How to prove this polynomial always has integer values at all integers

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Abstract

The following problem was posed by user "Kevin" on Mathoverflow. How to prove this polynomial always has integer values at all integers?
$P_m(x) = \sum_{i=0}^m \sum_{j=0}^m \binom{x+j}{j}\binom{x-1}{j}\binom{j}{i}\binom{m}{i}\binom{i}{m-j}\frac{3}{(2i-1)(2j+1)(2m-2i-1)}$.
We provide an answer.

So

$$P_m(x) = \sum_{i=0}^{m}\sum_{j=0}^{m}\binom{x+j}{j}\binom{x-1}{j}\binom{j}{i}\binom{m}{i}\binom{i}{m-j}\frac{3}{(2i-1)(2j+1)(2m-2i-1)}.$$

Our task is to show it takes integer values on integers.

As Kevin explains at

[question 209140](http://mathoverflow.net/q/209140)

$P_m(x)$ is an even polynomial of degree $2m$ and he could show that $xP_m(x)$ always has integer values at all integers.

Folowing Wadim Zudilin we put

$$B_k(x) = \binom{x+k}{2k} + \binom{-x+k}{2k}.$$

For $k \geq 0$ the B_k are even polynomials of degree $2k$ that take integer values on integers. One has $B_k(k) = 1$ for $k \geq 1$, but $B_0(0) = 2$. Further $B_k(i) = 0$ for $|i| < k$. So the matrix

$$(B_k(i))_{0\leq k\leq m}^{0\leq i\leq m}$$

is triangular.

Every even polynomial $f(x)$ of degree $2j$ is clearly a linear combination of $B_0, \ldots, B_j$ and the coefficients are determined by $f(0), \ldots, f(j)$. When $f(0) = 0$ it is actually a linear combination of $B_1, \ldots, B_j$.

Rewrite $P_m(x)$ as

$$P_m(x) = \sum_k d(m,k) B_k(x)$$

with $d(m,k) \in \mathbb{Q}$. As explained by Kevin, $P_m(k)$ vanishes if $m > 2|k| - 2 \geq 0$ because all terms in the sum vanish. It can also be shown that $P_m(0) = 0$ for $m \geq 2$, but that is more tricky. Indeed we will show that $d(m,0) = 0$ for $m \geq 2$.

Note that $P_m(x)$ visibly lies in the local ring $\mathbb{Z}_{(2)}$ for integer x. So it suffices to show that $d(m,k)$ lies in $\mathbb{Z}_{(p)}$ for any odd prime p. In fact we will find that the $d(m,k)$ are integers for $m \geq 1$. And $d(0,0) = 3/2$ lies in $\mathbb{Z}_{(p)}$ for our odd prime p. For m not too large one may simply compute all $d(m,k)$. The matrix

$$(d(m,k))_{0 \leq m \leq 10}^{0 \leq k \leq 10}$$

looks like this

$$\begin{pmatrix}
\frac{3}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 24 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 4 & 118 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 60 & 696 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 12 & 720 & 4824 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 336 & 8288 & 38240 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 60 & 6516 & 95928 & 336822 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2520 & 109872 & 1131732 & 3215544 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 392 & 67904 & 1735320 & 13647840 & 32651544
\end{pmatrix}.$$

We will tacitly use it to deal with small values of m.

We will study the set

$$V_p = \{(m,k) \in \mathbb{Z} \times \mathbb{Z} \mid d(m,k) \in \mathbb{Z}_{(p)}\}.$$

Using a method of Zeilberger we will prove relations between the $d(m,k)$ that were first discovered experimentally. One relation allows us to rewrite $m(m-1)(1+2m)d(m,k)$ in such a manner that we can use the method of Floors described in

[question 26336](http://mathoverflow.net/q/26336).

With that method we show that $m(m-1)(1+2m)d[m,k]$ is an integer multiple of $3m(m-1)$. Together with the relations this will allow us to show that V_p fills all of $\mathbb{Z}\times\mathbb{Z}$ for odd primes p.

Our variables i,j,k,m,n,q will take integer values only.

As in the A=B book [1] we use the convention that $\binom{x}{j}$ is a polynomial in x for fixed j. And it is the zero polynomial if $j<0$. So $\binom{i}{j}$ is defined for all integers i, j. It also vanishes if $j>i\geq 0$. Of course $\binom{i}{j}$ agrees with the usual binomial coefficient if $0\leq j\leq i$.

By inspecting the values at $x=0,\ldots,j$, we see that

$$(-1)^j\binom{x+j}{j}\binom{x-1}{j}-(-1)^{j-1}\binom{x+j-1}{j-1}\binom{x-1}{j-1}$$

equals $(-1)^j\binom{2j}{j}B_j(x)/2$ for $j\geq 0$. Taking the telescoping sum over j gives

$$(-1)^j\binom{x+j}{j}\binom{x-1}{j}=\sum_{k=0}^{j}(-1)^k\binom{2j}{j}B_k(x)/2$$

for $j\geq 0$. (Valid for all j, actually).

This allows us to conclude that

$$d(m,k)=\sum_{i=0}^{m}\sum_{j=k}^{m}\frac{3(-1)^{k+j}\binom{2k}{k}\binom{j}{i}\binom{m}{i}\binom{i}{m-j}}{2(2i-1)(2j+1)(2m-2i-1)}.$$

In particular $d(m,k)=0$ for $m<0$ and for $k>m$. We will see that $m(m-1)d(m,k)$ also vanishes for $2k-2<m$.

Let us use the notation $[\text{statement}]=\begin{cases}1, & \text{if statement is true;}\\ 0, & \text{otherwise.}\end{cases}$

Then

$$d(m,k)=\sum_{i,j}[j\geq k\geq 0]\text{term}(m,k,i,j), \tag{Σij}$$

where

$$\text{term}(m,k,i,j)=[m\geq 0]\frac{3(-1)^{k+j}\binom{2k}{k}\binom{j}{i}\binom{m}{i}\binom{i}{m-j}}{2(2i-1)(2j+1)(2m-2i-1)}.$$

Put

$$\begin{aligned}
&\mathrm{rel1}(m,k) = \\
&\quad -32(3-2k)^2(-k+m+1)(-k+m+2)d(m,k-2) \\
&\quad +4(-k+m+1)\left(2km^2-2(k-1)(8k-9)m+(2k-3)(8(k-2)k+9)\right)d(m,k-1) \\
&\quad +k(-2k+m+2)(-2k+m+3)(-2k+2m+1)d(m,k),
\end{aligned}$$

$$\begin{aligned}
&\mathrm{rel2}(m,k) = \\
&\quad -4\left((m-1)^2-1\right)d(m-1,k-1) \\
&\quad -4(2(k-1)+m+1)(-k+m+1)d(m,k-1) \\
&\quad +k(2k-m-2)d(m,k)
\end{aligned}$$

Key results

- $\mathrm{rel1}(m,k)$ vanishes.
- $m(m-1)d(m,k)$ vanishes for $2k-2<m$.
- $\mathrm{rel2}(m,k)$ vanishes.
- $m(m-1)(2m+1)d(m,k)$ is an integer multiple of $3m(m-1)$.

Before proving the Key results, let us draw conclusions from them. Let $m \geq 2$. As $d(m,0)=0$, we have $P_m(0)=0$ and the $d(m,k)$ are determined by $P_m(1)\ldots,P_m(m)$. Now the integral matrix

$$(B_k(i))_{1\leq k\leq m}^{1\leq i\leq m}$$

is triangular with ones on the diagonal. We conclude that $d(m,k)\in\mathbb{Z}_{(2)}$ for $m\geq 2$.

Let p be a prime, $p\geq 5$, and let $m\geq 2$. If p does not divide $2m+1$, then $d(m,k)\in\mathbb{Z}_{(p)}$ because $m(m-1)(2m+1)d(m,k)\in 3m(m-1)\mathbb{Z}_{(p)}$. Now assume p divides $2m+1$. Then it does not divide $2m+3$, so then $d(m+1,j)\in\mathbb{Z}_{(p)}$ for all j. Also, p does not divide $(m-1)(m+1)$, so it follows from $\mathrm{rel2}(m+1,k+1)=0$ that $d(m,k)\in\mathbb{Z}_{(p)}$. We have shown that $d(m,k)\in\mathbb{Z}_{(p)}$ if p is prime, $p\geq 5$, $m\geq 2$.

Remains $p=3$. Let $m\geq 2$ again.

If 3 does not divide $2m+1$, then $d(m,k)\in 3\mathbb{Z}_{(3)}$ because $m(m-1)(2m+1)d(m,k)\in 3m(m-1)\mathbb{Z}_{(3)}$.

If $m\equiv 1 \bmod 9$, or $m\equiv 7 \bmod 9$, then $(2m+1)/3$ is prime to 3 and $d(m,k)\in\mathbb{Z}_{(3)}$ because $m(m-1)((2m+1)/3)d(m,k)\in m(m-1)\mathbb{Z}_{(3)}$.

If $m \equiv 4 \bmod 9$, then $(m-1)(m+1)/3$ is prime to 3 and $d(m,k) \in \mathbb{Z}_{(3)}$ because $\mathrm{rel2}(m+1,k+1)=0$ shows $((m-1)(m+1)/3)d(m,k)$ is an integer linear combination of the integers $d(m+1,j)/3$.

We conclude that $d(m,k) \in \mathbb{Z}_{(3)}$ for $m \geq 2$. So the $d(m,k)$ are integers for $m \geq 2$ and P_m takes integer values on integers for $m \geq 2$. Recall that P_0, P_1 also take integer values. **Done**.

So we still have to prove the Key results.

First a technical issue. If $x > 0$ then $\binom{x}{j} = \frac{\Gamma(1+x)}{\Gamma(1+j)\Gamma(1+x-j)}$ and the bimeromorphic function

$$f(x,y) = \frac{\Gamma(1+x)}{\Gamma(1+y)\Gamma(1+x-y)}$$

is continuous at (x,j). However, if $i < 0$ then f has an indeterminate value at (i,j). For example, $\binom{i}{i}$ equals 1 if $i \geq 0$, but it vanishes for $i < 0$. At $(-1,-1)$ both 0 and 1 are values of f. Indeed Mathematica can be steered to give either answer.

`Binomial[i, j] /. i -> -1 /. j -> -1` gives 1 and

`Binomial[i, j] /. j -> -1 /. i -> -1` gives 0.

And `FullSimplify[Binomial[i, i] == Binomial[i - 1, i - 1]]` yields `True`. This answer is correct, but it tells only that for generic complex numbers i the identity holds.

Thus we need to make case distinctions when using identities between multimeromorphic functions, explicitly or implicitly, to prove identities involving the $\binom{i}{j}$.

We start proving that $\mathrm{rel1}(m,k)$ vanishes.

As $[j \geq k+1]\big(2(2k+1)\mathrm{term}(m,k,i,j) + (k+1)\mathrm{term}(m,k+1,i,j)\big) = 0$, we get from (Σij) that

$$2(2k+1)d(m,k) + (k+1)d(m,k+1) = \sum_i \mathrm{iterm}(m,k,i) \qquad (\Sigma i)$$

where

$$\mathrm{iterm}(m,k,i) = 2(2k+1)\mathrm{term}(m,k,i,k).$$

Now we use the

```
Fast Zeilberger Package version 3.61
written by Peter Paule, Markus Schorn, and Axel Riese
Copyright 1995-2015, Research Institute for Symbolic Computation (RISC),
Johannes Kepler University, Linz, Austria.
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It suggests to put

$$g(m,k,i) = \frac{3\times 2^{2k+3} m(-2i+m+1)\Gamma\left(k+\frac{3}{2}\right)\binom{k+1}{i-1}\binom{m-1}{k+1}\binom{k+1}{m-i}}{\Gamma\left(\frac{1}{2}\right)\Gamma(k+2)}$$

and show that

$$\begin{aligned}
&-32(1+2k)(3+2k)(k-m)(1+k-m)\mathrm{iterm}(m,k,i)\\
&-4(1+k-m)(57+110k+72k^2+16k^3-34m-46km-16k^2m+4m^2+2km^2)\\
&\qquad\times \mathrm{iterm}(m,k+1,i)\\
&-(2+k)(5+2k-2m)(3+2k-m)(4+2k-m)\mathrm{iterm}(m,k+2,i)\\
&\qquad -g(m,k,i+1)+g(m,k,i)=0
\end{aligned}$$

for $m \geq 0$. So we do that and then sum over i, using (Σi). The g terms drop out by telescoping and we get a relation

$$\begin{aligned}
&-32(1+2k)(3+2k)(k-m)(1+k-m)(2(2k+1)d(m,k)+(k+1)d(m,k+1))\\
&-4(1+k-m)(57+110k+72k^2+16k^3-34m-46km-16k^2m+4m^2+2km^2)\\
&\qquad\times(2(2k+3)d(m,k+1)+(k+2)d(m,k+2))\\
&-(2+k)(5+2k-2m)(3+2k-m)(4+2k-m)\\
&\times(2(2k+5)d(m,k+2)+(k+3)d(m,k+3))\\
&\qquad=0
\end{aligned}$$

valid for all m, as it is obvious for $m<0$. We may rewrite it as a recursion for rel1:

$$2(3+2k)\mathrm{rel1}(m,k+2)+(2+k)\ \mathrm{rel1}(m,k+3)=0.$$

As $d(m,k)$ vanishes for $k>m$, it follows from the recursion that $\mathrm{rel1}(m,k)$ vanishes for all k.

So we have established the vanishing of $\mathrm{rel1}(m,k)$.

Put

$$\mathrm{pterm}(m,x,i,j) = \binom{x+j}{j}\binom{x-1}{j}\binom{j}{i}\binom{m}{i}\binom{i}{m-j}\frac{3}{(2i-1)(2j+1)(2m-2i-1)},$$

so that

$$P_m(x) = \sum_{i,j} \mathrm{pterm}(m,x,i,j).$$

If $k \geq 1$ and pterm$[m,k,i,j]$ is nonzero, then $k-1 \geq j$ and $m \geq j \geq i \geq m-j$. We see that

$$P_m(k) = 0 \text{ if } 0 \leq 2k-2 < m,$$

because all the pterm(m,k,i,j) vanish. In particular we get

$$0 = P_m(1) = \sum_k d(m,k)B_k(1) = 2d(m,0) + d(m,1),$$

and

$$0 = P_m(2) = \sum_k d(m,k)B_k(2) = 2d(m,0) + 4d(m,1) + d(m,2)$$

for $m \geq 3$. So then $d(m,1) = -2d(m,0)$ and $d(m,2) = 6d(m,0)$. Substitute this into rel1$(m,2) = 0$ and you find

$$4m(m-1)(-2m-1)d(m,0) = 0.$$

This means that $d(m,0) = 0$ for $m \geq 3$. As $d(2,0)$ also vanishes, we now know that $m(m-1)P_m(k)$ vanishes if $m > 2|k| - 2$. As the matrix

$$(B_k(i))_{0 \leq k \leq m}^{0 \leq i \leq m}$$

is triangular, we now conclude that

$$m(m-1)d(m,k) \text{ vanishes for } m > 2k-2. \qquad \text{(SSE)}$$

So we have established the vanishing of $m(m-1)d(m,k)$ **for** $m > 2k-2$.

Before turning to rel2(m,k) we compute $d(2k-2,k)$ and $d(2k-3,k)$ for $k \geq 3$. These are the values that help to compute all $d(m,k)$ recursively with the recursion given by rel1(m,k)=0. As $d(2k-2,j)$ vanishes for $j < k$, one has

$$d(2k-2,k) = P_{2k-2}(k) = \text{pterm}(2k-2,k,k-1,k-1)$$

and similarly

$$d(2k-3,k) = P_{2k-3}(k) = \text{pterm}(2k-3,k,k-1,k-2) + \text{pterm}(2k-3,k,k-1,k-1).$$

So we know $d(m,k)$ for $m \geq 2k-3 \geq 3$. By (SSE) we also know $d(m,k)$ for $k \leq 1$ and any m. Using these values we get rel2$(m,k) = 0$ by inspection for $m \geq 2k-3$ or $k \leq 1$. Notice that $(-7+2k)$rel2$(2k-4,k) -$ rel1$(2k-4,k)$ is a combination of the known terms $d(-5+2k,-1+k)$, $d(-4+2k,-2+k)$, $d(-4+2k,-1+k)$. It also vanishes by inspection, so we now have that rel2$(m,k) = 0$ for $m \geq 2k-4$ or $k \leq 1$.

By substituting the definitions and expanding we check that

$$
\begin{aligned}
&(-1+k)(-1+2k-2m)(-3+2k-m)(4-2k+m)\mathrm{rel2}(m,k)\\
&+4(-1+(-1+m)^2)\mathrm{rel1}(m-1,k-1)\\
&-32(5-2k)^2(-2+k-m)(-1+k-m)\mathrm{rel2}(m,k-2)\\
&+4(1-k+m)\\
&\times(-99+16k^3-2m(32+m)-8k^2(11+2m)+2k(81+m(31+m)))\mathrm{rel2}(m,k-1)\\
&-(-1+k)(-4+2k-m)\ \mathrm{rel1}(m,k)\\
&-4(-1+k-m)(-5+2k+m)\mathrm{rel1}(m,k-1)\\
&=0
\end{aligned}
$$

As rel1 vanishes, this leads to the following recursion for rel2.

$$
\begin{aligned}
&(-1+k)(-1+2k-2m)(-3+2k-m)(4-2k+m)\mathrm{rel2}(m,k)\\
&-32(5-2k)^2(-2+k-m)(-1+k-m)\mathrm{rel2}(m,k-2)\\
&+4(1-k+m)\\
&\times(-99+16k^3-2m(32+m)-8k^2(11+2m)+2k(81+m(31+m)))\mathrm{rel2}(m,k-1)\\
&=0
\end{aligned}
$$

As $\mathrm{rel2}(m,k)=0$ for $2k-4\leq m$ or $k\leq 1$, the recursion shows by induction on k that $\mathrm{rel2}(m,k)=0$ for all m, k.

So we have also established the vanishing of $\mathrm{rel2}(m,k)$ and it is time to show the Key result that $m(m-1)(2m+1)d(m,k)$ is an integer multiple of $3m(m-1)$. This is obvious for $m<2$, so we further assume $m\geq 2$. Then we know that $d(m,0)=0$ and we have seen this implies $d(m,k)\in\mathbb{Z}_{(2)}$. So it suffices to show that $m(m-1)(2m+1)d(m,k)\in 3m(m-1)\mathbb{Z}[1/2]$.

Using relation (Σi) we may rewrite $\mathrm{rel1}(m,k)=0$ as

$$
\begin{aligned}
&2(m-1)m(2m+1)d(m,k-1)\\
&+(2-2k+m)(3-2k+m)(1-2k+2m)\sum_i \mathrm{iterm}(m,k-1,i)\\
&+16(3-2k)(-2+k-m)(-1+k-m)\sum_i \mathrm{iterm}(m,k-2,i)\\
&=0
\end{aligned}
$$

We claim that

$$\begin{aligned}&(2-2k+m)(3-2k+m)(1-2k+2m)\mathrm{iterm}(m,k-1,i)\\&+16(3-2k)(-2+k-m)(-1+k-m)\mathrm{iterm}(m,k-2,i)\end{aligned}$$

lies in $3m(m-1)\mathbb{Z}[1/2]$.

That will prove that the $(m-1)m(2m+1)d(m,k-1)$ are integer multiples of $3m(m-1)$.

Put

$$\mathrm{frac1}(m,k,i)=\frac{3(m-1)m\binom{2(k-1)}{k-1}(-2k+2m+1)\binom{k-1}{i}\binom{m}{i}\binom{i}{-k+m+1}}{(2i-1)(2m-2i-1)}$$

and

$$\mathrm{frac2}(m,k,i)=6(k-m-1)\binom{2(k-1)}{k-1}\binom{k-1}{i}\binom{m}{i}\binom{i}{-k+m+1}.$$

Then $\mathrm{frac1}(m,k,i)+\mathrm{frac2}(m,k,i)$ equals

$$\begin{aligned}&(2-2k+m)(3-2k+m)(1-2k+2m)\mathrm{iterm}(m,k-1,i)\\&+16(3-2k)(-2+k-m)(-1+k-m)\mathrm{iterm}(m,k-2,i),\end{aligned}$$

so it suffices to show that $\mathrm{frac1}(m,k,i)/(6m(m-1))$ and $\mathrm{frac2}(m,k,i)/(6m(m-1))$, which make sense for $m\geq 2$, lie in $\mathbb{Z}[1/2]$ for $m\geq 2$. Recall that the Catalan numbers

$$C(i)=\frac{\binom{2i}{i}}{i+1}$$

are integers. See

[A000108](https://oeis.org/A000108)

We now look at $\mathrm{frac1}(m,k,i)/(6m(m-1))$.

If $\mathrm{frac1}(m,k,i)$ is nonzero then $m\geq k-1\geq i\geq m+1-k\geq 0$. We distinguish two cases: $m=k-1\geq i\geq 0$ and $m>k-1\geq i\geq m+1-k\geq 0$.

First let $m=k-1\geq i\geq 0$. If $i=k-1$, then

$$\mathrm{frac1}(m,k,i)/(6m(m-1))=\mathrm{frac1}(k-1,k,k-1)/(6(k-1)(k-2))=C(k-2).$$

Similarly $\mathrm{frac1}(k-1,k,0)/(6(k-1)(k-2))=C(k-2)$.

So we may assume $0<i<m=k-1$. Then

$\mathrm{frac1}(m,k,i)/(6m(m-1))=\mathrm{frac1}(m,m+1,i)/(6m(m-1))$ equals

$$\frac{-(2i-2)!(2m)!(-2i+2m-2)!}{2(i!)^2(2i-1)!((m-i)!)^2(-2i+2m-1)!}$$

and we must show it takes values in $\mathbb{Z}[1/2]$.

This is the kind of expression to which one may apply the method of Floors explained in

[question 26336](http://mathoverflow.net/q/26336).

It is based on

$$\operatorname{ord}_p n! = \left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \dots$$

According to the method it suffices to check that $\text{test}(m, i, 2n+1) \geq 0$ for $n \geq 1$, where

$$\begin{aligned}\text{test}(m, i, q) = \\ & -2\left\lfloor \frac{m-i}{q} \right\rfloor + \left\lfloor \frac{-2i+2m-2}{q} \right\rfloor - \left\lfloor \frac{-2i+2m-1}{q} \right\rfloor \\ & -2\left\lfloor \frac{i}{q} \right\rfloor + \left\lfloor \frac{2i-2}{q} \right\rfloor - \left\lfloor \frac{2i-1}{q} \right\rfloor + \left\lfloor \frac{2m}{q} \right\rfloor .\end{aligned}$$

This is a tedious puzzle. For fixed q the function $\text{test}(m, i, q)$ is periodic of period q in both variables i and m. So for fixed q one may simply compute all values. We do it for $3 \leq q = 2n+1 < 17$. The results are nonnegative. But if q is large we need to be more efficient. If both $q = 2n+1$ and m are fixed, then $\text{test}(m, i, q)$ can only change value where at least one of the Floors jumps as a function of i. So it suffices to sample around the jumping points (modulo q). We know where they are. More specifically, we only need to consider the 15 cases where one of $i-1$, i, $i+1$ lies in $\{0, 1, -1+m, m, -2+m-n, -1+m-n, 1+n\}$. So we can eliminate i at the expense of having 15 cases. Similarly we can eliminate m for each of those cases, ending up with 153 test functions that depend on n only. Each test function is a linear combination of seven Floors. Each of the Floors stabilises after n has reached an easily computable bound. For instance $\left\lfloor -\frac{8}{2n+1} \right\rfloor$ is constant for $n \geq 4$. In fact the bound 5 suffices for all 7×153 Floors. Compute the 153 stable values. They are nonnegative. This solves the puzzle; the check for $3 \leq q = 2n+1 < 17$ was overkill.

So we now turn to the case $m > k-1 \geq i \geq m+1-k \geq 0$. Then

$\text{frac1}(m, k, i)/(6m(m-1)C(i-1)) =$

$$\frac{i!(2k-2)!m!(-2i+2m-2)!(-2k+2m+1)!}{(2i)!(k-1)!(-i+k-1)!(m-i)!(-2i+2m-1)!(-k+m+1)!(2m-2k)!(i+k-m-1)!}$$

We use the method of Floors again to show that $\text{frac1}(m, k, i)/(6m(m-1)C(i-1)) \in \mathbb{Z}[1/2]$. This time we eliminate k, m, i in that order and take $n \geq 6$ as bound where all 13×3508 Floors are stable.

So we have shown that $\mathrm{frac1}(m,k,i)/(6m(m-1))$ lies in $\mathbb{Z}[1/2]$ for $m \geq 2$. Remains showing that $\mathrm{frac2}(m,k,i)/(6m(m-1))$ lies in $\mathbb{Z}[1/2]$ for $m \geq 2$.

If $\mathrm{frac2}(m,k,i)$ is nonzero then $m > k-1 \geq i \geq m+1-k > 0$ and $\mathrm{frac2}(m,k,i)/(6m(m-1))$ equals

$$\frac{-(2k-2)!(m-2)!}{i!(k-1)!(-i+k-1)!(m-i)!(m-k)!(i+k-m-1)!}.$$

This can be treated like the previous case. We eliminate k, m, i in that order and take $n \geq 6$ as bound where all 8×1278 Floors are stable.

Done

References

[1] Marko Petkovšek, Herbert S. Wilf, Doron Zeilberger, A=B. With a foreword by Donald E. Knuth. A K Peters, Ltd., Wellesley, MA, 1996. xii+212 pp. ISBN: 1-56881-063-6