New self-dual additive $\mathbb{F}_4$-codes constructed from circulant graphs

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Abstract

In order to construct quantum $[[n, 0, d]]$ codes for $(n, d) = (56, 15)$, $(57, 15)$, $(58, 16)$, $(63, 16)$, $(67, 17)$, $(70, 18)$, $(71, 18)$, $(79, 19)$, $(83, 20)$, $(87, 20)$, $(89, 21)$, $(95, 20)$, we construct self-dual additive $\mathbb{F}_4$-codes of length n and minimum weight d from circulant graphs. The quantum codes with these parameters are constructed for the first time.

1 Introduction

Let $\mathbb{F}_4 = \{0, 1, \omega, \bar{\omega}\}$ be the finite field with four elements, where $\bar{\omega} = \omega^2 = \omega + 1$. An *additive* $\mathbb{F}_4$-code of length n is an additive subgroup of $\mathbb{F}_4^n$. An element of C is called a *codeword* of C. An additive $(n, 2^k)$ $\mathbb{F}_4$-code is an additive $\mathbb{F}_4$-code of length n with 2^k codewords. The (Hamming) weight of a codeword x of C is the number of non-zero components of x. The minimum non-zero weight of all codewords in C is called the *minimum weight* of C.

Let C be an additive $\mathbb{F}_4$-code of length n. The symplectic dual code C^* of C is defined as $\{x \in \mathbb{F}_4^n \mid x * y = 0 \text{ for all } y \in C\}$ under the trace inner product:

$$x * y = \sum_{i=1}^{n} (x_i y_i^2 + x_i^2 y_i)$$

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for $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n) \in \mathbb{F}_4^n$. An additive $\mathbb{F}_4$-code C is called (symplectic) *self-orthogonal* (resp. *self-dual*) if $C \subset C^*$ (resp. $C = C^*$).

Calderbank, Rains, Shor and Sloane [3] gave the following useful method for constructing quantum codes from self-orthogonal additive $\mathbb{F}_4$-codes (see [3] for more details on quantum codes). A self-orthogonal additive $(n, 2^{n-k})$ $\mathbb{F}_4$-code C such that there is no element of weight less than d in $C^* \setminus C$, gives a quantum $[[n, k, d]]$ code, where $k \neq 0$. In addition, a self-dual additive $\mathbb{F}_4$-code of length n and minimum weight d gives a quantum $[[n, 0, d]]$ code. Let $d_{\max}(n, k)$ denote the maximum integer d such that a quantum $[[n, k, d]]$ code exists. It is a fundamental problem to determine the value $d_{\max}(n, k)$ for a given (n, k). A table on $d_{\max}(n, k)$ is given in [3, Table III] for $n \leq 30$, and an extended table is available online [5].

In this note, we construct self-dual additive $\mathbb{F}_4$-codes of length n and minimum weight d for

$$\begin{aligned}(n, d) = {} & (56, 15), (57, 15), (58, 16), (63, 16), (67, 17), \\ & (70, 18), (71, 18), (79, 19), (83, 20), (87, 20), (89, 21), (95, 20). \end{aligned} \tag{1}$$

These codes are obtained from adjacency matrices of some circulant graphs. The above self-dual additive $\mathbb{F}_4$-codes allow us to construct quantum $[[n, 0, d]]$ codes for the (n, d) given in (1). These quantum codes improve the previously known lower bounds on $d_{\max}(n, 0)$ for the above n.

The data of these new quantum codes has already been included in [5]. All computer calculations in this note were performed using Magma [1].

2 Self-dual additive $\mathbb{F}_4$-codes from circulant graphs

A *graph* Γ consists of a finite set V of vertices together with a set of edges, where an edge is a subset of V of cardinality 2. All graphs in this note are simple, that is, graphs are undirected without loops and multiple edges. The *adjacency matrix* of a graph Γ with $V = \{x_1, x_2, \ldots, x_v\}$ is a $v \times v$ matrix $A_\Gamma = (a_{ij})$, where $a_{ij} = a_{ji} = 1$ if $\{x_i, x_j\}$ is an edge and $a_{ij} = 0$ otherwise. Let Γ be a graph and let A_Γ be the adjacency matrix of Γ. Let $C(\Gamma)$ denote the additive $\mathbb{F}_4$-code generated by the rows of $A_\Gamma + \omega I$, where I denotes the identity matrix. Then $C(\Gamma)$ is a self-dual additive $\mathbb{F}_4$-code [4].

Two additive $\mathbb{F}_4$-codes C_1 and C_2 of length n are *equivalent* if there is a map from $S_3^n \rtimes S_n$ sending C_1 onto C_2, where the symmetric group S_n acts on the set of the n coordinates and each copy of the the symmetric group S_3 permutes the non-zero elements $1, \omega, \bar{\omega}$ of the field in the respective coordinate. For any self-dual additive $\mathbb{F}_4$-code C, it was shown in [4, Theorem 6] that there is a graph Γ such that $C(\Gamma)$ is equivalent to C. Using this characterization, all self-dual additive $\mathbb{F}_4$-codes were classified for lengths up to 12 [4, Section 5].

An $n \times n$ matrix is *circulant* if it has the following form:

$$M = \begin{pmatrix} r_1 & r_2 & \cdots & r_{n-1} & r_n \\ r_n & r_1 & \cdots & r_{n-2} & r_{n-1} \\ r_{n-1} & r_n & \ddots & r_{n-3} & r_{n-2} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ r_2 & r_3 & \cdots & r_n & r_1 \end{pmatrix}. \tag{2}$$

Trivially, the matrix M is fully determined by its first row $(r_1, r_2, \ldots, r_n)$. A graph is called *circulant* if it has a circulant adjacency matrix. For a circulant adjacency matrix of the form (2), we have

$$r_1 = 0 \quad \text{and} \quad r_i = r_{n+2-i} \quad \text{for } i = 2, \ldots, \lfloor n/2 \rfloor. \tag{3}$$

Circulant graphs and their applications have been widely studied (see [7] for a recent survey on this subject). For example, it is known that the number of non-isomorphic circulant graphs is known for orders up to 47 (see the sequence A049287 in [8]). In this note, we concentrate on self-dual additive $\mathbb{F}_4$-codes $C(\Gamma)$ generated by the rows of $A_\Gamma + \omega I$, where A_Γ are the adjacency matrices of circulant graphs Γ. These codes were studied, for example, in [6] and [9].

3 New self-dual additive $\mathbb{F}_4$-codes and quantum codes from circulant graphs

3.1 Lengths up to 50

Throughout this section, let Γ denote a circulant graph with adjacency matrix A_Γ. Let $C(\Gamma)$ denote the self-dual additive $\mathbb{F}_4$-code generated by the rows of $A_\Gamma + \omega I$. Let $d^\Gamma_{\max}(n)$ denote the maximum integer d such that a self-dual additive $\mathbb{F}_4$-code $C(\Gamma)$ of length n and minimum weight d exists. Varbanov [9] gave a classification of self-dual additive $\mathbb{F}_4$-codes $C(\Gamma)$ for lengths $n = 13, 14, \ldots, 29, 31, 32, 33$ and determined the values $d^\Gamma_{\max}(n)$ for lengths up to 33.

Table 1: Self-dual additive $\mathbb{F}_4$-codes $C(\Gamma_n)$ of lengths $n = 34, 35, \ldots, 50$

n	$d^\Gamma_{\max}(n)$	Support of the first row of A_{Γ_n}	$d_{\max}(n, 0)$
34	10	2, 3, 6, 8, 9, 27, 28, 30, 33, 34	10–12
35	*10*	2, 4, 6, 7, 10, 27, 30, 31, 33, 35	11–13
36	*11*	2, 3, 4, 5, 7, 9, 13, 14, 24, 25, 29, 31, 33, 34, 35, 36	12–14
37	11	5, 6, 7, 9, 11, 12, 27, 28, 30, 32, 33, 34	11–14
38	12	2, 3, 5, 7, 10, 11, 20, 29, 30, 33, 35, 37, 38	12–14
39	11	2, 4, 5, 6, 7, 10, 11, 30, 31, 34, 35, 36, 37, 39	11–14
40	12	2, 3, 5, 8, 10, 21, 32, 34, 37, 39, 40	12–14
41	12	2, 3, 4, 5, 6, 10, 11, 13, 30, 32, 33, 37, 38, 39, 40, 41	12–15
42	12	2, 3, 13, 15, 16, 18, 21, 22, 23, 26, 28, 29, 31, 41, 42	12–16
43	*12*	3, 4, 7, 9, 10, 12, 33, 35, 36, 38, 41, 42	13–16
44	14	4, 5, 8, 10, 13, 17, 18, 21, 23, 25, 28, 29, 33, 36, 38, 41, 42	14–16
45	13	2, 4, 5, 9, 10, 12, 14, 15, 17, 18, 20, 27, 29, 30, 32, 33, 35, 37, 38, 42, 43, 45	13–16
46	14	4, 5, 7, 8, 9, 10, 11, 12, 13, 14, 15, 17, 19, 24, 29, 31, 33, 34, 35, 36, 37, 38, 39, 40, 41, 43, 44	14–16
47	13	4, 8, 11, 13, 14, 15, 34, 35, 36, 38, 41, 45	13–17
48	14	3, 4, 5, 10, 12, 14, 15, 16, 25, 34, 35, 36, 38, 40, 45, 46, 47	14–18
49	13	4, 5, 7, 8, 9, 10, 13, 14, 37, 38, 41, 42, 43, 44, 46, 47	13–18
50	14	3, 7, 8, 9, 11, 12, 13, 17, 20, 22, 24, 25, 26, 27, 28, 30, 32, 35, 39, 40, 41, 43, 44, 45, 49	14–18

For lengths $n = 13, 14, \ldots, 50$, by exhaustive search, we determined the largest minimum weights $d^\Gamma_{\max}(n)$. In Table 1, for lengths $n = 34, 35, \ldots, 50$, we list $d^\Gamma_{\max}(n)$ and an example of a self-dual additive $\mathbb{F}_4$-code $C(\Gamma_n)$ having minimum weight $d^\Gamma_{\max}(n)$, where the support of the first row of the circulant adjacency matrix A_{Γ_n} is given. Our present state of knowledge about the upper bound $d_{\max}(n, 0)$ on the minimum distance is also listed in the table. For most lengths, the self-dual additive $\mathbb{F}_4$-codes give quantum $[[n, 0, d]]$ codes such that $d = d_{\max}(n, 0)$ or d attains the currently known lower bound on $d_{\max}(n, 0)$; three exceptions (lengths $35, 36$ and 43) are typeset in *italics*.

Note that $d^\Gamma_{\max}(36) = 11$. For lengths $34, 35$ and 36, self-dual additive

Table 2: Weight distribution of $C(\Gamma_{36})$

i	A_i	i	A_i	i	A_i	i	A_i
0	1	17	16 145 280	24	5 144 050 296	31	3 388 554 144
11	1 584	18	51 147 440	25	7 408 053 504	32	1 588 252 581
12	9 936	19	145 391 760	26	9 402 473 952	33	577 571 712
13	52 992	20	370 815 624	27	10 446 604 880	34	152 925 552
14	265 392	21	847 669 248	28	10 073 332 800	35	26 213 616
15	1 168 032	22	1 733 647 968	29	8 336 897 280	36	2 179 688
16	4 578 786	23	3 165 414 336	30	5 836 058 352		

$\mathbb{F}_4$-codes $C(\Gamma)$ with minimum weight 10 were constructed in [9]. For length 36, we found a self-dual additive $\mathbb{F}_4$-code $C(\Gamma_{36})$ of length 36 and minimum weight 11 (see Table 1). The weight distribution of the code $C(\Gamma_{36})$ is listed in Table 2, where A_i denotes the number of codewords of weight i.

Proposition 1. *The largest minimum weight $d^{\Gamma}_{\max}(36)$ among all self-dual additive $\mathbb{F}_4$-codes $C(\Gamma)$ of length 36 from circulant graphs is 11.*

A self-dual additive $\mathbb{F}_4$-code is called Type II if it is even. It is known that a Type II additive $\mathbb{F}_4$-code must have even length. A self-dual additive $\mathbb{F}_4$-code, which is not Type II, is called Type I. Although the following proposition is somewhat trivial, we give a proof for completeness.

Proposition 2. *Let $C(\Gamma)$ be the self-dual additive $\mathbb{F}_4$-code of even length n generated by the rows of $A_\Gamma + \omega I$, where A_Γ is circulant. Let S be the support of the first row of A_Γ. Then $C(\Gamma)$ is Type II if and only if $n/2+1 \in S$.*

Proof. It was shown in [4, Theorem 15] that the codes $C(\Gamma)$ are Type II if and only if all the vertices of Γ have odd degree. For a circulant graph Γ, the degree of the vertices is constant and equals the size of the support S of the first row of A_Γ. From (3) it follows that the size of the support S is odd if and only if $r_{n/2+1} = 1$, i.e., $n/2+1 \in S$. □

Note that (3) also implies that the size of the support S of the first row of A_γ is always even when n is odd, i.e., self-dual codes of odd length from circulant graphs cannot be Type II.

By Proposition 2, the codes $C(\Gamma_n)$ $(n = 38, 40, 42, 44, 46, 48, 50)$ are Type II. In addition, the other codes in Table 1 are Type I. Let $d^{\Gamma}_{\max,I}(n)$

denote the maximum integer d such that a Type I additive $\mathbb{F}_4$-code $C(\Gamma)$ of length n and minimum weight d exists. By exhaustive search, we verified that $d^{\Gamma}_{\max,I}(44) = d^{\Gamma}_{\max}(44) - 2$, $d^{\Gamma}_{\max,I}(n) = d^{\Gamma}_{\max}(n) - 1$ $(n = 38, 40, 46, 48)$ and $d^{\Gamma}_{\max,I}(n) = d^{\Gamma}_{\max}(n)$ $(n = 42, 50)$. For $(n, d) = (42, 12)$ and $(50, 14)$, we list an example of Type I additive $\mathbb{F}_4$-code $C(\Gamma'_n)$ of length n and minimum weight d, where the support of the first row of the circulant adjacency matrix $A_{\Gamma'_n}$ is given in Table 3.

Table 3: Type I additive $\mathbb{F}_4$-codes $C(\Gamma'_n)$ of lengths $42, 50$

n	d	Support of the first row of $A_{\Gamma'_n}$
42	12	2, 3, 5, 6, 8, 11, 12, 13, 31, 32, 33, 36, 38, 39, 41, 42
50	14	5, 6, 7, 9, 10, 11, 12, 20, 32, 40, 41, 42, 43, 45, 46, 47

3.2 Sporadic lengths $n \geq 51$

For lengths $n \geq 51$, by non-exhaustive search, we tried to find self-dual additive $\mathbb{F}_4$-codes $C(\Gamma)$ with large minimum weight, where Γ is a circulant graph. By this method, we found new self-dual additive $\mathbb{F}_4$-codes $C(\Gamma_n)$ of length n and minimum weight d for

$$(n, d) = (56, 15), (57, 15), (58, 16), (63, 16), (67, 17),$$
$$(70, 18), (71, 18), (79, 19), (83, 20), (87, 20), (89, 21), (95, 20).$$

For each self-dual additive $\mathbb{F}_4$-code $C(\Gamma_n)$, the support of the first row of the circulant adjacency matrix A_{Γ_n} is listed in Table 4. Additionally, for $n = 51, \dots, 55, 59, 60, 64, 65, 66, 69, 72, \dots, 78, 81, 82, 84, 88, 94, 100$, we found self-dual additive $\mathbb{F}_4$-codes $C(\Gamma_n)$ from circulant graphs matching the known lower bound on the minimum distance of quantum codes $[[n, 0, d]]$. For the remaining lengths, our non-exhaustive computer search failed to discover a self-dual additive $\mathbb{F}_4$-code from a circulant graph matching the known lower bound.

For the codes $C(\Gamma_n)$ $(n = 56, 57, 58, 63, 67, 70, 71, 79)$, we give in Table 5 part of the weight distribution. Due to the computational complexity, we calculated the number A_i of codewords of weight i for only $i = 15, 16, \dots, 19$. As some basic properties of the graphs Γ_n, we give in Table 6 the valency

Table 4: New self-dual additive $\mathbb{F}_4$-codes $C(\Gamma_n)$

Code	Support of the first row of A_{Γ_n}
$C(\Gamma_{56})$	2, 3, 7, 8, 12, 14, 15, 16, 17, 20, 22, 26, 28, 30, 32, 36, 38, 41, 42, 43, 44, 46, 50, 51, 55, 56
$C(\Gamma_{57})$	7, 8, 10, 12, 17, 18, 22, 23, 24, 35, 36, 37, 41, 42, 47, 49, 51, 52
$C(\Gamma_{58})$	2, 3, 7, 10, 13, 14, 15, 17, 21, 25, 27, 29, 30, 31, 33, 35, 39, 43, 45, 46, 47, 50, 53, 57, 58
$C(\Gamma_{63})$	2, 5, 6, 9, 13, 14, 15, 16, 17, 19, 46, 48, 49, 50, 51, 52, 56, 59, 60, 63
$C(\Gamma_{67})$	4, 5, 6, 11, 12, 14, 15, 16, 17, 18, 21, 25, 26, 27, 28, 30, 39, 41, 42, 43, 44, 48, 51, 52, 53, 54, 55, 57, 58, 63, 64, 65
$C(\Gamma_{70})$	2, 6, 7, 8, 11, 12, 13, 14, 15, 17, 19, 20, 21, 22, 23, 24, 28, 29, 30, 32, 33, 35, 36, 37, 39, 40, 42, 43, 44, 48, 49, 50, 51, 52, 53, 55, 57, 58, 59, 60, 61, 64, 65, 66, 70
$C(\Gamma_{71})$	2, 3, 5, 11, 12, 15, 17, 20, 23, 26, 27, 28, 31, 34, 35, 38, 39, 42, 45, 46, 47, 50, 53, 56, 58, 61, 62, 68, 70, 71
$C(\Gamma_{79})$	2, 4, 7, 10, 13, 15, 18, 19, 20, 21, 23, 24, 25, 29, 30, 31, 32, 35, 36, 37, 39, 42, 44, 45, 46, 49, 50, 51, 52, 56, 57, 58, 60, 61, 62, 63, 66, 68, 71, 74, 77, 79
$C(\Gamma_{83})$	3, 4, 5, 7, 9, 11, 14, 19, 20, 21, 22, 23, 24, 27, 28, 30, 31, 32, 33, 34, 36, 38, 41, 44, 47, 49, 51, 52, 53, 54, 55, 57, 58, 61, 62, 63, 64, 65, 66, 71, 74, 76, 78, 80, 81, 82
$C(\Gamma_{87})$	7, 11, 12, 13, 14, 15, 20, 23, 24, 25, 27, 28, 29, 30, 31, 34, 35, 37, 40, 41, 42, 47, 48, 49, 52, 54, 55, 58, 59, 60, 61, 62, 64, 65, 66, 69, 74, 75, 76, 77, 78, 82
$C(\Gamma_{89})$	3, 4, 7, 10, 14, 15, 18, 19, 21, 23, 25, 26, 30, 32, 34, 35, 37, 39, 40, 45, 46, 51, 52, 54, 56, 57, 59, 61, 65, 66, 68, 70, 72, 73, 76, 77, 81, 84, 87, 88
$C(\Gamma_{95})$	4, 5, 6, 11, 12, 14, 15, 18, 19, 26, 27, 28, 30, 31, 32, 33, 34, 35, 36, 38, 40, 42, 43, 45, 47, 50, 52, 54, 55, 57, 59, 61, 62, 63, 64, 65, 66, 67, 69, 70, 71, 78, 79, 82, 83, 85, 86, 91, 92, 93

Table 5: Number A_i of codewords of weight i ($i = 15, 16, \ldots, 19$)

Code	d	A_{15}	A_{16}	A_{17}	A_{18}	A_{19}
$C(\Gamma_{56})$	15	4 032	25 508	173 264	1 124 648	6 839 224
$C(\Gamma_{57})$	15	1 938	18 126	120 783	838 451	5 093 409
$C(\Gamma_{58})$	16		24 882	0	1 205 240	0
$C(\Gamma_{63})$	16		2 142	12 726	113 568	757 575
$C(\Gamma_{67})$	17			2 278	23 785	193 429
$C(\Gamma_{70})$	18				15 260	0
$C(\Gamma_{71})$	18				6 745	43 949
$C(\Gamma_{79})$	19					1 343

$k(\Gamma_n)$, the diameter $d(\Gamma_n)$, the girth $g(\Gamma_n)$, the size $\omega(\Gamma_n)$ of the maximum clique and the order $|\operatorname{Aut}(\Gamma_n)|$ of the automorphism group. With the exception of $n = 53$, the automorphism group is the dihedral group on n points of order $2n$. Note, however, that the notion of equivalence for graphs and codes are different, i. e., the graph invariants are not preserved with respect to code equivalence [2]. By Proposition 2, the codes $C(\Gamma_{58})$ and $C(\Gamma_{70})$ are Type II.

Finally, by the method in [3], the existence of our self-dual additive $\mathbb{F}_4$-codes $C(\Gamma_n)$ yields the following:

Theorem 3. *There are a quantum* $[[n, 0, d]]$ *codes for*

$$(n, d) = (56, 15), (57, 15), (58, 16), (63, 16), (67, 17),\\ (70, 18), (71, 18), (79, 19), (83, 20), (87, 20), (89, 21), (95, 20).$$

The above quantum $[[n, 0, d]]$ codes improve the previously known lower bounds on $d_{\max}(n, 0)$ ($n = 56, 57, 58, 63, 67, 70, 71, 79, 87, 89$). More precisely,

Table 6: Properties of the graphs Γ_n

Graph	$d_{\min}(C(\Gamma_n))$	$k(\Gamma_n)$	$d(\Gamma_n)$	$g(\Gamma_n)$	$\omega(\Gamma_n)$	$\lvert \mathrm{Aut}(\Gamma_n)\rvert$
Γ_{51}	14	24	2	3	6	102
Γ_{52}	14	16	3	3	4	104
Γ_{53}	15	26	2	3	5	1378
Γ_{54}	16	29	2	3	8	108
Γ_{55}	14	14	3	3	4	110
Γ_{56}	15	26	2	3	19	112
Γ_{57}	15	18	2	3	5	114
Γ_{58}	16	25	2	3	7	116
Γ_{59}	15	30	2	3	8	118
Γ_{60}	16	31	2	3	6	120
Γ_{63}	16	20	2	3	5	126
Γ_{64}	16	43	2	3	12	128
Γ_{65}	16	28	2	3	6	130
Γ_{66}	16	33	2	3	6	132
Γ_{67}	17	32	2	3	6	134
Γ_{69}	17	38	2	3	7	138
Γ_{70}	18	45	2	3	10	140
Γ_{71}	18	30	2	3	6	142
Γ_{72}	18	27	2	3	6	144
Γ_{73}	18	40	2	3	8	146
Γ_{74}	18	32	2	3	6	148
Γ_{75}	18	34	2	3	6	150
Γ_{76}	18	37	2	3	8	152
Γ_{77}	18	48	2	3	10	154
Γ_{78}	18	35	2	3	7	156
Γ_{79}	19	42	2	3	8	158
Γ_{81}	19	40	2	3	7	162
Γ_{82}	20	43	2	3	7	164
Γ_{83}	20	46	2	3	9	166
Γ_{84}	20	25	2	3	6	168
Γ_{87}	20	42	2	3	7	174
Γ_{88}	20	37	2	3	6	176
Γ_{89}	21	40	2	3	6	178
Γ_{94}	20	44	2	3	10	188
Γ_{95}	20	50	2	3	7	190
Γ_{100}	20	48	2	3	7	200

we give our present state of knowledge about $d_{\max}(n,0)$ [5]:

$$\begin{aligned} 15 \le d_{\max}(56,0) \le 20, &\quad 15 \le d_{\max}(57,0) \le 20,\\ 16 \le d_{\max}(58,0) \le 20, &\quad 16 \le d_{\max}(63,0) \le 22,\\ 17 \le d_{\max}(67,0) \le 24, &\quad 18 \le d_{\max}(70,0) \le 24,\\ 18 \le d_{\max}(71,0) \le 25, &\quad 19 \le d_{\max}(79,0) \le 28,\\ 20 \le d_{\max}(83,0) \le 29, &\quad 20 \le d_{\max}(87,0) \le 30,\\ 21 \le d_{\max}(89,0) \le 31, &\quad 20 \le d_{\max}(95,0) \le 33. \end{aligned}$$

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