

# On the Existence and Frequency Distribution of the Shell Primes

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*In memory of Bill Joe Rodman Jr., brother, friend and mentor...*

## Abstract

This research presents the results of a study on the existence and frequency distribution of the shell primes defined herein as prime numbers that result from the calculation of the "half-shell" of an  $p$ -dimensional entity of the form  $n^p - (n - 1)^p$  where power  $p$  is prime and base  $n$  is the realm of the positive integers. Following the introduction of the shell primes, we will look at the results of a non-sieving application of the Euler zeta function to the prime shell function as well as to any integer-valued polynomial function in general which has the ability to produce prime numbers when power  $p$  is prime. One familiar with the Euler zeta function, which established the remarkable relationship between the prime and composite numbers, might naturally ponder the results of the application of this special function in cases where there is no known way to sieve composite numbers out of the product term in this famous equation. Such would be case when an infinite series of numbers to be analyzed are calculated by a polynomial expression that yields successively increasing positive integer values and which has as its input domain the positive integers themselves. In such cases there may not be an intuitive way to eliminate the composite terms from the product term in the Euler zeta function equation by either scaling a previous prime number calculation or by employing predictable values of the domain of the function which would render outputs of the polynomial prime. So the best one may be able to hope for in these cases is to calculate some value to be added or subtracted from unity in the numerator above the product term in the Euler Zeta function to make both sides of that equation equal with the expectation that that value could be used to predict the number of prime numbers that exist as outputs of the polynomial function for some limit less than or equal to  $x$  of the input domain.

This research introduces the results of a study on the existence and frequency distribution of the shell prime numbers. Following the introduction of the shell primes, we will take a look at non-sieving applications of the Euler zeta function for integer-valued polynomials in general.

We begin with a definition of the shell primes as they occur in nature. The shell primes are defined herein as prime numbers that result from the calculation of the "half-shell" of an  $a$ -dimensional entity of the form

$$n^a - (n - 1)^a$$

for positive integers  $n \geq 2$  and  $a \geq 2$ . This shell can intuitively be thought of as the partial outer boundary, or half-shell, of an  $a$ -dimensional entity of base  $n$ . The first two occurrences are illustrated in Figs. 1 and 2.



Fig. 1 - Half-shell of a 2-dimensional entity

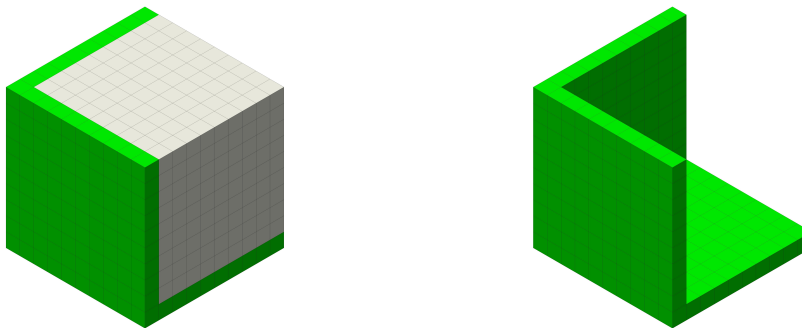


Fig. 2 - Half-shell of a 3-dimensional entity

The shell of an  $p$ -dimensional entity is further defined by the function  $n^p - (n - 1)^p$  for any positive integer base  $n \geq 2$  and any positive prime power  $p \geq 2$ . We will refer to the function  $n^p - (n - 1)^p$  as the "prime shell" function since the base  $n$  is taken to some prime power  $p$ . The "shell primes" thus are prime numbers that result from the calculation of the prime shell function of base  $n \geq 2$  for positive prime powers  $p \geq 2$ .

Within the framework of the above definitions, we now postulate the following theorem:

**Theorem 1** *Only prime shells, i.e., shells of the form  $n^p - (n - 1)^p$  for integer base  $n \geq 2$  and prime power  $p \geq 2$  will produce prime numbers. Prime numbers do not exist for shells of the form  $n^c - (n - 1)^c$ , where  $c$  is a composite number, and this holds true regardless of whether the base  $n$  is prime or composite.*

In other words, shell primes are generated from prime shells only, or shells which are calculated using prime powers. However, shell primes occur for both prime and composite values of the base  $n$  when the power  $p$  is prime, and they occur in a seemingly random fashion. Prime numbers, however, do not exist for shells which are calculated using composite powers in the shell function  $n^a - (n - 1)^a$  regardless of whether the base  $n$  is prime or composite.

The first few rows of the shell function are expanded in Fig. 3 using coefficients that are found in Paschal's triangle. The prime shells, i.e., shells which will yield prime numbers, are illustrated in red. Note that the leading coefficients of the prime shells are prime numbers themselves. Also, note how all the coefficients of the powers of base  $n$  in the shell functions in the prime rows are divisible by the first prime number in their row according to Fermat's Little Theorem.

$$\begin{aligned}
& 2n - 1 \\
& 3n^2 - 3n + 1 \\
& 4n^3 - 6n^2 + 4n - 1 \\
& 5n^4 - 10n^3 + 10n^2 - 5n + 1 \\
& 6n^5 - 15n^4 + 20n^3 - 15n^2 + 6n - 1 \\
& 7n^6 - 21n^5 + 35n^4 - 35n^3 + 21n^2 - 7n + 1 \\
& 8n^7 - 28n^6 - 56n^5 - 70n^4 + 56n^3 - 28n^2 + 8n - 1 \\
& 9n^8 - 36n^7 + 84n^6 - 126n^5 + 126n^4 - 84n^3 + 36n^2 - 9n + 1 \\
& 10n^9 - 45n^9 + 120n^7 - 210n^6 + 252n^5 - 210n^4 + 120n^3 - 45n^2 + 1 \\
& 11n^{10} - 55n^9 + 165n^8 - 330n^7 + 462n^6 - 462n^5 + 330n^4 - 165n^3 + 55n^2 - 11n + 1 \\
& 12n^{11} - 66n^{10} + 220n^9 - 495n^8 + 792n^7 - 924n^6 + 792n^5 - 495n^4 + 220n^3 - 66n^2 + 2n - 1 \\
& 13n^{12} - 78n^{11} + 286n^{10} - 715n^9 + 1287n^8 - 1716n^7 + 1716n^6 - 1287n^5 + 715n^4 - 286n^3 + 78n^2 - 13n + 1 \\
& \cdot \\
& \cdot \\
& \cdot
\end{aligned}$$

Fig. 3 - Expansion of the shell function  $n^a - (n - 1)^a$  for power  $a \geq 2$

It was observed that prime numbers generated by the prime shell function seem to arise in no less of a random fashion than do prime numbers on the real number line as can be seen in Figs. 4 and 5.

	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64	65	66	67	68	69	70	71	72	73	74	75	76	77	78	79	80	81	82	83	84	85	86	87	88	89	90	91	92	93	94	95	96	97	98	99	100																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																		
2	3	5	7	11	17	23	31	41	53	67	83	101	121	143	167	203	251	311	383	467	563	671	791	923	1067	1223	1391	1571	1763	1967	2183	2411	2651	2903	3167	3443	3731	4031	4343	4667	5003	5351	5711	6083	6467	6863	7271	7691	8123	8567	9023	9491	9971	10463	10967	11483	12011	12551	13103	13667	14243	14831	15431	16043	16667	17303	17951	18611	19283	19967	20663	21371	22091	22823	23567	24323	25091	25871	26663	27467	28283	29111	29951	30803	31667	32543	33431	34331	35243	36167	37103	38051	39011	39983	40967	41963	42971	43991	45023	46067	47123	48191	49271	50363	51467	52583	53711	54851	56003	57167	58343	59531	60731	61943	63167	64403	65651	66911	68183	69467	70763	72071	73391	74723	76067	77423	78791	80171	81563	82967	84383	85811	87251	88703	90167	91643	93131	94631	96143	97667	99203	100751	102311	103883	105467	107063	108671	110291	111923	113567	115223	116891	118571	120263	121967	123683	125411	127151	128903	130667	132443	134231	136031	137843	139667	141503	143351	145211	147083	148967	150863	152771	154691	156623	158567	160523	162491	164471	166463	168467	170483	172511	174551	176603	178667	180743	182831	184931	187043	189167	191303	193451	195611	197783	199967	202163	204371	206591	208823	211067	213323	215591	217871	220163	222467	224783	227111	229451	231803	234167	236543	238931	241331	243743	246167	248603	251051	253511	255983	258467	260963	263471	265991	268523	271067	273623	276191	278771	281363	283967	286583	289211	291851	294503	297167	299843	302531	305231	307943	310667	313403	316151	318911	321683	324467	327263	330071	332891	335723	338567	341423	344291	347171	350063	352967	355883	358811	361751	364703	367667	370643	373631	376631	379643	382667	385703	388751	391811	394883	397967	401063	404171	407291	410423	413567	416723	419891	423071	426263	429467	432683	435911	439151	442403	445667	448943	452231	455531	458843	462167	465503	468851	472211	475583	478967	482363	485771	489191	492623	496067	499523	502991	506471	509963	513467	516983	520511	524051	527603	531167	534743	538331	541931	545543	549167	552803	556451	560111	563783	567467	571163	574871	578591	582323	586067	589823	593591	597371	601163	604967	608783	612611	616451	620303	624167	628043	631931	635831	639743	643667	647603	651551	655511	659483	663467	667463	671471	675491	679523	683567	687623	691691	695771	699863	703967	708083	712211	716351	720503	724667	728843	733031	737231	741443	745667	749891	754123	758367	762613	766871	771143	775423	779711	784011	788323	792647	796983	801331	805691	810063	814443	818831	823231	827643	832067	836503	840951	845411	849883	854367	858863	863371	867891	872423	876967	881523	886091	890671	895263	899871	904491	909123	913767	918423	923091	927771	932463	937167	941883	946611	951351	956103	960867	965643	970431	975231	980043	984867	989703	994551	999411	1004283	1009171	1014071	1018983	1023903	1028831	1033771	1038723	1043683	1048651	1053631	1058623	1063627	1068643	1073671	1078711	1083763	1088827	1093903	1098991	1104091	1109203	1114327	1119467	1124623	1129791	1134971	1140163	1145367	1150583	1155811	1161051	1166303	1171567	1176843	1182131	1187431	1192743	1198067	1203403	1208751	1214111	1219483	1224867	1230263	1235671	1241091	1246531	1251983	1257447	1262923	1268411	1273911	1279423	1284947	1290483	1296031	1301591	1307163	1312747	1318343	1323951	1329571	1335203	1340847	1346503	1352171	1357851	1363543	1369247	1374963	1380691	1386431	1392183	1397947	1403723	1409511	1415311	1421123	1426947	1432783	1438631	1444491	1450363	1456247	1462143	1468051	1473971	1479903	1485847	1491803	1497771	1503751	1509743	1515747	1521763	1527791	1533831	1539883	1545947	1552023	1558111	1564211	1570323	1576447	1582583	1588731	1594891	1601063	1607247	1613443	1619651	1625871	1632103	1638347	1644603	1650871	1657151	1663443	1669747	1676063	1682391	1688731	1695083	1701447	1707823	1714211	1720611	1727023	1733447	1739883	1746331	1752791	1759263	1765747	1772243	1778751	1785271	1791803	1798347	1804903	1811471	1818051	1824643	1831247	1837863	1844491	1851131	1857783	1864447	1871123	1877811	1884511	1891223	1897947	1904683	1911431	1918191	1924963	1931751	1938551	1945363	1952187	1959023	1965871	1972731	1979603	1986487	1993383	2000291	2007211	2014143	2021087	2028043	2035011	2042

Fig. 4 - Spreadsheet matrix illustrating the distribution of the shell primes along the real number line calculated from the prime shells of the form  $n^p - (n - 1)^p$  for  $2 \leq n \leq 100$  (columns) and for prime power  $2 \leq p \leq 100$  (rows). Descending rows correspond to increasing shell power  $p$ , and ascending columns to the right correspond to increasing base  $n$ . Shells calculated with composite powers are omitted from the rows since they contain no prime numbers. The grayed out sections denote the range of numbers having a number of digits beyond the capability of the author's resources to check for primality (128-digit limit).[1]



A third observation is in regard to the repeating pattern of the last digits in the calculations yielded by the general shell function  $n^a - (n - 1)^a$  for base  $n \geq 2$  and power  $a \geq 2$ , prime and composite. Please see Fig. 6 for a snapshot of the repeating pattern of the last digits in the shell function calculations for base  $n \geq 2$  and power  $a \geq 2$ . Note that the sum of the digits in the inside boxes of the horizontal rows add up to 24. It is noted here that all shell function calculations that end in the digit 3 are found in the rows of shell functions with composite powers which yield no prime numbers.

1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
3	5	7	9	1	3	5	7	9	1	3	5	7	9	1	3	5	7	9	1
7	9	7	1	1	7	9	7	1	1	7	9	7	1	1	7	9	7	1	1
5	5	5	9	1	5	5	5	9	1	5	5	5	9	1	5	5	5	9	1
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
3	5	7	9	1	3	5	7	9	1	3	5	7	9	1	3	5	7	9	1
7	9	7	1	1	7	9	7	1	1	7	9	7	1	1	7	9	7	1	1
5	5	5	9	1	5	5	5	9	1	5	5	5	9	1	5	5	5	9	1
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

Fig. 6 - Repeating pattern of the last digits of shell calculations for base  $n \geq 2$  and power  $a \geq 2$

A fourth observation made from the charts in Figs. 4 and 5 is that there appear to be some rather large "holes" in the distribution of the prime numbers generated by the prime shell function within the range analyzed. Also, it was observed that there was a rather long delay in the appearance of the first prime numbers for the prime shell functions  $n^{37} - (n - 1)^{37}$  and  $n^{73} - (n - 1)^{73}$ .

In an effort to gain some level of understanding of the seemingly random occurrences of prime numbers generated by the prime shell function  $n^p - (n - 1)^p$ , one might naturally turn to an application of the prime counting functions introduced by Gauss and Riemann [4]. We recall Gauss' estimate of the number of primes less than some integer limit  $x$  as

$$\pi(x) \approx \frac{x}{\ln(x)}.$$

Or, as Gauss' more refined prediction allows,

$$\pi(x) \approx Li(x)$$

where

$$Li(x) = \int_2^x \frac{1}{\ln(t)} dt.$$

It is believed that Riemann improved on the logarithmic integral for predicting the number of prime numbers less than or equal to some positive integer limit  $x$  (at least for relatively small  $x$ ) by including in Gauss' estimate the powers of primes as illustrated in the following equation [4]:

$$\pi(x) + \frac{1}{2}\pi(\sqrt{x}) + \frac{1}{3}\pi(\sqrt[3]{x}) + \dots \approx Li(x)$$

so that

$$\pi(x) \approx Li(x) - \frac{1}{2}Li(\sqrt{x}) - \frac{1}{3}Li(\sqrt[3]{x}) - \dots = R(x)$$

in which the last term is the Riemann function more conveniently expressed as

$$R(x) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} Li(x^{\frac{1}{n}}).$$

But the function  $Li(x)$  can be approximated by the logarithmic sum as follows [4]:

$$Li(x) \approx Ls(x) = \frac{1}{\ln(2)} + \frac{1}{\ln(3)} + \frac{1}{\ln(4)} + \frac{1}{\ln(5)} + \dots + \frac{1}{\ln(x)}.$$

Calculations that employ the logarithmic sum illustrated above conveniently facilitate the spreadsheet computation and plotting of logarithmic curves that manifest the trend of prime number distribution along the number line of positive integers. It will be shown that this is no less the case for the calculation of the frequency distribution of the shell prime numbers which, when generated by the prime shell function with the appropriate mathematical operations applied, will approximate the number of shell primes less than or equal to some upper limit  $x$  of the base  $n$  for some fixed prime power  $p$ .

In an endeavor to formulate a function parallel to  $Li(x)$  which will approximate the frequency and distribution of the shell prime numbers over the realm of the positive integers, it is expedient to introduce a function  $M(x)$  that applies both logarithmic and power operations on the prime shell function prior to its integration or summation. For the integration, we define

$$Mi(x) = \int_2^x \frac{1}{g[f(n)]} dn$$

where the function  $g$  represents logarithmic and power operations performed on the prime shell function  $f(n)$  prior to its integration. As it turns out, the integration of  $g$  of  $f(n)$ , defined in terms of  $f(n)$ , will estimate the number of shell primes less than or equal to some upper limit  $x$  of the base  $n$  using the prime shell function  $n^p - (n-1)^p$ . Thus, the following approximation is proposed for calculating the number of shell prime numbers for base  $n \leq x$  by the prime shell function:

$$\Pi(x) \approx Mi(x)$$

where  $\Pi(x)$  represents the actual count of prime numbers that exist for the prime shell function for base  $n$  less than or equal to some upper limit  $x$ , and  $Mi(x)$  is the aforementioned integral that parallels  $Li(x)$  in the Gauss equation. It will be shown more clearly that  $Mi(x)$  is an integral of the prime shell function  $f(n) = n^p - (n-1)^p$  which has had logarithmic and power operations performed on it prior to its integration.

Substituting a logarithmic sum  $Ms(x)$  in place of the integral  $Mi(x)$  in the above equation, similar to how  $Ls(x)$  can be used to estimate  $Li(x)$ , one obtains

$$\Pi(x) \approx Mi(x) \approx Ms(x) = \frac{1}{g(f(2))} + \frac{1}{g(f(3))} + \frac{1}{g(f(4))} + \frac{1}{g(f(5))} + \dots + \frac{1}{g(f(x))}$$

where  $g(f(n))$  is, up to this point, some undefined function  $g$  of the prime shell function  $f(n) = n^p - (n-1)^p$ . As it turns out, a good approximation of  $Ms(x)$  using the logarithmic summation in place of the integral  $Mi(x)$  is found to be

$$Mi(x) \approx Ms(x) = \frac{1}{\sqrt[m]{\ln f(2)}} + \frac{1}{\sqrt[m]{\ln f(3)}} + \frac{1}{\sqrt[m]{\ln f(4)}} + \frac{1}{\sqrt[m]{\ln f(5)}} + \dots + \frac{1}{\sqrt[m]{\ln f(x)}}$$

where the  $m^{th}$  root has been empirically determined by the author to be approximately 1.68723 for the range of base  $n \leq x$  in this study of the prime shell function  $n^p - (n-1)^p$ . Please see Fig. 7 for a graph of the approximating curves generated by the  $Ms(x)$  function which were used to estimate the number of shell primes for base  $n \leq 100$  (bottom curve) and base  $n \leq 200$  (top curve) compared to the actual count of shell prime numbers  $\Pi_{n \leq 100}$  and  $\Pi_{n \leq 200}$ , respectively.

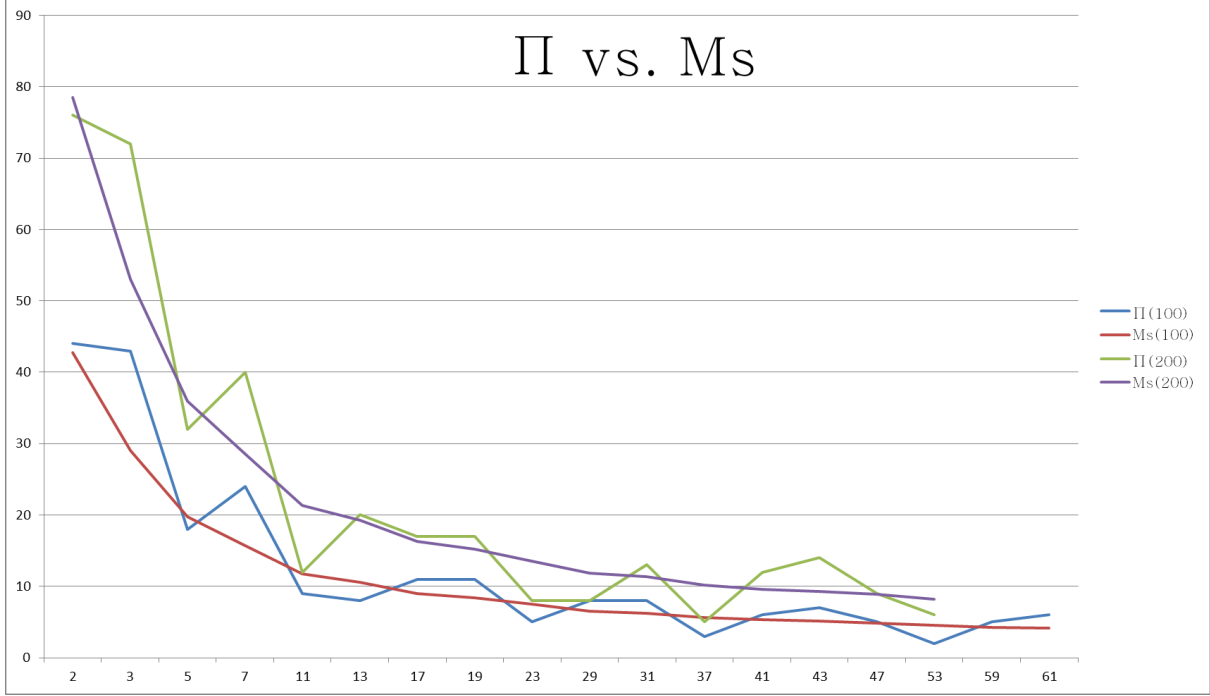


Fig. 7 - Graph of the approximating curves generated by the  $Ms(x)$  function which estimate the number of shell prime numbers for base  $n \leq 100$  (bottom curve) and base  $n \leq 200$  (top curve) for the prime shell function compared to the actual count of shell primes numbers  $\Pi_{n \leq 100}$  and  $\Pi_{n \leq 200}$ , respectively

It is noted at this point that it would certainly be possible to find constants A and B which could be applied to the  $f(n)$  terms of the logarithmic summation function  $Ms(x)$  that would appear to provide a better fit of the  $Ms(x)$  sum curve to the plot of the actual count of prime numbers within the range of this analysis. For example, an equation such as

$$\frac{1}{A * \sqrt{\ln(f(n))} + B}$$

could be formulated to generate a curve that may more closely follow the path of the plot of the actual number of primes generated by the prime shell function within the range of this analysis. However, it was determined to be more effective to optimize a logarithmic root function to apply to the  $f(n)$  terms and omit any additive constants in approximating the average path of the plot of the actual number of primes generated by the prime shell functions. Although the  $f(n)$  terms in the  $Ms(x)$  summation that apply constants A and B could create an equation that would appear to be better formed to the points of the actual count of prime numbers less than or equal to some upper limit  $x$ ,  $f(n)$  terms that omit any such additive constants but which employ an optimized root of the natural logarithm of the function  $f(n)$  will ensure that the curve generated by the  $Ms(x)$  sum will remain asymptotic to the x-axis at infinity. As an example, it was found that an additive constant B could in some cases cause the curve of the  $Ms(x)$  function to dip below the x-axis into negative values for the number of estimated primes thus rendering the  $Ms(x)$  curve estimate to be non-asymptotic and not a good representation of the diminishing frequency of prime numbers generated as the prime power  $p$  of the prime shell function approaches infinity.

Please see Table 1 for the actual count of shell primes  $\Pi$  for base  $n \leq 100$  and base  $n \leq 200$  for prime powers  $p \leq 61$  compared to the  $Ms(x)$  estimates that employ the  $m$ th root of the natural log of its terms.

prime power $p$	$\Pi \leq f(100)$	$Ms \leq f(100)$	$\Pi \leq f(200)$	$Ms \leq f(200)$
2	44	42.75969	76	78.48273
3	43	29.01307	72	53.06455
5	18	19.71488	32	35.92022
7	24	15.71077	40	28.56513
11	9	11.77596	12	21.36330
13	8	10.61689	20	19.24779
17	11	9.00845	17	16.31698
19	11	8.42015	17	15.24648
23	5	7.50235	8	13.57796
29	8	6.52712	8	11.80714
31	8	6.27155	13	11.34343
37	3	5.64216	5	10.20206
41	6	5.30697	12	9.59457
43	7	5.15842	14	9.32540
47	5	4.89221	9	8.84316
53	2	4.55460	6	8.23180
59	5	4.27319	—	—
61	6	4.18934	—	—

Table 1 - Actual count of the shell prime numbers  $\Pi$  for base  $n \leq 100$  and for  $\leq 200$  for prime powers  $p \leq 61$  compared to the estimates yielded by the logarithmic sum curves generated with the root value  $m = 1.68723$

It was proposed that a good approximation for the  $m^{th}$  root of the natural log of the prime shell function  $f(n)$  which will produce  $Ms(x)$  curves that estimate the number of prime numbers was empirically determined to be 1.68723, and this was illustrated graphically in Fig. 7 for the prime shell function  $n^p - (n-1)^p$  for the cases of  $x = 100$  and  $x = 200$  for  $p \leq 61$  and  $p \leq 53$ , respectively. The author arrived at this empirical estimate by graphing several curves with incremental values of the root  $m$  as is illustrated in Fig. 8. From this graph, one can also determine values of the root  $m$  of the natural log of the prime shell function which will generate curves that provide upper and lower bounds for the actual number of primes generated by the prime shell function. In Fig. 8, the upper bounding curve has a value of  $m_{upper} = 2.00000$  in the root function  $g(f(x))$ , and the lower bounding curve has a value of  $m_{lower} = 1.35759$ . The approximation of  $m \approx 1.68723$  was determined to be the best fit  $Ms(x)$  curve for the case of  $n \leq 100$ , and it is anticipated that this value of  $m$  will remain fairly constant as the prime power continues to increase in the prime shell function  $n^p - (n-1)^p$  and as the range  $x$  of base  $n$  in the prime shell function continues toward  $\infty$ .



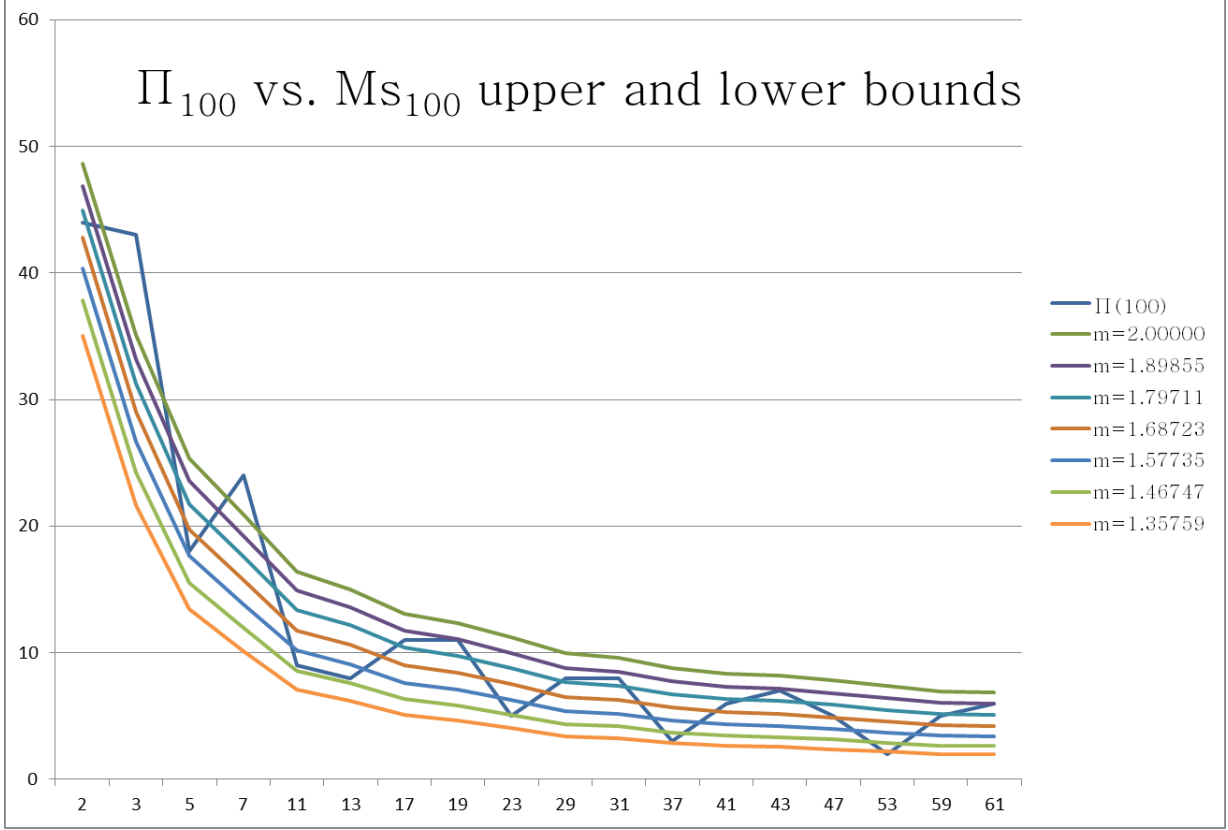


Fig. 8 - Logarithmic curves generated by varying the value of the root  $m$  to create upper and lower bounds for the estimates of the number of shell primes for base  $n \leq 100$  in the prime shell function  $n^p - (n-1)^p$  for prime power  $p \leq 61$

If one were to include the powers of primes in the equation for the calculation of the shell primes, similar to what Riemann did to improve the Gauss' *Li* equation (at least for relatively small  $x$ ), we would get an equation that parallels the Riemann prime counting function such as follows[4]:

$$Mi(x) \approx \Pi(x) + \frac{1}{2}\Pi(\sqrt{x}) + \frac{1}{3}\Pi(\sqrt[3]{x}) + \frac{1}{5}\Pi(\sqrt[5]{x}) - \frac{1}{6}\Pi(\sqrt[6]{x})\dots$$

which implies

$$\Pi(x) \approx Mi(x) - \frac{1}{2}Mi(\sqrt{x}) - \frac{1}{3}Mi(\sqrt[3]{x}) - \frac{1}{5}Mi(\sqrt[5]{x}) + \frac{1}{6}Mi(\sqrt[6]{x})\dots$$

To facilitate the calculation of this improved estimate, one could replace the first few integrals in the above equation with  $Ms(x)$  logarithmic sums to get a more accurate approximation of the Riemann method for calculating  $\Pi(x)$ . That exercise, however, will be deferred in this research.

One who is familiar with Euler's zeta function might naturally ponder the application of that special function to the case of the prime shell function. We recall the famous equation [5]

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \dots = \frac{1}{1 - \frac{1}{2^s}} \cdot \frac{1}{1 - \frac{1}{3^s}} \cdot \frac{1}{1 - \frac{1}{5^s}} \cdot \frac{1}{1 - \frac{1}{7^s}} \cdot \dots \quad (1)$$

in which Euler sieved the prime numbers out from the sum on the left hand side of Eq. 1 to include in the product on the right-hand side of that equation. However, when one replaces the integer values in the Euler zeta function in Eq. 1 with the values calculated by the prime shell function  $f(n) = n^p - (n-1)^p$ , one will find that there is not an efficient way to sieve prime numbers out from the sum on the left hand side of

Eq. 1 to create the prime product term on the right-hand side of the equation by either scaling a previous prime number calculation or by employing values of the base  $n$  which would predictably make subsequent outputs prime. So the best one may be able to hope for in this case is to determine some value to be added or subtracted from unity in the numerator on the right-hand side of Eq. 1 to balance both sides of the equation. Thus, the author introduces the following equation for the non-sieving application of the Euler zeta function to the special case of the prime shell function:

$$\mathbf{Z}(s) = \sum_{n=1}^{\infty} \frac{1}{[n^p - (n-1)^p]^s} = \frac{1 + \prod_{i=2}^{\infty} [(-1)^{i-1}] \sum^i}{\prod_{n=2}^{\infty} 1 - \frac{1}{[n^p - (n-1)^p]^s}} \quad (2)$$

where

$$\begin{aligned} \prod_{i=2}^{\infty} [(-1)^{i-1}] \sum^i &= -\Sigma_1 \Sigma_2 + \Sigma_1 \Sigma_2 \Sigma_3 - \Sigma_1 \Sigma_2 \Sigma_3 \Sigma_4 + \Sigma_1 \Sigma_2 \Sigma_3 \Sigma_4 \Sigma_5 - \Sigma_1 \Sigma_2 \Sigma_3 \Sigma_4 \Sigma_5 \Sigma_6 + \dots \implies \\ \Sigma_1 \Sigma_2 &= \sum_{i=2}^{\infty} \sum_{j=i}^{\infty} \frac{1}{f(i)} \frac{1}{f(j)} \\ \Sigma_1 \Sigma_2 \Sigma_3 &= \sum_{i=2}^{\infty} \sum_{j=i}^{\infty} \sum_{k=j+1}^{\infty} \frac{1}{f(i)} \frac{1}{f(j)} \frac{1}{f(k)} \\ \Sigma_1 \Sigma_2 \Sigma_3 \Sigma_4 &= \sum_{i=2}^{\infty} \sum_{j=i}^{\infty} \sum_{k=j+1}^{\infty} \sum_{l=k+1}^{\infty} \frac{1}{f(i)} \frac{1}{f(j)} \frac{1}{f(k)} \frac{1}{f(l)} \\ \Sigma_1 \Sigma_2 \Sigma_3 \Sigma_4 \Sigma_5 &= \sum_{i=2}^{\infty} \sum_{j=i}^{\infty} \sum_{k=j+1}^{\infty} \sum_{l=k+1}^{\infty} \sum_{m=l+1}^{\infty} \frac{1}{f(i)} \frac{1}{f(j)} \frac{1}{f(k)} \frac{1}{f(l)} \frac{1}{f(m)} \\ \Sigma_1 \Sigma_2 \Sigma_3 \Sigma_4 \Sigma_5 \Sigma_6 &= \sum_{i=2}^{\infty} \sum_{j=i}^{\infty} \sum_{k=j+1}^{\infty} \sum_{l=k+1}^{\infty} \sum_{m=l+1}^{\infty} \sum_{n=m+1}^{\infty} \frac{1}{f(i)} \frac{1}{f(j)} \frac{1}{f(k)} \frac{1}{f(l)} \frac{1}{f(m)} \frac{1}{f(n)} \\ &\quad \cdot \\ &\quad \cdot \\ &\quad \cdot \end{aligned}$$

As intimated by the first row of Fig. 9, if one applies the realm of positive integers  $i$  to the Euler zeta function using the non-sieving method so that the product term in the denominator on the right-hand side of Eq. 1 includes all the prime and composite terms, then the numerator on the right-hand side of Eq. 1 will tend to zero as the limit  $x$  tends to infinity. Thus, for the case of the positive integers, one might associate the value of all the composite terms in the product term in the denominator of the right-hand side of the non-sieving Euler zeta function, if they were included, to  $-1$  in the numerator on that side of the equation, because when all the composite terms are included in the product term in the denominator of Eq. 1, the numerator on the right-hand side of that equation tends to zero (i.e.,  $1 + (M) = 0$ ) as  $x$  approaches  $\infty$ . Compare that to the value of unity attained in the numerator of the right-hand side Eq. 1 when all the composite terms are sieved out of the denominator of the product term on the right-hand side of the equation as Euler did when he discovered this famous relationship between the composite and prime numbers. So one might postulate that the total effect of sieving the composite terms out of the product term in the denominator of the right-hand side of the Euler zeta function in Eq. 1 is that the numerator on the right-hand side of the equation changes from zero to unity as the composite-numbered terms are eliminated. It is observed that while the "M-series"

$$\prod_{i=2}^{\infty} [(-1)^{i-1}] \sum^i$$

in the numerator of the right-hand side of the non-sieving zeta function represents an infinite sum of infinite sums, its value is constrained between 0 and  $-1$  throughout the application of this function to the realm of positive integers as well as to the range of integer calculations yielded by the prime shell function  $n^p - (n-1)^p$ .

		$M_{x=100}$		$M_{x=200}$	
$i$	$n$	sum: 5.18738 product: 0.01000	-0.94812622482360	sum: 5.87803 product: 0.00500	-0.97060984525939
2	$2n-1$	sum: 3.28434 product: 0.08873	-0.70856869191073	sum: 3.63091 product: 0.06270	-0.77232394108548
3	$3n^2-3n+1$	sum: 1.30195 product: 0.72955	-0.05016737946525	sum: 1.30362 product: 0.72833	-0.05053523893596
5	$5n^4-10n^3+10n^2-5n+1$	sum: 1.03927 product: 0.96097	-0.00129463514931	sum: 1.03927 product: 0.96097	-0.00129463735049
7	$6-21n^5+35n^4-35n^3+21n^2-7$	sum: 1.00845 product: 0.99155	-0.00006682330849	sum: 1.00845 product: 0.99155	-0.00006682330851

Fig. 9 - M-series calculations for the realm of the positive integers  $i$  and for the first four prime powers  $p = 2, 3, 5, 7$  of the prime shell function  $n^p - (n-1)^p$  for  $x = 100$  and for  $x = 200$

In Figure 9, descending rows below row  $i$  depict the M-series values  $M_{x=100}$  and  $M_{x=200}$ , respectively, that result from the non-sieving application of the zeta function to the prime shell function of powers 2, 3, 5 and 7, respectively. Included to the left of each M-series value are the  $M_s$  (sum) and  $\Pi$  (product) calculations for that row of the prime shell function that were used to calculate the M-series values under the columns  $M_{x=100}$  and  $M_{x=200}$ . Fig. 10 is a graph plotted that superimposes the trends of  $M_{x=100}$  and  $M_{x=200}$  for the case of the non-sieving application of the Euler zeta function, and it illustrates the change in the value of the M-series as the prime power in the prime shell function increases from 2 to 7 and as the range of the base  $n$  of the prime shell function increases from  $x = 100$  to  $x = 200$ . Notice the trend is that the M-series increases from  $-1$  to 0 asymptotically as the power of the prime shell equation increases and that the increase in the M-series value between  $x = 100$  to  $x = 200$  is very slight. It is expected that as  $x$  approaches  $\infty$  for row  $i$  in the case of the application of the non-sieving zeta function to the realm of positive integers that the value of the M-series will approach  $-1$  thereby rendering the numerator of the right-hand side of Eq. 2 to zero. It is also expected that the limit of the M-series in the descending rows below row  $i$  which summarize the results of the non-sieving application of the Euler zeta function to the prime shell functions of powers 2, 3, 5 and 7 will approach stable limits as the range  $x$  of the base  $n$  approaches  $\infty$ . From the trends manifested by the graphs in Fig. 10, it is anticipated that the M-series values will approach the following approximate limits as  $x$  approaches  $\infty$ :

$$\begin{aligned}
 M_{i_{x=\infty}} &\approx -1 \\
 M_{2_{(x=\infty)}} &\approx -0.8 \\
 M_{3_{(x=\infty)}} &\approx -0.051 \\
 M_{5_{(x=\infty)}} &\approx -0.0013 \\
 M_{7_{(x=\infty)}} &\approx -0.000067
 \end{aligned}$$

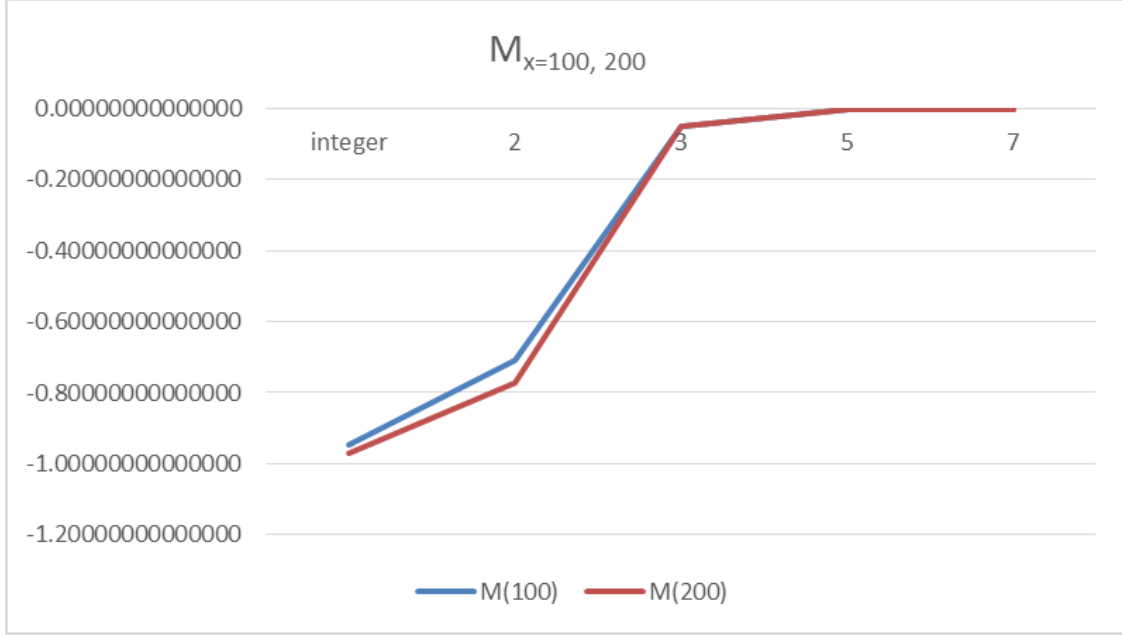


Fig. 10 - Graph of the M-series superimposed for  $x = 100$  and  $x = 200$  for  $i = \text{integer}$  and for the first four prime powers  $p = 2, 3, 5, 7$  of the shell prime function. Note that the M-series is asymptotic to 0 as the prime power  $p$  in the prime shell function  $n^p - (n - 1)^p$  increases from 2 to 7.

Table 2 tabulates the values of  $M_{x=100}$  and  $M_{x=200}$  for the non-sieving application of the Euler zeta function to the positive integers  $i$  and to the prime shell function with powers 2, 3, 5 and 7. It is anticipated that such calculations will be able to tell us something about the density of shell prime numbers less than a given limit  $x$  of the range of base  $n$  for any given prime power  $p$  in the prime shell function. Further study should reveal more insight into the properties of this special series and its potential for predicting the number of primes generated by the prime shell function and for any other integer-valued polynomial in general which has the capability to generate prime numbers and which has as its input domain the realm of positive integers.

	M $x=100$	M $x=200$
integer $i$	-0.94812622482360	-0.97060984525939
$p = 2$	-0.70856869191073	-0.77232394108548
$p = 3$	-0.05016737946525	-0.05053523893596
$p = 5$	-0.00129463514931	-0.00129463735049
$p = 7$	-0.00006682330849	-0.00006682330851

Table 2

As an exercise to promote the notion of a non-sieving application of the Euler zeta function to integer-valued polynomial functions in general, we shall apply the non-sieving method to any integer-generating polynomial for which the input domain of the function is the positive integers that generate successive integer values by the polynomial for processing in the zeta function. Thus, we introduce the following "big Zeta" equation for this generalized application:

$$\mathbf{Z} = 1 + \sum_{n=1}^{\infty} \frac{1}{f(n)} = \frac{1 + \prod_{i=1}^{\infty} (-1)^i \sum^{i+1}}{\prod_{n=1}^{\infty} 1 - \frac{1}{f(n)}}. \quad (3)$$

where  $f(n)$  represents any polynomial function which yields successively increasing positive integer values when the input domain of the function is the positive integers and where  $f(1) \neq 1$ .

We begin our example of the non-sieving application of the Euler zeta function to integer-valued polynomials in general by writing

$$Z = 1 + \frac{1}{f(1)} + \frac{1}{f(2)} + \frac{1}{f(3)} + \frac{1}{f(4)} + \frac{1}{f(5)} + \frac{1}{f(6)} + \frac{1}{f(7)} + \frac{1}{f(8)} + \frac{1}{f(9)} + \dots \quad (4)$$

We first multiply both sides of Eq. 4 by the first fraction on the right hand side of that equation which yields

$$\frac{1}{f(1)} \cdot Z = \frac{1}{f(1)} + \frac{1}{f(1)} \cdot \frac{1}{f(1)} + \frac{1}{f(1)} \cdot \frac{1}{f(2)} + \frac{1}{f(1)} \cdot \frac{1}{f(3)} + \frac{1}{f(1)} \cdot \frac{1}{f(4)} + \frac{1}{f(1)} \cdot \frac{1}{f(5)} + \frac{1}{f(1)} \cdot \frac{1}{f(6)} + \frac{1}{f(1)} \cdot \frac{1}{f(7)} + \frac{1}{f(1)} \cdot \frac{1}{f(8)} + \frac{1}{f(1)} \cdot \frac{1}{f(9)} + \dots$$

and when we subtract this result from both sides of Eq. 4, we get

$$\left(1 - \frac{1}{f(1)}\right) \cdot Z = 1 + \frac{1}{f(2)} + \frac{1}{f(3)} + \frac{1}{f(4)} + \frac{1}{f(5)} + \frac{1}{f(6)} + \frac{1}{f(7)} + \frac{1}{f(8)} + \frac{1}{f(9)} + \dots - \sum_{n=1}^{\infty} \frac{1}{f(1) \cdot f(n)}. \quad (5)$$

Continuing with the multiplication of Eq. 5 by the next fraction in the sequence,  $\frac{1}{f(2)}$ :

$$\begin{aligned} \frac{1}{f(2)} \cdot \left(1 - \frac{1}{f(1)}\right) \cdot Z &= \frac{1}{f(2)} + \frac{1}{f(2)} \cdot \frac{1}{f(2)} + \frac{1}{f(2)} \cdot \frac{1}{f(3)} + \frac{1}{f(2)} \cdot \frac{1}{f(4)} + \frac{1}{f(2)} \cdot \frac{1}{f(5)} + \frac{1}{f(2)} \cdot \frac{1}{f(6)} + \frac{1}{f(2)} \cdot \frac{1}{f(7)} + \frac{1}{f(2)} \cdot \frac{1}{f(8)} + \frac{1}{f(2)} \cdot \frac{1}{f(9)} + \dots \\ &\quad - \sum_{n=1}^{\infty} \frac{1}{f(1) \cdot f(2) \cdot f(n)} \end{aligned}$$

so that when we subtract this result from both sides of Eq. 5, we end up with

$$\begin{aligned} \left(1 - \frac{1}{f(2)}\right) \left(1 - \frac{1}{f(1)}\right) \cdot Z &= 1 + \frac{1}{f(3)} + \frac{1}{f(4)} + \frac{1}{f(5)} + \frac{1}{f(6)} + \frac{1}{f(7)} + \frac{1}{f(8)} + \frac{1}{f(9)} + \dots \\ &\quad - \sum_{n=1}^{\infty} \frac{1}{f(1) \cdot f(n)} - \sum_{n=2}^{\infty} \frac{1}{f(2) \cdot f(n)} + \sum_{n=1}^{\infty} \frac{1}{f(1) \cdot f(2) \cdot f(n)}. \end{aligned}$$

Continuing with the multiplication of this new equation by the next term in the sequence,  $\frac{1}{f(3)}$ :

$$\begin{aligned} \frac{1}{f(3)} \cdot \left(1 - \frac{1}{f(2)}\right) \cdot \left(1 - \frac{1}{f(1)}\right) \cdot Z &= \frac{1}{f(3)} + \frac{1}{f(3)} \cdot \frac{1}{f(3)} + \frac{1}{f(3)} \cdot \frac{1}{f(4)} + \frac{1}{f(3)} \cdot \frac{1}{f(5)} + \frac{1}{f(3)} \cdot \frac{1}{f(6)} + \frac{1}{f(3)} \cdot \frac{1}{f(7)} + \frac{1}{f(3)} \cdot \frac{1}{f(8)} + \frac{1}{f(3)} \cdot \frac{1}{f(9)} + \dots \\ &\quad - \sum_{n=1}^{\infty} \frac{1}{f(1) \cdot f(3) \cdot f(n)} - \sum_{n=2}^{\infty} \frac{1}{f(2) \cdot f(3) \cdot f(n)} + \sum_{n=1}^{\infty} \frac{1}{f(1) \cdot f(2) \cdot f(3) \cdot f(n)} \end{aligned}$$

so that

$$\begin{aligned} \left(1 - \frac{1}{f(3)}\right) \left(1 - \frac{1}{f(2)}\right) \left(1 - \frac{1}{f(1)}\right) \cdot Z &= 1 + \frac{1}{f(4)} + \frac{1}{f(5)} + \frac{1}{f(6)} + \frac{1}{f(7)} + \frac{1}{f(8)} + \frac{1}{f(9)} + \dots \\ &\quad - \sum_{n=1}^{\infty} \frac{1}{f(1) \cdot f(n)} - \sum_{n=2}^{\infty} \frac{1}{f(2) \cdot f(n)} - \sum_{n=3}^{\infty} \frac{1}{f(3) \cdot f(n)} \\ &\quad + \sum_{n=1}^{\infty} \frac{1}{f(1) \cdot f(2) \cdot f(n)} + \sum_{n=1}^{\infty} \frac{1}{f(1) \cdot f(3) \cdot f(n)} + \sum_{n=2}^{\infty} \frac{1}{f(2) \cdot f(3) \cdot f(n)} - \sum_{n=1}^{\infty} \frac{1}{f(1) \cdot f(2) \cdot f(3) \cdot f(n)}. \end{aligned}$$

Continuing with the multiplication of this new equation by the next term in the sequence,  $\frac{1}{f(4)}$ :

$$\begin{aligned} \frac{1}{f(4)} \cdot \left(1 - \frac{1}{f(3)}\right) \cdot \left(1 - \frac{1}{f(2)}\right) \cdot \left(1 - \frac{1}{f(1)}\right) \cdot Z &= \frac{1}{f(4)} + \frac{1}{f(4)} \cdot \frac{1}{f(4)} + \frac{1}{f(4)} \cdot \frac{1}{f(4)} \cdot \frac{1}{f(5)} + \frac{1}{f(4)} \cdot \frac{1}{f(4)} \cdot \frac{1}{f(6)} + \frac{1}{f(4)} \cdot \frac{1}{f(4)} \cdot \frac{1}{f(7)} + \frac{1}{f(4)} \cdot \frac{1}{f(4)} \cdot \frac{1}{f(8)} + \frac{1}{f(4)} \cdot \frac{1}{f(4)} \cdot \frac{1}{f(9)} + \dots \\ &- \sum_{n=1}^{\infty} \frac{1}{f(1) \cdot f(4) \cdot f(n)} - \sum_{n=2}^{\infty} \frac{1}{f(2) \cdot f(4) \cdot f(n)} - \sum_{n=3}^{\infty} \frac{1}{f(3) \cdot f(4) \cdot f(n)} + \sum_{n=1}^{\infty} \frac{1}{f(1) \cdot f(2) \cdot f(4) \cdot f(n)} \\ &+ \sum_{n=1}^{\infty} \frac{1}{f(1) \cdot f(3) \cdot f(4) \cdot f(n)} + \sum_{n=2}^{\infty} \frac{1}{f(2) \cdot f(3) \cdot f(4) \cdot f(n)} - \sum_{n=1}^{\infty} \frac{1}{f(1) \cdot f(2) \cdot f(3) \cdot f(4) \cdot f(n)} \end{aligned}$$

so that

$$\begin{aligned} \left(1 - \frac{1}{f(4)}\right) \left(1 - \frac{1}{f(3)}\right) \left(1 - \frac{1}{f(2)}\right) \left(1 - \frac{1}{f(1)}\right) \cdot Z &= 1 + \frac{1}{f(5)} + \frac{1}{f(6)} + \frac{1}{f(7)} + \frac{1}{f(8)} + \frac{1}{f(9)} + \dots \\ &- \sum_{n=1}^{\infty} \frac{1}{f(1) \cdot f(n)} - \sum_{n=2}^{\infty} \frac{1}{f(2) \cdot f(n)} - \sum_{n=3}^{\infty} \frac{1}{f(3) \cdot f(n)} - \sum_{n=4}^{\infty} \frac{1}{f(4) \cdot f(n)} \\ &+ \sum_{n=1}^{\infty} \frac{1}{f(1) \cdot f(2) \cdot f(n)} + \sum_{n=1}^{\infty} \frac{1}{f(1) \cdot f(3) \cdot f(n)} + \sum_{n=2}^{\infty} \frac{1}{f(2) \cdot f(3) \cdot f(n)} - \sum_{n=1}^{\infty} \frac{1}{f(1) \cdot f(2) \cdot f(3) \cdot f(n)} \\ &+ \sum_{n=1}^{\infty} \frac{1}{f(1) \cdot f(4) \cdot f(n)} + \sum_{n=2}^{\infty} \frac{1}{f(2) \cdot f(4) \cdot f(n)} + \sum_{n=3}^{\infty} \frac{1}{f(3) \cdot f(4) \cdot f(n)} - \sum_{n=1}^{\infty} \frac{1}{f(1) \cdot f(2) \cdot f(4) \cdot f(n)} \\ &- \sum_{n=1}^{\infty} \frac{1}{f(1) \cdot f(3) \cdot f(4) \cdot f(n)} - \sum_{n=2}^{\infty} \frac{1}{f(2) \cdot f(3) \cdot f(4) \cdot f(n)} + \sum_{n=1}^{\infty} \frac{1}{f(1) \cdot f(2) \cdot f(3) \cdot f(4) \cdot f(n)}. \end{aligned}$$

Continuing with the multiplication of of this new equation by the next term in the sequence,  $\frac{1}{f(5)}$ :

$$\begin{aligned} \frac{1}{f(5)} \cdot \left(1 - \frac{1}{f(4)}\right) \cdot \left(1 - \frac{1}{f(3)}\right) \cdot \left(1 - \frac{1}{f(2)}\right) \cdot \left(1 - \frac{1}{f(1)}\right) \cdot Z &= \frac{1}{f(5)} + \frac{1}{f(5)} \cdot \frac{1}{f(5)} + \frac{1}{f(5)} \cdot \frac{1}{f(5)} \cdot \frac{1}{f(6)} + \frac{1}{f(5)} \cdot \frac{1}{f(5)} \cdot \frac{1}{f(7)} + \frac{1}{f(5)} \cdot \frac{1}{f(5)} \cdot \frac{1}{f(8)} + \frac{1}{f(5)} \cdot \frac{1}{f(5)} \cdot \frac{1}{f(9)} + \dots \\ &- \sum_{n=1}^{\infty} \frac{1}{f(1) \cdot f(5) \cdot f(n)} - \sum_{n=2}^{\infty} \frac{1}{f(2) \cdot f(5) \cdot f(n)} - \sum_{n=3}^{\infty} \frac{1}{f(3) \cdot f(5) \cdot f(n)} \\ &- \sum_{n=4}^{\infty} \frac{1}{f(4) \cdot f(5) \cdot f(n)} + \sum_{n=1}^{\infty} \frac{1}{f(1) \cdot f(2) \cdot f(5) \cdot f(n)} + \sum_{n=1}^{\infty} \frac{1}{f(1) \cdot f(3) \cdot f(5) \cdot f(n)} \\ &+ \sum_{n=2}^{\infty} \frac{1}{f(2) \cdot f(3) \cdot f(5) \cdot f(n)} - \sum_{n=1}^{\infty} \frac{1}{f(1) \cdot f(2) \cdot f(3) \cdot f(5) \cdot f(n)} + \sum_{n=1}^{\infty} \frac{1}{f(1) \cdot f(4) \cdot f(5) \cdot f(n)} \\ &+ \sum_{n=2}^{\infty} \frac{1}{f(2) \cdot f(4) \cdot f(5) \cdot f(n)} + \sum_{n=3}^{\infty} \frac{1}{f(3) \cdot f(4) \cdot f(5) \cdot f(n)} - \sum_{n=1}^{\infty} \frac{1}{f(1) \cdot f(2) \cdot f(4) \cdot f(5) \cdot f(n)} \\ &- \sum_{n=1}^{\infty} \frac{1}{f(1) \cdot f(3) \cdot f(4) \cdot f(5) \cdot f(n)} - \sum_{n=2}^{\infty} \frac{1}{f(2) \cdot f(3) \cdot f(4) \cdot f(5) \cdot f(n)} + \sum_{n=1}^{\infty} \frac{1}{f(1) \cdot f(2) \cdot f(3) \cdot f(4) \cdot f(5) \cdot f(n)} \end{aligned}$$

so that

$$\begin{aligned}
& \left(1 - \frac{1}{f(5)}\right) \left(1 - \frac{1}{f(4)}\right) \left(1 - \frac{1}{f(3)}\right) \left(1 - \frac{1}{f(2)}\right) \left(1 - \frac{1}{f(1)}\right) \cdot Z = 1 + \frac{1}{f(6)} + \frac{1}{f(7)} + \frac{1}{f(8)} + \frac{1}{f(9)} + \dots \\
& - \sum_{n=1}^{\infty} \frac{1}{f(1) \cdot f(n)} - \sum_{n=2}^{\infty} \frac{1}{f(2) \cdot f(n)} - \sum_{n=3}^{\infty} \frac{1}{f(3) \cdot f(n)} - \sum_{n=4}^{\infty} \frac{1}{f(4) \cdot f(n)} - \sum_{n=5}^{\infty} \frac{1}{f(5) \cdot f(n)} \\
& + \sum_{n=1}^{\infty} \frac{1}{f(1) \cdot f(2) \cdot f(n)} + \sum_{n=1}^{\infty} \frac{1}{f(1) \cdot f(3) \cdot f(n)} + \sum_{n=2}^{\infty} \frac{1}{f(2) \cdot f(3) \cdot f(n)} - \sum_{n=1}^{\infty} \frac{1}{f(1) \cdot f(2) \cdot f(3) \cdot f(n)} \\
& + \sum_{n=1}^{\infty} \frac{1}{f(1) \cdot f(4) \cdot f(n)} + \sum_{n=2}^{\infty} \frac{1}{f(2) \cdot f(4) \cdot f(n)} + \sum_{n=3}^{\infty} \frac{1}{f(3) \cdot f(4) \cdot f(n)} - \sum_{n=1}^{\infty} \frac{1}{f(1) \cdot f(2) \cdot f(4) \cdot f(n)} \\
& - \sum_{n=1}^{\infty} \frac{1}{f(1) \cdot f(3) \cdot f(4) \cdot f(n)} - \sum_{n=2}^{\infty} \frac{1}{f(2) \cdot f(3) \cdot f(4) \cdot f(n)} + \sum_{n=1}^{\infty} \frac{1}{f(1) \cdot f(2) \cdot f(3) \cdot f(4) \cdot f(n)} \\
& + \sum_{n=1}^{\infty} \frac{1}{f(1) \cdot f(5) \cdot f(n)} + \sum_{n=2}^{\infty} \frac{1}{f(2) \cdot f(5) \cdot f(n)} + \sum_{n=3}^{\infty} \frac{1}{f(3) \cdot f(5) \cdot f(n)} + \sum_{n=4}^{\infty} \frac{1}{f(4) \cdot f(5) \cdot f(n)} \\
& - \sum_{n=1}^{\infty} \frac{1}{f(1) \cdot f(2) \cdot f(5) \cdot f(n)} - \sum_{n=1}^{\infty} \frac{1}{f(1) \cdot f(3) \cdot f(5) \cdot f(n)} - \sum_{n=2}^{\infty} \frac{1}{f(2) \cdot f(3) \cdot f(5) \cdot f(n)} \\
& + \sum_{n=1}^{\infty} \frac{1}{f(1) \cdot f(2) \cdot f(3) \cdot f(5) \cdot f(n)} - \sum_{n=1}^{\infty} \frac{1}{f(1) \cdot f(4) \cdot f(5) \cdot f(n)} - \sum_{n=2}^{\infty} \frac{1}{f(2) \cdot f(4) \cdot f(5) \cdot f(n)} \\
& - \sum_{n=3}^{\infty} \frac{1}{f(3) \cdot f(4) \cdot f(5) \cdot f(n)} + \sum_{n=1}^{\infty} \frac{1}{f(1) \cdot f(2) \cdot f(4) \cdot f(5) \cdot f(n)} + \sum_{n=1}^{\infty} \frac{1}{f(1) \cdot f(3) \cdot f(4) \cdot f(5) \cdot f(n)} \\
& + \sum_{n=2}^{\infty} \frac{1}{f(2) \cdot f(3) \cdot f(4) \cdot f(5) \cdot f(n)} - \sum_{n=1}^{\infty} \frac{1}{f(1) \cdot f(2) \cdot f(3) \cdot f(4) \cdot f(5) \cdot f(n)}.
\end{aligned}$$

We continue in this fashion until all the fraction terms on the right-hand side of the big Zeta function in Eq. 4 are eliminated. This, of course, will only occur at  $\infty$ . But as we continue to eliminate the terms on the right-hand side of the big Zeta function in Eq. 4, a value for what we will call the "M-series" function

$$\prod_{i=1}^{\infty} (-1)^i \sum^{i+1}$$

emerges which adds to unity in the numerator on the right-hand side of Eq. 3. It is observed that the value of this M-series function increases between the bounds of  $-1$  and  $0$  as the terms, both composite and prime, are eliminated from the right-hand side of the big Zeta function Eq. 4. To evaluate this M-series for the operations we have performed thus far with the big Zeta function on the generalized polynomial function  $f(n)$  in this example, we rearrange the summation terms to obtain

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{1}{f(n)} \cdot \left( -\frac{1}{f(1)} + \frac{1}{f(1) \cdot f(2)} + \frac{1}{f(1) \cdot f(3)} - \frac{1}{f(1) \cdot f(2) \cdot f(3)} + \frac{1}{f(1) \cdot f(4)} - \frac{1}{f(1) \cdot f(2) \cdot f(4)} - \frac{1}{f(1) \cdot f(3) \cdot f(4)} \right. \\
& + \frac{1}{f(1) \cdot f(2) \cdot f(3) \cdot f(4)} + \frac{1}{f(1) \cdot f(5)} - \frac{1}{f(1) \cdot f(2) \cdot f(5)} - \frac{1}{f(1) \cdot f(3) \cdot f(5)} + \frac{1}{f(1) \cdot f(2) \cdot f(3) \cdot f(5)} \\
& \left. - \frac{1}{f(1) \cdot f(4) \cdot f(5)} + \frac{1}{f(1) \cdot f(2) \cdot f(4) \cdot f(5)} + \frac{1}{f(1) \cdot f(3) \cdot f(4) \cdot f(5)} - \frac{1}{f(1) \cdot f(2) \cdot f(3) \cdot f(4) \cdot f(5)} \dots \right)
\end{aligned}$$

$$\begin{aligned}
& \sum_{n=2}^{\infty} \frac{1}{f(n)} \cdot \left( -\frac{1}{f(2)} + \frac{1}{f(2) \cdot f(3)} + \frac{1}{f(2) \cdot f(4)} - \frac{1}{f(2) \cdot f(3) \cdot f(4)} + \frac{1}{f(2) \cdot f(5)} - \frac{1}{f(2) \cdot f(3) \cdot f(5)} \right. \\
& \qquad \qquad \qquad \left. - \frac{1}{f(2) \cdot f(4) \cdot f(5)} + \frac{1}{f(2) \cdot f(3) \cdot f(4) \cdot f(5)} \dots \right) \\
& \sum_{n=3}^{\infty} \frac{1}{f(n)} \cdot \left( -\frac{1}{f(3)} + \frac{1}{f(3) \cdot f(4)} + \frac{1}{f(3) \cdot f(5)} - \frac{1}{f(3) \cdot f(4) \cdot f(5)} \dots \right) \\
& \sum_{n=4}^{\infty} \frac{1}{f(n)} \cdot \left( -\frac{1}{f(4)} + \frac{1}{f(4) \cdot f(5)} \dots \right) \\
& \sum_{n=5}^{\infty} \frac{1}{f(n)} \cdot \left( -\frac{1}{f(5)} \dots \right) \\
& \sum_{n=6}^{\infty} \frac{1}{f(n)}.
\end{aligned}$$

Once the summation terms are thus grouped in this fashion according to their lower limits, we begin collecting like terms across the groups to organize the sums according to the number of terms which their product will contain when they are expanded with their coefficients. Referring to the group above, we begin by extracting the summations with the fewest terms which have been highlighted in red , i.e.,

$$\begin{aligned}
& -\frac{1}{f(1)} \cdot \sum_{n=1}^{\infty} \frac{1}{f(n)} - \frac{1}{f(2)} \cdot \sum_{n=2}^{\infty} \frac{1}{f(n)} - \frac{1}{f(3)} \cdot \sum_{n=3}^{\infty} \frac{1}{f(n)} - \frac{1}{f(4)} \cdot \sum_{n=4}^{\infty} \frac{1}{f(n)} - \frac{1}{f(5)} \cdot \sum_{n=5}^{\infty} \frac{1}{f(n)} - \dots = \\
& -\frac{1}{f(1)} \cdot \left( \frac{1}{f(1)} + \frac{1}{f(2)} + \frac{1}{f(3)} + \frac{1}{f(4)} + \frac{1}{f(5)} + \dots \right) \\
& -\frac{1}{f(2)} \cdot \left( \frac{1}{f(2)} + \frac{1}{f(3)} + \frac{1}{f(4)} + \frac{1}{f(5)} + \dots \right) \\
& -\frac{1}{f(3)} \cdot \left( \frac{1}{f(3)} + \frac{1}{f(4)} + \frac{1}{f(5)} + \dots \right) \\
& -\frac{1}{f(4)} \cdot \left( \frac{1}{f(4)} + \frac{1}{f(5)} + \dots \right) \\
& -\frac{1}{f(5)} \cdot \left( \frac{1}{f(5)} + \dots \right) \\
& - \dots
\end{aligned}$$



which yields the first term in the infinite series,

$$\prod_{i=1}^{\infty} (-1)^i \sum^{i+1}$$

or M-series of degree 2, previously defined as

$$-\Sigma_1 \Sigma_2 = - \sum_{i=1}^{\infty} \sum_{j=i}^{\infty} \frac{1}{f(i)} \frac{1}{f(j)}.$$

The second term of the infinite series

$$\prod_{i=1}^{\infty} (-1)^i \sum^{i+1}$$

is also formed by collecting like terms across the original grouping (again highlighted in red )

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{1}{f(n)} \cdot \left( -\frac{1}{f(1)} + \frac{1}{f(1) \cdot f(2)} + \frac{1}{f(1) \cdot f(3)} - \frac{1}{f(1) \cdot f(2) \cdot f(3)} + \frac{1}{f(1) \cdot f(4)} - \frac{1}{f(1) \cdot f(2) \cdot f(4)} - \frac{1}{f(1) \cdot f(3) \cdot f(4)} \right. \\ & + \frac{1}{f(1) \cdot f(2) \cdot f(3) \cdot f(4)} + \frac{1}{f(1) \cdot f(5)} - \frac{1}{f(1) \cdot f(2) \cdot f(5)} - \frac{1}{f(1) \cdot f(3) \cdot f(5)} + \frac{1}{f(1) \cdot f(2) \cdot f(3) \cdot f(5)} \\ & \left. - \frac{1}{f(1) \cdot f(4) \cdot f(5)} + \frac{1}{f(1) \cdot f(2) \cdot f(4) \cdot f(5)} + \frac{1}{f(1) \cdot f(3) \cdot f(4) \cdot f(5)} - \frac{1}{f(1) \cdot f(2) \cdot f(3) \cdot f(4) \cdot f(5)} \dots \right) \\ & \sum_{n=2}^{\infty} \frac{1}{f(n)} \cdot \left( -\frac{1}{f(2)} + \frac{1}{f(2) \cdot f(3)} + \frac{1}{f(2) \cdot f(4)} - \frac{1}{f(2) \cdot f(3) \cdot f(4)} + \frac{1}{f(2) \cdot f(5)} - \frac{1}{f(2) \cdot f(3) \cdot f(5)} \right. \\ & \left. - \frac{1}{f(2) \cdot f(4) \cdot f(5)} + \frac{1}{f(2) \cdot f(3) \cdot f(4) \cdot f(5)} \dots \right) \\ & \sum_{n=3}^{\infty} \frac{1}{f(n)} \cdot \left( -\frac{1}{f(3)} + \frac{1}{f(3) \cdot f(4)} + \frac{1}{f(3) \cdot f(5)} - \frac{1}{f(3) \cdot f(4) \cdot f(5)} \dots \right) \\ & \sum_{n=4}^{\infty} \frac{1}{f(n)} \cdot \left( -\frac{1}{f(4)} + \frac{1}{f(4) \cdot f(5)} \dots \right) \\ & \sum_{n=5}^{\infty} \frac{1}{f(n)} \cdot \left( -\frac{1}{f(5)} \dots \right) \\ & \sum_{n=6}^{\infty} \frac{1}{f(n)} \end{aligned}$$

and these terms can be extracted from the original grouping and organized in the same fashion:

$$\left( \frac{1}{f(1) \cdot f(2)} + \frac{1}{f(1) \cdot f(3)} + \frac{1}{f(1) \cdot f(4)} + \frac{1}{f(1) \cdot f(5)} + \dots \right) \cdot \sum_{n=1}^{\infty} \frac{1}{f(n)} +$$

$$\left(\frac{1}{f(2) \cdot f(3)} + \frac{1}{f(2) \cdot f(4)} + \frac{1}{f(2) \cdot f(5)} + \dots\right) \cdot \sum_{n=2}^{\infty} \frac{1}{f(n)} +$$

$$\left(\frac{1}{f(3) \cdot f(4)} + \frac{1}{f(3) \cdot f(5)} + \dots\right) \cdot \sum_{n=3}^{\infty} \frac{1}{f(n)} +$$

$$\left(\frac{1}{f(4) \cdot f(5)} + \dots\right) \cdot \sum_{n=4}^{\infty} \frac{1}{f(n)} + \dots$$

=

$$\left(\frac{1}{f(1) \cdot f(2)} + \frac{1}{f(1) \cdot f(3)} + \frac{1}{f(1) \cdot f(4)} + \frac{1}{f(1) \cdot f(5)} + \dots\right) \cdot \left(\frac{1}{f(1)} + \frac{1}{f(2)} + \frac{1}{f(3)} + \frac{1}{f(4)} + \frac{1}{f(5)} + \dots\right) +$$

$$\left(\frac{1}{f(2) \cdot f(3)} + \frac{1}{f(2) \cdot f(4)} + \frac{1}{f(2) \cdot f(5)} + \dots\right) \cdot \left(\frac{1}{f(2)} + \frac{1}{f(3)} + \frac{1}{f(4)} + \frac{1}{f(5)} + \dots\right) +$$

$$\left(\frac{1}{f(3) \cdot f(4)} + \frac{1}{f(3) \cdot f(5)} + \dots\right) \cdot \left(\frac{1}{f(3)} + \frac{1}{f(4)} + \frac{1}{f(5)} + \dots\right) +$$

$$\left(\frac{1}{f(4) \cdot f(5)} + \dots\right) \cdot \left(\frac{1}{f(4)} + \frac{1}{f(5)} + \dots\right) + \dots$$

which yields the second term in the infinite series,

$$\mathbb{M}_{i=2}^{(-1)^i \sum^{i+1}}$$

or M-series of degree 3, previously defined as

$$+\Sigma_1 \Sigma_2 \Sigma_3 = + \sum_{i=1}^{\infty} \sum_{j=i}^{\infty} \sum_{k=j+1}^{\infty} \frac{1}{f(i)} \frac{1}{f(j)} \frac{1}{f(k)}.$$

The third term of the infinite series

$$\mathbb{M}_{i=1}^{(-1)^i \sum^{i+1}}$$

is also formed by collecting the next level of terms across the summations in the original grouping

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{1}{f(n)} \cdot \left( -\frac{1}{f(1)} + \frac{1}{f(1) \cdot f(2)} + \frac{1}{f(1) \cdot f(3)} - \frac{1}{f(1) \cdot f(2) \cdot f(3)} + \frac{1}{f(1) \cdot f(4)} - \frac{1}{f(1) \cdot f(2) \cdot f(4)} \right. \\ & - \frac{1}{f(1) \cdot f(3) \cdot f(4)} + \frac{1}{f(1) \cdot f(2) \cdot f(3) \cdot f(4)} + \frac{1}{f(1) \cdot f(5)} - \frac{1}{f(1) \cdot f(2) \cdot f(5)} - \frac{1}{f(1) \cdot f(3) \cdot f(5)} \\ & + \frac{1}{f(1) \cdot f(2) \cdot f(3) \cdot f(5)} - \frac{1}{f(1) \cdot f(4) \cdot f(5)} + \frac{1}{f(1) \cdot f(2) \cdot f(4) \cdot f(5)} + \frac{1}{f(1) \cdot f(3) \cdot f(4) \cdot f(5)} \\ & \left. - \frac{1}{f(1) \cdot f(2) \cdot f(3) \cdot f(4) \cdot f(5)} \dots \right) \end{aligned}$$

$$\begin{aligned}
& \sum_{n=2}^{\infty} \frac{1}{f(n)} \cdot \left( -\frac{1}{f(2)} + \frac{1}{f(2) \cdot f(3)} + \frac{1}{f(2) \cdot f(4)} - \frac{1}{f(2) \cdot f(3) \cdot f(4)} + \frac{1}{f(2) \cdot f(5)} - \frac{1}{f(2) \cdot f(3) \cdot f(5)} \right. \\
& \qquad \qquad \qquad \left. - \frac{1}{f(2) \cdot f(4) \cdot f(5)} + \frac{1}{f(2) \cdot f(3) \cdot f(4) \cdot f(5)} \dots \right) \\
& \sum_{n=3}^{\infty} \frac{1}{f(n)} \cdot \left( -\frac{1}{f(3)} + \frac{1}{f(3) \cdot f(4)} + \frac{1}{f(3) \cdot f(5)} - \frac{1}{f(3) \cdot f(4) \cdot f(5)} \dots \right) \\
& \sum_{n=4}^{\infty} \frac{1}{f(n)} \cdot \left( -\frac{1}{f(4)} + \frac{1}{f(4) \cdot f(5)} \dots \right) \\
& \sum_{n=5}^{\infty} \frac{1}{f(n)} \cdot \left( -\frac{1}{f(5)} \dots \right) \\
& \sum_{n=6}^{\infty} \frac{1}{f(n)}
\end{aligned}$$

which are extracted in the same fashion:

$$\begin{aligned}
& \left( -\frac{1}{f(1) \cdot f(2) \cdot f(3)} - \frac{1}{f(1) \cdot f(2) \cdot f(4)} - \frac{1}{f(1) \cdot f(3) \cdot f(4)} - \frac{1}{f(1) \cdot f(2) \cdot f(5)} - \frac{1}{f(1) \cdot f(3) \cdot f(5)} \right. \\
& \qquad \qquad \qquad \left. - \frac{1}{f(1) \cdot f(4) \cdot f(5)} - \dots \right) \cdot \sum_{n=1}^{\infty} \frac{1}{f(n)} + \\
& \left( -\frac{1}{f(2) \cdot f(3) \cdot f(4)} - \frac{1}{f(2) \cdot f(3) \cdot f(5)} - \frac{1}{f(2) \cdot f(4) \cdot f(5)} - \dots \right) \cdot \sum_{n=2}^{\infty} \frac{1}{f(n)} + \\
& \left( -\frac{1}{f(3) \cdot f(4) \cdot f(5)} - \dots \right) \cdot \sum_{n=3}^{\infty} \frac{1}{f(n)} + \dots \\
& = \\
& \left( -\frac{1}{f(1) \cdot f(2) \cdot f(3)} - \frac{1}{f(1) \cdot f(2) \cdot f(4)} - \frac{1}{f(1) \cdot f(3) \cdot f(4)} - \frac{1}{f(1) \cdot f(2) \cdot f(5)} - \frac{1}{f(1) \cdot f(3) \cdot f(5)} \right. \\
& \qquad \qquad \qquad \left. - \frac{1}{f(1) \cdot f(4) \cdot f(5)} - \dots \right) \cdot \left( \frac{1}{f(1)} + \frac{1}{f(2)} + \frac{1}{f(3)} + \frac{1}{f(4)} + \frac{1}{f(5)} + \dots \right) + \\
& \left( -\frac{1}{f(2) \cdot f(3) \cdot f(4)} - \frac{1}{f(2) \cdot f(3) \cdot f(5)} - \frac{1}{f(2) \cdot f(4) \cdot f(5)} - \dots \right) \cdot \left( \frac{1}{f(2)} + \frac{1}{f(3)} + \frac{1}{f(4)} + \frac{1}{f(5)} + \dots \right) +
\end{aligned}$$

$$\left(-\frac{1}{f(3) \cdot f(4) \cdot f(5)}\right) \cdot \left(\frac{1}{f(3)} + \frac{1}{f(4)} + \frac{1}{f(5)} + \dots\right) + \dots$$

which yields the third term in the infinite series,

$$\prod_{i=3}^{\infty} (-1)^i \sum^{i+1}$$

or M-series of degree 4, previously defined as

$$-\Sigma_1 \Sigma_2 \Sigma_3 \Sigma_4 = -\sum_{i=1}^{\infty} \sum_{j=i}^{\infty} \sum_{k=j+1}^{\infty} \sum_{l=k+1}^{\infty} \frac{1}{f(i)} \frac{1}{f(j)} \frac{1}{f(k)} \frac{1}{f(l)}.$$

We now see that a pattern emerges in which it is clear that an infinite sum of infinite sums will be obtained which will add to unity in the numerator of the right-hand side of Eq. 3 if one continues to eliminate terms from the right-hand side of Eq. 4 using the non-sieving method. It is hypothesized that once all terms are sieved out of the right hand side of the big Zeta function in Eq. 4, a constant will emerge for the M-series in Eq. 3 which takes on a finite value between

$$-1 \leq \prod_{i=1}^{\infty} (-1)^i \sum^{i+1} \leq 0$$

which may reveal something about the frequency of prime numbers yielded by an integer-valued polynomial function  $f(n)$  for some limit of the input domain  $n \leq x$ .

Table 3 depicts the M-series values of  $M_{x=100}$  and  $M_{x=200}$  for the non-sieving application of the Euler zeta function to the domain of the integers  $2 \leq i \leq 100$  and  $2 \leq i \leq 200$ . It is observed from this table that as the upper limit increases from  $x = 100$  to  $x = 200$  that the value of the M-series approaches  $-1$ . Table 4 includes the values of  $M_{x=100}$  and  $M_{x=200}$  for the prime shell function  $n^p - (n-1)^p$  for prime powers 2, 3, 5 and 7, and it appears that these values will stabilize to some limit as  $x$  approaches  $\infty$ .

prime number $\geq 2$	$\pi \leq 100$	$L_s \leq 100$	$M_{x=100}$	$\pi \leq 200$	$L_s \leq 200$	$M_{x=200}$
$i$	25	29.99144	-0.94812622482360	46	50.04329	-0.97060984525939

Table 3

prime power $p$	$\Pi \leq f(100)$	$M_s \leq f(100)$	$M_{x=100}$	$\Pi \leq f(200)$	$M_s \leq f(200)$	$M_{x=200}$
2	44	42.75969	-0.70856869191073	76	78.48273	-0.77232394108548
3	43	29.01307	-0.05016737946525	72	53.06455	-0.05053523893596
5	18	19.71488	-0.00129463514931	32	35.92022	-0.00129463735049
7	24	15.71077	-0.00006682330849	40	28.56513	-0.00006682330851

Table 4

It is anticipated that the finite values yielded by the M-series function in non-sieving applications of the Euler zeta function in the case of integer-valued polynomials in general may tell us something about how many prime numbers exist within the ranges that those polynomials were evaluated compared to the number of primes that exist on the real number line less than or equal to some upper limit  $x$  that served as the input domain for the base  $n$  for those functions. Thus, the following theorem is proposed:

**Theorem 3** *When a non-sieving application of the Euler zeta function is applied to process values generated by an integer-valued polynomial, then there is an infinite series*

$$\prod_{i=2}^{\infty} [(-1)^{i-1}] \sum^i$$

*that arises which adds to unity in the numerator of the product term in the zeta function to make both sides of the equation equal. The value of this M-series is bounded by  $-1$  and  $0$  in the non-sieving application of the Euler zeta function.*

It is the author's hope that the results of this study will motivate further research into the behavior of prime number frequency among the prime shell function and any other integer-generating polynomial function in general which has the capability to generate prime numbers.

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