Equivalence classes of ballot paths modulo strings of length 2 and 3

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Abstract

Two paths are equivalent modulo a given string τ, whenever they have the same length and the positions of the occurrences of τ are the same in both paths. This equivalence relation was introduced for Dyck paths in [1], where the number of equivalence classes was evaluated for any string of length 2.

In this paper, we evaluate the number of equivalence classes in the set of ballot paths for any string of length 2 and 3, as well as in the set of Dyck paths for any string of length 3.

1 Introduction

Throughout this paper, a *path* is considered to be a lattice path on the integer plane, consisting of steps $\mathrm{u} = (1,1)$ (called *rises*) and $\mathrm{d} = (1,-1)$ (called *falls*). Since the sequence of steps of a path is encoded by a word in $\{\mathrm{u},\mathrm{d}\}^*$, we will make no distinction between these two notions. The *length* $|\alpha|$ of a path α is the number of its steps. The *height of a point* of a path is its y-coordinate. For any path α and $i \geq 1$, we define $\alpha^i = \alpha\alpha\cdots\alpha$ (i times), and $\alpha^0 = \varepsilon$, where ε is the empty path.

A *Dyck path* is a path that starts and ends at the same height and lies weakly above this height. It is convenient to consider that the starting point of a Dyck path is the origin of a pair of axes.

The set of Dyck paths of semilength n is denoted by $\mathcal{D}_n$, and we set $\mathcal{D} = \bigcup_{n\geq 0} \mathcal{D}_n$, where $\mathcal{D}_0 = \{\varepsilon\}$. It is well known that $|\mathcal{D}_n| = C_n$, where $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the n-th Catalan number (sequence A000108 in [10]). The generating function of the Catalan numbers is denoted by $C(x)$, where

$$C(x) = 1 + xC^2(x) = \frac{1-\sqrt{1-4x}}{2x} = \sum_{n\geq 0} C_n x^n.$$

We will also use the generating function $M(x)$ of the Motzkin numbers (sequence A001006 in [10]), where

$$M(x) = 1 + xM(x) + x^2M^2(x) = \frac{1-x-\sqrt{1-2x+3x^2}}{2x^2}.$$

A path which is a prefix of a Dyck path, is called *ballot path*. The set of ballot paths of length n is denoted by $\mathcal{P}_n$ and $\mathcal{P} = \bigcup_{n\geq 0} \mathcal{P}_n$.

A path $\tau \in \{\mathrm{u},\mathrm{d}\}^*$, called in this context *string*, *occurs* in a path α if $\alpha = \beta\tau\gamma$, for some $\beta,\gamma \in \{\mathrm{u},\mathrm{d}\}^*$. The number of occurrences of the string τ in α, is denoted by $|\alpha|_\tau$. In particular, $|\alpha|_{\mathrm{u}}$ and $|\alpha|_{\mathrm{d}}$ are the number of rises and falls of α respectively. We also use the notation $h(\alpha) = |\alpha|_{\mathrm{u}} - |\alpha|_{\mathrm{d}}$, which corresponds to the height of the last point of α, whenever α starts at height zero.

We say that an occurrence of the string τ in the path α is at height $j \geq 0$, whenever the minimum height of the points of τ in this occurrence is equal to j. Furthermore, we say that an occurrence of the string τ in a path α of length n is at position $i \in [n] = \{1, 2, \ldots, n\}$, whenever the first step of this occurrence is the i-th step of α.

Many articles dealing with the number of occurrences of strings in Dyck paths have appeared in the literature (e.g., [3, 6, 7, 8, 9]). More general results on this subject are given in [4, 5].

In another direction, taking into account not only the number of occurrences of the string τ in a path, but also the positions of these occurrences, we consider the equivalence relation $\underset{\tau}{\sim}$ on the set of paths defined by

$$\alpha \underset{\tau}{\sim} \alpha' \text{ iff } |\alpha| = |\alpha'| \text{ and the positions of the occurrences of } \tau \text{ in } \alpha \text{ and } \alpha' \text{ are the same.}$$

Recently, Baril and Petrossian in [1] and [2], introduced this equivalence relation on the set of Dyck paths and the set of Motzkin paths respectively, and they evaluated the number of τ-equivalence classes (i.e., the classes with respect to $\underset{\tau}{\sim}$) in terms of generating functions for any string τ of length at most 2.

In this paper, we examine this equivalence relation on the set of ballot paths and we evaluate the number of τ-equivalence classes of $\mathcal{P}_n$, for every string τ of length 2 or 3, as well as the number of *Dyck classes* (i.e., classes containing at least one Dyck path), for every string τ of length 3.

Firstly, for each case of τ, we decompose the paths $p \in \mathcal{P}$ as

$$p = a_0 c_1 a_1 c_2 a_2 \cdots c_k a_k, \qquad k \geq 0, \tag{1.1}$$

where the c_i's are the maximal clusters of τ in p and the a_i's are the τ-*components*, i.e., the subpaths of p avoiding τ and lying between these clusters. We also use the notation $a(p) = a_k$. Clearly two ballot paths $p = a_0 c_1 a_1 c_2 a_2 \cdots c_k a_k$ and $p' = a'_0 c'_1 a'_1 c'_2 a'_2 \cdots c'_{k'} a'_{k'}$ are τ-equivalent iff $k = k'$ and $|a_i| = |a'_i|$, $c_i = c'_i$, for all $i \leq k$.

Next, by assuming several conditions for the τ-components, we define a set $\mathcal{A}_n^{(\tau)}$ of representatives of length n, where $\mathcal{A}^{(\tau)} = \bigcup_{n \geq 0} \mathcal{A}_n^{(\tau)}$ and $\mathcal{A}_0^{(\tau)} = \{\varepsilon\}$, and we prove the following result:

Proposition 1.1. *For every $p \in \mathcal{P}_n$, there exists a unique $p' \in \mathcal{A}_n^{(\tau)}$, such that $p \underset{\tau}{\sim} p'$, $n \geq 0$.*

Finally, by enumerating the set of representatives, we evaluate the number of τ-equivalence classes in both cases of ballot and Dyck paths. As a result, in several cases we obtain some well known sequences, e.g., the Fibonacci numbers f_n (see sequence A000045 in [10]), whereas in other cases we introduce some new ones.

Throughout this paper, we denote by h_i, $i \in [k]$, the height of the last point of c_i and we define $h_0 = 0$ and h_{k+1} to be the height of the last point of a_k. We can easily check that, for any string τ, if $p, p' \in \mathcal{P}$, such that $p \sim p'$, and p, p' are decomposed according to (1.1), i.e.,

$$p = a_0 c_1 a_1 \cdots c_k a_k, \qquad p' = a'_0 c_1 a'_1 \cdots c_k a'_k,$$

then

$$h_{i+1} - h'_{i+1} = h_i - h'_i + 2(|a_i|_{\mathrm{u}} - |a'_i|_{\mathrm{u}}), \tag{1.2}$$

which gives that h_i and h'_i, have the same parity, $0 \leq i \leq k+1$.

Whenever there is no danger of confusion, we will write for simplicity $\sim$, $\mathcal{A}$, "component" and "equivalent" instead of $\underset{\tau}{\sim}$, $\mathcal{A}^{(\tau)}$, "τ-component" and "τ-equivalent" respectively. All sets are denoted by a capital calligraphic letter, while their corresponding generating functions with respect to the length are denoted by the same capital plain letter. The bar sign always denotes the intersection of the set with $\mathcal{D}$ (e.g., $\bar{\mathcal{A}} = \mathcal{A} \cap \mathcal{D}$). Furthermore, for any path α, we use the subscript α in the notation of a set to denote its subset of paths starting with α, e.g., $\mathcal{B}_\alpha = \{p \in \mathcal{B} : p \text{ starts with } \alpha\}$ and B_α denotes the corresponding generating function. We also use the Iverson notation $[P]$, which Proposition P is equal to 1, if P is true, or 0, if P is false.

There are several different options for the definition of the set $\mathcal{A}$ of representatives. In general, we prefer to define $\mathcal{A}$ so that the sequence (h_i) of each representative is the maximum in its class, because this makes the task of enumeration much easier. When this is the case, the proof of Proposition 1.1 is obvious and it is omitted. However, when we are also interested in evaluating the number of Dyck classes, it is more convenient to define the set of representatives so that this sequence is the minimum in its class (see the cases uuu, uud, duu). Then, the set of representatives of Dyck classes is the set $\bar{\mathcal{A}}$. Unfortunately, this is not the case for the string udu, where $\mathcal{A}$ is defined so that this sequence is the lexicographically minimum in its class and the representatives of Dyck classes are not always Dyck paths. This makes this case far more complex than the others.

In sections 2 and 3, we enumerate the τ-equivalence classes of ballot paths for any string τ of length 2 and 3 respectively. In section 4, we enumerate the τ-equivalence classes of Dyck paths for any string τ of length 3.

The next two Tables summarize the results of the following sections, where the entries in the last column refer to the corresponding sequence in [10] (OEIS), whereas they are left blank whenever this sequence is a new one. The first three rows of Table 2 were given in [1].

$\tau \setminus n$	1	2	3	4	5	6	7	8	9	10	11	12	OEIS
du	1	1	2	3	5	8	13	21	34	55	89	144	A000045
ud	1	2	3	5	8	13	21	34	55	89	144	233	du shifted
uu	1	2	3	5	8	14	24	42	73	128	224	393	
dd	1	1	1	2	3	5	7	12	18	31	47	81	A191385
uuu	1	1	2	3	5	8	13	21	34	55	89	144	A000045
ddd	1	1	1	1	1	2	3	5	7	10	13	20	
uud	1	1	2	3	4	6	9	13	19	28	41	60	A000930
duu	1	1	1	2	3	4	6	9	13	19	28	41	uud shifted
udd	1	1	1	2	3	4	5	8	12	17	23	35	
ddu	1	1	1	1	2	3	4	5	8	12	17	23	udd shifted
udu	1	1	2	3	5	8	13	21	34	55	89	144	A000045
dud	1	1	1	2	3	5	7	11	16	26	39	63	

Table 1: Number of equivalence classes of ballot paths of length n for various strings.

2 Strings of length 2

2.1 The strings ud and du

Every $p \in \mathcal{P}$ is uniquely decomposed according to (1.1) as $p = a_0\mathrm{ud}a_1\mathrm{ud}a_2 \cdots \mathrm{ud}a_k$.

Similarly, every p is uniquely decomposed as $p = a_0\mathrm{du}a_1\mathrm{du}a_2 \cdots \mathrm{du}a_k$.

$\tau \setminus n$	1	2	3	4	5	6	7	8	9	10	11	12	OEIS
uu, dd	1	2	4	9	22	56	147	393	1065	2915	8042	22330	A244886
ud	1	2	5	14	41	121	354	1021	2901	8130	22513	61713	A244885
du	1	2	5	13	34	89	233	610	1597	4181	10946	28657	A001519
uuu, ddd	1	1	2	4	8	17	37	81	180	405	917	2090	
uud, udd	1	2	4	8	17	35	75	157	337	712	1529	3248	
duu, ddu	1	1	2	4	8	17	35	75	157	337	712	1529	uud shifted
udu, dud	1	2	4	9	22	54	134	335	843	2132	5409	13761	

Table 2: Number of equivalence classes of Dyck paths of semilength n for various strings.

Let $\mathcal{A}^{(\mathrm{ud})}$ (resp. $\mathcal{A}^{(\mathrm{du})}$) be the set of ballot paths, the components of which have the form $a_i = \mathrm{u}^s, \quad s \geq 0$.

Equivalently, $\mathcal{A}^{(\mathrm{ud})}$ (resp. $\mathcal{A}^{(\mathrm{du})}$) is the set of ballot paths avoiding dd (resp. avoiding dd and ending with u). Hence, by deleting the last rise of every path in $\mathcal{A}^{(\mathrm{du})}_{n+1}$, we obtain bijectively every path in $\mathcal{A}^{(\mathrm{ud})}_n$, so that

$$|\mathcal{A}^{(\mathrm{du})}_{n+1}| = |\mathcal{A}^{(\mathrm{ud})}_n|. \tag{2.1}$$

Then, we have the following result:

Proposition 2.1. *The number of* ud *(resp.* du*)-equivalence classes of ballot paths of length $n \geq 1$ is equal to f_{n+1} (resp. f_n).*

Proof. Clearly, $|\mathcal{A}^{(\mathrm{ud})}_1| = 1$ and $|\mathcal{A}^{(\mathrm{ud})}_2| = 2$. Furthermore, from each path $p \in \mathcal{A}^{(\mathrm{ud})}_{n+2}$, by deleting its last ud, if $a(p) = \varepsilon$, or its last step, if $a(p) \neq \varepsilon$, we obtain respectively a path $q_1 \in \mathcal{A}^{(\mathrm{ud})}_n$ or a path $q_2 \in \mathcal{A}^{(\mathrm{ud})}_{n+1}$. Since this procedure is clearly reversible, we obtain that $|\mathcal{A}^{(\mathrm{ud})}_{n+2}| = |\mathcal{A}^{(\mathrm{ud})}_{n+1}| + |\mathcal{A}^{(\mathrm{ud})}_n|$, $n \geq 1$.

The result for the du-equivalence classes then follows from relation (2.1). ∎

2.2 The string uu

Every $p \in \mathcal{P}$ is uniquely decomposed according to (1.1) as $p = a_0\mathrm{u}^{r_1}a_1\mathrm{u}^{r_2}a_2\cdots\mathrm{u}^{r_k}a_k$, $r_i \geq 2$, $i \in [k]$. Notice that $a_0 = (\mathrm{ud})^t$, $t \geq 0$, whereas a_0 can also be equal to $(\mathrm{ud})^t\mathrm{u}$, $t \geq 0$, for $k = 0$.

Let $\mathcal{A}$ be the set of ballot paths, the components of which have either one of the following forms:

$$a_i = \mathrm{d}^s, \qquad a_i = \mathrm{d}^{h_i}(\mathrm{ud})^t, \qquad a_i = \mathrm{d}^{h_i - 1}(\mathrm{ud})^t,$$

where $i, s, t \geq 1$, and a_k can also be empty.

Equivalently, $\mathcal{A}$ is the set of ballot paths p avoiding dudd, avoiding dud at height greater than 1, and not ending with du, unless $p = (\mathrm{ud})^t\mathrm{u}$, $t \geq 1$.

Proof of Proposition 1.1 for $\tau =$ uu. Given a ballot path $p = a_0\mathrm{u}^{r_1}a_1\mathrm{u}^{r_2}a_2\cdots\mathrm{u}^{r_k}a_k$, setting $h'_0 = 0$, $a'_0 = a_0$ and, for $i \in [k]$, $a'_i = \mathrm{d}^{s_i}(\mathrm{du})^{t_i}$ and $h'_{i+1} = h'_i + h(a'_i)$, where

$$s_i = \begin{cases} |a_i|, & |a_i| \leq h'_i, \\ h'_i, & |a_i| - h'_i > 0 \text{ is even}, \\ h'_i - 1, & |a_i| - h'_i > 0 \text{ is odd}, \end{cases}$$

and $2t_i = |a_i| - s_i$, we obtain inductively a sequence of paths a'_i. Let $p' = a'_0\mathrm{u}^{r_1}a'_1\mathrm{u}^{r_2}a'_2\cdots\mathrm{u}^{r_k}a'_k$. It is easy to check that $p' \in \mathcal{A}$ and $p' \sim p$. Furthermore, since no two paths of $\mathcal{A}$ are equivalent, we obtain the required result. □

Proposition 2.2. *The number of* uu-*equivalence classes of ballot paths of length n is equal to the n-th coefficient of the generating function*

$$A(x) = \frac{1-x-x^4}{1-2x+x^3-x^4+x^5}.$$

Proof. For the enumeration of the set $\mathcal{A}$, we define the sets

$$\mathcal{B} = \{p \in \mathcal{A} : p \text{ avoids dud at height greater than zero}\}$$

and

$$\mathcal{G} = \{p \in \mathcal{A} : p \text{ avoids dud}\}.$$

We first consider the following decomposition of a path $p \in \mathcal{A} \setminus \bar{\mathcal{A}}$:

$$p = (\mathrm{ud})^i\mathrm{u}, i \geq 0, \quad \text{or} \quad p = \alpha\mathrm{u}\beta, \quad \text{or} \quad p = \alpha\mathrm{uu}, \quad \text{or} \quad p = \alpha\mathrm{u}\beta'\mathrm{u}q,$$

where $\alpha \in \bar{\mathcal{A}}$, $\beta \in \bar{\mathcal{B}} \setminus \{\varepsilon\}$, $\beta' \in \bar{\mathcal{B}}$ and $q \in \mathcal{G} \setminus \{\varepsilon, \mathrm{udu}\}$. It follows that

$$A - \bar{A} = \frac{x}{1-x^2} + x\bar{A}(\bar{B}-1) + x^2\bar{A} + x^2\bar{A}\bar{B}(G-1-x^3),$$

so that

$$A = \frac{x}{1-x^2} + \bar{A}\left(1 + x(\bar{B}-1) + x^2 + x^2\bar{B}(G-1-x^3)\right). \tag{2.2}$$

Next, by considering the decomposition $\alpha = \beta\mathrm{u}\gamma\mathrm{d}$ of a path $\alpha \in \bar{\mathcal{A}} \setminus \{\varepsilon\}$, where $\beta \in \bar{A}$, $\gamma \in \bar{\mathcal{B}}$ and γ does not end with dud, we obtain that $\bar{A} - 1 = x^2\bar{A}(\bar{B} - x^2(\bar{B}-1))$, which gives

$$\bar{A} = \frac{1}{1-x^2(x^2+\bar{B}-x^2\bar{B})} = \frac{1}{1-x^4-x^2(1-x^2)\bar{B}}, \tag{2.3}$$

while, by considering the same decomposition for the set $\bar{\mathcal{B}} \setminus \{\varepsilon\}$, where $\beta \in \bar{B}$, $\gamma \in \bar{\mathcal{G}}$, we obtain that $\bar{B} - 1 = x^2\bar{B}\bar{G}$, which gives

$$\bar{B} = \frac{1}{1-x^2\bar{G}}. \tag{2.4}$$

Furthermore, by considering the following decomposition of a path $q \in \mathcal{G} \setminus \bar{\mathcal{G}}$:

$$q = \mathrm{u}r, \quad \text{or} \quad q = \mathrm{udu}r, \quad \text{or} \quad q = \gamma\mathrm{u}r',$$

where $r \in \mathcal{G} \setminus \{\mathrm{udu}\}$, $\gamma \in \bar{\mathcal{G}} \setminus \{\varepsilon, \mathrm{ud}\}$ and $r' \in \mathcal{G} \setminus \{\varepsilon, \mathrm{udu}\}$, we deduce that

$$G - \bar{G} = (x+x^3)(G-x^3) + x(\bar{G}-1-x^2)(G-1-x^3),$$

which gives

$$G - 1 - x^3 = \frac{\bar{G}-1+x}{1-x\bar{G}}. \tag{2.5}$$

Finally, since $\bar{G} = 1 + x^2M(x^2)$ (see [11]), using relations (2.2), (2.3), (2.4) and (2.5), after some simple calculations, we obtain the required result. □

2.3 The string dd

Every $p \in \mathcal{P}$ is uniquely decomposed according to (1.1) as $p = a_0 \mathrm{d}^{r_1} a_1 \mathrm{d}^{r_2} a_2 \cdots \mathrm{d}^{r_k} a_k$, $r_i \geq 2$, $i \in [k]$.

Let $\mathcal{A}$ be the set of ballot paths, the nonempty components of which have the form $a_i = \mathrm{u}^s$, $s \geq 1$, whereas a_k can also be empty.

Equivalently, $\mathcal{A}$ is the set of ballot paths avoiding udu and not ending with ud.

Proposition 2.3. *The number of* dd*-equivalence classes of ballot paths of length n is equal to the n-th coefficient of the generating function*

$$A(x) = \frac{1 + x^2 M(x^2)}{1 - x + x^2 - x^3 M(x^2)}.$$

Proof. A path $p \in \mathcal{A} \setminus \bar{\mathcal{A}}$ is decomposed as $p = \alpha \mathrm{u} q$, where $\alpha \in \bar{\mathcal{A}}$, $q \in \mathcal{A}$. Hence, $A = \bar{A} + x\bar{A}A$, so that

$$A = \frac{\bar{A}}{1 - x\bar{A}}. \tag{2.6}$$

Clearly, every path $\beta \in \mathcal{D}$ avoiding udu either belongs to $\bar{\mathcal{A}}$ or it has the form $\beta = \alpha \mathrm{ud}$, $\alpha \in \bar{\mathcal{A}}$. Then, since Dyck paths avoiding udu are enumerated by $1 + x^2 M(x^2)$ (see [11]), we have that

$$1 + x^2 M(x^2) = \bar{A} + x^2 \bar{A}. \tag{2.7}$$

From relations (2.6) and (2.7), we obtain the required result. □

Remark: The coefficients of $A(x)$ form the sequence A191385 in [10], where a different combinatorial interpretation is given: $[x^n]A(x)$ is the number of Motzkin paths of length n with no horizontal steps h at positive height and no ascents of length 1.

More generally, the number of d^μ-equivalence classes, $\mu \geq 2$, of ballot paths of length n is equal to the number of Motzkin paths of length n with no h at positive height and no ascents of length less than μ. To verify this, we consider the mapping ϕ between the set of ballot paths and the set of Motzkin paths with no h at positive height, defined recursively as follows:

$$\phi(\varepsilon) = \varepsilon, \qquad \phi(\mathrm{u}p) = \mathrm{h}\phi(p), \qquad \phi(\mathrm{u}\alpha\mathrm{d}p) = \mathrm{u}r(\alpha)\mathrm{d}\phi(p),$$

where $p \in \mathcal{P}$, $\alpha \in \mathcal{D}$ and $r(\alpha)$ is the reversed path of α (i.e., its symmetrical path with respect to a vertical axis). It is easy to check that ϕ is a length preserving bijection. Furthermore, since $\mathcal{A}^{(\mathrm{d}^\mu)}$ is the set of ballot paths avoiding $\mathrm{ud}^i\mathrm{u}$, $i \in [\mu - 1]$, ending with u or d^μ, we can easily deduce that $\phi\left(\mathcal{A}^{(\mathrm{d}^\mu)}\right)$ is equal to the set of Motzkin paths of length n with no h at positive height and no ascents of length less than μ, giving the required result.

3 Strings of length 3

3.1 The string uuu

Every $p \in \mathcal{P}$ is uniquely decomposed according to (1.1) as $p = a_0 \mathrm{u}^{r_1} a_1 \mathrm{u}^{r_2} a_2 \cdots \mathrm{u}^{r_k} a_k$, $r_i \geq 3$, $i \in [k]$.

Let $\mathcal{A}$ be the set of ballot paths, with their components a_i, $i \in [k]$, having either one of the following forms:

$$a_i = \mathrm{d}^s, \qquad a_i = \mathrm{d}^{h_i}(\mathrm{ud})^t, \qquad a_i = \mathrm{d}^{h_i - 1}(\mathrm{ud})^t,$$

where $s,t\geq 1$, while a_k can also be empty and $a_0\in\{\varepsilon,(\mathrm{ud})^t,\mathrm{u}(\mathrm{ud})^t : t\geq 1\}$.

Equivalently, $\mathcal{A}$ is the set of ballot paths p not starting with uudd, avoiding dudd and duud, avoiding dud at height greater than 1, and ending with uuu or with d, unless $p=\mathrm{u}$.

Proof of Proposition 1.1 for $\tau=\mathrm{uuu}$. Given a ballot path $p=a_0\mathrm{u}^{r_1}a_1\mathrm{u}^{r_2}a_2\cdots\mathrm{u}^{r_k}a_k$, setting $h'_0=0$, $a'_0=(\mathrm{ud})^{|a_0|/2}$ if $|a_0|$ is even or $a'_0=\mathrm{u}(\mathrm{ud})^{(|a_0|-1)/2}$ if $|a_0|$ is odd, $a'_i=\mathrm{d}^{s_i}(\mathrm{ud})^{t_i}$, $i\in[k]$, and $h'_{i+1}=h'_i+h(a'_i)+r_{i+1}$, $0\leq i\leq k$, where $r_{k+1}=0$ and, for $i\geq 1$,

$$s_i=\begin{cases}|a_i|, & |a_i|\leq h'_i,\\ h'_i, & |a_i|-h'_i>0 \text{ is even},\\ h'_i-1, & |a_i|-h'_i>0 \text{ is odd},\end{cases}$$

and $2t_i=|a_i|-s_i$, we obtain inductively a sequence of paths a'_i. Let $p'=a'_0\mathrm{u}^{r_1}a'_1\mathrm{u}^{r_2}a'_2\cdots\mathrm{u}^{r_k}a'_k$. It is easy to check that $p'\in\mathcal{A}$ and $p'\sim p$. Furthermore, since no two paths of $\mathcal{A}$ are equivalent, we obtain the required result. □

Proposition 3.1. *The number of* uuu*-equivalence classes of ballot paths of length $n\geq 1$ is equal to f_n.*

Proof. Clearly, $|\mathcal{A}_1|=|\mathcal{A}_2|=1$. Let $\mathcal{S}_n$ (resp. $\mathcal{T}_n$) be the set of all $p\in\mathcal{A}_n$ with $a(p)=\varepsilon$ (resp. $a(p)\neq\varepsilon$). We first show that

$$|\mathcal{T}_{n+1}|=|\mathcal{A}_n|,\qquad n\geq 0. \tag{3.1}$$

Indeed, for $p=a_0\mathrm{u}^{r_1}a_1\cdots\mathrm{u}^{r_k}a_k\in\mathcal{T}_{n+1}$, so that $a_k\neq\varepsilon$, we define a path $q\in\mathcal{A}_{n+1}$ as follows:

$$q=\begin{cases}a_0\mathrm{u}^{r_1}a_1\cdots\mathrm{u}^{r_k}a'_k, & k>0,\\ a'_0, & k=0,\end{cases}$$

where

$$a'_k=\begin{cases}\mathrm{d}^{s-1}, & a_k=\mathrm{d}^s,\\ \mathrm{d}^{h_k}(\mathrm{ud})^{t-1}, & a_k=\mathrm{d}^{h_k-1}(\mathrm{ud})^t,\\ \mathrm{d}^{h_k-1}(\mathrm{ud})^t, & a_k=\mathrm{d}^{h_k}(\mathrm{ud})^t,\end{cases}\qquad\text{and}\qquad a'_0=\begin{cases}\mathrm{u}(\mathrm{ud})^{t-1}, & a_0=(\mathrm{ud})^t,\\ (\mathrm{ud})^t, & a_0=\mathrm{u}(\mathrm{ud})^t.\end{cases}$$

Since the mapping $p\mapsto q$ is clearly a bijection, we obtain relation (3.1).

Next, we show that

$$|\mathcal{S}_{n+2}|=|\mathcal{S}_n|+|\mathcal{T}_{n-2}|+|\mathcal{T}_{n-1}|,\qquad n\geq 1. \tag{3.2}$$

Indeed, let $p=a_0\mathrm{u}^{r_1}a_1\cdots\mathrm{u}^{r_{k-1}}a_{k-1}\mathrm{u}^{r_k}\in\mathcal{S}_{n+2}$. If $r_k\geq 5$, then, by deleting the last 2 rises, we obtain a path of $\mathcal{S}_n$, whereas, if $r_k=4$ (resp. $r_k=3$), then, by deleting u^{r_k} we obtain a path in $\mathcal{T}_{n-2}$ (resp. $\mathcal{T}_{n-1}$). Since this procedure is reversible, we obtain relation (3.2).

Then, using induction and relations (3.1) and (3.2), we show that $|\mathcal{A}_{n+2}|=|\mathcal{A}_{n+1}|+|\mathcal{A}_n|$. Indeed,

$$\begin{aligned}|\mathcal{A}_{n+2}|&=|\mathcal{T}_{n+2}|+|\mathcal{S}_{n+2}|=|\mathcal{A}_{n+1}|+|\mathcal{S}_n|+|\mathcal{T}_{n-2}|+|\mathcal{T}_{n-1}|\\&=|\mathcal{A}_{n+1}|+|\mathcal{S}_n|+|\mathcal{A}_{n-3}|+|\mathcal{A}_{n-2}|=|\mathcal{A}_{n+1}|+|\mathcal{S}_n|+|\mathcal{A}_{n-1}|\\&=|\mathcal{A}_{n+1}|+|\mathcal{S}_n|+|\mathcal{T}_n|=|\mathcal{A}_{n+1}|+|\mathcal{A}_n|.\end{aligned}$$

□

3.2 The string ddd

Every $p \in \mathcal{P}$ is uniquely decomposed according to (1.1) as $p = a_0\mathrm{d}^{r_1}a_1\mathrm{d}^{r_2}a_2\cdots\mathrm{d}^{r_k}a_k$, $r_i \geq 3$, $i \in [k]$.

Let $\mathcal{A}$ be the set of ballot paths, the nonempty components of which have the form $a_i = \mathrm{u}^s$, $s \geq 1$, whereas a_k can also be empty.

Equivalently, $\mathcal{A}$ is the set of ballot paths avoiding udu and uddu, and ending with u or ddd.

Proposition 3.2. *The number of* ddd*-equivalence classes of ballot paths of length n is equal to the n-th coefficient of the generating function $A = A(x)$, satisfying the relation*

$$x^2(1-2x+x^2-x^4)A^3 + 2x(1-x)^2A^2 + (1-3x+x^2)A - 1 = 0.$$

Proof. A path $p \in \mathcal{A} \setminus \bar{\mathcal{A}}$ is decomposed as $p = \alpha \mathrm{u} q$, where $\alpha \in \bar{\mathcal{A}}$, $q \in \mathcal{A}$. Hence, $A = \bar{A} + x\bar{A}A$, so that

$$\bar{A} = \frac{A}{1+xA}. \tag{3.3}$$

Moreover, a nonempty path $\alpha \in \bar{\mathcal{A}}$ is decomposed as

$$\alpha = \beta\mathrm{u}\gamma'\mathrm{d}, \quad \text{or} \quad \alpha = \beta\mathrm{u}\gamma\mathrm{u}\delta\mathrm{uddd}, \qquad \beta,\gamma,\delta \in \bar{\mathcal{A}}, \gamma' \in \bar{\mathcal{A}} \setminus \{\varepsilon\}.$$

Hence,

$$\bar{A} - 1 = x^2\bar{A}(\bar{A}-1) + x^6\bar{A}^3. \tag{3.4}$$

From relations (3.3) and (3.4), we obtain the required result. □

3.3 The strings uud and duu

Every $p \in \mathcal{P}$ is uniquely decomposed according to (1.1) as $p = a_0\mathrm{uud}a_1\mathrm{uud}a_2\cdots\mathrm{uud}a_k$.

Similarly, every p is uniquely decomposed as $p = a_0\mathrm{duu}a_1\mathrm{duu}a_2\cdots\mathrm{duu}a_k$.

Let $\mathcal{A}^{(\mathrm{uud})}$ be the set of ballot paths with their components a_i having either one of the forms:

$$a_i = \mathrm{d}^s, s \leq h_i - 1, \qquad a_i = \mathrm{d}^{h_i}(\mathrm{ud})^t, \qquad a_i = \mathrm{d}^{h_i}(\mathrm{ud})^t\mathrm{u},$$

where $s, t \geq 0$, and let $\mathcal{A}^{(\mathrm{duu})}$ be the set of ballot paths with their components a_i having either one of the forms:

$$a_0 = (\mathrm{ud})^t\mathrm{u}, \qquad a_0 = (\mathrm{ud})^t\mathrm{u}^2, k > 0, \qquad a_0 = (\mathrm{ud})^t, k = 0,$$
$$a_i = \mathrm{d}^s, s \leq h_i - 2, \qquad a_i = \mathrm{d}^{h_i-1}(\mathrm{ud})^t, \qquad a_i = \mathrm{d}^{h_i-1}(\mathrm{ud})^t\mathrm{u},$$

where $0 < i < k$ and $s, t \geq 0$,

$$a_k = \mathrm{d}^s, s \leq h_k - 1, \qquad a_k = \mathrm{d}^{h_k}(\mathrm{ud})^t, \qquad a_k = \mathrm{d}^{h_k}(\mathrm{ud})^t\mathrm{u},$$

where $k > 0$ and $s, t \geq 0$.

We will give the proof of Proposition 1.1 for the string duu only, since the proof for uud it is similar and slightly easier.

Proof of Proposition 1.1 for $\tau = \mathrm{duu}$. Given a ballot path $p = a_0\mathrm{duu}a_1\mathrm{duu}a_2\cdots\mathrm{duu}a_k$, setting $h'_0 = 0$, $a'_0 = (\mathrm{ud})^{(|a_0|-1)/2}\mathrm{u}$, if $|a_0|$ is odd, $a'_0 = (\mathrm{ud})^{|a_0|/2}\mathrm{u}$, if $|a_0|$ is even and $k = 0$, and $a'_0 = (\mathrm{ud})^{(|a_0|-2)/2}\mathrm{u}^2$, if $|a_0|$ is even and $k > 0$, and $h'_{i+1} = h'_i + h(a'_i) + [i < k]$, $i \in [k]$, and also setting

$$a'_i = \begin{cases} \mathrm{d}^{|a_i|}, & |a_i| \le h'_i - 1, \\ \mathrm{d}^{h'_i-1}(\mathrm{ud})^{(|a_i|-h'_i)/2}\mathrm{u}, & |a_i| - h'_i > 0 \text{ is even}, \\ \mathrm{d}^{h'_i-1}(\mathrm{ud})^{(|a_i|-h'_i+1)/2}, & |a_i| - h'_i > 0 \text{ is odd}, \end{cases} \qquad 1 \le i < k,$$

and

$$a'_k = \begin{cases} \mathrm{d}^{|a_k|}, & |a_k| \le h'_k, \\ \mathrm{d}^{h'_k}(\mathrm{ud})^{(|a_k|-h'_k)/2}, & |a_k| - h'_k > 0 \text{ is even}, \\ \mathrm{d}^{h'_k}(\mathrm{ud})^{(|a_k|-h'_k-1)/2}\mathrm{u}, & |a_k| - h'_k > 0 \text{ is odd}, \end{cases} \qquad k > 0,$$

we obtain inductively a sequence of paths a'_i. Let $p' = a'_0\mathrm{duu}a'_1\mathrm{duu}a'_2\cdots\mathrm{duu}a'_k$. It is easy to check that $p' \in \mathcal{A}^{(\mathrm{duu})}$ and $p' \underset{\mathrm{duu}}{\sim} p$. Furthermore, since no two paths of $\mathcal{A}^{(\mathrm{duu})}$ are equivalent, we obtain the required result. □

We note that

$$|\mathcal{A}_1^{(\mathrm{uud})}| = |\mathcal{A}_2^{(\mathrm{uud})}| = 1 \qquad \text{and} \qquad |\mathcal{A}_3^{(\mathrm{uud})}| = 2,$$
$$|\mathcal{A}_1^{(\mathrm{duu})}| = |\mathcal{A}_2^{(\mathrm{duu})}| = |\mathcal{A}_3^{(\mathrm{duu})}| = 1,$$

and

$$|\mathcal{A}_{n+1}^{(\mathrm{duu})}| = |\mathcal{A}_n^{(\mathrm{uud})}|, \qquad n \ge 3. \tag{3.5}$$

For the proof of the last equality, consider the bijection which maps each

$$p = a_0\mathrm{duu}a_1\mathrm{duu}a_2\cdots\mathrm{duu}a_{k-1}\mathrm{duu}a_k \in \mathcal{A}^{(\mathrm{duu})}$$

to the path

$$q = a'_0\mathrm{udd}a_1\mathrm{udd}a_2\cdots\mathrm{udd}a_{k-1}\mathrm{udd}a'_k \in \mathcal{A}^{(\mathrm{uud})},$$

where a'_0 is obtained by deleting the last step of a_0 and, for $k > 0$,

$$a'_k = \begin{cases} a_k, & a_k = \mathrm{d}^s, 0 \le s \le h_k - 1, \\ \mathrm{d}^{h_k-1}(\mathrm{ud})^{t+1}, & a_k = \mathrm{d}^{h_k}(\mathrm{ud})^t\mathrm{u}, t \ge 0, \\ \mathrm{d}^{h_k-1}(\mathrm{ud})^t\mathrm{u}, & a_k = \mathrm{d}_k^h(\mathrm{ud})^t, t \ge 0. \end{cases}$$

Proposition 3.3. *The number $|\mathcal{A}_n^{(\tau)}|$ of τ-equivalence classes of ballot paths of length $n \ge 1$ satisfies the recurrence relation*

$$|\mathcal{A}_{n+3}^{(\tau)}| = |\mathcal{A}_{n+2}^{(\tau)}| + |\mathcal{A}_n^{(\tau)}|,$$

where $\tau \in \{\mathrm{uud}, \mathrm{duu}\}$.

Proof. From each path $p \in \mathcal{A}_{n+3}^{(\mathrm{uud})}$, by deleting its last uud, if $a(p) = \varepsilon$, or its last step, if $a(p) \ne \varepsilon$, we obtain respectively a path $q \in \mathcal{A}_n^{(\mathrm{uud})}$ or a path $q \in \mathcal{A}_{n+2}^{(\mathrm{uud})}$. Since this procedure is clearly reversible, we obtain the required relation for $\tau = \mathrm{uud}$.

The result for the string duu then follows from relation (3.5). □

3.4 The strings udd and ddu

Every $p \in \mathcal{P}$ is uniquely decomposed according to (1.1) as $p = a_0\mathrm{udd}a_1\mathrm{udd}a_2 \cdots \mathrm{udd}a_k$.

Similarly, every p is uniquely decomposed as $p = a_0\mathrm{ddu}a_1\mathrm{ddu}a_2 \cdots \mathrm{ddu}a_k$.

Let $\mathcal{A}^{(\mathrm{udd})}$ (resp. $\mathcal{A}^{(\mathrm{ddu})}$) be the set of ballot paths, the udd-components (resp. ddu-components) of which have the form $a_i = \mathrm{u}^s$, $s \geq 0$.

Equivalently, $\mathcal{A}^{(\mathrm{udd})}$ (resp. $\mathcal{A}^{(\mathrm{ddu})}$) is the set of ballot paths avoiding udu and ddd, and ending with udd or u (resp. with u).

Hence, by deleting the last rise of every path in $\mathcal{A}^{(\mathrm{ddu})}_{n+1}$, we obtain bijectively every path in $\mathcal{A}^{(\mathrm{udd})}_{n}$, so that

$$|\mathcal{A}^{(\mathrm{ddu})}_{n+1}| = |\mathcal{A}^{(\mathrm{udd})}_{n}|, \qquad n \geq 0. \tag{3.6}$$

Proposition 3.4. *The number of* udd *(resp.* ddu*)-equivalence classes of ballot paths of length $n \geq 1$ is equal to the n-th (resp. $(n-1)$-th) coefficient of the generating function*

$$\frac{C(x^4)}{1 - xC(x^4)},$$

where

$$[x^n]\frac{C(x^4)}{1 - xC(x^4)} = \sum_{i=0}^{\lfloor n/4 \rfloor} \frac{n-4i+1}{n-3i+1}\binom{n-2i}{i}.$$

Proof. First assume that $\mathcal{A} = \mathcal{A}^{(\mathrm{udd})}$. A path $p \in \mathcal{A} \setminus \bar{\mathcal{A}}$ is decomposed as $p = \alpha\mathrm{u}q$, where $\alpha \in \bar{\mathcal{A}}$, $q \in \mathcal{A}$. Hence, $A = \bar{A} + x\bar{A}A$, so that

$$A = \frac{\bar{A}}{1 - x\bar{A}}.$$

Moreover, a nonempty path $\alpha \in \bar{\mathcal{A}}$ is decomposed as $\alpha = \beta\mathrm{u}\gamma\mathrm{udd}$, where $\beta, \gamma \in \bar{\mathcal{A}}$. Hence, $\bar{A} - 1 = x^4\bar{A}^2$, which implies that $\bar{A} = C(x^4)$.

It follows that

$$A(x) = \frac{C(x^4)}{1 - xC(x^4)}.$$

Then by expanding $A(x)$ to a geometric series and using the well known formula

$$[x^n]C^s(x) = \frac{s}{2n+s}\binom{2n+s}{n}, \qquad n \geq 0, s > 0, \tag{3.7}$$

we can easily obtain the formula for $|\mathcal{A}^{(\mathrm{udd})}_{n}|$. The formula for $|\mathcal{A}^{(\mathrm{ddu})}_{n}|$ then follows from relation (3.6). $\square$

3.5 The string udu

Every $p \in \mathcal{P}$ is uniquely decomposed according to (1.1) as $p = a_0(\mathrm{ud})^{r_1}\mathrm{u}a_1(\mathrm{ud})^{r_2}\mathrm{u}a_2 \cdots (\mathrm{ud})^{r_k}\mathrm{u}a_k$, $r_i \geq 1$, $i \in [k]$.

Let $\mathcal{A}^{(\mathrm{udu})}$ be the set of ballot paths, the components of which have the form $a_i = \mathrm{u}^s$, $s \geq 0$.

Equivalently, $\mathcal{A}^{(\mathrm{udu})}$ is the set of ballot paths avoiding dd and ending with u.

It follows that $\mathcal{A}^{(\mathrm{udu})} = \mathcal{A}^{(\mathrm{du})}$, so that by Proposition 2.1 we have the following result:

Proposition 3.5. *The number of* udu*-equivalence classes of ballot paths of length $n \geq 1$ is equal to f_n.*

3.6 The string dud

Every $p \in \mathcal{P}$ is uniquely decomposed according to (1.1) as $p = a_0\mathrm{d}(\mathrm{ud})^{r_1}a_1\mathrm{d}(\mathrm{ud})^{r_2}a_2 \cdots \mathrm{d}(\mathrm{ud})^{r_k}\mathrm{u}a_k$, $r_i \geq 1$, $i \in [k]$.

Let $\mathcal{A}$ be the set of ballot paths, the nonempty components of which have either one of the following forms:

$$a_i = \mathrm{u}^s, \quad s \geq 1 + [0 < i < k], \qquad\qquad a_i = \mathrm{d}, \quad 0 < i < k.$$

Equivalently, $\mathcal{A}$ is the set of ballot paths p avoiding d^4, $\mathrm{u}^2\mathrm{d}^2$, $\mathrm{d}^2\mathrm{u}^2$, $\mathrm{u}^2\mathrm{d}\mathrm{u}^2$, starting with uu or udud and ending with uu, dud or dudu, unless $p \in \{\varepsilon, \mathrm{u}\}$.

Proposition 3.6. *The number of* dud*-equivalence classes of ballot paths of length n is equal to the n-th coefficient of the generating function $A = A(x)$, satisfying the relation*

$$x^2(1-x-x^2)A^3+2x(1-\frac{3}{2}x-x^2+x^3+x^4-x^5)A^2+(1-3x-x^2+3x^3+x^4-3x^5)A-1+x^2-x^4=0.$$

Proof. A path $p \in \mathcal{A} \setminus \bar{\mathcal{A}}$ is decomposed as $p = \alpha\mathrm{u}q$, where $\alpha \in \bar{\mathcal{A}}$, $q \in \mathcal{A}$. Hence, $A = \bar{A} + x\bar{A}A$, so that

$$\bar{A} = \frac{A}{1+xA}. \tag{3.8}$$

We set $\bar{\mathcal{B}}$ to be the set of Dyck paths β avoiding d^4, $\mathrm{u}^2\mathrm{d}^2$, $\mathrm{d}^2\mathrm{u}^2$, $\mathrm{u}^2\mathrm{d}\mathrm{u}^2$ and starting with uu or udud, unless $\beta = \mathrm{ud}$. Then, since every nonempty path $\alpha \in \bar{\mathcal{A}}$ is uniquely decomposed as $\alpha = \beta\mathrm{ud}$, $\beta \in \bar{\mathcal{B}}$, we have that

$$\bar{A} - 1 = x^2\bar{B}. \tag{3.9}$$

A path $\beta \in \bar{\mathcal{B}}$ is uniquely decomposed as

$$\beta = \gamma\mathrm{ud}, \quad \text{or} \quad \beta = \alpha\mathrm{u}\beta'\mathrm{udd}, \quad \text{or} \quad \beta = \alpha\mathrm{u}\alpha'\mathrm{u}\beta'\mathrm{uddd}, \qquad \alpha, \alpha' \in \bar{\mathcal{A}}, \gamma \in \bar{\mathcal{B}} \cup \{\varepsilon\}, \beta' \in \bar{\mathcal{B}}.$$

It follows that

$$\bar{B} = x^2 + x^2\bar{B} + x^4\bar{A}\bar{B} + x^6\bar{A}^2\bar{B},$$

which, combined with (3.9), gives

$$\bar{A} - 1 = x^4 + x^2(\bar{A}-1) + x^4\bar{A}(\bar{A}-1) + x^6\bar{A}^2(\bar{A}-1),$$

so that

$$x^6\bar{A}^3 + x^4(1-x^2)\bar{A}^2 - (1-x^2+x^4)\bar{A} + 1 - x^2 + x^4 = 0. \tag{3.10}$$

From relations (3.8) and (3.10), we obtain the required result. □

4 Enumeration of Dyck classes

In this section, we evaluate the number of τ-equivalence classes of Dyck paths of semilength n, for any string τ of length 3. By symmetry, it is enough to deal only with the strings uuu, uud, duu and udu. For the first 3 cases, we use the set $\mathcal{A}^{(\tau)}$ of representatives defined for the corresponding τ's in the previous section, whereas for the case of udu, we define a new one.

In the sequel, all generating functions used for the enumeration of the Dyck classes are defined with respect to the semilength of the Dyck paths.

Proposition 4.1. *The set $\bar{\mathcal{A}}^{(\tau)} = \mathcal{A}^{(\tau)} \cap \mathcal{D}$ is equal to the set of representatives of the Dyck classes, for $\tau \in \{\mathrm{uuu}, \mathrm{uud}, \mathrm{duu}\}$.*

Proof. It is enough to show that if $p \in \mathcal{P}$, $p' \in \mathcal{A}^{(\tau)}$ and $p \underset{\tau}{\sim} p'$, then $h(p') \leq h(p)$, since then, if $p \in \mathcal{D}$, it follows that $h(p') = h(p) = 0$, so that $p' \in \mathcal{D}$, which gives the required result.

We restrict ourselves to the string $\tau = \mathrm{uuu}$, since the proofs for the other cases is similar.

Let $p = a_0 \mathrm{u}^{r_1} a_1 \mathrm{u}^{r_2} a_2 \cdots \mathrm{u}^{r_k} a_k \in \mathcal{P}$ and $p' = a'_0 \mathrm{u}^{r_1} a'_1 \mathrm{u}^{r_2} a'_2 \cdots \mathrm{u}^{r_k} a'_k \in \mathcal{A}^{(\mathrm{uuu})}$, decomposed according to (1.1), such that $p \underset{\tau}{\sim} p'$. We will show by induction that $h'_i \leq h_i$, for all $i \leq k+1$, which in particular gives that $h(p') \leq h(p)$.

For $0 \leq i \leq k$, we have that

$$\begin{aligned} h'_{i+1} = h'_i + h(a'_i) + r_{i+1} = r_{i+1} + &\begin{cases} h'_i - |a_i|, & |a_i| \leq h'_i \\ 0, & |a_i| - h'_i > 0 \text{ is even} \\ 1, & |a_i| - h'_i > 0 \text{ is odd} \end{cases} \\ \leq r_{i+1} + &\begin{cases} h_i + h(a_i), & |a_i| \leq h'_i \\ 0, & |a_i| - h'_i > 0 \text{ is even} \\ 1, & |a_i| - h'_i > 0 \text{ is odd} \end{cases} \\ \leq h_i + h(a_i) + r_{i+1} = h_{i+1}, & \end{aligned}$$

because, if $h'_{i+1} = 1 + r_{i+1}$ and since h_{i+1}, h'_{i+1} have the same parity, we must have that $h_{i+1} \geq 1 + r_{i+1}$. ◻

4.1 The string uuu

In this section we use the set $\mathcal{A}$ of representatives defined in section 3.1

Proposition 4.2. *The number of* uuu*-equivalence classes of Dyck paths of semilength n is equal to the n-coefficient of the generating function*

$$F(x) = \frac{1 + xG}{1 - x(G-1)^2},$$

where $G = G(x)$ satisfies the relation

$$xG^3 - (1+2x)G^2 + (1+3x)G - x = 0.$$

Proof. In view of Proposition 4.1, for the enumeration of the classes of Dyck paths, it is enough to enumerate the set $\bar{\mathcal{A}}$ of Dyck paths not starting with uudd, and avoiding dudd, duud, and dud at height greater than 1.

For this, we consider the generating function $F = F(x) = \sum_{p \in \bar{\mathcal{A}}} x^{|p|_{\mathrm{u}}}$. We also define the sets

$$\mathcal{B} = \{\alpha \in \mathcal{D} : \alpha \text{ avoids dudd, duud, and dud at height greater than } 0\}$$

and

$$\mathcal{G} = \{\alpha \in \mathcal{D} : \alpha \text{ avoids dud, duud}\}.$$

Using the last return decomposition of a nonempty Dyck path α:

$$\alpha = \mathrm{u}\beta\mathrm{d}, \qquad \text{or} \qquad \alpha = \alpha'\mathrm{u}\beta'\mathrm{d}, \qquad \alpha', \beta, \beta' \in \mathcal{D}, \alpha' \neq \varepsilon,$$

we have that:

i) $\alpha \in \bar{\mathcal{A}} \setminus \{\varepsilon\}$ iff $\alpha' \in \bar{\mathcal{A}} \setminus \{\varepsilon\}$, $\beta \in \mathcal{B}$, $\beta' \in \mathcal{B}_{\mathrm{uu}} \cup \{\varepsilon\}$ and β, β' do not end with ud, which gives that
$$F - 1 = x(B - xB) + x(F-1)(1 + B_{\mathrm{uu}} - xB_{\mathrm{uu}}),$$
so that
$$F - 1 = \frac{xB}{1 - xB_{\mathrm{uu}}}. \tag{4.1}$$

ii) $\alpha \in \mathcal{B} \setminus \{\varepsilon\}$ iff $\alpha' \in \mathcal{B} \setminus \{\varepsilon\}$, $\beta \in \mathcal{G}$ and $\beta' \in \mathcal{G}_{\mathrm{uu}} \cup \{\varepsilon\}$, which gives
$$B - 1 = xG + x(B-1)(1 + G_{\mathrm{uu}}),$$
so that
$$B - 1 = \frac{xG}{1 - x(1 + G_{\mathrm{uu}})}. \tag{4.2}$$

iii) $\alpha \in \mathcal{B}_{\mathrm{uu}}$ iff $\alpha' \in \mathcal{B}_{\mathrm{uu}}$, $\beta \in \mathcal{G} \setminus \{\varepsilon\}$ and $\beta' \in \mathcal{G}_{\mathrm{uu}} \cup \{\varepsilon\}$, which gives
$$B_{\mathrm{uu}} = x(G-1) + xB_{\mathrm{uu}}(1 + G_{\mathrm{uu}}),$$
so that
$$B_{\mathrm{uu}} = \frac{x(G-1)}{1 - x(1 + G_{\mathrm{uu}})}. \tag{4.3}$$

iv) $\alpha \in \mathcal{G} \setminus \{\varepsilon\}$ iff $\alpha' \in \mathcal{G} \setminus \{\varepsilon\}$, $\beta \in \mathcal{G}$ and $\beta' \in \mathcal{G}_{\mathrm{uu}}$, which gives
$$G - 1 = xG + x(G-1)G_{\mathrm{uu}},$$
so that
$$G - 1 = \frac{xG}{1 - xG_{\mathrm{uu}}}. \tag{4.4}$$

v) $\alpha \in \mathcal{G}_{\mathrm{uu}}$ iff $\alpha' \in \mathcal{G}_{\mathrm{uu}}$, $\beta \in \mathcal{G} \setminus \{\varepsilon\}$ and $\beta' \in \mathcal{G}_{\mathrm{uu}}$, which gives
$$G_{\mathrm{uu}} = x(G-1) + xG_{\mathrm{uu}}^2,$$
so that
$$G_{\mathrm{uu}} = \frac{x(G-1)}{1 - xG_{\mathrm{uu}}}. \tag{4.5}$$

From relations (4.4) and (4.5), it follows that
$$G_{\mathrm{uu}} = (G-1)^2/G \qquad \text{and} \qquad x(G-1)^3 - (G-1)G + xG^2 = 0.$$
Thus, substituting in relations (4.2) and (4.3), we obtain
$$B_{\mathrm{uu}} = (G-1)^2 \qquad \text{and} \qquad B = (G-1)^2 + G = B_{\mathrm{uu}} + G.$$
Finally, substituting in relation (4.1), we obtain the required result. □

4.2 The strings uud and duu

In this section, we use the sets of representatives $\mathcal{A}^{(\mathrm{uud})}$ and $\mathcal{A}^{(\mathrm{duu})}$ defined in section 3.3. Then, we have that

$$|\bar{\mathcal{A}}^{(\mathrm{duu})}_{2n+2}| = |\bar{\mathcal{A}}^{(\mathrm{uud})}_{2n}|, \qquad n \geq 0. \tag{4.6}$$

For the proof of the above equality, consider the bijection

$$p \mapsto q, \qquad p = a_0\mathrm{duu}a_1\mathrm{duu}a_2\cdots\mathrm{duu}a_k \in \bar{\mathcal{A}}^{(\mathrm{duu})}, q = a_0'\mathrm{uud}a_1\mathrm{uud}a_2\cdots\mathrm{uud}a_k' \in \bar{\mathcal{A}}^{(\mathrm{uud})},$$

where a_0' is obtained by deleting the last ud of a_0, if $k=0$, whereas, if $k>0$, a_0' is obtained by deleting the last u of a_0, and

$$a_k' = \begin{cases} \mathrm{d}^{s-1}, & a_k = \mathrm{d}^s, 2 \leq s \leq h_k, \\ \mathrm{d}^{h_k-1}(\mathrm{ud})^t, & a_k = \mathrm{d}^{h_k}(\mathrm{ud})^t, t \geq 1. \end{cases}$$

Proposition 4.3. *The number of* uud *(resp.* duu*)-equivalence classes of Dyck paths of semilength $n \geq 1$ is equal to the n-th (resp. $(n-1)$-th) coefficient of the generating function*

$$\frac{1}{1-x(1+x)C(x^2)},$$

where

$$[x^n]\frac{1}{1-x(1+x)C(x^2)} = \sum_{i=0}^{\lfloor (n-1)/2\rfloor}\sum_{j=0}^{\lfloor (n-2i)/2\rfloor}\frac{n-2i-j}{n-j}\binom{n-i}{j}\binom{n-2i-j}{j}.$$

Proof. Let $F = F(x) = \sum_{p\in\bar{\mathcal{A}}} x^{|\alpha|_{\mathrm{u}}}$.

Using the definition of $\mathcal{A}^{(\mathrm{uud})}$ in section 3.3, we can easily check that $\bar{\mathcal{A}}^{(\mathrm{uud})}$ is the set of Dyck paths avoiding both uuu and dud at height greater than 0. It follows that every nonempty path $\alpha \in \bar{\mathcal{A}}^{(\mathrm{uud})}$ is decomposed as $p = \mathrm{u}\beta\mathrm{d}\gamma$, where $\gamma \in \bar{\mathcal{A}}^{(\mathrm{uud})}$ and β belongs to the set $\mathcal{B}$ of Dyck paths avoiding uuu and dud. Hence

$$F - 1 = xBF. \tag{4.7}$$

Clearly, $\mathcal{B} = \{\varepsilon\} \cup \mathcal{B}_{\mathrm{ud}} \cup \mathcal{B}_{\mathrm{uud}}$, so that

$$B = 1 + B_{\mathrm{ud}} + B_{\mathrm{uud}}. \tag{4.8}$$

A path $\beta \in \mathcal{B}_{\mathrm{ud}}$ is decomposed as $\beta = \mathrm{ud}\beta_1$, where $\beta_1 \in \mathcal{B}_{\mathrm{uud}} \cup \{\varepsilon\}$. Hence,

$$B_{\mathrm{ud}} = x(1 + B_{\mathrm{uud}}). \tag{4.9}$$

On the other hand, a path $\beta \in \mathcal{B}_{\mathrm{uud}}$ is decomposed as $\beta = \mathrm{uud}\beta_1\mathrm{d}\beta_2$, where $\beta_1, \beta_2 \in \mathcal{B}_{\mathrm{uud}} \cup \{\varepsilon\}$. Hence,

$$B_{\mathrm{uud}} = x^2(1 + B_{\mathrm{uud}})^2.$$

This shows that $B_{\mathrm{uud}} = C(x^2) - 1$ and, using relations (4.7), (4.8) and (4.9), we obtain that

$$F(x) = \frac{1}{1-x(1+x)C(x^2)}.$$

Then, expanding $F(x)$ to a geometric series and using formula (3.7), we obtain the required formula for $|\bar{\mathcal{A}}^{(\mathrm{uud})}_{2n}|$. The formula for $|\bar{\mathcal{A}}^{(\mathrm{duu})}_{2n}|$ then follows from relation (4.6). $\square$

4.3 The string udu

The set of representatives used in section 3.5 does not contain any Dyck paths, and therefore it is not convenient for the present case. For this reason, we will consider a more suitable set of representatives, as follows:

We define $\mathcal{A}$ to be the set of ballot paths $p = a_0(\mathrm{ud})^{r_1}\mathrm{u}a_1(\mathrm{ud})^{r_2}\mathrm{u}a_2\cdots(\mathrm{ud})^{r_k}\mathrm{u}a_k$, $r_i \geq 1, i \in [k]$, the nonempty components of which have either one of the following forms:

$$a_0 = \mathrm{u}^s\mathrm{d}^s, \quad a_0 = \mathrm{u}^s\mathrm{d}^{s-1}, s \geq 1, \qquad \text{if } k = 0,$$

$$a_0 = \mathrm{u}^s, s \in [3], \quad a_0 = \mathrm{u}^s\mathrm{d}^s \text{ or } \mathrm{u}^{s+1}\mathrm{d}^s, s \geq 2, \qquad \text{if } k > 0,$$

and, for $i \in [k]$,

$$a_i = \mathrm{u}, i < k, \quad a_i = \mathrm{u}^2, h_i = 1, i < k, \quad a_i = \mathrm{d}^s, s \geq 1 + [i < k], \quad a_i = \mathrm{u}^s\mathrm{d}^{s+h_i} \text{ or } \mathrm{u}^s\mathrm{d}^{s+h_i-1}, s \geq 1.$$

Proof of Proposition 1.1 for $\tau = \mathrm{udu}$. Given a Dyck path $p = a_0(\mathrm{ud})^{r_1}\mathrm{u}a_1(\mathrm{ud})^{r_2}\mathrm{u}a_2\cdots(\mathrm{ud})^{r_k}\mathrm{u}a_k$, setting $h'_0 = 0$, $a'_i = \mathrm{u}^{s_i}\mathrm{d}^{t_i}$, $h'_{i+1} = h'_i + h(a'_i) + [i < k]$, where

$$s_0 = \begin{cases} |a_0|, & k > 0, |a_0| \leq 3 \\ \lceil |a_0|/2 \rceil, & \text{otherwise,} \end{cases}$$

$$s_i = \left(\left\lceil \frac{|a_i| - h'_i}{2} \right\rceil \right)^+ + [i < k]\left([|a_i| = 2][h'_i = 1] + [|a_i| = 1]\right), \qquad i \in [k]$$

and

$$t_i = |a_i| - s_i, \qquad i \geq 0,$$

we obtain inductively a sequence of paths a_i. Let $p' = a'_0(\mathrm{ud})^{r_1}\mathrm{u}a'_1(\mathrm{ud})^{r_2}\mathrm{u}a'_2\cdots(\mathrm{ud})^{r_k}\mathrm{u}a'_k$. It is easy to check that $p' \in \mathcal{A}$ and $p' \sim p$. Furthermore, since no two paths of $\mathcal{A}$ are equivalent, we obtain the required result. □

In the next result, we characterize the elements of $\mathcal{A}$ which are representatives of Dyck classes.

Proposition 4.4. *A path $p' \in \mathcal{A}$ is equivalent to a Dyck path p iff it has either one of the following forms:*

1. *p' is a Dyck path or*
2. *$p' = \alpha(\mathrm{ud})^r\mathrm{u}^2\beta$, where $r \geq 1$, $\alpha, \beta \in \mathcal{D}$, β starts with u^2 and α ends with d^2 and it has either an occurrence of d^3 or an occurrence of d^2 before an occurrence of $\mathrm{u}^2\mathrm{d}^2$.*

Proof. Using the same notation as in the proof of Proposition 1.1, we consider

$$p = a_0(\mathrm{ud})^{r_1}\mathrm{u}a_1(\mathrm{ud})^{r_2}\mathrm{u}a_2\cdots(\mathrm{ud})^{r_k}\mathrm{u}a_k \text{ and } p' = a'_0(\mathrm{ud})^{r_1}\mathrm{u}a'_1(\mathrm{ud})^{r_2}\mathrm{u}a'_2\cdots(\mathrm{ud})^{r_k}\mathrm{u}a'_k,$$

with $p \in \mathcal{D}$, $p' \in \mathcal{A}$ and $p \sim p'$.

We will show that either $p' \in \mathcal{D}$, or p' is of the form $\alpha(\mathrm{ud})^r\mathrm{u}^2\beta$.

Clearly, if $k = 0$, then $p' = (\mathrm{ud})^{|a_0|/2} \in \mathcal{D}$, so that we may assume that $k > 0$.

Let t be the greatest element of $[k]$ such that $h'_t = 1$, if such an element exists, or $t = 0$, otherwise. Clearly, if $t = k$, since h_{k+1}, h'_{k+1} have the same parity and $h_{k+1} = 0$, it follows that $h'_{k+1} = 0$, so that $p' \in \mathcal{D}$.

In the sequel, we assume that $t < k$. We first show the following result:

Lemma. *If there exists $i \in [t+1, k]$, such that $h'_i \leq h_i$, then $p' \in \mathcal{D}$.*

For this, we show by induction that $h'_j \leq h_j$ for every $j \in [i, k+1]$. Let $j \in [i, k]$ with $h'_j \leq h_j$. We will show that $h'_{j+1} \leq h_{j+1}$.

If $|a_j| = 1$, then $a'_j = a_j$ and the result follows easily.

On the other hand, if $|a_j| \neq 1$, then $|a'_j|_{\mathrm{u}} = \Big(\lceil (|a_j| - h'_j - 1)/2 \rceil\Big)^+$. We can easily check that

$$\frac{|a_j| - h_j\prime}{2} \leq |a_j|_{\mathrm{u}} + \frac{h_j - h'_j}{2},$$

which, since $(h_j - h'_j)/2$ is a nonnegative integer, gives that

$$|a'_j|_{\mathrm{u}} \leq |a_j|_{\mathrm{u}} + \frac{h_j - h'_j}{2}.$$

Then using relation (1.2), we deduce that

$$h_{j+1} - h'_{j+1} \geq h_j - h'_j + 2\left(|a_j|_{\mathrm{u}} - |a_j|_{\mathrm{u}} - \frac{h_j - h'_j}{2}\right) = 0.$$

This shows that, $h'_j \leq h_j$, for every $j \in [i, k+1]$. In particular, $h'_{k+1} \leq h_{k+1} = 0$, so that $p' \in \mathcal{D}$, completing the proof of the lemma.

If $t = 0$, then, since $h'_1 \leq h_1$, by the lemma it follows that $p' \in \mathcal{D}$, so that in the sequel we may assume that $t > 0$.

If now $h'_i = 2$, for some $i \in [t+1, k]$, then by the lemma it follows that $p' \in \mathcal{D}$; so that we can restrict ourselves to the case where $h'_i \geq 3$ for every $i \in [t+1, k]$. It follows that $|a_t| = 1$ or $|a_t| = 2$.

If $a_t = \mathrm{u}$, or $a_t = \mathrm{u}^2$, or $a_t = \mathrm{d}^2$ with $h_t \geq 5$, we can easily check that $h'_{t+1} \leq h_{t+1}$, so that by the lemma it follows that $p' \in \mathcal{D}$.

We finally consider the remaining case where $a_t = \mathrm{d}^2$ and $h_t = 3$. Then, since $a'_t = \mathrm{u}^2$, $h'_{t+1} = 4$, $h_{t+1} = 2$, applying relation (1.2) for every $i \in [t+1, k-1]$, we obtain that

$$h_k - h'_k = h_{t+1} - h'_{t+1} + 2\sum_{i=t+1}^{k-1} (|a_i|_{\mathrm{u}} - |a'_i|_{\mathrm{u}}) = -2 + 2\sum_{\substack{i \in [t+1, k-1] \\ |a_i| \geq 2}} |a_i|_{\mathrm{u}},$$

since $|a'_i|_{\mathrm{u}} = 0$, for every $i \in [t+1, k-1]$ with $|a_i| \geq 2$.

We consider two subcases.

If $|a_i|_{\mathrm{u}} \geq 1$, for some $i \in [t+1, k-1]$, then $h_k - h'_k \geq 0$, so that by the lemma we obtain that $p' \in \mathcal{D}$.

If, on the other hand, $|a_i|_{\mathrm{u}} = 0$ for every $i \in [t+1, k-1]$, we have that $h_k - h'_k = -2$. If $p' \notin \mathcal{D}$, then $|a'_k|_{\mathrm{d}} = |a_k|$, so that

$$2 \leq h'_{k+1} = h'_k - |a_k| = h_k + 2 - |a_k| \leq h_k - h(a_k) + 2 = h_{k+1} + 2 = 2.$$

This shows that $h'_{k+1} = 2$, that is p' ends at height 2.

Furthermore, since $h'_i \geq 3$ for every $i \in [t+1, k]$, we obtain that the subpath β of p' starting from the second rise of a'_t and ending at the end of p' is a Dyck path starting with u^2.

If we set $a = a'_0(\mathrm{ud})^{r_1}\mathrm{u}a'_1(\mathrm{ud})^{r_2}\mathrm{u}a'_2\cdots(\mathrm{ud})^{r_{t-1}}\mathrm{u}a'_{t-1}$ and $r = r_t \geq 1$, we obtain that α is a Dyck path ending with d^2 and $p' = \alpha(\mathrm{ud})^r\mathrm{u}^2\beta$.

In the sequel, assuming that α avoids d^3, we will show that α has an occurrence of d^2 followed by an occurrence of $\mathrm{u}^2\mathrm{d}^2$.

Firstly, we note that, since in this case $h_t = 3$, we obtain

$$\sum_{i=0}^{t-1}(h(a_i) - h(a'_i)) = 2. \tag{4.10}$$

Clearly, since α avoids d^3, each a'_i, $i \geq 0$, has one of the following forms:

$$a'_0 = \mathrm{u}^s, s \in \{0, 1, 2, 3\}, \qquad \text{or} \qquad a'_0 = \mathrm{u}^s\mathrm{d}^2, s \in \{2, 3\}$$

and, for $i \in [t-1]$,

$$a'_i = \mathrm{u}^s, \qquad \text{or} \qquad a'_i = \mathrm{u}^s\mathrm{d}^2, \qquad s \in \{0, 1, 2\}.$$

It is easy to check that in each case $h(a_i) - h(a'_i) \in \{-4, -2, 0, 4\}$. Then, in view of relation (4.10), it follows that there exists $i \in [t-1]$ such that $h(a_i) - h(a'_i) = -2$. This can occur only when $a'_i = \mathrm{u}^s\mathrm{d}^2$ and $a_i = \mathrm{u}^{s-1}\mathrm{d}^3$ or $a_i = \mathrm{d}^3\mathrm{u}^{s-1}$, where $s \in \{1, 2\}$, which ensures the existence of $\mathrm{u}^2\mathrm{d}^2$ in α.

If there is no occurrence of d^2 on the left of a'_i in α, since $h'_i \in \{1, 2\}$, we can easily check that $i \in \{1, 2\}$ and that $a_j = a'_j$ for all $j < i$, so that $h_i = h'_i$. Then, since the last point of a'_i in p' lies at height 0 or 1 and $h(a_i) - h(a'_i) = -2$, it follows that the last point of a_i in p lies at height -2 or -1, which is a contradiction since $p \in \mathcal{D}$.

For the converse, it is enough to show that if $p' = \alpha(\mathrm{ud})^r\mathrm{u}^2\beta \in \mathcal{A}$ with the above properties, then there exists $p \in \mathcal{D}$ such that $p \sim p'$.

Firstly, assuming that α has an occurrence of d^3, we set γ to be the path obtained from α, by changing the first fall of the leftmost such occurrence into a rise.

On the other hand, if α avoids d^3 and has an occurrence of $\mathrm{u}^2\mathrm{d}^2$ and an occurrence of d^2 on the left of it, we set γ to be the path obtained from α by changing $\mathrm{u}^2\mathrm{d}^2$ into ud^3 and the leftmost d^2 into u^2.

In both cases, we can easily check that $\gamma \in \mathcal{P}$, $\alpha \sim \gamma$ and γ ends at height 2. Now, since β starts with u^2, there exist $\beta_1, \beta_2 \in \mathcal{D}$ with $\beta_1 \neq \varepsilon$ such that $\beta = \mathrm{u}\beta_1\mathrm{d}\beta_2$. If we set $p = \gamma(\mathrm{ud})^r\mathrm{ud}^2\beta_1\mathrm{d}\beta_2$, it is easy to see that $p \in \mathcal{D}$ and $p \sim p'$. $\square$

Proposition 4.5. *The number of* udu*-equivalence classes of Dyck paths of semilength n is equal to the n-th coefficient of the generating function*

$$F(x) = \frac{x(1-x)^2(1+G+xG) + x^5(1+xG)G^2}{(1-x)((1-x)^2 + (x-2)x^2G)} - \frac{x^4(1-x+x^3)(1+xG)GT}{(1-x)^2(1-x+x^3-xT)}$$

where the generating functions $G = G(x)$ and $T = T(x)$ are given by

$$G = x(1+G) + x^2(1+xG)(1+x(1+xG))G$$

and

$$x^3T^3 + x^2T^2 + (x-1)T + x^2 = 0.$$

Proof. In order to enumerate the udu-equivalence classes of $\mathcal{D}$, according to Proposition 4.4, it is enough to evaluate the generating function $F = F(x)$ of the set of paths p' having either one of the two forms described in Proposition 4.4.

For the first form, we note that $\bar{\mathcal{A}}$ is the set of Dyck paths avoiding $\mathrm{d}^2\mathrm{u}^2$, $\mathrm{d}^2\mathrm{ud}^2$, u^4 at height greater than 0, $\mathrm{u}^2\mathrm{d}^s\mathrm{u}$, $s \geq 2$, at height greater than 1, and not ending with $\mathrm{d}^2\mathrm{ud}$.

Let $\mathcal{B}$ (resp. $\mathcal{G}$) be the set of paths in $\bar{\mathcal{A}} \setminus \{\varepsilon\}$ also avoiding u^4 at height 0 and $\mathrm{u}^2\mathrm{d}^s\mathrm{u}$, $s \geq 2$, at height 1 (resp. $\mathrm{u}^2\mathrm{d}^2$), and let $\mathcal{K}$ (resp. $\mathcal{L}$) be the set of paths in $\bar{\mathcal{A}}$ (resp. $\mathcal{G}$) ending with d^2. We denote by H, B, G, K, L the generating functions, with respect to the semilength, of the sets $\bar{\mathcal{A}}, \mathcal{B}, \mathcal{G}, \mathcal{K}, \mathcal{L}$ respectively.

Clearly, a path $p \in \bar{\mathcal{A}} \setminus \{\varepsilon\}$ (resp. $\mathcal{G}$) is decomposed as

$$p = (\mathrm{ud})^s, \quad \text{or} \quad p = \delta, \quad \text{or} \quad p = \delta(\mathrm{ud})^{s+1}, \qquad s \geq 1,$$

where $\delta \in \mathcal{K}$ (resp. $\delta \in \mathcal{L}$).

It follows that, $H - 1 = \dfrac{x}{1-x} + K + \dfrac{x^2}{1-x}K$, which gives

$$K = \frac{-1 + (1-x)H}{1 - x + x^2}. \tag{4.11}$$

Similarly, we obtain that

$$L = \frac{G - x - xG}{1 - x + x^2}. \tag{4.12}$$

A path $p \in \mathcal{K}$ is decomposed as $p = \alpha\mathrm{u}\beta\mathrm{d}$ where $\alpha\mathrm{ud} \in \bar{\mathcal{A}} \setminus (\mathcal{K} \cup \{\varepsilon\})$ and $\beta \in \mathcal{B}$, so that $K = (H - 1 - K)B$, which combined with (4.11) gives

$$H = 1 + xH + x(1 + x(H-1))B. \tag{4.13}$$

Next, given a path $p \in \mathcal{B} \setminus \mathcal{G}$, we consider the last $\mathrm{u}^2\mathrm{d}^s$ decomposition $p = \beta\gamma$, where $\beta \in \mathcal{B}$ ends with $\mathrm{u}^2\mathrm{d}^s$, $s \geq 2$, and $\gamma \in \mathcal{G}_{\mathrm{udu}} \cup \{\varepsilon\}$.

Clearly, by deleting the last peak (i.e., ud) of β, we obtain a path $\beta' \in \mathcal{B}$. Since the mapping $\beta \mapsto \beta'$ is clearly a bijection, we obtain that the set of paths β is enumerated by xB and hence

$$B - G = xB(1 + xG). \tag{4.14}$$

Moreover, if a path $p \in \mathcal{G}$ is decomposed as $p = \gamma\mathrm{u}\gamma_1\mathrm{d}$, we have that $\gamma\mathrm{ud} \in \mathcal{G} \setminus \mathcal{L}$, $\gamma_1 \in \mathcal{G}$ and starts with udu or $\mathrm{ud}^2\mathrm{u}$. If γ_1 starts with $\mathrm{ud}^2\mathrm{u}$, then it is decomposed as $\gamma_1 = \mathrm{u}\gamma_2\mathrm{d}\gamma_3$, where $\gamma_2 \in \mathcal{G}_{\mathrm{udu}}$ and $\gamma_3 \in \mathcal{G}_{\mathrm{udu}} \cup \{\varepsilon\}$.

It follows that $L = (G - L)\,(xG + xxG(1 + xG))$, which combined with relation (4.12) gives

$$G = x(1 + G) + x^2(1 + xG)(1 + x(1 + xG))G. \tag{4.15}$$

Finally, from relations (4.13), (4.15), (4.14), we deduce that

$$H = \frac{1 - x + x(1-2x)G)}{(1-x)^2 + (x-2)x^2G}. \tag{4.16}$$

Next, we deal with the second form of paths $p' = \alpha(\mathrm{ud})^r\mathrm{u}\beta$ described in Proposition 4.4.

For the paths α, we define the set $\mathcal{E}$ of Dyck paths with no occurrence of d^3 and with no occurrence of d^2 followed by an occurrence of $\mathrm{u}^2\mathrm{d}^2$ and we denote by H_1, B_1, G_1 the generating functions of the sets $\mathcal{E}\cap\mathcal{K}$, $B\cap\mathcal{E}\cap\mathcal{K}$, $\mathcal{G}\cap\mathcal{E}\cap\mathcal{K}$ respectively.

Then, we proceed to the enumeration of the set $\mathcal{K}\setminus\mathcal{E}$ consisting of the above mentioned paths α, with corresponding generating function $K-H_1$.

Every path $p\in\mathcal{E}\cap\mathcal{K}$ is decomposed as

$$p=(\mathrm{ud})^s\mathrm{u}\beta\mathrm{d},\qquad \text{or}\qquad p=\delta(\mathrm{ud})^{s+1}\mathrm{u}\gamma\mathrm{d},$$

where $s\geq 0$, $\beta\in(\mathcal{B}\cap\mathcal{E})\setminus\mathcal{K}$, $\delta\in\mathcal{E}\cap\mathcal{K}$, $\gamma\in\mathcal{G}\cap\mathcal{E}\setminus(\mathcal{K}\cup\{\mathrm{ud}\})$.

Moreover, since every path $\beta\in(\mathcal{B}\cap\mathcal{E})\setminus\mathcal{K}$ (resp. $\gamma\in(\mathcal{G}\cap\mathcal{E})\setminus(\mathcal{K}\cup\{\mathrm{ud}\})$) has either one of the forms

$$(\mathrm{ud})^t\qquad\text{or}\qquad\sigma(\mathrm{ud})^{t+1},$$

where $t\geq 1$ and $\sigma\in\mathcal{B}\cap\mathcal{E}\cap\mathcal{K}$ (resp. $\sigma\in\mathcal{G}\cap\mathcal{E}\cap\mathcal{K}$), we obtain that the set $(\mathcal{B}\cap\mathcal{E})\setminus\mathcal{K}$ (resp. $(\mathcal{G}\cap\mathcal{E})\setminus\mathcal{K}$) is enumerated by $\frac{x}{1-x}(1+xB_1)$ (resp. $\frac{x}{1-x}(1+xG_1)$).

Thus, we obtain that

$$H_1=\frac{x}{1-x}\frac{x(1+xB_1)}{1-x}+H_1\frac{x^2}{1-x}\left(\frac{x(1+xG_1)}{1-x}-x\right),$$

which gives

$$H_1=\frac{x^2\left(1+xB_1+x^2H_1(1+G_1)\right)}{(1-x)^2}. \tag{4.17}$$

Similarly, every path $p\in\mathcal{G}\cap\mathcal{E}\cap\mathcal{K}$, is decomposed as

$$p=(\mathrm{ud})^s\mathrm{u}\gamma\mathrm{d},\qquad\text{or}\qquad p=\delta(\mathrm{ud})^{s+1}\mathrm{u}\gamma\mathrm{d},$$

where $s\geq 0$, $\delta\in\mathcal{G}\cap\mathcal{E}\cap\mathcal{K}$, $\gamma\in(\mathcal{G}_{\mathrm{udu}}\cap\mathcal{E})\setminus\mathcal{K}$ or $\gamma=\mathrm{u}\gamma_1\mathrm{d}\gamma_2\mathrm{d}$, $\gamma_1,\gamma_2\in(\mathcal{G}_{\mathrm{udu}}\cap\mathcal{E})\setminus\mathcal{K}$.

As before, we find that the set $(\mathcal{G}_{\mathrm{udu}}\cap\mathcal{E})\setminus\mathcal{K}$ is enumerated by $\frac{x^2}{1-x}(1+xG_1)$.

Hence,

$$\begin{aligned}G_1&=\left(\frac{1}{1-x}+\frac{x}{1-x}G_1\right)x\left(\frac{x^2}{1-x}(1+xG_1)+\frac{x^5}{(1-x)^2}(1+xG_1)^2\right)\\&=\frac{x^3}{(1-x)^2}(1+xG_1)^2\left(1+\frac{x^3}{1-x}(1+xG_1)\right).\end{aligned} \tag{4.18}$$

Every path $p\in(\mathcal{B}\setminus\mathcal{G})\cap\mathcal{E}\cap\mathcal{K}$ is decomposed as

$$p=(\mathrm{ud})^s\mathrm{u}^2\mathrm{d}^2\gamma,\qquad s\geq 0,\quad \gamma\in(\mathcal{G}_{\mathrm{udu}}\cap\mathcal{E}\cap\mathcal{K})\cup\{\varepsilon\}.$$

Hence,

$$B_1-G_1=\frac{x^2}{1-x}(1+xG_1). \tag{4.19}$$

From relations (4.17), (4.18) and (4.19), setting $T=B_1-G_1$, we obtain

$$x^3T^3+x^2T^2+(x-1)T+x^2=0 \tag{4.20}$$

and

$$H_1 = \frac{(1-x+x^3)T}{(1-x)(1-x+x^3-xT)}. \tag{4.21}$$

Finally, the paths β of the second form described in Proposition 4.5 are decomposed as

$$\beta = \mathrm{u}^2\mathrm{d}\gamma_1\mathrm{d}\gamma_2, \qquad \gamma_1 \in \mathcal{G}, \gamma_2 \in \mathcal{G}_{\mathrm{udu}} \cup \{\varepsilon\},$$

and hence they are enumerated by the generating function $x^2G(1+xG)$.

In conclusion, the generating function F is given by the equality

$$F = H + \frac{x^2}{1-x}\,(K - H_1)\, x^2G(1+xG),$$

which, combined with relations (4.11), (4.16) and (4.21), gives the required result. $\square$

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