A note on enumerating colored integer partitions

Ran Pan
Department of Mathematics
University of California, San Diego
La Jolla, CA 92093-0112. USA
r1pan@ucsd.edu

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Abstract

In this short note, we give basic enumerative results on colored integer partitions.

1 Introduction

An integer partition of n is a way of writing an integer n as sum of positive integers. For example, $\lambda = (4, 2, 1)$ is a partition of 7. By convention, elements in a partition are written in decreasing order. If $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k)$ is a partition of n, we use the notation $\lambda \vdash n$.

The number of partitions of n is counted by the partition function $P(n)$. For example, $P(4) = 5$ because $4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1$. In general, the generating function of $P(n)$ is as follows,

$$f(x) = 1 + \sum_{n=1}^{\infty} P(n)x^n = \prod_{j=1}^{\infty} \frac{1}{1 - x^j}. \tag{1}$$

Now we want to use k colors to color each element in a partition, that is, we label elements in the partition with $\{1, 2, \cdots, k\}$. For example, there are 6 2-clored partitions of 2. In the matrices, the top rows are labellings of colors and the bottom rows are integer partitions.

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}$$

Here we are not discussing about Ramanujan's n-color partitions. Details of Ramanujan's n-color partitions can be found in [1].

2 k-colored and exact k-colored

From the preceding example, it is clear that number of k-colored partitions is related to the number of parts of a partition. In other words, if a partition λ has m parts, then we have k^m ways to color this partition.

Suppose $P(n, m)$ is the number of partitions of n into m parts, recall that

$$\sum_{n \geq 1, m \geq 1} P(n, m)x^n y^m = \prod_{j=1}^{\infty} \frac{1}{1 - x^j y}.$$

We let $CP_k(n,m)$ denote the number of k-colored partitions of n with m parts and let $CP_k(n)$ denote the number of k-colored partitions of n. Clearly, $CP_k(n,m) = k^m P(n,m)$. Then for the generating function of $CP_k(n,m)$, we have

$$\begin{aligned}\sum_{n\geq 1, m\geq 1} CP_k(n,m)x^n y^m &= \sum_{n\geq 1, m\geq 1} k^m P(n,m)x^n y^m \\ &= \sum_{n\geq 1, m\geq 1} P(n,m)x^n (ky)^m \\ &= \prod_{j=1}^{\infty} \frac{1}{1-kx^j y}.\end{aligned}$$

Setting $y = 1$, we have following theorem.

Theorem 1.

$$1 + \sum_{n\geq 1} CP_k(n)x^n = \prod_{j=1}^{\infty} \frac{1}{1-kx^j},$$

where $CP_k(n)$ is the number of k-colored partitions of n.

$CP_k(n)$	A few initial terms, $n \geq 1$	OEIS
$k = 1$	1, 2, 3, 5, 7, 11, 15, 22, 30, 42 ,$\cdots$	A000041
$k = 2$	2, 6, 14, 34, 74, 166, 350, 746, 1546, $\cdots$	A070933
$k = 3$	3, 12, 39, 129, 399, 1245, 3783, 11514, $\cdots$	A242587
$k = 4$	4, 20, 84, 356, 1444, 5876, 23604, 94852, $\cdots$	A246936

Now we consider the number of partitions that have exact k colors, that is, for any partition, there are exactly k colors that are used. For example, there are only two 2-colored partitions of 2 and they are

$$\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

We let $EP_k(n)$ denote the number of the exact k-colored partitions of n. Then we have the following recursion

$$EP_k(n) = CP_k(n) - \sum_{j=1}^{k-1} \binom{k}{j} EP_j(n)$$

and the base case

$$EP_2(n) = CP_2(n) - \binom{2}{1} EP_1(n) = CP_2(n) - 2CP_1(n),$$

$\{EP_2\}_{n\geq 1} = \{0, 2, 8, 24, 60, 144, 320, \cdots\}$.

Based on the recursion above, we have following corollary.

Corollary 2.

$$EP_k(n) = \sum_{j=0}^{k-1} (-1)^j \binom{k}{j} CP_{k-j}(n),$$

which implies

$$1+\sum_{n\geq 1} EP_k(n)x^n = \sum_{j=0}^{k-1}(-1)^j\binom{k}{j}\prod_{i=1}^{\infty}\frac{1}{1-(k-j)x^i}.$$

For example, when $k=3$,

$$\begin{aligned}1+\sum_{n\geq 1} EP_3(n)x^n &= \sum_{j=0}^{2}(-1)^j\binom{3}{j}\prod_{i=1}^{\infty}\frac{1}{1-(3-j)x^i}\\ &= \prod_{i=1}^{\infty}\frac{1}{1-3x^i}-3\prod_{i=1}^{\infty}\frac{1}{1-2x^i}+3\prod_{i=1}^{\infty}\frac{1}{1-x^i}.\end{aligned}$$

A few initial terms of $EP_3(n)$ for $n\geq 1$ are 0, 0, 6, 42, 198, 780, 2778, $\cdots$.

$EP_k(n)$	A few initial terms, $n \geq 1$	OEIS
$k=2$	0, 2, 8, 24, 60, 144, 320, 702, 1486, $\cdots$	A255970
$k=3$	0, 0, 6, 42, 198, 780, 2778, 9342, 30186, $\cdots$	A255970
$k=4$	0, 0, 0, 24, 264, 1848, 10512, 53184, $\cdots$	A255970
$k=5$	0, 0, 0, 0, 120, 1920, 18840, 146760, $\cdots$	A255970

3 Adjacent parts having different colors

Now we want to color parts in a partition so that adjacent parts have different colors. We let $D_k(n)$ denote the number of k-colored partitions of n satisfying above condition. For example, $D_2(3)=6$. In following matrices, bottom rows are partitions and top rows are labellings of colors.

$$\begin{pmatrix}1\\3\end{pmatrix}\begin{pmatrix}2\\3\end{pmatrix}\begin{pmatrix}1&2\\2&1\end{pmatrix}\begin{pmatrix}2&1\\2&1\end{pmatrix}\begin{pmatrix}1&2&1\\1&1&1\end{pmatrix}\begin{pmatrix}2&1&2\\1&1&1\end{pmatrix}$$

We let $D_k(n,m)$ denote the number of k-colored partitions of n having m parts satisfying the above condition. Similar to the previous section, it's not hard to see that $D_k(m,n) = k(k-1)^{m-1}P(n,m)$, where $P(n,m)$ is the number of partitions of n into m parts. Then for the generating function of $D_k(n,m)$, we have

$$\begin{aligned}\sum_{n\geq 1, m\geq 1} D_k(n,m)x^n y^m &= \sum_{n\geq 1, m\geq 1} = k(k-1)^{m-1}P(n,m)x^n y^m\\ &= \frac{k}{k-1}\sum_{n\geq 1, m\geq 1} P(n,m)x^n\left((k-1)y\right)^m\\ &= \frac{k}{k-1}\prod_{j=1}^{\infty}\frac{1}{1-(k-1)x^j y}.\end{aligned}$$

Setting $y=1$, we have following theorem.

Theorem 3.

$$1+\sum_{n\geq 1} D_k(n)x^n = \frac{k}{k-1}\prod_{j=1}^{\infty}\frac{1}{1-(k-1)x^j},$$

where $D_k(n)$ is the number of k-colored partitions of n so that no adjacent parts have the same color.

Clearly, $D_1(n) = \{1, 1, 1, 1, 1, 1, \cdots\}$. For instance, when $k = 3$, $D_3(n) = \frac{3}{2}CP_2(n)$ and the initial terms of $D_3(n)$ for $n \geq 1$ are 3, 9, 21, 51, 111, 249, 525, 1119, 2319, $\cdots$.

$D_k(n)$	A few initial terms, $n \geq 1$	OEIS
$k = 2$	2, 4, 6, 10, 14, 22, 30, 44, 60, $\cdots$	A139582
$k = 3$	3, 9, 21, 51, 111, 249, 525, 1119, 2319, $\cdots$	not found
$k = 4$	4, 16, 52, 172, 532, 1660, 5044, 15352, $\cdots$	not found
$k = 5$	5, 25, 105, 445, 1805, 7345, 29505, 118565, $\cdots$	not found

Similarly, we have exact k-colored partitions of n so that no adjacent parts have the same color. We let $ED_k(n)$ denote the number of such partitions. Based on the idea of recursion, we have the following corollary.

Corollary 4.

$$ED_k(n) = \sum_{j=0}^{k-1}(-1)^j \binom{k}{j} D_{k-j}(n),$$

which implies

$$1 + \sum_{n\geq 1} ED_k(n)x^n = \sum_{j=0}^{k-1}(-1)^j \binom{k}{j} \frac{k-j}{k-j-1} \prod_{i=1}^{\infty} \frac{1}{1-(k-j-1)x^i}.$$

Clearly, $ED_1(n) = \{1, 1, 1, 1, 1, 1, \cdots\}$.

$ED_k(n)$	A few initial terms, $n \geq 1$	OEIS
$k = 2$	0, 2, 4, 8, 12, 20, 28, 42, 58, $\cdots$	not found
$k = 3$	0, 0, 6, 24, 72, 186, 438, 990, 2142, $\cdots$	not found
$k = 4$	0, 0, 0, 24, 168, 792, 3120, 11136, $\cdots$	not found
$k = 5$	0, 0, 0, 0, 120, 1320, 9240, 52560, $\cdots$	not found

4 Partitions with distinct parts

In this section, we focus on partitions with distinct parts. We let $d(n, m)$ denote the number of partitions of n into m distinct parts, then we recall that

$$1 + \sum_{n\geq 1, m\geq 1} d(n, m)x^n y^m = \prod_{j=1}^{\infty}(1 + x^j y).$$

Similar to previous sections, we can get following results accordingly.

Corollary 5.

$$1 + \sum_{n\geq 1} Cd_k(n)x^n = \prod_{j=1}^{\infty}(1 + kx^j),$$

where $Cd_k(n)$ denotes the number of k-colored partitions of n with distinct parts.

$Cd_k(n)$	A few initial terms, $n \geq 1$	OEIS
$k=1$	1, 1, 2, 2, 3, 4, 5, 6, 8, 10, 12, 15, 18 ,$\cdots$	A000009
$k=2$	2, 2, 6, 6, 10, 18, 22, 30, 42, 66, 78, 110,$\cdots$	A032302
$k=3$	3, 3, 12, 12, 21, 48, 57, 84, 120, 228, $\cdots$	A032308
$k=4$	4, 4, 20, 20, 36, 100, 116, 180, 260, 580, 660, $\cdots$	A261568
$k=5$	5, 5, 30, 30, 55, 180, 205, 330, 480, 1230, 1380, $\cdots$	A261569

We let $Ed_k(n)$ denote the number of the exact k-colored partitions of n with distinct parts. Then we have the following recursion

$$Ed_k(n) = Cd_k(n) - \sum_{j=1}^{k-1} \binom{k}{j} Ed_j(n)$$

and the base case

$$Ed_2(n) = Cd_2(n) - \binom{2}{1} Ed_1(n) = Cd_2(n) - 2Cd_1(n),$$

$\{Ed_2\}_{n\geq 1} = \{0, 0, 2, 2, 4, 10, 12, 18, 26, 46, 54, \cdots\}$.

Corollary 6.

$$Ed_k(n) = \sum_{j=0}^{k-1} (-1)^j \binom{k}{j} Cd_{k-j}(n),$$

which implies

$$1 + \sum_{n\geq 1} Ed_k(n)x^n = \sum_{j=0}^{k-1} (-1)^j \binom{k}{j} \prod_{i=1}^{\infty} \left(1 + (k-j)x^i\right).$$

$Ed_k(n)$	A few initial terms, $n \geq 1$	OEIS
$k=2$	0, 0, 2, 2, 4, 10, 12, 18, 26, $\cdots$	not found
$k=3$	0, 0, 0, 0, 0, 6, 6, 12, 18, 60, $\cdots$	not found
$k=4$	0, 0, 0, 0, 0, 0, 0, 0, 0, 24, 24, 48, 72, 120 1848, 10512, 53184, $\cdots$	not found
$k=5$	0, 0, 0, 0, 0, 0, 0, 0, 0,0, 0, 0, 0, 0, 120, 120, 240, 360, 600, 840, $\cdots$	not found

Now we let $Dd_k(n)$ denote the number of k-colored partitions of n with distinct parts so that no adjacent parts have the same color. For example, $Dd_3(3) = 9$ and they are

$$\begin{pmatrix} 1 \\ 3 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} \begin{pmatrix} 3 \\ 3 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}.$$

Corollary 7.

$$1 + \sum_{n\geq 1} Dd_k(n)x^n = \frac{k}{k-1} \prod_{j=1}^{\infty} \left(1 + (k-1)x^j\right),$$

where $Dd_k(n)$ is the number of k-colored partitions of n with distinct parts so that no adjacent parts have the same color.

Clearly, $Dd_1(n) = \{1, 1, 1, 1, 1, \cdots\}$. For example, $Dd_4(n) = \frac{4}{3}Cd_3(n)$ and then hence a few initial terms of $Dd_3(n)$ for $n \geq 1$ are 4, 4, 16, 16, 28, 64, 76, 112, 160, 304, $\cdots$.

$Dd_k(n)$	A few initial terms, $n \geq 1$	OEIS
$k=2$	2, 2, 4, 4, 6, 8, 10, 12, 16, 20, 24, 30, 36 $,\cdots$	not found
$k=3$	3, 3, 9, 9, 15, 27, 33, 45, 63, 99, 117, 165, $\cdots$	not found
$k=4$	4, 4, 16, 16, 28, 64, 76, 112, 160, 304, 352, 532, $\cdots$	not found
$k=5$	5, 5, 25, 25, 45, 125, 145, 225, 325, 725, 825, $\cdots$	not found

Now we consider to use exact k colors. We use $EDd_k(n)$ to denote the number of exact k-colored partitions of n with distinct parts so that no adjacent parts have the same color. Then we have

Corollary 8.

$$EDd_k(n) = \sum_{j=0}^{k-1} (-1)^j \binom{k}{j} Dd_{k-j}(n),$$

which implies

$$1 + \sum_{n \geq 1} EDd_k(n) x^n = \sum_{j=0}^{k-1} (-1)^j \binom{k}{j} \frac{k-j}{k-j-1} \prod_{i=1}^{\infty} \left(1 + (k-j-1)x^i\right).$$

$EDd_k(n)$	A few initial terms, $n \geq 1$	OEIS
$k=2$	0, 0, 2, 2, 4, 6, 8, 10, 14, 18 $,\cdots$	not found
$k=3$	0, 0, 0, 0, 0, 6, 6, 12, 18, 42, 48, 78, $\cdots$	not found
$k=4$	0, 0, 0, 0, 0, 0, 0, 0, 0, 24, 24, 48, $\cdots$	not found

5 Remarks

Since for k-colored partitions, the coloring can be treated as a word over alphabet $\{1, 2, 3, \cdots, k\}$, we could have more general conditions on colorings. A natural question is that how many k-colored partitions of n such that the coloring avoids some given substring. Duane and Remmel studied pattern matching in colored permutations in [3]. So we could ask how about pattern matching in colored integer partitions.

References

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