A RECURRENCE FOR AN EXPRESSION INVOLVING DOUBLE FACTORIALS

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ABSTRACT. We find a closed-form solution to the recurrence $z_{n+2} = \frac{1}{z_{n+1}} + z_n$, where $n \in \mathbb{Z}_{\geq 1}$ and $z_1 \in \mathbb{R}_{>0},\ z_2 \in \mathbb{R}_{>0}$. As a corollary, we derive an alternate proof of a recurrence for an expression involving double factorials which first appeared in the 2004 Putnam Contest. We subsequently pose several open questions suitable for motivated undergraduate students.

1. Recurrence Relation

The general recurrence we wish to solve is:

$$z_1 \in \mathbb{R}_{>0},\ z_2 \in \mathbb{R}_{>0},\ z_{n+2} = \frac{1}{z_{n+1}} + z_n. \tag{1.0.1}$$

The solution will be shown to be (for $n \geq 3$ and $\epsilon(n)$ defined in 1.1)

$$z_n = z_{\epsilon(n)} \cdot \prod_{k=1}^{\lceil \frac{n}{2} \rceil - 1} \frac{n - 2k + z_1 z_2}{n - 2k - 1 + z_1 z_2} \tag{1.0.2}$$

As a corollary we obtain another proof of

$$\forall n \in \mathbb{Z}_{\geq 3}, \frac{n!!}{(n-1)!!} = \frac{1}{\left(\frac{(n-1)!!}{(n-2)!!}\right)} + \frac{(n-2)!!}{(n-3)!!}, \tag{1.0.3}$$

which is a result originally coming from Problem A3 of the 2004 Putnam Contest, according to [5].

Not only does equation 1.0.1 bear a vague similarity to the one defining the Fibonacci numbers, but the sequence we derive will conveniently satisfy an identity similar to Catalan's Identity for the Fibonacci numbers.

The ratio $\frac{(n+1)!!}{(n)!!}$ arose in [2], wherein it is shown that the ratio's reciprocal (up to a constant factor) is equal to the average distance from a random point in the unit n-sphere along a random ray to the intersection of that ray with the sphere.

To address the proposition, we'll first need to develop a few tools.

Definition 1.1. For $n \in \mathbb{Z}$ let

$$\epsilon(n) = \begin{cases} 2 & n \equiv 0 \pmod 2 \\ 1 & n \equiv 1 \pmod 2 \end{cases}$$

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Definition 1.2. Let $\Gamma_1 = z_1$ (from eqn. 1.0.1), $\Gamma_2 = z_2$.

For $j \in \mathbb{Z}_{\geq 1}$ let

$$G(j) = \frac{j + \Gamma_1\Gamma_2}{j - 1 + \Gamma_1\Gamma_2}$$

and for $n \in \mathbb{Z}_{\geq 3}$ let

$$\Gamma_n = \Gamma_{\epsilon(n)} \cdot \prod_{k=1}^{\lceil \frac{n}{2} \rceil - 1} G(n - 2k)$$

.

Remark 1.1. Note that $G(j) > 0$ and therefore by closure of $\mathbb{R}_{>0}$ under multiplication it follows that $\Gamma_n > 0$.

Then our proposed solution to (1.0.1) can be stated as

Proposition 1.2 (Translated Double Factorial Recurrence)**.** $\{\Gamma_n\}_{n \geq 1}$ *as defined in (1.2) solves the recurrence (1.0.1). That is, letting* $\Gamma_1 = z_1 \in \mathbb{R}_{>0}$ *and letting* $\Gamma_2 = z_2 \in \mathbb{R}_{>0}$, *then we have* $\Gamma_n = \frac{1}{\Gamma_{n-1}} + \Gamma_{n-2}$ $(\forall n \in \mathbb{Z}_{\geq 3})$.

Lemma 1.3. *First, notice that* $(\forall n \in \mathbb{Z}_{\geq 3})$, $\Gamma_n = G(n-2) \cdot \Gamma_{n-2}$

Proof. Since $\lceil \frac{n}{2} \rceil - 2 = \lceil \frac{n-2}{2} \rceil - 1$ and $\epsilon(n-2) = \epsilon(n)$, it follows that for $n \geq 3$,

$$\begin{aligned}
\Gamma_n &= \Gamma_{\epsilon(n)} \cdot \prod_{k=1}^{\lceil \frac{(n)}{2} \rceil - 1} G(n - 2k) \\
&= \Gamma_{\epsilon(n-2)} \cdot G(n-2) \cdot \prod_{k=1}^{\lceil \frac{(n-2)}{2} \rceil - 1} G((n-2) - 2k) \\
&= G(n-2) \cdot \left(\Gamma_{\epsilon(n-2)} \cdot \prod_{k=1}^{\lceil \frac{(n-2)}{2} \rceil - 1} G((n-2) - 2k) \right) \\
&= G(n-2) \cdot \Gamma_{n-2}
\end{aligned}$$

□

The proof of the proposition then hinges on a Catalan-type identity satisfied by consecutive elements of the Γ_* sequence.

Lemma 1.4. $\forall n \geq 2$, $\Gamma_n\Gamma_{n-1} = n - 2 + \Gamma_1\Gamma_2$.

Proof. Proceed by induction; we prove two base cases: firstly, for n = 2,

$$\Gamma_2\Gamma_1 = 2 - 2 + \Gamma_2\Gamma_1$$

Secondly, for n = 3,

$$\Gamma_3\Gamma_2 = \left(\frac{1 + \Gamma_1\Gamma_2}{\Gamma_1\Gamma_2} \Gamma_1 \right) \Gamma_2 = 1 + \Gamma_1\Gamma_2 = (3 - 2) + \Gamma_1\Gamma_2$$

Now assume that the statement is true for $n = N$, where $N \in \mathbb{Z}_{\geq 3}$. Then for $n = N + 1$,

$$\Gamma_{N+1}\Gamma_N = (G(N-1)\cdot\Gamma_{N-1})\,(G(N-2)\cdot\Gamma_{N-2}) \tag{1.4.1}$$

(by Lemma 1.3)

$$= (G(N-1)\cdot G(N-2))\,(\Gamma_{N-1}\cdot\Gamma_{N-2}) \tag{1.4.2}$$

$$= \left(\frac{N-1+\Gamma_1\Gamma_2}{N-2+\Gamma_1\Gamma_2}\cdot\frac{N-2+\Gamma_1\Gamma_2}{N-3+\Gamma_1\Gamma_2}\right)(\Gamma_{N-1}\cdot\Gamma_{N-2}) \tag{1.4.3}$$

$$= \left(\frac{N-1+\Gamma_1\Gamma_2}{N-3+\Gamma_1\Gamma_2}\right)(\Gamma_{N-1}\cdot\Gamma_{N-2}) \tag{1.4.4}$$

$$= \left(\frac{N-1+\Gamma_1\Gamma_2}{N-3+\Gamma_1\Gamma_2}\right)(N-3+\Gamma_1\Gamma_2) \tag{1.4.5}$$

(per our induction assumption)

$$= (N-1+\Gamma_1\Gamma_2) \tag{1.4.6}$$

□

We now have all the tools we need in order to prove the proposition.

Proof of Prop. 1.2. For $n \in \mathbb{Z}_{\geq 3}$,

$$\frac{1}{\Gamma_{n-1}} + \Gamma_{n-2} = \Gamma_{n-2}\cdot\left(\frac{1}{\Gamma_{n-1}\Gamma_{n-2}}+1\right) \tag{1.4.7}$$

(Recall $\Gamma_* > 0$ (Remark 1.1))

$$= \Gamma_{n-2}\cdot\left(\frac{1}{n-3+\Gamma_1\Gamma_2}+1\right) \tag{1.4.8}$$

(by (1.4))

$$= \Gamma_{n-2}\cdot\left(\frac{n-2+\Gamma_1\Gamma_2}{n-3+\Gamma_1\Gamma_2}\right) \tag{1.4.9}$$

$$= \Gamma_{n-2}\cdot G(n-2) \tag{1.4.10}$$

$$= \Gamma_n \tag{1.4.11}$$

(by (1.3))

□

Question 1.5. Can the definition of Γ_n be extended to Γ_x (where x may be any positive real number) such that Γ_x is continuous, while still satisfying the recurrence formula (replacing n with x)?

Question 1.6. If such a function exists, does it have any other interesting properties or satisfy any other identities?

Question 1.7. What if we additionally require the function to be differentiable on its domain?

2. Double Factorial Expression

It's now time to consider the expression $\frac{(n+1)!!}{n!!}$ (recall that $(2n)!! = (2n)\cdot(2n-2)\cdots 2$ and $(2n+1)!! = (2n+1)\cdot(2n-1)\cdots 1$, and that $0!! = 1$).

It just so happens that letting $\Gamma_1 = 1$ and $\Gamma_2 = 1$ yields the desired sequence of expressions. That is,

$$G(j) = \frac{j + \Gamma_1\Gamma_2}{j - 1 + \Gamma_1\Gamma_2}$$

becomes (using $'$-notation to remind ourselves that we have made substitutions)

$$G'(j) := \frac{j + 1 \cdot 1}{j - 1 + 1 \cdot 1} = \frac{j+1}{j}$$

,

and thus

$$\Gamma_n = \Gamma_{\epsilon(n)} \cdot \prod_{k=1}^{\lceil \frac{n}{2} \rceil - 1} G(n - 2k)$$

becomes (noting that since we have substituted $\Gamma_1 = \Gamma_2 = 1$, it follows that $\Gamma_{\epsilon(n)} = 1\ \forall n$)

$$\Gamma'_n = 1 \cdot \prod_{k=1}^{\lceil \frac{n}{2} \rceil - 1} G'(n - 2k) \tag{2.0.1}$$

$$= \prod_{k=1}^{\lceil \frac{n}{2} \rceil - 1} \frac{(n - 2k) + 1}{(n - 2k)} \tag{2.0.2}$$

$$= \frac{\prod_{k=1}^{\lceil \frac{n}{2} \rceil - 1} (n - 2k + 1)}{\prod_{k=1}^{\lceil \frac{n}{2} \rceil - 1} (n - 2k)} \tag{2.0.3}$$

$$= \frac{(n-1)!!}{(n-2)!!} \tag{2.0.4}$$

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Thus we have an alternate derivation of the double factorial formula:

Corollary 2.1. $\dfrac{n!!}{(n-1)!!} = \dfrac{1}{\left(\frac{(n-1)!!}{(n-2)!!}\right)} + \dfrac{(n-2)!!}{(n-3)!!}, \quad \forall n \in \mathbb{Z}_{\geq 3}.$

Proof. We may replace Γ wherever it occurs in Proposition 1.2 with Γ' (this is simply the result of setting $\Gamma_1 = \Gamma_2 = 1$ as shown in 2.0.1–2.0.4). □

References

[1] David Callan, *A combinatorial survey of identities for the double factorial* (2009).

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