{w}_0 = (s_{i_1}, \dots, s_{i_N})$. The only thing we need to show is that $*$ -compositions of $[\tilde{w}_0]$, $r_{i_N}[\tilde{w}_0]$ and $[\tilde{w}_0]r_{i_1}$ are same. If $r_{i_N}[\tilde{w}_0] = [\tilde{w}'_0]$ then $(s_{i_N^*}, s_{i_1}, \dots, s_{i_{N-1}}) \in [\tilde{w}'_0]$. Hence $*$ -compositions of $[\tilde{w}_0]$ and $[\tilde{w}'_0]$ are same. Similarly, $*$ -compositions of $[\tilde{w}_0]r_{i_1}$ and $[\tilde{w}_0]$ are same. Hence we proved the proposition. \square

Example 3.17.

(1) Let \tilde{w}_0 be a reduced expression of w_0 of A_n adapted to

$$Q = \circ \xleftarrow{1} \circ \xleftarrow{2} \cdots \cdots \circ \xleftarrow{n-1} \circ \xleftarrow{n}$$

Let $\sigma = *$. Then the σ -composition of $[\tilde{w}_0]$ consists of $\lceil \frac{n+1}{2} \rceil$ components such that

$$(3.7) \quad \begin{cases} (n+1, \dots, n+1) & \text{if } n \text{ is even} \\ (n+1, \dots, n+1, \frac{n+1}{2}) & \text{if } n \text{ is odd.} \end{cases}$$

It is well known that all the adapted reduced expressions of w_0 are in this r -cluster point and all of equivalent classes in this r -cluster point are adapted to some Dynkin quiver.

- (2) In type A_k ($k \leq 4$) (resp. D_4), there is a one-to-one correspondence between r -cluster points and $*$ -compositions (resp. σ -composition). (See Appendix A for A_4 .)
- (3) In type A_5 , there are at least 8 r -cluster points.

Remark 3.18. The number of commutation classes of reduced expressions $[\tilde{w}_0]$ of type A_n increases exponentially as the n increases [19, A006245]. However, in this paper, we claim that commutation classes in the same cluster point are closely related to each other. Hence classifying cluster points can be an interesting problem. (See Conjecture 1 below.)

Conjecture 1. *As a generalization of Example 3.17 (2), we conjecture that the σ -Coxeter composition classifies all cluster points, where σ is non-trivial.*

4. LABELING OF COMBINATORIAL AR QUIVERS

In this section, we discuss about finding labels of combinatorial AR quivers. For A_n type, there are more efficiency way to find the label of each vertex in Γ_Q than direct computations. Similarly, for the labeling in $\Upsilon_{[\tilde{w}]}$ of other finite types, there exists analogues way to avoid large amount of computations (see Remark 2.2 (1)). We first discuss combinatorial AR quivers of type A and generalize the argument to classical finite types.

Let Q be an AR quiver of type A_n . Recall that the subquiver $B(Q)$ of the repetition quiver $\mathbb{Z}Q$ induces a coordinate of the AR quiver Γ_Q . We denote by $(\alpha)_C$ for $\alpha \in \Phi^+$ the coordinate corresponding to α . On the other hand, we denote $\alpha \in \Phi^+$ by $(a, b)_\Phi$ when $(\alpha)_C = (a, b)$.

Lemma 4.1. [3, 10] *We call the vertex k in the Dynkin quiver Q a left intermediate if Q has the subquiver $\begin{array}{ccc} \circ & \xrightarrow{\quad} & \circ \\ k-1 & & k \end{array}$ and call the vertex k in the Dynkin quiver Q a right intermediate if Q has the subquiver $\begin{array}{ccc} \circ & \xleftarrow{\quad} & \circ \\ k-1 & & k \end{array}$. Then we have the following properties.*

(1) For a simple root α_k , we have

$$(4.1) \quad (\alpha_k)_C = \begin{cases} (k, \xi_k), & \text{if } k \text{ is a sink in } Q, \\ (n+1-k, \xi_k - n + 1), & \text{if } k \text{ is a source in } Q, \\ (1, \xi_k - k + 1), & \text{if } k \text{ is a right intermediate,} \\ (n, \xi_k - n + k), & \text{if } k \text{ is a left intermediate.} \end{cases}$$

(2) If $\beta \rightarrow \alpha$ is an arrow in Γ_Q for $\alpha, \beta \in \Phi^+$ then $(\beta, \alpha) = 1$.

Here ξ is the height function such that $\max\{\xi_k \mid k = 1, \dots, n\} = 0$.

After all, the following theorem shows how to find vertices in Γ_Q associated to a (non-simple) positive root in an efficient way. In order to introduce such methods, we distinguish types of sectional paths in AR quivers.

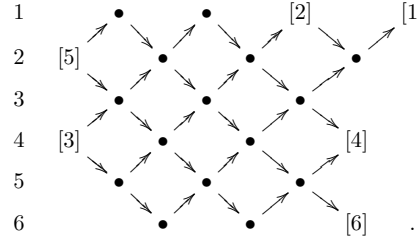
Definition 4.2. (cf. [22, Definition 3.3]) In an AR quiver Γ_Q , a sectional path is called *N-sectional* if the path is upwards. On the other hand, if a sectional path is downwards, it is said to be an *S-sectional* path.

Theorem 4.3. [20] For a positive root $\alpha = \sum_{j=k_1}^{k_2} \alpha_j$ of type A_n , let us call α_{k_1} by the left end and α_{k_2} by the right end of α .

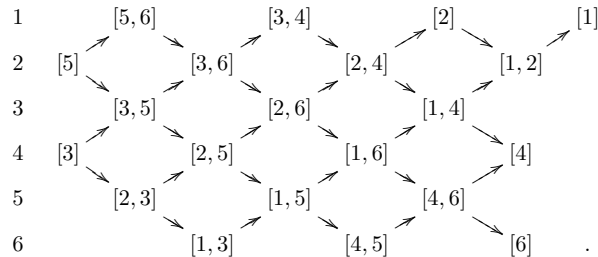
- (a) Every vertex in an N-sectional path in Γ_Q shares its left end.
- (b) Every vertex in an S-sectional path in Γ_Q shares its right end.

Now we know how to draw the AR quiver Γ_Q associated to the Dynkin quiver Q of A_n purely combinatorially. We summarize the procedure with the example below.

Example 4.4. For $Q = \circ_1 \rightarrow \circ_2 \rightarrow \circ_3 \leftarrow \circ_4 \leftarrow \circ_5 \leftarrow \circ_6$ of type A_6 , Lemma 4.1 tells that Γ_Q can be drawn with partial labels:



Finally, using Theorem 4.3, we can complete whole labels of Γ_Q



Now, we generalize the above arguments in Γ_Q . In order to find analogous results in $\Upsilon_{[\tilde{w}]}$ of every finite type, we introduce the notion of *component*:

Definition 4.5. Let $\alpha = \sum_{i \in J} c_i \epsilon_i$ and $\beta = \sum_{i \in J} d_i \epsilon_i$. (Note that J need not to be the same as I .)

- (1) If $i \in I$ satisfies $c_i \neq 0$ then ϵ_i is called a component of α .
- (2) If $i \in I$ satisfies $c_i > 0$ (resp. $c_i < 0$) then ϵ_i is called a *positive component* (resp. *negative component*) of α .
- (3) We say α and β share a component if there is $i \in I$ such that ϵ_i is a positive component to both α and β or a negative component to both α and β .

Remark 4.6. In A_n type, we have $[i, j] = \epsilon_i - \epsilon_{j+1}$. Hence Theorem 4.3 can be restated as follows: An N -sectional (resp. S -sectional) path in Γ_Q shares a positive (resp. negative) component. In short, each sectional path in Γ_Q shares a component.

For type A_n , recall that the action s_i on Φ^+ can be described as follows:

$$(4.2) \quad [j, k] \mapsto \begin{cases} [j, k-1] & \text{if } j < k = i, \\ [j+1, k] & \text{if } j = i < k, \\ [j, k+1] & \text{if } j < k = i-1, \\ [j-1, k] & \text{if } j = i+1 < k, \\ -[i] & \text{if } i = j = k, \\ [j, k] & \text{otherwise.} \end{cases}$$

Then the following lemma is an easy consequence induced from the action of simple reflection on Φ^+ :

Lemma 4.7. Let s_t be a simple reflection on W of type A_n and $[i, j] := \sum_{k=i}^j \alpha_k$ for $i, j \in I$.

- (1) If $s_t[i, k], s_t[j, k] \in \Phi^+$ then $s_t[i, k] = [i', k']$ and $s_t[j, k] = [j', k']$ for some $i', j' \leq k' \in \{1, 2, \dots, n\}$.
- (2) If $s_t[i, j], s_t[i, k] \in \Phi^+$ then $s_t[i, j] = [i', j']$ and $s_t[i, k] = [i', k']$ for some $i' \leq j', k' \in \{1, 2, \dots, n\}$.

Proposition 4.8. Let $\tilde{w} = (s_{i_1}, s_{i_2}, \dots, s_{i_N})$ be a reduced expression of $w \in W$ of type A_n and $\Gamma_{[\tilde{w}]}$ be the combinatorial AR quiver.

- (a) If there is an arrow from $\beta_{k_1}^{\tilde{w}}$ in the l -th residue to $\beta_{k_2}^{\tilde{w}}$ in the $(l-1)$ -th residue, then the corresponding positive roots $[i_1, j_1]$ and $[i_2, j_2]$ to $\beta_{k_1}^{\tilde{w}}$ and $\beta_{k_2}^{\tilde{w}}$ satisfy $i_1 = i_2$.
- (b) If there is an arrow from $\beta_{k_1}^{\tilde{w}}$ in the l -th residue to $\beta_{k_2}^{\tilde{w}}$ in the $(l+1)$ -th residue, then the corresponding positive roots $[i_1, j_1]$ and $[i_2, j_2]$ to $\beta_{k_1}^{\tilde{w}}$ and $\beta_{k_2}^{\tilde{w}}$ satisfy $j_1 = j_2$.

Proof. (a) The arrow from $\beta_{k_1}^{\tilde{w}}$ in the l -th residue to $\beta_{k_2}^{\tilde{w}}$ on the $(l-1)$ -th residue implies that $k_1 > k_2$ and

(4.3) the vertices $\{\beta_k^{\tilde{w}} \mid k = k_2 + 1, \dots, k_1 - 1\}$ in $\Upsilon_{[\tilde{w}]}$ are not on the l -th or $(l-1)$ -th residue.

Denote $\tilde{w}_{\leq k_2-1} = s_{i_1} s_{i_2} \dots s_{i_{k_2-1}}$. Then $[i_1, j_1] = \tilde{w}_{\leq k_2-1} s_{i_{k_2}} s_{i_{k_2+1}} \dots s_{i_{k_1-1}} (\alpha_{i_{k_1}} = [l])$ and $[i_2, j_2] = \tilde{w}_{\leq k_2-1} (\alpha_{i_{k_2}} = [l-1])$. Using (4.2) and (4.3), we have

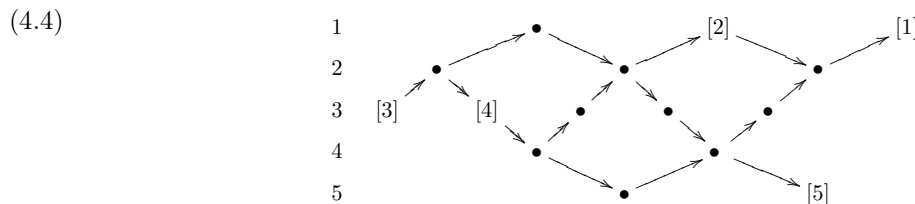
$$s_{i_{k_2}} s_{i_{k_2+1}} \dots s_{i_{k_1-1}} (\alpha_{i_{k_1}}) = [l-1, j]$$

for some $j \geq l$. Then the first assertion follows from Lemma 4.7.

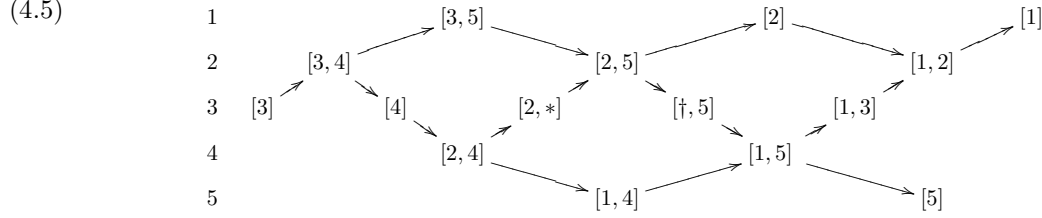
(b) The same argument as that in the proof of (a) works. \square

Theorem 4.9. For any $\Upsilon_{[\tilde{w}]}$ of type A , if two roots α and β are in an N -sectional (resp. S -sectional) path then α and β share their positive (resp. negative) components.

Example 4.10. Let $\tilde{w}_0 = (s_1, s_2, s_1, s_3, s_5, s_4, s_3, s_2, s_3, s_5, s_4, s_1, s_3, s_2, s_3)$ of A_5 . We can easily find that labels of sinks and sources of the quiver $\Upsilon_{[\tilde{w}_0]}$ are [1], [5] and [3]. In addition, one can compute that $\beta_3^{\tilde{w}_0} = \alpha_2$ and $\beta_{13}^{\tilde{w}_0} = -w_0(s_3 s_2(\alpha_3)) = \alpha_4$ easily. Hence $\Gamma_{[\tilde{w}]}$ has the form



By Proposition 4.8, we can find almost all labels of $\Upsilon_{[\tilde{w}_0]}$ as follows:



Finally, we can conclude that $* = 3$ and $\dagger = 4$, since the labels of $\Upsilon_{[\tilde{w}_0]}$ coincide with Φ^+ .

By applying similar arguments of Lemma 4.7 and Proposition 4.8, we have the following theorem for classical finite types ABCD:

Theorem 4.11. *For any $\Upsilon_{[\tilde{w}]}$ of classical finite types, a sectional path shares a component; that is, if two roots α and β are in a sectional path then α and β share one component.*

We do not know whether the above theorem holds for exceptional types E and F_4 or not. However, we can observe the following proposition without consideration of types:

Proposition 4.12. *For α and β in a sectional path in $\Upsilon_{[\tilde{w}]}$, there exists no $\{\gamma_i \mid 1 \leq i \leq r\}$ in the same sectional path such that*

$$\sum_{i=1}^r \gamma_i = \alpha + \beta \quad \text{and} \quad \gamma_i \neq \alpha, \beta \quad \text{for all } 1 \leq i \leq r.$$

Proof. Our assertion for classical types follows from the previous theorem. By considering a sectional path

$$\left\{ w(\alpha_{i_1}) \rightarrow ws_{i_1}(\alpha_{i_2}) \rightarrow \cdots \rightarrow w \prod_{s=1}^k s_{i_s}(\alpha_{i_{k+1}}) \right\}$$

for any w of finite type, one can check our assertion in general. \square

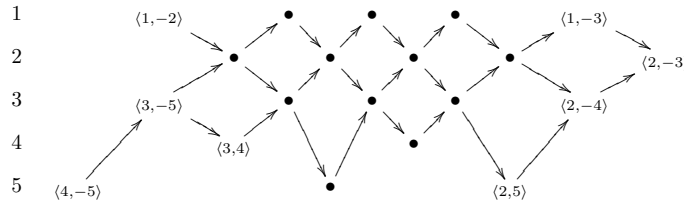
Example 4.13. Recall that the set of positive roots can be expressed as

$$\{\epsilon_i \pm \epsilon_j \mid 1 \leq i < j \leq n\}.$$

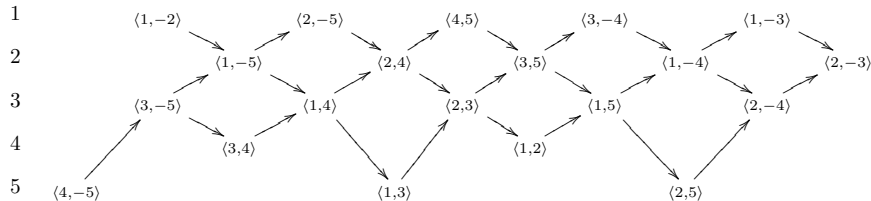
For type D_5 , consider the reduced expression

$$\tilde{w}_0 = (s_2, s_1, s_3, s_2, s_1, s_5, s_3, s_2, s_1, s_4, s_3, s_2, s_1, s_5, s_3, s_2, s_1, s_4, s_3, s_5).$$

The combinatorial AR quiver $\Upsilon_{[\tilde{w}_0]}$ has the form of



Here $\epsilon_i \pm \epsilon_j$ is denoted by $\langle i, \pm j \rangle$. Note that the labels filled in the previous quiver are not hard to find by direct computations. Now, by Theorem 4.11, we can complete to find all labels in $\Upsilon_{[\tilde{w}_0]}$.



Example 4.14. In Example 2.18, $\Upsilon_{[\tilde{w}_0]}$ can be also labeled in terms of orthonormal basis:

$$\Upsilon_{[\tilde{w}_0]} = \begin{array}{cccccccc} 1 & & & & & & & & \\ & \epsilon_1 - \epsilon_2 & & & & & & & \\ & & \searrow & & & & & & \\ 2 & & & \epsilon_1 - \epsilon_3 & & \epsilon_1 + \epsilon_3 & & \epsilon_1 + \epsilon_2 & \\ & & & & \nearrow & & \searrow & & \\ & & & & & & & \epsilon_2 - \epsilon_3 & \\ & & & & & & & & \nearrow & \\ 3 & & & & & & & & & \epsilon_2 + \epsilon_3 & \\ & & & & & & & & & & \searrow & \\ & & & & & & & & & & & 2\epsilon_3 \end{array},$$

which implies Theorem 4.11. Note that, for any reduced expression, every positive root of the form $2\epsilon_i$ has residue n and any positive root has residue n is of the form $2\epsilon_i$.

5. APPLICATION TO KLR ALGEBRAS AND PBW BASES

In this section, we apply our results in previous sections to the representation theory of KLR algebras which were introduced by Khovanov-Lauda [12] and Rouquier [26], independently.

5.1. KLR algebra. Let I be an index set. A *symmetrizable Cartan datum* D is a quintuple $(A, P, \Pi, P^\vee, \Pi^\vee)$ consisting of (a) an integer-valued matrix $A = (a_{ij})_{i,j \in I}$, called the *symmetrizable generalized Cartan matrix*, (b) a free abelian group P , called the *weight lattice*, (c) $\Pi = \{\alpha_i \in P \mid i \in I\}$, called the set of *simple roots*, (d) $P^\vee := \text{Hom}(P, \mathbb{Z})$, called the *coweight lattice*, (e) $\Pi^\vee = \{h_i \mid i \in I\} \subset P^\vee$, called the set of *simple coroots*, satisfying

$$\langle h_i, \alpha_j \rangle = a_{ij} \text{ for all } i, j \in I \text{ and } \Pi \text{ is linearly independent.}$$

The free abelian group $Q := \bigoplus_{i \in I} \mathbb{Z}\alpha_i$ is called the *root lattice*. Set $Q^+ = \sum_{i \in I} \mathbb{Z}_{\geq 0}\alpha_i$.

Let \mathbf{k} be a commutative ring. For $i, j \in I$ such that $i \neq j$ and let us take a family of polynomials $(Q_{ij})_{i,j \in I}$ in $\mathbf{k}[u, v]$ which are of the form

$$(5.1) \quad Q_{ij}(u, v) = \delta(i \neq j) \sum_{\substack{(p,q) \in \mathbb{Z}_{\geq 0}^2 \\ d_i \times p + d_j \times q = -d_i \times a_{ij}}} t_{i,j;p,q} u^p v^q$$

with $t_{i,j;p,q} \in \mathbf{k}$, $t_{i,j;p,q} = t_{j,i;q,p}$ and $t_{i,j;-a_{ij},0} \in \mathbf{k}^\times$. Thus we have $Q_{i,j}(u, v) = Q_{j,i}(v, u)$.

We denote by $\mathfrak{S}_n = \langle \mathfrak{s}_1, \dots, \mathfrak{s}_{n-1} \rangle$ the symmetric group on n letters, where $\mathfrak{s}_i := (i, i+1)$ is the transposition of i and $i+1$. Then \mathfrak{S}_n acts on I^n by place permutations.

For $n \in \mathbb{Z}_{\geq 0}$ and $\beta \in Q^+$ such that $\text{ht}(\beta) = n$, we set

$$I^\beta = \{\nu = (\nu_1, \dots, \nu_n) \in I^n \mid \alpha_{\nu_1} + \dots + \alpha_{\nu_n} = \beta\}.$$

Definition 5.1. For $\beta \in Q^+$ with $|\beta| = n$, the *Khovanov-Lauda-Rouquier algebra* $R(\beta)$ at β associated with a symmetrizable Cartan datum $(A, P, \Pi, P^\vee, \Pi^\vee)$ and a matrix $(Q_{ij})_{i,j \in I}$ is the \mathbb{Z} -gradable \mathbf{k} -algebra generated by the elements $\{e(\nu)\}_{\nu \in I^\beta}$, $\{x_k\}_{1 \leq k \leq n}$, $\{\tau_m\}_{1 \leq m \leq n-1}$ satisfying the following defining relations:

$$\begin{aligned} e(\nu)e(\nu') &= \delta_{\nu, \nu'} e(\nu), \quad \sum_{\nu \in I^\beta} e(\nu) = 1, \quad x_k x_m = x_m x_k, \quad x_k e(\nu) = e(\nu) x_k, \\ \tau_m e(\nu) &= e(\mathfrak{s}_m(\nu)) \tau_m, \quad \tau_k \tau_m = \tau_m \tau_k \text{ if } |k-m| > 1, \quad \tau_k^2 e(\nu) = Q_{\nu_k, \nu_{k+1}}(x_k, x_{k+1}) e(\nu), \\ (\tau_k x_m - x_{\mathfrak{s}_k(m)} \tau_k) e(\nu) &= \begin{cases} -e(\nu) & \text{if } m = k, \nu_k = \nu_{k+1}, \\ e(\nu) & \text{if } m = k+1, \nu_k = \nu_{k+1}, \\ 0 & \text{otherwise,} \end{cases} \\ (\tau_{k+1} \tau_k \tau_{k+1} - \tau_k \tau_{k+1} \tau_k) e(\nu) &= \begin{cases} \frac{Q_{\nu_k, \nu_{k+1}}(x_k, x_{k+1}) - Q_{\nu_k, \nu_{k+1}}(x_{k+2}, x_{k+1})}{x_k - x_{k+2}} e(\nu) & \text{if } \nu_k = \nu_{k+2}, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

For $\beta, \gamma \in Q^+$ with $\text{ht}(\beta) = m$, $\text{ht}(\gamma) = n$, set

$$e(\beta, \gamma) = \sum_{\substack{\nu \in I^{m+n}, \\ (\nu_1, \dots, \nu_m) \in I^\beta, \\ (\nu_{m+1}, \dots, \nu_{m+n}) \in I^\gamma}} e(\nu) \in R(\beta + \gamma).$$

Then $e(\beta, \gamma)$ is an idempotent. Let

$$(5.2) \quad R(\beta) \otimes R(\gamma) \rightarrow e(\beta, \gamma)R(\beta + \gamma)e(\beta, \gamma)$$

be the \mathbf{k} -algebra homomorphism given by

$$\begin{aligned} e(\mu) \otimes e(\nu) &\mapsto e(\mu * \nu) \quad (\mu \in I^\beta), \\ x_k \otimes 1 &\mapsto x_k e(\beta, \gamma) \quad (1 \leq k \leq m), \quad 1 \otimes x_k \mapsto x_{m+k} e(\beta, \gamma) \quad (1 \leq k \leq n), \\ \tau_k \otimes 1 &\mapsto \tau_k e(\beta, \gamma) \quad (1 \leq k < m), \quad 1 \otimes \tau_k \mapsto \tau_{m+k} e(\beta, \gamma) \quad (1 \leq k < n), \end{aligned}$$

where $\mu * \nu$ is the concatenation of μ and ν ; i.e., $\mu * \nu = (\mu_1, \dots, \mu_m, \nu_1, \dots, \nu_n)$.

For a $R(\beta)$ -module M and a $R(\gamma)$ -module N , we define the *convolution product* $M \circ N$ by

$$M \circ N := R(\beta + \gamma)e(\beta, \gamma) \otimes_{R(\beta) \otimes R(\gamma)} (M \otimes N).$$

For a graded $R(\beta)$ -module $M = \bigoplus_{k \in \mathbb{Z}} M_k$, we define $qM = \bigoplus_{k \in \mathbb{Z}} (qM)_k$, where

$$(qM)_k = M_{k-1} \quad (k \in \mathbb{Z}).$$

We call q the *grading shift functor* on the category of graded $R(\beta)$ -modules.

Let $\text{Rep}(R(\beta))$ be the category consisting of finite dimensional graded $R(\beta)$ -modules and $[\text{Rep}(R(\beta))]$ be the Grothendieck group of $\text{Rep}(R(\beta))$. Then $[\text{Rep}(R)] := \bigoplus_{\beta \in \mathbf{Q}^+} [\text{Rep}(R(\beta))]$ has a natural $\mathbb{Z}[q, q^{-1}]$ -algebra structure induced by the convolution product \circ and the grading shift functor q . In this paper, we usually ignore grading shifts.

For an $R(\beta)$ -module M and an $R(\gamma_k)$ -module M_k ($1 \leq k \leq n$), we denote by

$$M^{\circ 0} := \mathbf{k}, \quad M^{\circ r} = \underbrace{M \circ \dots \circ M}_r, \quad \bigcirc_{k=1}^n M_k = M_1 \circ \dots \circ M_n.$$

Theorem 5.2 ([12, 26]). *For a given symmetrizable Cartan datum \mathbf{D} , let $U_{\mathbb{Z}[q, q^{-1}]}(\mathfrak{g})^\vee$ the dual of the integral form of the negative part of the quantum group $U_q(\mathfrak{g})$ associated with \mathbf{D} and R be the KLR algebra associated with \mathbf{D} and $(Q_{ij}(u, v))_{i, j \in I}$. Then we have*

$$(5.3) \quad U_{\mathbb{Z}[q, q^{-1}]}^-(\mathfrak{g})^\vee \simeq [\text{Rep}(R)].$$

From now on, we shall deal with the representation theory of KLR algebras which are associated to the Cartan matrix \mathbf{A} of finite types.

Definition 5.3. [18, §2.1]. For a convex total order $<$ on $\Phi(w)$, a pair (α, β) with $\alpha < \beta$ is called a *minimal pair* of $\gamma \in \Phi(w)$ with respect to the convex total order $<$ if

- $\gamma = \alpha + \beta \in \Phi(w)$,
- there exist no pair $(\alpha', \beta') \in (\Phi(w))^2$ such that $\gamma = \alpha' + \beta'$ and $\alpha < \alpha' < \gamma < \beta' < \beta$.

Convention 5.4. For a reduced expression \tilde{w} of $w \in \mathbf{W}$, we fix a labeling of $\Phi(w)$ as $\{\beta_k^{\tilde{w}} \mid 1 \leq k \leq \ell(w)\}$.

- (i) We identify a sequence $\underline{m}_{\tilde{w}} = (m_1, m_2, \dots, m_{\ell(w)}) \in \mathbb{Z}_{\geq 0}^{\ell(w)}$ with

$$(m_1 \beta_1^{\tilde{w}}, m_2 \beta_2^{\tilde{w}}, \dots, m_{\ell(w)} \beta_{\ell(w)}^{\tilde{w}}) \in (\mathbf{Q}^+)^{\ell(w)}.$$

- (ii) For a sequence $\underline{m}_{\tilde{w}}$ and another reduced expression \tilde{w}' of w , $\underline{m}_{\tilde{w}'}$ is a sequence in $\mathbb{Z}_{\geq 0}^{\ell(w)}$ by considering $\underline{m}_{\tilde{w}}$ as a sequence of positive roots, rearranging with respect to $<_{\tilde{w}'}$ and applying the convention (i).

- (iii) For a sequence $\underline{m}_{\tilde{w}} \in \mathbb{Z}_{\geq 0}^{\ell(w)}$, the weight $\text{wt}(\underline{m}_{\tilde{w}})$ of $\underline{m}_{\tilde{w}}$ is defined by $\sum_{i=1}^{\ell(w)} m_i \beta_i^{\tilde{w}} \in \mathbf{Q}^+$.

We usually drop the script \tilde{w} if there is no fear of confusion.

Definition 5.5 ([18, 22]). For sequences $\underline{m}, \underline{m}' \in \mathbb{Z}_{\geq 0}^{\ell(w)}$, we define an order $\leq_{\tilde{w}}^b$ as follows:

$\underline{m}' = (m'_1, \dots, m'_{\ell(w)}) <_{\tilde{w}}^b \underline{m} = (m_1, \dots, m_{\ell(w)})$ if and only if $\text{wt}(\underline{m}) = \text{wt}(\underline{m}')$ and there exist integers k, s such that $1 \leq k \leq s \leq \ell(w)$, $m'_t = m_t$ ($t < k$), $m'_k < m_k$, and $m'_t = m_t$ ($s < t \leq \ell(w)$), $m'_s < m_s$.

The following order on sequences of positive roots was introduced in [22].

Definition 5.6. [22] For sequences $\underline{m}, \underline{m}' \in \mathbb{Z}_{\geq 0}^{\ell(w)}$, we define an order $\prec_{[\tilde{w}]}^b$ as follows:

(5.4) $\underline{m}' = (m'_1, \dots, m'_{\ell(w)}) \prec_{[\tilde{w}]}^b \underline{m} = (m_1, \dots, m_{\ell(w)})$ if and only if $\underline{m}'_{\tilde{w}'} <_{\tilde{w}'}^b \underline{m}_{\tilde{w}'}$ for all reduced expression $\tilde{w}' \in [\tilde{w}]$.

Note that $\prec_{[\tilde{w}]}^b$ is far coarser than $<_{\tilde{w}}^b$.

Definition 5.7. A pair (α, β) of positive roots is $[\tilde{w}_0]$ -simple if there exists no sequence $\underline{m} \in \mathbb{Z}_{\geq 0}^N$ such that

(5.5)
$$\underline{m} \prec_{[\tilde{w}_0]}^b (\alpha, \beta).$$

For a module M , we denote by $\text{hd}(M)$ the head of M and by $\text{soc}(M)$ the socle of M .

Theorem 5.8. [6, 18] Let R be the KLR algebra corresponding to a Cartan matrix \mathbf{A} of finite type. For each positive root $\beta \in \Phi^+$, there exists a simple module $S_{\tilde{w}_0}(\beta)$ satisfying the following properties:

- (a) $S_{\tilde{w}_0}(\beta)^{\circ m}$ is a simple $R(m\beta)$ -module.
- (b) For every $\underline{m}_{\tilde{w}_0} \in \mathbb{Z}_{\geq 0}^{\ell(w_0)}$, there exists a non-zero R -module homomorphism

(5.6)
$$\begin{aligned} \mathbf{r}_{\underline{m}} : \vec{S}_{\tilde{w}_0}(\underline{m}) &:= S_{\tilde{w}_0}(\beta_1)^{\circ m_1} \circ \dots \circ S_{\tilde{w}_0}(\beta_{\ell(w_0)})^{\circ m_{\ell(w_0)}} \\ &\rightarrow \overleftarrow{S}_{\tilde{w}_0}(\underline{m}) := S_{\tilde{w}_0}(\beta_{\ell(w_0)})^{\circ m_{\ell(w_0)}} \circ \dots \circ S_{\tilde{w}_0}(\beta_1)^{\circ m_1}. \end{aligned}$$

such that

$$\text{Hom}_{R(\text{wt}(\underline{m}))}(\vec{S}_{\tilde{w}_0}(\underline{m}), \overleftarrow{S}_{\tilde{w}_0}(\underline{m})) = \mathbf{k} \cdot \mathbf{r}_{\underline{m}}$$

and

$$\text{Im}(\mathbf{r}_{\underline{m}}) \simeq \text{hd}\left(\vec{S}_{\tilde{w}_0}(\underline{m})\right) \simeq \text{soc}\left(\overleftarrow{S}_{\tilde{w}_0}(\underline{m})\right) \text{ is simple.}$$

- (c) For any sequence $\underline{m}_{\tilde{w}_0} \in \mathbb{Z}_{\geq 0}^{\ell(w_0)}$, we have

(5.7)
$$[\vec{S}_{\tilde{w}_0}(\underline{m})] \in [\text{Im}(\mathbf{r}_{\underline{m}})] + \sum_{\substack{\underline{m}' <_{\tilde{w}_0}^b \underline{m} \\ \text{wt}(\underline{m}') = \text{wt}(\underline{m})}} \mathbb{Z}_{\geq 0}[q^{\pm 1}][\text{Im}(\mathbf{r}_{\underline{m}'})].$$

- (d) For any sequence $\underline{m}_{\tilde{w}_0} \in \mathbb{Z}_{\geq 0}^{\ell(w_0)}$, $\vec{S}_{\tilde{w}_0}(\underline{m})$ has a unique simple head $\text{hd}\left(\vec{S}_{\tilde{w}_0}(\underline{m})\right)$ and $\vec{S}_{\tilde{w}_0}(\underline{m}) \not\cong$

$\vec{S}_{\tilde{w}_0}(\underline{m}')$ if $\underline{m} \neq \underline{m}'$.

- (e) For every simple R -module M , there exists a unique sequence $\underline{m} \in \mathbb{Z}_{\geq 0}^N$ such that $M \simeq \text{Im}(\mathbf{r}_{\underline{m}}) \simeq \text{hd}\left(\vec{S}_{\tilde{w}_0}(\underline{m})\right)$.

- (f) For any minimal pair $(\beta_k^{\tilde{w}_0}, \beta_l^{\tilde{w}_0})$ of $\beta_j^{\tilde{w}_0} = \beta_k^{\tilde{w}_0} + \beta_l^{\tilde{w}_0}$ with respect to $<_{\tilde{w}_0}$, there exists an exact sequence

$$0 \rightarrow S_{\tilde{w}_0}(\beta_j) \rightarrow S_{\tilde{w}_0}(\beta_k) \circ S_{\tilde{w}_0}(\beta_l) \xrightarrow{\mathbf{r}_{\underline{m}}} S_{\tilde{w}_0}(\beta_l) \circ S_{\tilde{w}_0}(\beta_k) \rightarrow S_{\tilde{w}_0}(\beta_j) \rightarrow 0,$$

where $\underline{m}_{\tilde{w}_0} \in \mathbb{Z}_{\geq 0}^{\ell(w_0)}$ such that $m_k = m_l = 1$ and $m_i = 0$ for all $i \neq k, l$.

Note that the set $\text{Irr}(R)$ of isomorphism classes of all simple R -modules forms a natural basis of $[\text{Rep}(R)]$ and does *not* depend on the choice of reduced expression \tilde{w}_0 of w_0 .

We also note that Theorem 5.8 implies that

- (i) the subset $\vec{S}_{\tilde{w}_0}(R) := \left\{ [\vec{S}_{\tilde{w}_0}(\underline{m})] \mid \underline{m}_{\tilde{w}_0} \in \mathbb{Z}_{\geq 0}^{\ell(w_0)} \right\}$ of isomorphism classes of R -modules forms another basis of $[\text{Rep}(R)]$,
- (ii) $\prec_{\tilde{w}_0}^b$ can be interpreted as a unitriangular matrix which plays the role of the transition matrix between $\vec{S}_{\tilde{w}_0}(R)$ and $\text{Irr}(R)$ for *any* reduced expression \tilde{w}_0 of w_0 .

5.2. $\vec{S}_{[\tilde{w}_0]}(R)$ and $\prec_{[\tilde{w}_0]}^b$. In this subsection, we will apply the observations in the previous sections to the representation theory of KLR-algebras and PBW-bases:

Theorem 5.9. [22, Theorem 5.13] *For any \tilde{w}_0 of w_0 and $\underline{m}_{\tilde{w}_0} \in \mathbb{Z}_{\geq 0}^{\ell(w_0)}$, we can define the module $\vec{S}_{[\tilde{w}_0]}(\underline{m})$; i.e.,*

$$\vec{S}_{\tilde{w}_0}(\underline{m}_{\tilde{w}_0}) \simeq \vec{S}_{\tilde{w}'_0}(\underline{m}_{\tilde{w}'_0}) \quad \text{for all } \tilde{w}_0, \tilde{w}'_0 \in [\tilde{w}_0].$$

Moreover, we can refine the transition matrix between $\vec{S}_{[\tilde{w}_0]}(R) := \{ \vec{S}_{[\tilde{w}_0]}(\underline{m}) \mid \underline{m} \in \mathbb{Z}_{\geq 0}^{\ell(w_0)} \}$ and $\text{Irr}(R)$ by replacing $\prec_{\tilde{w}_0}^b$ with the far coarser order $\prec_{[\tilde{w}_0]}^b$.

Remark 5.10. For any $\tilde{w}_0, \tilde{w}'_0 \in [\tilde{w}_0]$, Theorem 5.8 tells that

$$S_{\tilde{w}_0}(\beta) \simeq S_{\tilde{w}'_0}(\beta) \quad \text{for all } \beta \in \Phi^+.$$

Thus we denote by $S_{[\tilde{w}_0]}(\beta)$ the simple module $S_{\tilde{w}'_0}(\beta)$ for any $\tilde{w}'_0 \in [\tilde{w}_0]$ and $\beta \in \Phi^+$.

Proposition 5.11. *Let (α, β) be an incomparable pair of positive roots with respect to the order $\prec_{[\tilde{w}_0]}$. Then (α, β) is $[\tilde{w}_0]$ -simple and we have*

$$S_{[\tilde{w}_0]}(\alpha) \circ S_{[\tilde{w}_0]}(\beta) \simeq S_{[\tilde{w}_0]}(\beta) \circ S_{[\tilde{w}_0]}(\alpha) \text{ is simple.}$$

Proof. By Corollary 2.20, there exist $\tilde{w}'_0 \in [\tilde{w}_0]$ and $k \in \mathbb{Z}_{\geq 1}$ such that $\alpha = \beta_k^{\tilde{w}'_0}$ and $\beta = \beta_{k+1}^{\tilde{w}'_0}$. Let us denote by (α, β) the sequence $\underline{m}_{\tilde{w}'_0}$ such that $m_k = m_{k+1} = 1$ and $m_i = 0$ for all $i \neq k, k+1$. Then there is no $\underline{m}_{\tilde{w}'_0}$ such that $\underline{m} \prec_{\tilde{w}'_0}^b (\alpha, \beta)$. Hence Theorem 5.8 (c) tells that the composition series of $S_{[\tilde{w}_0]}(\alpha) \circ S_{[\tilde{w}_0]}(\beta)$ consists of $\text{Im}(\mathbf{r}_{(\alpha, \beta)})$. Then our assertion follows from Theorem 5.8 (b). \square

Remark 5.12. Proposition 5.11 tells that $S_{[\tilde{w}_0]}(\alpha)$ and $S_{[\tilde{w}_0]}(\beta)$ commutes up to grading shift (or q -commutes) if α and β are incomparable with respect to $\prec_{[\tilde{w}_0]}$. However, the converse is not true. In Proposition 5.13 below, we will show that for comparable pair (α, β) , $S_{[\tilde{w}_0]}(\alpha)$ and $S_{[\tilde{w}_0]}(\beta)$ commutes if they lie in the same sectional path in $\Upsilon_{[\tilde{w}_0]}$, which is a generalization of [22, Proposition 4.2].

Proof of Theorem 5.9. By proposition 5.11, the isomorphism class of the module $\vec{S}_{\tilde{w}_0}(\underline{m}_{\tilde{w}_0})$ and the homomorphism $\mathbf{r}_{\underline{m}_{\tilde{w}_0}}$ does not depend on the choice of $\tilde{w}_0 \in [\tilde{w}_0]$. Thus our first assertion follows. By applying the first assertion to (5.7) for all $\tilde{w}'_0 \in [\tilde{w}_0]$, we have

$$\vec{S}_{[\tilde{w}_0]}(\underline{m}) \in [\text{Im}(\mathbf{r}_{\underline{m}})] + \sum_{\substack{\underline{m}' \prec_{\tilde{w}'_0}^b \underline{m} \text{ for all } \tilde{w}'_0 \in [\tilde{w}_0] \\ \text{wt}(\underline{m}') = \text{wt}(\underline{m})}} \mathbb{Z}_{\geq 0}[q^{\pm 1}][\text{Im}(\mathbf{r}_{\underline{m}'})].$$

Thus our second assertion follows from the definition of $\prec_{[\tilde{w}_0]}^b$; that is,

$$(5.8) \quad \vec{S}_{[\tilde{w}_0]}(\underline{m}) \in [\text{Im}(\mathbf{r}_{\underline{m}})] + \sum_{\substack{\underline{m}' \prec_{[\tilde{w}_0]}^b \underline{m} \\ \text{wt}(\underline{m}') = \text{wt}(\underline{m})}} \mathbb{Z}_{\geq 0}[q^{\pm 1}][\text{Im}(\mathbf{r}_{\underline{m}'})]. \quad \square$$

REFERENCES

1. M. Auslander, I. Reiten and S. Smalø, *Representation theory of Artin algebras*, Cambridge studies in advanced mathematics **36**, Cambridge 1995.
2. I. Assem, D. Simson and A. Skowroński, *Elements of the representation theory of associative algebras. Vol.1*, London Math. Soc. Student Texts **65**, Cambridge 2006.
3. R. Bedard, *On commutation classes of reduced words in Weyl groups*, European J. Combin. **20** (1999), 483–505.
4. G. Birkhoff, *Lattice Theory*, Amer. Math. Soc., Providence, 1967.
5. N. Bourbaki. *Éléments de mathématique. Fasc. XXXIV. Groupes et algèbres de Lie. Chapitres IV–VI*. Actualités Scientifiques et Industrielles, No. 1337. Hermann,
6. J. Brundan, A. Kleshchev and P. J. McNamara, *Homological properties of finite Khovanov-Lauda-Rouquier algebras*, Duke Math. J., **163** (2014), 1353–1404.
7. J. Claxton and P. Tingley, *Young tableaux, multisegments, and PBW bases*, Sémin. Lothar. Combin. 73 (2015), Article B73c
8. P. Gabriel, *Auslander-Reiten sequences and Representation-finite algebras*, Lecture notes in Math., vol. 831, Springer-Verlag, Berlin and New York, (1980), pp.1-71.
9. D. Hernandez and B. Leclerc, *Quantum Grothendieck rings and derived Hall algebras*, arXiv:1109.0862v2 [math.QA], J. Reine Angew. Math. **701** (2015), 77–126.
10. S.-J. Kang, M. Kashiwara and M. Kim, *Symmetric quiver Hecke algebras and R-matrices of quantum affine algebras II*, Duke Math. J. **164**(8), 1549–1602.
11. S. Kato, *Poincaré-Birkhoff-Witt bases and Khovanov-Lauda-Rouquier algebras*, Duke Math. J. **163**, 3 (2014), 619–663.
12. M. Khovanov and A. D. Lauda, *A diagrammatic approach to categorification of quantum groups I*, Represent. Theory **13** (2009), 309–347.
13. H. Krause, *Representations of quivers via reflection functors*, arXiv:0804.1428 (2008).
14. G. Lusztig *Canonical Bases Arising from Quantized Enveloping Algebras*, J. Amer. Math. Soc. Vol. 3, No. 2, (1990), 447-498.
15. ———, *Quantum groups at roots of 1*, Geom. Dedicata, **33** (1990), 89-113.
16. ———, *Introduction to Quantum Groups*, Birkhäuser, 1993.
17. ———, *“Canonical bases and Hall algebras” in Representation Theories and Algebraic Geometry (Montreal, PQ, 1997)*, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci. **514**, Kluwer Acad. Publ., Dordrecht, (1998), 365–399.
18. P. McNamara, *Finite dimensional representations of Khovanov-Lauda-Rouquier algebras I: finite type*, J. Reine Angew. Math. **707** (2015), 103–124.
19. N. J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences*, published electronically at <http://oeis.org>.
20. S.-j. Oh, *Auslander-Reiten quiver of type A and generalized quantum affine Schur-Weyl duality*, Trans. Amer. Math. Soc. **369** (2017), 1895–1933
21. ———, *Auslander-Reiten quiver of type D and generalized quantum affine Schur-Weyl duality*, J. Algebra **460** (2016), 203-252.
22. ———, *Auslander-Reiten quiver and representation theories related to KLR-type Schur-Weyl duality*, arXiv:1509.04949.
23. P. Papi, *A characterization of a special ordering in a root system*, Proc. Amer. Math. **120** (1994), 661–665.
24. C.M. Ringel, *Tame algebras*, Proceedings ICRA 2, Springer LNM 831, (1980), 137–87.
25. ———, *PBW-bases of quantum groups*, J. Reine Angew. Math. **470** (1996), pp. 51–88.
26. R. Rouquier, *2 Kac-Moody algebras*, arXiv:0812.5023 (2008).
27. ———, *Quiver Hecke algebras and 2-Lie algebras*, Algebra Colloq. **19** (2012), no. 2, 359–410.
28. Y. Saito, *PBW basis of quantized universal enveloping algebras*, Publ. RIMS, Kyoto Univ. **30** (1994), 209–232.
29. R. Schiffler, *Quiver Representations*, CMS Books in Mathematics, Springer Verlag, 2014.
30. M. Varagnolo and E. Vasserot, *Canonical bases and KLR algebras*, J. reine angew. Math. **659** (2011), 67–100.
31. D.P. Zelobenko, *Extremal cocycles on Weyl groups*, Funkz. Analiz i ego Pril. **21** (1987), 11–21

DEPARTMENT OF MATHEMATICS, EWhA WOMANS UNIVERSITY, SEOUL 120-750, SOUTH KOREA
E-mail address: sejin092@gmail.com

DEPARTMENT OF MATHEMATICAL SCIENCES, KAIST, 291 DAEHAK-RO YUSEONGGU DAEJEON, 305-701 SOUTH KOREA
E-mail address: uhrisu@gmail.com