

# Set partitions and integrable hierarchies

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## Abstract

We demonstrate that statistics for several types of set partitions are described by generating functions which appear in the theory of integrable equations.

*Key words:* set partition,  $B$ -type partition, non-overlapping partition, atomic partition, Bell polynomial, Dowling number, Bessel number, generating function, integrable hierarchy

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## 1 Introduction

In combinatorics, a typical problem is to establish algorithms for generating and counting of all objects of a given type. We are interested in the situation when:

- such an algorithm can be described in terms of generating operations, which make possible to build the objects under scrutiny recursively from the objects with lesser number of elements;
- each object can be associated with a number or a monomial in formal variables, in such a way that the generating operations correspond to certain algebraic or differential operations.

As a result, the set of all objects is mapped into a sequence of numbers or polynomials governed by certain recurrence relations. A remarkable fact is that in some cases the generating functions for these sequences appear also in the theory of integrable equations. We mention the relation between the number partitions and statistical mechanics [34], the combinatorics of Painlevé transcendents [11], of KdV solitons [16] and of KP tau-function [1], as just few examples of interconnection of both theories. A well known example, which is more close to the subject of this paper, is given by the Bell polynomials which describe the statistics of set partitions and also define the potential Burgers hierarchy [23, 24]. Our goal is to generalize this observation for several special types of set partitions with restrictions on the structure of blocks.

In sections 2, 4 we consider the generic set partitions and  $B$  type partitions related, respectively, with the Burgers and Ibragimov–Shabat linearizable hierarchies. Section 3 is the only one, where set partitions are replaced by other combinatorial objects; the corresponding equations are related with the Burgers hierarchy by the hodograph

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transformation. In all these cases the definitions of monomials rely upon counting of elements in blocks.

In section 5, we study the non-overlapping partitions associated with the KdV hierarchy; section 6 is devoted to the atomic partitions and the Kaup–Broer hierarchy (a gauge equivalent version of the nonlinear Schrödinger hierarchy). Here, the combinatorics becomes more complicated, as well as its statistics: the definition of the monomial corresponding to a partition is based on the sequence of operations which bring to this partition, rather than on the size of its blocks. Moreover, the combinatorics becomes more disguised: in the Burgers case, the generating function is related intermediately with the higher flows, while in the KdV and NLS cases it becomes the asymptotic series in powers of the spectral parameter for the logarithmic derivative of  $\psi$ -function, governed by Riccati equation.

The structure of all sections is uniform: first, we define the generating operations for the combinatorial objects under consideration, next, we introduce the polynomials which describe their statistics and derive the recurrence relations, finally, a comparison with an integrable hierarchy is given.

## 2 Set partitions

### 2.1 Basic notions

Recall that a *set partition* is a set of nonempty, pairwise disjoint subsets (*blocks*) of the set, such that their union gives the whole set. The notation  $\pi \vdash [n]$  means that  $\pi$  is a partition of the set  $[n] = \{1, \dots, n\}$ . The set of all partitions of  $[n]$  is denoted  $\Pi_n$  and its subset consisting of all partitions with  $k$  blocks is denoted  $\Pi_{n,k}$ ,  $1 \leq k \leq n$ . As a rule, we denote the blocks by the same letter as the partition, with a subscript:  $\pi = \{\pi_1, \dots, \pi_k\}$ . For  $n = 0$ , it is convenient to accept that the set  $[0] = \emptyset$  admits the unique partition  $\emptyset$  which contains no blocks, that is,  $\Pi_0 = \Pi_{0,0} = \{\emptyset\}$ .

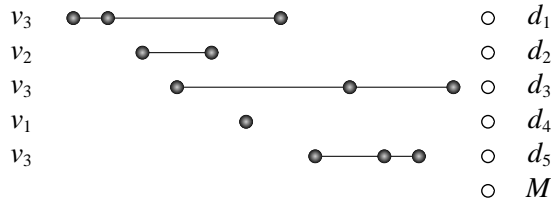
Let  $|A|$  denote the number of elements of the set  $A$ . Then  $|\pi|$  is the number of blocks in the set partition  $\pi$ , and  $|\pi_j|$  is the size of the block  $\pi_j$ . A block is called a singleton if  $|\pi_j| = 1$ , and a multiplet if  $|\pi_j| > 1$ . The total number of elements in the partitioned set will be denoted  $\|\pi\|$ .

By definition, a partition is an unsorted set, that is, the order of blocks and order of elements in blocks do not matter. Nonetheless, the ordering defined by the enumeration of elements can be used for a definition of unique representation of partitions. In the *canonical form*, a block of a partition is represented by a list of elements sorted by increase, and the partition itself is represented by a list of blocks, sorted by increase of their first (minimal) elements. For short, it is common to write, for instance,  $\pi = 1, 2, 7|3, 5|4, 9, 12|6|8, 10, 11$ , with the blocks separated by vertical bars.

Alternatively, a set partition in the canonical form is represented by the sequence  $s = (s_1, \dots, s_n)$ , where  $s_m$  is the index of the block containing the element  $m$ ; in the above example, it is  $(1, 1, 2, 3, 2, 4, 1, 5, 3, 5, 5, 3)$ . This gives rise to the integer sequences characterized by the *restricted growth* condition:

$$s_1 = 1, \quad 1 \leq s_m \leq \max(s_1, \dots, s_{m-1}) + 1. \quad (1)$$

The set of such sequences of length  $n$  will be denoted  $R_n$ .



**Figure 1.** Generating operations  $d_j$ ,  $M$  append new element at one of the vacancies marked by empty circles. A block of size  $l$  is associated with the variable  $v_l$ .

Throughout the paper, we use the graphical representation of set partitions as illustrated by fig. 1. The elements are enumerated from left to right, the rows represent the blocks sorted from top to bottom by increase of minimal elements. The line between the outermost elements of a block is called its *support*:  $\text{supp } \pi_j = [\min \pi_j, \max \pi_j]_{\mathbb{R}}$ . This notion is used in sections 5, 6, devoted to the so-called *non-overlapping* and *atomic* partitions.

## 2.2 Generating operations and generating functions

The problem of construction of all set partitions  $\Pi_n$  admits the following solution. Let us consider the operations

$$d_j : \Pi_{n,k} \rightarrow \Pi_{n+1,k}, \quad 1 \leq j \leq k, \quad M : \Pi_{n,k} \rightarrow \Pi_{n+1,k+1}$$

defined as adding of a new element either to one of existing blocks of the partition  $\pi = \{\pi_1, \dots, \pi_k\}$ , or as a new singleton:

$$d_j \pi = \{\pi_1, \dots, \pi_j \cup \{n+1\}, \dots, \pi_k\}, \quad M \pi = \{\pi_1, \dots, \pi_k, \{n+1\}\}. \quad (2)$$

These operations generate partitions in the canonical form automatically, if the element  $n+1$  (or the set  $\{n+1\}$ ) is appended to the corresponding list, see fig. 1, where the column on the right contains all possible vacancies for the new element.

**Statement 1.** *Any partition of  $[n]$  is generated, in a unique way, by operations  $d_j, M$  applied to the seed partition  $\emptyset$ .*

*Proof.* The sequence of operations is recovered uniquely by deleting the elements in the reverse order, from  $n$  to 1.  $\square$

In terms of the restricted growth sequences  $s = (s_1, \dots, s_n) \in R_n$ , the generating operations amount to appending of an integer  $s_{n+1} \in [k+1]$ , where  $k = \max s$ , and the operation  $M$  corresponds to the value  $k+1$ . In fact,  $s$  encodes the sequence of operations which generate a given partition. Alternatively, it can be written as follows. Let  $\Phi_n$  denote the set of words consisting of  $n$  characters  $d_j$  and  $M$ , such that the subscript for any character  $d$  may take values in the range from 1 to the number of occurrences of  $M$  to the right from this character. Statement 1 establishes the bijection  $\varphi \mapsto \varphi(\emptyset)$  between  $\Phi_n$  and  $\Pi_n$ . Although we do not use this representation of set partitions in this

section, its generalizations will be important in the study of special partition in sections 5, 6.

The generating operations provide a plain and obvious construction algorithm for the set partitions, which is implemented, in particular, in the *Combinatorica* package [27]. It should be mentioned, that there exist also more effective algorithms based on the Gray codes [21, Ch.7.2.1.5], [25].

Let us turn to the problem of counting of the set partitions. Let the variable  $v_l$  be assigned to any block  $\pi_j$  with  $l$  elements. In the corresponding restricted growth sequence,  $l$  is the multiplicity of the entry  $j$ . A partition is associated with the monomial equal to the product of variables assigned to all blocks:

$$p(\pi) = \prod_{j=1}^{|\pi|} v_{|\pi_j|}. \quad (3)$$

Summation over all partitions of  $[n]$  defines the (*complete exponential*) *Bell polynomials* (see e.g. [10])

$$Y_n(v_1, \dots, v_n) = \sum_{\pi \in \Pi_n} \prod_{j=1}^{|\pi|} v_{|\pi_j|} = \sum_{s \in R_n} \prod_{j=1}^{\max s} v_{\{i: s_i=j\}}. \quad (4)$$

In the polynomial  $Y_n$ , the degrees in any monomial  $v_1^{k_1} \dots v_r^{k_r}$  are related by equality  $n = k_1 + 2k_2 + \dots + rk_r$  (that is, the polynomial is homogeneous with respect to the weight  $w(v_l) = l$ ), while the coefficient of the monomial is equal to the number of partitions of  $[n]$  into  $k_1$  blocks with 1 element,  $\dots$ ,  $k_r$  blocks with  $r$  elements, for instance:

$$\begin{array}{r}
 Y_1 = v_1 \\
 n = 1 \\
 1
 \end{array}
 \quad
 \begin{array}{r}
 Y_2 = v_2 + v_1^2 \\
 n = 2 = 1 + 1 \\
 12 \quad 1|2
 \end{array}
 \quad
 \begin{array}{r}
 Y_3 = v_3 + 3v_1v_2 + v_1^3 \\
 n = 3 = 1 + 2 = 1 + 1 + 1 \\
 123 \quad 1|23 \quad 1|2|3 \\
 \quad \quad 12|3 \\
 \quad \quad 13|2
 \end{array}$$
  

$$\begin{array}{r}
 Y_4 = v_4 + 4v_1v_3 + 3v_2^2 + 6v_1^2v_2 + v_1^4 \\
 n = 4 = 1 + 3 = 2 + 2 = 1 + 1 + 2 = 1 + 1 + 1 + 1 \\
 1234 \quad 1|234 \quad 12|34 \quad 1|2|34 \quad 1|2|3|4 \\
 \quad \quad 123|4 \quad 13|24 \quad 1|23|4 \\
 \quad \quad 124|3 \quad 14|23 \quad 1|24|3 \\
 \quad \quad 134|2 \quad \quad \quad 12|3|4 \\
 \quad \quad \quad \quad \quad \quad 13|2|4 \\
 \quad \quad \quad \quad \quad \quad 14|2|3
 \end{array}$$

At  $n = 0$ , we have  $Y_0 = p(\emptyset) = 1$ , in accordance with the common convention that a product over empty set is equal to 1.

The number of set partitions with blocks of prescribed size is computed explicitly and we arrive to the formula

$$Y_n = \sum_{k_1+2k_2+\dots+rk_r=n} \frac{n!}{(1!)^{k_1} \dots (r!)^{k_r} k_1! \dots k_r!} v_1^{k_1} \dots v_r^{k_r}, \quad (5)$$

where the sum is taken over all *partitions of the number  $n$* . The following statement provides a more effective computation method. In this statement and further on,  $D = \partial_x$  denotes the differentiation with respect to the variable  $x$ , and variables  $v_l$  are interpreted as derivatives of a function  $v = v_0 = v(x)$ .

**Statement 2.** *Polynomials  $Y_n$  in variables  $v_l = D^l(v)$  are governed by the recurrence relation*

$$Y_0 = 1, \quad Y_{n+1} = (D + v_1)(Y_n), \quad n \geq 0. \quad (6)$$

The exponential generating function of the sequence  $Y_n$  is

$$\sum_{n=0}^{\infty} Y_n \frac{z^n}{n!} = \exp \left( \sum_{n=1}^{\infty} v_n \frac{z^n}{n!} \right). \quad (7)$$

*Proof.* Let  $\pi \in \Pi_n$ . The action of the differentiation  $D$  on the monomial  $p(\pi)$  amounts, according to the Leibniz rule, to replacement of  $v_l$  by  $v_{l+1}$  for each factor in turn, taking the multiplicity into account. In the partitions language, this corresponds to adding of a new element to one of the blocks in turn. As the result, we obtain the sum of monomials  $p(d_j \pi)$  with all admissible values of  $j$ . The multiplication of monomial  $p(\pi)$  by  $v_1$  yields the monomial  $p(M\pi)$ .

Thus,  $(D + v_1)(p(\pi)) = \sum_{\pi'} p(\pi')$ , where the sum is taken over the partitions  $\pi'$  obtained from  $\pi$  by adding the element  $n + 1$  in all possible ways. Summation over  $\pi$  brings to equation (6), taking Statement 1 into account. This equation can be cast to the form  $Y_{n+1} = e^{-v} D e^v (Y_n)$  which implies  $Y_n = e^{-v} D^n (e^v)$ . For the exponential generating function, we obtain

$$\sum_{n=0}^{\infty} Y_n \frac{z^n}{n!} = e^{-v} \sum_{n=0}^{\infty} D^n (e^v) \frac{z^n}{n!} = e^{v(x+z)-v(x)}$$

and expansion of  $v(x + z)$  into the Taylor series with respect to  $z$  gives (7).  $\square$

The few first polynomials  $Y_n$  are written down in table 1. The terms of degree  $k$  called partial Bell polynomials  $Y_{n,k}$  appear in the Faà di Bruno formula for the  $n$ -th derivative of a composite function:

$$D^n(f(v)) = f'(v)Y_{n,1} + f''(v)Y_{n,2} + \cdots + f^{(n)}(v)Y_{n,n} \quad (8)$$

(see discussion in [19]). By summing up the coefficients of  $Y_{n,k}$ , which is equivalent to identifying all variables  $v_l$ , we forget about the sizes of blocks and consider just their number in a given partition. This gives rise to the Bell–Touchard polynomials in one variable

$$B_n(v) = Y_n(v, \dots, v) = \sum_{k=0}^n \left\{ \begin{matrix} n \\ k \end{matrix} \right\} v^k,$$

where the coefficient of  $v^k$ , the *Stirling number of the second kind*, is equal to the number of partitions of  $[n]$  into  $k$  blocks [31, A048993]. By definition,  $\left\{ \begin{matrix} n \\ 0 \end{matrix} \right\} = 0$  at  $n > 0$  and  $\left\{ \begin{matrix} 0 \\ 0 \end{matrix} \right\} = 1$ , in accordance with the convention  $\Pi_{0,0} = \{\emptyset\}$ . The substitution  $v_l = v$  turns the recurrence relation (6) into

$$B_0(v) = 1, \quad B_{n+1}(v) = (v\partial_v + v)(B_n(v)) \quad \Leftrightarrow \quad \left\{ \begin{matrix} n+1 \\ k \end{matrix} \right\} = \left\{ \begin{matrix} n \\ k-1 \end{matrix} \right\} + k \left\{ \begin{matrix} n \\ k \end{matrix} \right\}$$

$$\begin{aligned}
Y_0 &= 1 \\
Y_1 &= v_1 \\
Y_2 &= v_2 + v_1^2 \\
Y_3 &= v_3 + 3v_1v_2 + v_1^3 \\
Y_4 &= v_4 + (4v_1v_3 + 3v_2^2) + 6v_1^2v_2 + v_1^4 \\
Y_5 &= v_5 + (5v_1v_4 + 10v_2v_3) + (10v_1^2v_3 + 15v_1v_2^2) + 10v_1^3v_2 + v_1^5
\end{aligned}$$

$n \setminus k$	0	1	2	3	4	5	6	7	$B_n$
0	1								1
1	0	1							1
2	0	1	1						2
3	0	1	3	1					5
4	0	1	7	6	1				15
5	0	1	15	25	10	1			52
6	0	1	31	90	65	15	1		203
7	0	1	63	301	350	140	21	1	877

**Table 1.** Complete exponential Bell polynomials; the terms of the same degree are collected in parentheses. Stirling numbers of the second kind  $\{n \atop k\} = |\Pi_{n,k}|$ ; sums over rows are Bell numbers  $B_n = |\Pi_n|$ .

and the generating function (7) takes the form

$$\sum_{n=0}^{\infty} B_n(v) \frac{z^n}{n!} = \exp(v(e^z - 1)).$$

The total number of partitions of a set with  $n$  elements, the *Bell* or *exponential number* [31, A000110], is defined by equations

$$B_n = B_n(1) = Y_n(1, \dots, 1) = \sum_{k=0}^n \{n \atop k\}, \quad \sum_{n=0}^{\infty} B_n \frac{z^n}{n!} = e^{e^z - 1}.$$

### 2.3 Potential Burgers hierarchy

Let us recall the notion of generalized symmetry from the theory of integrable equations. Let  $v_{,t} = \partial_t(v)$  and  $v_i = D^i(v)$  denote, as before, the derivatives with respect to the distinguished variable  $x$ . The evolution equation

$$v_{,t} = f(v_0, \dots, v_m)$$

admits the symmetry  $v_{,\tau} = g(v_0, \dots, v_n)$  (classical at  $n \leq 1$  and generalized, or higher, at  $n > 1$ ), if the corresponding flows commute, that is, the equality

$$f_*(g) = g_*(f)$$

holds identically with respect to  $v_i$ , where  $f_*$  is the differential operator

$$f_* = \partial_{v_0}(f) + \cdots + \partial_{v_m}(f)D^m.$$

The equation is considered integrable if it admits a sequence (hierarchy) of symmetries of arbitrarily large order. Moreover, in a typical situation, the higher symmetries commute with each other. The potential Burgers hierarchy

$$v_{,t_n} = Y_n(v_1, \dots, v_n), \quad n = 0, 1, 2, \dots \quad (9)$$

satisfies this commutativity property. This follows easily from its equivalence to the linear hierarchy

$$\psi_{,t_n} = \psi_n, \quad (10)$$

namely, the change  $\psi = e^v$  results in

$$v_{,t_n} = \psi^{-1}D^n(\psi) = e^{-v}D^n(e^v) = (D + v_1)^n(1) = Y_n(v_1, \dots, v_n).$$

The mutual commutativity of the flows (10) is obvious ( $D_n(\psi_m) = D_m(\psi_n)$ ) and it is not difficult to prove that this property is preserved under the point transformations, see e.g. [17].

*Remark 1.* More generally, in the theory of integrability it is common to consider equivalent the equations related by point or contact transformations and also some non-invertible differential substitutions, because these changes do not affect such properties as the existence of higher symmetries, conservation laws, Lax pairs and so on. The combinatorial interpretation is, however, not invariant. In contrast, we have seen that meaningful combinatorics may appear just from nothing, as a result of simple change of variables between equations (10) and (9). Further examples of this kind are given in sections 3.2, 4.2. Thus, a combinatorics is associated with a concrete form of a hierarchy.

The commutativity follows also from the identity (12) below, which coincides with the recurrence relation (6) at  $m = 1$ ,  $Y_1 = v_1$ ,  $(Y_1)_* = D$ . The general case can be easily proven by induction with respect to  $m$ , but instead we will generalize the combinatorial proof of equation (6). First, let us introduce the following notation.

The *concatenation* of partitions  $\pi \vdash [n]$ ,  $\rho \vdash [m]$  is the partition

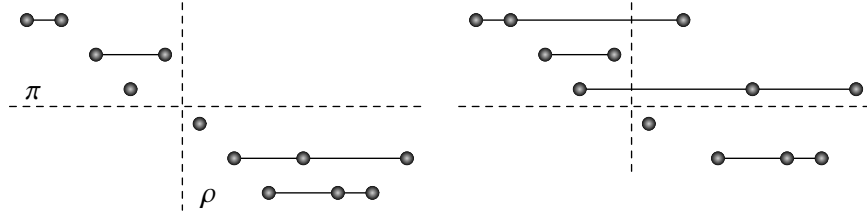
$$\pi|\rho = \pi \cup (\rho + n) \vdash [m + n], \quad (11)$$

where  $\rho + n$  denotes the partition of  $[n + 1, n + m]$  obtained from  $\rho$  by adding  $n$  to all elements. (Compare with the *Mathematica* language convention that adding of a scalar to a list acts on each entry:  $(a_1, \dots, a_k) + b = (a_1 + b, \dots, a_k + b)$ .)

**Statement 3.** *The Bell polynomials satisfy the identity*

$$Y_{m+n} = (Y_m)_*(Y_n) + Y_m Y_n. \quad (12)$$

*Proof.* Given  $\pi \vdash [n]$ ,  $\rho \vdash [m]$ , let us construct the partition  $\sigma = \pi|\rho$  (fig. 2 on the left). The corresponding monomials (3) satisfy the relation  $p(\sigma) = p(\pi)p(\rho)$ . Next, let us consider the partitions obtained from  $\sigma$  by deleting of one of the size  $l$  blocks of  $\rho + n$  and adding its elements to blocks of  $\pi$ , in all possible ways (one such partition



**Figure 2.** Towards the proof of identity (12).

is shown on the right of fig. 2). The sum of the corresponding monomials is equal to  $D^l(p(\pi))\partial_{v_l}(p(\rho))$ .

Vise versa, given  $\sigma \vdash [m+n]$ , let us consider the blocks with the minimal element not greater than  $n$ . Their intersections with  $[n]$  uniquely define the partition  $\pi$ , while the block formed from the rests of these blocks and the rest blocks of  $\sigma$  constitute the partition  $\rho+n$ . Therefore, the above operations give all partitions  $\Pi_{m+n}$ , without repetitions, when  $\pi$  and  $\rho$  run, respectively, over  $\Pi_n$  and  $\Pi_m$ . The summation of monomials gives

$$\sum_{\sigma \in \Pi_{m+n}} p(\sigma) = \sum_{\pi \in \Pi_n} \sum_{\rho \in \Pi_m} \left( p(\pi)p(\rho) + \sum_l D^l(p(\pi))\partial_{v_l}(p(\rho)) \right),$$

as required. □

## 2.4 Burgers hierarchy

The right hand sides of equations (9) do not contain  $v$ , which makes possible the substitution  $u = u_0 = v_1$ . This brings to the Burgers hierarchy

$$u_{,t_n} = D(Y_n(u_0, \dots, u_{n-1})), \quad n = 1, 2, \dots, \quad (13)$$

homogeneous with respect to the weight  $w(u_j) = j + 1$ . The few first equations are

$$\begin{aligned} u_{,t_1} &= u_1, \\ u_{,t_2} &= u_2 + 2uu_1, \\ u_{,t_3} &= u_3 + (3uu_2 + 3u_1^2) + 3u^2u_1, \\ u_{,t_4} &= u_4 + (4uu_3 + 10u_1u_2) + (6u^2u_2 + 12uu_1^2) + 4u^3u_1, \\ u_{,t_5} &= u_5 + (5uu_4 + 15u_1u_3 + 10u_2^2) + (10u^2u_3 + 50uu_1u_2 + 15u_1^3) \\ &\quad + (10u^3u_2 + 30u^2u_1^2) + 5u^4u_1, \quad \dots \end{aligned}$$

What is the combinatorial interpretation of the coefficients in this case? The differentiation in equation (13) corresponds to adding, turn by turn, of an element to blocks of partitions  $\Pi_n$ . Since we do not add a new block in this case, hence the partitions under consideration are constructed in the same manner as in section 2.2, but the operation  $M$  is not used on the last step. As a result, only those of partitions  $\Pi_{n+1}$  are constructed,



which do not contain element  $n + 1$  as a singleton; for instance, at  $n = 3$ :

$$\begin{array}{rcc}
 u_2 + 3uu_1 + u^3 & \xrightarrow{D} & u_3 + 3uu_2 + 3u_1^2 + 3u^2u_1 + 0u^4 \\
 123 & 1|23 & 1|2|3 & 1234 & 1|234 & 12|34 & 1|2|34 & \cancel{1|2|3|4} \\
 & & 12|3 & & \cancel{123|4} & 13|24 & \cancel{1|23|4} \\
 & & 13|2 & & 124|3 & 14|23 & 1|24|3 \\
 & & & & 134|2 & & \cancel{1|2|3|4} \\
 & & & & & & \cancel{13|2|4} \\
 & & & & & & 14|2|3
 \end{array}$$

Renumbering makes possible to replace the element  $n + 1$  with any other one, and we arrive to the following statement.

**Statement 4.** *In the Burgers hierarchy, the coefficient of the monomial  $u_0^{k_0} \dots u_r^{k_r}$  is equal to the number of partitions of the set  $[k_0 + 2k_1 + \dots + (r + 1)k_r]$  into  $k_0$  blocks with 1 element,  $\dots$ ,  $k_r$  blocks with  $r + 1$  elements, under the additional condition that a distinguished element of the set is not a singleton.*

The total number of partitions of  $[n + 1]$  such that a distinguished element is not a singleton is found by setting all variables equal to 1:

$$\begin{aligned}
 D(Y_n(u_0, \dots, u_{n-1}))|_{u_j=1} &= B'_n(1) = \\
 &= \sum_{k=0}^n k \binom{n}{k} = \sum_{k=0}^{n+1} \binom{n+1}{k} - \sum_{k=0}^n \binom{n}{k} = B_{n+1} - B_n, \quad n \geq 0.
 \end{aligned}$$

These integers form the sequence

$$0, 1, 3, 10, 37, 151, 674, 3263, 17007, 94828, 562595, \dots$$

According to [31, A005493], it can be characterized also as the total number of blocks in all partitions of  $[n]$ . Indeed, the partitions of  $[n + 1]$  under consideration are obtained from partitions of  $[n]$  by enlarging of one of the blocks, and this operation is applied exactly as many times as many blocks there are in all partitions.

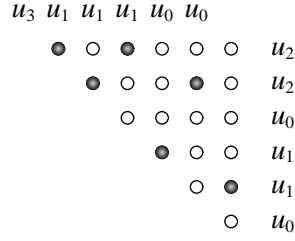
### 3 Natural growth sequences

#### 3.1 Statistics

Let  $T_n$  denote the set of integer sequences  $a = (a_1, \dots, a_n)$  satisfying the *natural growth* condition  $1 \leq a_i \leq i$ . The sequence  $a$  can be matched to an upper triangular  $n \times n$  matrix, with one unit in each column and rest elements equal to zero, see fig. 3. It is clear that  $T_n = [1] \times [2] \times \dots \times [n]$ , that is, the structure of this set is simpler comparing to the set  $R_n$  of the restricted growth sequences (1). In particular,  $|T_n| = n!$ , instead of the Bell numbers for  $|R_n|$ . Nevertheless, these sequences also support a quite meaningful generating function.

For any sequence  $a \in T_n$ , let us count the occurrences of integers from 1 to  $n$  and form the monomial

$$p(a) = u_{|\{i: a_i=1\}|} \cdots u_{|\{i: a_i=n\}|}. \tag{14}$$



**Figure 3.** Sequence  $a = (1, 2, 1, 4, 2, 5)$ . Distribution by rows gives  $p(a) = u^2 u_1^2 u_2^2$ , distribution by diagonals gives  $\bar{p}(a) = u^2 u_1^3 u_3$ .

Summation over all sequences defines the polynomials

$$h_n(u_0, \dots, u_n) = \sum_{a \in T_n} \prod_{j=1}^n u_{|\{i: a_i=j\}|}, \quad (15)$$

analogous to the definition of the Bell polynomials (4) for the restricted growth sequences. In both cases, the definition of monomials is related with counting of elements in the level sets, which are the rows of the upper triangular matrix or the partition blocks. Notice, that one can count the units in the diagonals of the matrix instead of the rows, resulting in the monomial  $\bar{p}(a) = \prod_{j=1}^n u_{|\{i: a_i=i-j+1\}|}$ . In general,  $\bar{p}(a) \neq p(a)$ , but, nevertheless, the summation over all  $a$  gives the same result. Indeed, if we reverse a part of each column, from the first element to the diagonal one, then a matrix of the same type appears, and this transform is bijective and takes the rows into the diagonals. For the sequences, it is given by the map  $a \rightarrow (2, \dots, n+1) - a$ .

In order to derive the recurrence relation for  $h_n$  we assume, as before, that  $u = u_0 = u(x)$ ,  $D = \partial_x$ .

**Statement 5.** *The polynomials  $h_n$  in variables  $u_l = D^l(u)$  satisfy the recurrence relation*

$$h_1 = u_1, \quad h_{n+1} = D(uh_n), \quad n = 1, 2, \dots, \quad (16)$$

that is,  $h_n = (Du)^n(1)$ .

*Proof.* Let us interpret the sequence  $a$  as a distribution of enumerated balls over enumerated boxes, possibly empty, such that  $i$ -th ball is allowed to be only in first  $i$  boxes. The variable  $u_l$  is assigned to each box with  $l$  balls, and the monomial (14) is equal to their product. Generating operations for  $T_n$  are extremely simple: the passage to  $T_{n+1}$  amounts to appending of an arbitrary integer  $a_{n+1} \in [n+1]$ . Let  $d_j a = (a_1, \dots, a_n, j)$ ,  $j = 1, \dots, n+1$ . These operations are interpreted as follows: first, we add the empty box number  $n+1$ , this multiplies  $p(a)$  by  $u = u_0$ ; next, we put the ball number  $n+1$  into the box number  $j$ , this replaces the variable  $u_l$  assigned to this box with  $u_{l+1}$ . As a result,  $p(a)$  is mapped into the sum of monomials over all sequences  $a'$  with the same first  $n$  elements as in  $a$ :

$$\sum_{a'} p(a') = \sum_{j=1}^{n+1} p(d_j a) = \sum_{l=0}^n u_{l+1} \partial_{u_l} (u p(a)) = D(u p(a)).$$

Summation over  $a \in T_n$  completes the proof. □

$$\begin{aligned}
h_1 &= u_1 \\
h_2 &= uu_2 + u_1^2 \\
h_3 &= u^2u_3 + 4uu_1u_2 + u_1^3 \\
h_4 &= u^3u_4 + u^2(7u_1u_3 + 4u_2^2) + 11uu_1^2u_2 + u_1^4 \\
h_5 &= u^4u_5 + u^3(11u_1u_4 + 15u_2u_3) + u^2(32u_1^2u_3 + 34u_1u_2^2) + 26uu_1^3u_2 + u_1^5
\end{aligned}$$

$n \setminus k$	1	2	3	4	5	6	7
1	1						
2	1	1					
3	1	4	1				
4	1	11	11	1			
5	1	26	66	26	1		
6	1	57	302	302	57	1	
7	1	120	1191	2416	1191	120	1

**Table 2.** Polynomials  $h_n$ . Euler numbers  $\langle n \rangle_k$ .

In contrast to the Bell polynomials,  $h_n$  are homogeneous not only with respect to the weight  $w(u_l) = l$ , but also with respect to the degree. The few first polynomials are written down in table 2, with the terms collected by powers of the distinguished variable  $u = u_0$ . Let the rest variables be equal to  $v$  and denote  $h_n(u, v, \dots, v) = vE_n(u, v)$ , this gives the homogeneous polynomials in two variables

$$E_n(u, v) = \sum_{k=1}^n \langle n \rangle_k u^{n-k} v^{k-1} = u^{n-1} + \dots + v^{n-1},$$

where the *Euler number*  $\langle n \rangle_k$  is equal to the number of sequences from  $T_n$  which take  $k$  different values [31, A008292]. The total sum of the coefficients is, obviously,  $E_n(1, 1) = |T_n| = n!$ .

The above substitution replaces the operator  $D$  with  $v(\partial_u + \partial_v)$  and the recurrence relation (16) takes the form

$$E_1 = 1, \quad E_{n+1} = (\partial_u + \partial_v)(uvE_n), \quad n = 1, 2, \dots,$$

which implies the symmetry property and the recurrence relation for the Euler numbers

$$\langle n \rangle_k = \langle n+1-k \rangle_n, \quad \langle n \rangle_k = (n+1-k)\langle n-1 \rangle_{k-1} + k\langle n-1 \rangle_k.$$

*Remark 2.* It should be mentioned, that the Euler numbers admit also many other interpretations, related with the permutations of  $n$  elements, in particular,  $\langle n \rangle_k$  is equal to the number of permutations  $\sigma = (\sigma_1, \dots, \sigma_n) \in S_n$  with  $k-1$  descents (that is, positions  $i$ , such that  $\sigma_i > \sigma_{i+1}$ ), and also to the number of permutations with  $k-1$  exceedances (positions  $i$ , such that  $\sigma_i > i$ ). In fact, the permutations admit a wide variety of combinatorial interpretations and representations, resulting in diversity of

associated statistics, see e.g. [32, 6]. The natural growth sequences also appears in the theory of permutations, as the inversion vectors and Lehmer codes. Recall, that the inversion vector is a sequence  $b = (b_1, \dots, b_n)$ , such that  $b_i$  is equal to the number of elements  $\sigma$  on the left of  $i$  and greater than  $i$ , that is  $b_i = |\{j : j < (\sigma^{-1})_i \wedge \sigma_j > i\}|$ . It is clear that  $b_i \in [0, n - i]$ , therefore, adding 1 to the entries of an inversion vector and taking its reverse bring to a sequence from  $T_n$ . This mapping  $S_n \rightarrow T_n$  is a bijection, see e.g. [32, 27].

### 3.2 Hierarchy of equation $u_t = u^2 u_2$

The polynomials  $h_n$  appear, up to some denotation, in the computation of the inverse function derivatives, like the Bell polynomials appear in the formula for the derivatives of a composite function (8). Indeed, let  $u = 1/D(v(x))$ ,  $D = d/dx$ , then

$$\frac{d^n x}{dv^n} = (uD)^n(x) = u(Du)^{n-1}(1) = u h_{n-1}(u, \dots, u_{n-1})$$

(cf. with the definition of  $h_n$  in [31, A145271]). Essentially, this formula is used in the hodograph type transformation of the linear hierarchy (10) which we rewrite now in the form

$$\partial_{\tilde{t}_n}(\tilde{v}) = \partial_{\tilde{x}}^n(\tilde{v}).$$

Under the change of variables  $v = \tilde{x}$ ,  $x = \tilde{v}$ ,  $t_n = \tilde{t}_n$ , the derivatives are computed according to the rule

$$\partial_{\tilde{x}} = \frac{1}{v_1} D, \quad \partial_{\tilde{t}_n} = \partial_{t_n} - \frac{v, t_n}{v_1} D, \quad D = \partial_x, \quad v_1 = D(v)$$

and the equation takes the form  $v, t_n = -(D \frac{1}{v_1})^{n-1}(1)$ . Further differential substitution  $u = 1/v_1$  brings to equations

$$u, t_n = u^2 D(Du)^{n-1}(1) = u^2 D(h_{n-1}(u, \dots, u_{n-1})), \quad n = 2, 3, \dots, \quad (17)$$

in particular, first two flows are

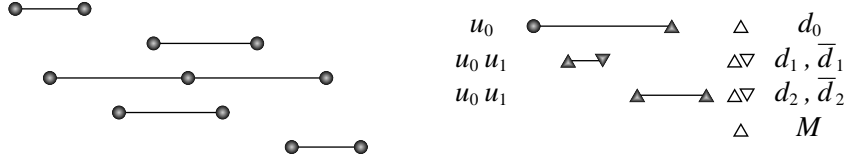
$$u, t_2 = u^2 u_2, \quad u, t_3 = u^3 u_3 + 3u^2 u_1 u_2$$

(a discussion of this example can be found in [17, sect.20.1]).

The commutativity of equations (17) follows, like in section 2.3, either from their relation to equations (10) or intermediately from the identity

$$h_{m+n+1} = (h_m)_* u^2 D(h_n) + D(u h_m h_n),$$

which can be easily proved by induction with respect to  $m$ . It would be interesting to find its combinatorial meaning, like in Statement 3.



**Figure 4.** On the left: full representation of a  $B$  type partition; the blocks are sorted downwards by increase of elements with minimal absolute value. On the right: representation of the canonical form; negative elements are marked  $\blacktriangledown$ , positive  $\blacktriangle$ . Generating operations add element to one of the vacant positions marked by empty triangles. The corresponding monomials (18) are written near to the blocks.

## 4 $B$ type set partitions

### 4.1 Generating operations and statistics

Set partitions of  $B$  type [12], see also [3, 29, 33], appear if we employ the reflection symmetry of the set under consideration.

**Definition 1.** A partition  $\pi$  of the set  $\{-n, \dots, n\}$  is called  $B$  type partition if:

- 1)  $\pi = -\pi$ , that is, for any block  $\beta \in \pi$  also  $-\beta \in \pi$ ;
- 2)  $\pi$  contains just one block  $\pi_0$  (called 0-block), such that  $\pi_0 = -\pi_0$ .

All such partitions are denoted  $\Pi_n^B$ , and partitions with  $k$  block pairs are denoted  $\Pi_{n,k}^B$ ,  $0 \leq k \leq n$ . The canonical form of  $B$  type partition is defined as follows: in 0-block, all negative elements are removed; in all blocks, the elements are sorted by increase of the absolute value; from any block pair we keep only the block with positive first element; the blocks are sorted by increase of their first elements. Moreover, it is convenient to write  $\bar{j}$  instead of  $-j$ . For instance, the partition

$$\{\{-5, -3\}, \{-1, 2\}, \{-4, 0, 4\}, \{-2, 1\}, \{3, 5\}\}$$

is encoded as  $04|1\bar{2}|35$ . The pictorial representation is clear from fig. 4.

Notice, that deleting elements  $\pm n$  from any partition  $\Pi_n^B$  yields a partition from  $\Pi_{n-1}^B$ . Therefore,  $\Pi_n^B$  is obtained from  $\Pi_{n-1}^B$  by adding  $\pm n$  in all possible ways, namely:

$$d_0 : \Pi_{n-1,k}^B \rightarrow \Pi_{n,k}^B, \text{ adding both elements } \pm n \text{ to 0-block;}$$

$$d_j : \Pi_{n-1,k}^B \rightarrow \Pi_{n,k}^B, \text{ adding } \pm n \text{ to the blocks } \pm\pi_j, j = 1, \dots, k;$$

$$\bar{d}_j : \Pi_{n-1,k}^B \rightarrow \Pi_{n,k}^B, \text{ adding } \pm n \text{ to the blocks } \mp\pi_j, j = 1, \dots, k;$$

$$M : \Pi_{n-1,k}^B \rightarrow \Pi_{n,k+1}^B, \text{ adding a new block pair } \{-n\}, \{n\}.$$

These operations generate, in a unique way, any  $B$  type partition starting from the trivial partition of the set  $\{0\}$ .

Let us define the mapping  $p$  from  $\Pi_n^B$  into the set of monomials of variables  $u_j$ . Given a block  $\beta$ , let  $\langle\beta\rangle$  denote the number of its positive elements:

$$\langle\beta\rangle = |\{i \in \beta : i > 0\}|.$$

It is clear that the number of negative elements in the block is equal to  $\langle -\beta \rangle$ . Let a partition  $\pi \in \Pi_{n,k}^B$  consists of 0-block  $\pi_0$  and block pairs  $\pi_1, -\pi_1, \dots, \pi_k, -\pi_k$ , such that the elements with minimal absolute value are positive in blocks  $\pi_j$ . We assign the monomial of degree  $2k + 1$

$$p(\pi) = u_{\langle \pi_0 \rangle} u_{\langle \pi_1 \rangle - 1} u_{\langle -\pi_1 \rangle} \cdots u_{\langle \pi_k \rangle - 1} u_{\langle -\pi_k \rangle} \quad (18)$$

to such a partition, so that all partitions  $\Pi_n^B$  are associated with the polynomial

$$a_n(u_0, u_1, \dots, u_n) = \sum_{\pi \in \Pi_n^B} u_{\langle \pi_0 \rangle} \prod_{j=1}^{(|\pi| - 1)/2} u_{\langle \pi_j \rangle - 1} u_{\langle -\pi_j \rangle} \quad (19)$$

which serves as  $\mathbb{Z}_2$ -analog of the Bell polynomial  $Y_n$ . As an example, let us write down all partitions  $\Pi_3^B$ , collecting together the partitions which correspond to one and the same monomial (as usual,  $u = u_0$ ):

$$a_3 = u_3 + 5u^2u_2 + 8uu_1^2 + 9u^4u_1 + u^7$$

0123	0 123	0 12 $\bar{3}$	0 12 3	0 1 2 3
	0 1 $\bar{2}$ $\bar{3}$	0 1 $\bar{2}$ 3	0 1 $\bar{2}$  3	
	012 3	01 23	0 13 2	
	013 2	01 2 $\bar{3}$	0 1 $\bar{3}$  2	
	023 1	02 13	0 23 1	
		02 1 $\bar{3}$	0 2 $\bar{3}$  1	
		03 12	01 2 3	
		03 1 $\bar{2}$	02 1 3	
			03 1 2	

**Statement 6.** *Polynomials (19) in variables  $u_i = D^i(u)$ ,  $D = \partial_x$ , satisfy the recurrence relation*

$$a_0 = u, \quad a_{n+1} = (D + u^2)(a_n), \quad n \geq 0. \quad (20)$$

*Proof.* Let  $\pi \in \Pi_{n-1,k}^B$  and let us track the monomial  $p(\pi)$  under the action of generating operations:

- $d_0$ : the factor  $u_{\langle \pi_0 \rangle}$  is replaced with  $u_{\langle \pi_0 \rangle + 1}$ ;
- $d_j$ : the factor  $u_{\langle \pi_j \rangle - 1}$  is replaced with  $u_{\langle \pi_j \rangle}$ ;
- $\bar{d}_j$ : the factor  $u_{\langle -\pi_j \rangle}$  is replaced with  $u_{\langle -\pi_j \rangle + 1}$ ;
- $M$ : two factors  $u$  are added.

Therefore, the application of all possible operations  $\pi \rightarrow \pi'$  replaces the monomial  $p(\pi)$  with  $\sum_{\pi'} p(\pi') = (D + u^2)(p(\pi))$ ; the summation over  $\pi$  completes the proof.  $\square$

The few first polynomials  $a_n$  are given in table 3. Let us simplify the statistics by merging the terms of the same degree; this brings to the polynomials  $a_n(u, \dots, u) = (u\partial_u + u^2)^n(u)$ . The coefficients of  $u^{2k+1}$  are the numbers of partitions  $\Pi_{n,k}^B$ , that is, the analogs of the Stirling numbers of the 2nd kind for  $B$  type partitions [31, A039755]. The total coefficient sums  $a_n(1, \dots, 1)$  give the total numbers of partitions  $\Pi_n^B$  which constitute the sequence of  $B$ -analogs of the Bell numbers, or the *Dowling numbers* [31, A007405].

$$\begin{aligned}
a_0 &= u \\
a_1 &= u_1 + u^3 \\
a_2 &= u_2 + 4u^2u_1 + u^5 \\
a_3 &= u_3 + (5u^2u_2 + 8uu_1^2) + 9u^4u_1 + u^7 \\
a_4 &= u_4 + (6u^2u_3 + 26uu_1u_2 + 8u_1^3) + (14u^4u_2 + 44u^3u_1^2) + 16u^6u_1 + u^9 \\
a_5 &= u_5 + (7u^2u_4 + 38uu_1u_3 + 26uu_2^2 + 50u_1^2u_2) + (20u^4u_3 + 170u^3u_1u_2 \\
&\quad + 140u^2u_1^3) + (30u^6u_2 + 140u^5u_1^2) + 25u^8u_1 + u^{11}
\end{aligned}$$

$n \setminus k$	0	1	2	3	4	5	6	7
0	1							1
1	1	1						2
2	1	4	1					6
3	1	13	9	1				24
4	1	40	58	16	1			116
5	1	121	330	170	25	1		648
6	1	364	1771	1520	395	36	1	4088
7	1	1093	9219	12411	5075	791	49	28640

**Table 3.** Polynomials  $a_n$  are homogeneous with respect to the weight  $w(u_j) = 2j + 1$ ; parentheses separate the terms of the same degree.  $B$ -analogs of the Stirling numbers of the 2nd kind  $|\Pi_{n,k}^B|$ ; sums in rows are equal to the Dowling numbers  $|\Pi_n^B|$ .

## 4.2 Ibragimov–Shabat hierarchy

The diagram

$$\begin{array}{ccc}
\psi_{,t_3} = \psi_3 & & u_{,t_3} = u_3 + 3u^2u_2 + 9uu_1^2 + 3u^4u_1 \\
\updownarrow \psi^2 = s & & \updownarrow u^2 = v \\
s_{,t_3} = D\left(s_2 - \frac{3s_1^2}{4s}\right) & & v_{,t_3} = D\left(v_2 - \frac{3v_1^2}{4v} + 3vv_1 + v^3\right) \\
\uparrow s = q_1 & & \uparrow v = w_1 \\
q_{,t_3} = q_3 - \frac{3q_2^2}{4q_1} & \xleftrightarrow{q=e^{2w}} & w_{,t_3} = w_3 - \frac{3w_2^2}{4w_1} + 3w_1w_2 + w_1^3
\end{array}$$

describes the chain of point transformations and substitutions of introducing a potential type which linearizes the Ibragimov–Shabat equation [18]. The whole hierarchy consists of equations related by the same transformations with equations  $\psi_{,t_{2m+1}} = \psi_{2m+1}$ . The absence of even-order flows is explained by the fact that, although the changes look quite harmless, they partially destroy the symmetry algebra. For  $\partial_{t_{2m+1}}$ , the change  $\psi^2 = s$  brings to an equation with the total derivative in the right hand side:

$$s_{,t_{2m+1}} = 2\psi\psi_{2m+1} = D(2\psi\psi_{2m} - 2\psi_1\psi_{2m-1} + 2\psi_2\psi_{2m-2} + \cdots \pm \psi_m^2). \quad (21)$$

In contrast, the analogous equation for  $s_{,t_{2m}}$  contains the term  $\psi_m^2$  outside the parentheses, that is  $s_{,t_{2m}} \notin \text{Im } D$ ; so that the further substitution  $s = q_1$  leads out of the class of evolution equations. The structure of odd flows is described by the following statement.

**Statement 7.** Denote  $D_t = \partial_{t_1} + z^2 \partial_{t_3} + z^4 \partial_{t_5} + \dots$ ,  $A = A(z) = a_0 + a_1 z + a_2 z^2 + \dots$ ,  $\bar{A} = A(-z)$ , then the Ibragimov–Shabat hierarchy is equivalent to equations

$$D_t(u) = \frac{1}{2u} D(A\bar{A}) = \frac{1}{2z} (A - \bar{A}) - uA\bar{A}, \quad (22)$$

$$z(D + u^2)(A) = A - u, \quad (23)$$

and the coefficients  $a_n$  coincide with the polynomials (20).

*Proof.* Let us consider the generating function

$$\Psi = \psi + \psi_1 z + \psi_2 z^2 + \dots$$

and define  $A$  by equation  $\Psi = \sqrt{2}e^w A$ . The change of variables  $\psi = \sqrt{q_1} = \sqrt{2e^{2w} w_1} = \sqrt{2}e^w u$  maps the identity  $zD(\Psi) = \Psi - \psi$  into equation (23), which is exactly equivalent to recurrence relations (20).

Next, let  $\bar{\Psi} = \Psi(-z)$ , then (cf. with (21))

$$\begin{aligned} D(\Psi\bar{\Psi}) &= z^{-1}(\Psi - \psi)\bar{\Psi} - z^{-1}\Psi(\bar{\Psi} - \psi) \\ &= z^{-1}\psi(\Psi - \bar{\Psi}) = 2\psi(\psi_1 + \psi_3 z^2 + \dots) = 2\psi D_t(\psi) = D_t(s). \end{aligned}$$

By applying  $D^{-1}$ , we obtain  $\Psi\bar{\Psi} = D_t(q) = 2e^{2w} D_t(w)$  which implies

$$2uD_t(u) = D_t(v) = DD_t(w) = \frac{1}{2}D(e^{-2w}\Psi\bar{\Psi}) = D(A\bar{A}).$$

The second equality in (22) is obtained by eliminating of derivatives in virtue of (23).  $\square$

## 5 Non-overlapping partitions

### 5.1 Generating operations

The definition of non-overlapping partitions [14] makes use of the ordering of the set  $[n]$ .

**Definition 2.** Blocks  $\alpha$  and  $\beta$  of a partition  $\pi \vdash [n]$  overlap, if

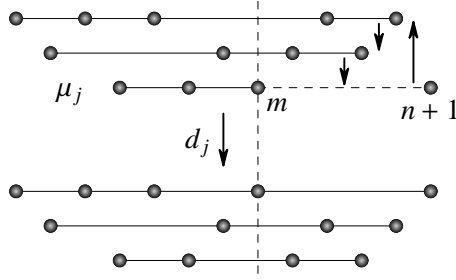
$$\min \alpha < \min \beta < \max \alpha < \max \beta.$$

A partition is called *non-overlapping* if no pair of its blocks overlap. Equivalently, the supports of any two blocks either do not intersect or one of them contains another one.

For instance, the partition in the fig. 1 does not satisfy this requirement, because the blocks  $\pi_1, \pi_3$  overlap (as well as  $\pi_2, \pi_3$ ; the other pairs do not).

All non-overlapping partitions of  $[n]$  are denoted  $\Pi_n^*$ , and those which contain  $k$  multiplets (and any number of singletons) will be denoted  $\Pi_{n,k}^*$ ,  $0 \leq k \leq n/2$ . It is natural to consider  $\emptyset$  non-overlapping partition, that is,  $\Pi_0^* = \Pi_{0,0}^* = \{\emptyset\}$ . Also, let  $\tilde{\Pi}_n^*$  denote the set of non-overlapping partitions of  $[n]$  without singletons.





**Figure 5.** Operation  $d_j$  for non-overlapping partitions.

*Remark 3.* In papers [14, 9],  $S_{n,k}^*$  denotes the number of non-overlapping partitions of  $[n]$  with  $k$  blocks of any size. Counting multiplets only turns out to be more convenient in our construction method for partitions. In this approach, one may collect, in the mind's eye, all singletons of a partition into one distinguished block (possibly, empty). Notice, that this establishes a bijection between  $\Pi_n^*$  and a subset of  $\tilde{\Pi}_{n+2}^*$  which consists of partitions with elements 1 and  $n+2$  in one block.

The generating operations for non-overlapping partitions are more complicated than in the previous sections. These include the operations

$$d_j : \Pi_{n,k}^* \rightarrow \Pi_{n+1,k}^*, \quad 0 \leq j \leq k, \quad P : \Pi_{n,k}^* \times \Pi_{m,l}^* \rightarrow \Pi_{n+m+2,k+l+1}^*$$

which we now describe.

*Operation  $d_0$ ,* adding of the singleton  $\{n+1\}$  to a partition. This coincides with the operation  $M$  for generic set partitions from section 2.2, but we use another notation and interpret  $d_0$  as adding the element into the team of singletons. Notice, that  $d_0\emptyset = \{\{1\}\}$ .

*Operation  $d_j$ ,*  $1 \leq j \leq k$ , adding the element  $n+1$  to a multiplet. Let  $\mu_1, \dots, \mu_k$  be all multiplets in  $\pi \in \Pi_{n,k}^*$ , sorted by increase of their minimal elements. If we just add the element  $n+1$  to  $\mu_j$  then the multiplets with the supports embracing  $\mu_j$  will overlap with this block (see fig. 5 above). Let these multiplets are those with the indices  $j_1 < \dots < j_s = j$ . Let us divide them into the left and right parts with respect to  $m = \max \mu_j$ :

$$\mu_{j_r}^- = \{i \in \mu_{j_r} : i < m\}, \quad \mu_{j_r}^+ = \{i \in \mu_{j_r} : i \geq m\},$$

and form new multiplets by cyclic permutation of the right parts (fig. 5 below):

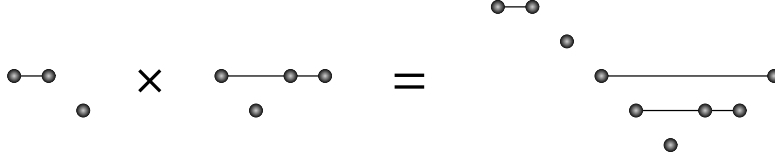
$$\tilde{\mu}_{j_1} = \mu_{j_1}^- \cup \{m, n+1\}, \quad \tilde{\mu}_{j_r} = \mu_{j_r}^- \cup \mu_{j_{r-1}}^+, \quad r = 2, \dots, s.$$

Notice, that the block  $\tilde{\mu}_{j_1}$ , which acquires the element  $n+1$ , contains 3 elements or more. It is easy to see that the described procedure brings to the blocks which do not overlap with each other and with the rest multiplets.

*Operation  $P$ .* Let  $\pi \in \Pi_{n,k}^*$ ,  $\rho \in \Pi_{m,l}^*$ , then

$$P(\pi, \rho) = \pi \cup \{\{n+1, m+n+2\}\} \cup (\rho + n+1) \in \Pi_{m+n+2,k+l+1}^*. \quad (24)$$

This is a kind of concatenation (11), with the wrapping of the second partition by a doublet (see fig. 6). In particular,  $P(\emptyset, \emptyset) = \{\{1, 2\}\}$ .



**Figure 6.** Operation  $P$  for non-overlapping partitions.

**Statement 8.** *Applying all admissible operations  $d_j, P$  to the seed partition  $\emptyset$  generates all non-overlapping partitions:*

$$\Pi_0^* = \{\emptyset\}, \quad \Pi_{n+1}^* = \bigcup_{k=0}^{\lfloor n/2 \rfloor} \bigcup_{j=0}^k d_j \Pi_{n,k}^* \cup \bigcup_{j=0}^{n-1} P(\Pi_j^*, \Pi_{n-1-j}^*), \quad n \geq 0, \quad (25)$$

moreover, any partition appears only once.

*Proof.* The last of operations leading to a given partition  $\pi \in \Pi_{n+1}^*$  is defined by the block  $\beta$  with the element  $n+1$ :

- if  $|\beta| = 1$  then  $\pi = d_0 \rho$ ;
- if  $|\beta| = 2$  then  $\pi = P(\rho, \sigma)$ ;
- if  $|\beta| > 2$  then  $\pi = d_j \rho$ , where  $j$  is the maximal index of the multiplet with the support containing the element  $m \in \beta$  which precedes  $n+1$ .

In all cases, the partitions  $\rho$  or  $\rho, \sigma$  are determined uniquely and consist of lesser number of elements, so that the induction can be applied.  $\square$

The generating of a given partition is encoded by expression  $\varphi(u)$ , builded from a formal variable  $u$  by the binary operation  $P(a, b)$  and operations  $d_j a$ ,  $0 \leq j \leq \deg a - 1$ , where  $\deg a$  is equal to the number of occurrences of  $u$  in  $a$ . Let the set  $\Phi_n^*$  consist of all such expressions containing  $n+1$  characters  $u, d, P$ , and the set  $\Phi_{n,k}^*$  include those of them which contain  $k+1$  characters  $u$ . Then

$$d_j : \Phi_{n,k}^* \rightarrow \Phi_{n+1,k}^*, \quad 0 \leq j \leq k, \quad P : \Phi_{n,k}^* \times \Phi_{m,l}^* \rightarrow \Phi_{n+m+2,k+l+1}^*.$$

Moreover,  $0 \leq k \leq n/2$ , because the number of occurrences of  $P$  in  $\varphi$  is always one less than the number of occurrences of  $u$  (that is, it is equal to  $\deg \varphi - 1$ ). A comparison with the action of the operations on the partitions leads to the following bijection, taking the Statement 8 into account.

**Corollary 9.** *The substitution  $\varphi(u) \mapsto \varphi(\emptyset)$  is a one-to-one mapping  $\Phi_{n,k}^* \rightarrow \Pi_{n,k}^*$ , for all  $n, k$ .*

Table 4 contains a complete list of expressions  $\Phi_n^*$  and respective partitions  $\Pi_n^*$  for  $n \leq 4$ . Notice, that  $\Pi_n^* = \Pi_n$  at  $n \leq 3$ , and in  $\Pi_4$  only the partition 13|24 is eliminated.

The non-overlapping partitions without singletons  $\tilde{\Pi}_n^*$  are builded by the same scheme, but with  $d_0$  excluded from the set of generating operations. The corresponding set  $\tilde{\Phi}_n^*$  consists of all expressions  $\Phi_n^*$  with no occurrence of  $d_0$ .

$n$	$\varphi(u)$	$\varphi(\emptyset)$	$p(\varphi)$	$n$	$\varphi(u)$	$\varphi(\emptyset)$	$p(\varphi)$
0	$u$	$\emptyset$	$u$	4	$d_0d_0d_0d_0u$	$1 2 3 4$	$u_4$
					$d_0d_0P(u, u)$	$12 3 4$	$uu_2$
1	$d_0u$	1	$u_1$		$d_1d_0P(u, u)$	$124 3$	$u_1^2$
					$d_0d_1P(u, u)$	$123 4$	$u_1^2$
2	$d_0d_0u$	$1 2$	$u_2$		$d_1d_1P(u, u)$	$1234$	$uu_2$
	$P(u, u)$	12	$u^2$		$d_0P(u, d_0u)$	$13 2 4$	$u_1^2$
					$d_1P(u, d_0u)$	$134 2$	$uu_2$
3	$d_0d_0d_0u$	$1 2 3$	$u_3$		$d_0P(d_0u, u)$	$1 23 4$	$uu_2$
	$d_0P(u, u)$	$12 3$	$uu_1$		$d_1P(d_0u, u)$	$1 234$	$u_1^2$
	$d_1P(u, u)$	123	$uu_1$		$P(u, d_0d_0u)$	$14 2 3$	$uu_2$
	$P(u, d_0u)$	$13 2$	$uu_1$		$P(u, P(u, u))$	$14 23$	$u^3$
	$P(d_0u, u)$	$1 23$	$uu_1$		$P(d_0u, d_0u)$	$1 24 3$	$u_1^2$
					$P(d_0d_0u, u)$	$1 2 34$	$uu_2$
					$P(P(u, u), u)$	$12 34$	$u^3$

**Table 4.** Sets  $\Phi_n^*$ ,  $\Pi_n^*$  and monomials  $p(\varphi)$  for  $n \leq 4$ .

## 5.2 Statistics

In sections 2, 4 we assigned a monomial to any partition by counting the elements (or positive elements) in the blocks and by multiplying the corresponding variables  $u_l$ . For the non-overlapping partitions this does not bring to a meaningful result, because the operations  $d_j$  change the sizes of blocks unpredictably. In this case, a suitable definition comes from the encoding of partitions by expressions  $\varphi \in \Phi_n^*$ . We define the monomial  $p(\varphi)$  as the value of expression  $\varphi$ , computed according to the following rules:

(i) operation  $d_j a$  adds 1 to the subscript of  $u$  for  $j + 1$ -th occurrence of  $u$  in  $a$ , counting from the left (as usual, we assume that  $u = u_0$ );

(ii) all operations  $P(a, b)$  are applied after  $d_j$ , and their result is equal to the product  $ab$ .

Let us define the polynomials

$$f_n(u, u_1, \dots, u_n) = \sum_{\varphi \in \Phi_n^*} p(\varphi) \quad (26)$$

by summation over all expressions  $\Phi_n^*$ , and consider the generating function

$$f = -z/2 + f_0 z^{-1} + \dots + f_{n-1} z^{-n} + \dots \quad (27)$$

**Statement 10.** *Polynomials  $f_n$  in variables  $u_i = D^i(u)$ ,  $D = \partial_x$ , are governed by recurrence relations*

$$f_0 = u, \quad f_{n+1} = D(f_n) + \sum_{s=0}^{n-1} f_s f_{n-1-s}, \quad n = 0, 1, 2, \dots, \quad (28)$$

*equivalent to the Riccati equation*

$$D(f) + f^2 = z^2/4 - u. \quad (29)$$

*Proof.* Equation (28) follows from (25) and identities

$$\sum_{j=0}^{\deg \varphi - 1} p(d_j \varphi) = D(p(\varphi)), \quad p(P(\varphi, \phi)) = p(\varphi)p(\phi),$$

where  $\varphi \in \Phi_n^*$ ,  $\phi \in \Phi_m^*$ . The second identity follows from the rule (ii). In order to prove the first one, let us preserve the same order of variables  $u_i$  in a monomial, as they appear after applying of operations  $d_j$  according to the rule (i). Then, if a monomial corresponding to expression  $\varphi \in \Phi_{n,k}^*$  is equal to  $p(\varphi) = u_{i_0} \cdots u_{i_k}$ , then  $p(d_j \varphi) = u_{i_0} \cdots u_{i_{j+1}} \cdots u_{i_k}$ ,  $0 \leq j \leq k$ , and we only have to sum over  $j$ .  $\square$

In this proof, the operation  $d_j$  is identified with the action of differentiation  $D$  onto one of the factors according to the Leibniz rule. This leads to the following simple description: *in polynomials  $f_n$ , the coefficient of any monomial is equal to the number of ways to obtain this monomial by multiplication and differentiation.* For instance, there are 4 ways for the monomial  $uu_1$  (see table 4): first multiply  $u$  by  $u$  and next apply  $D$  to the first or second factor, or first apply  $D$  to  $u$  and then multiply by  $u$  from left or from right. The few first polynomials  $f_n$  are written down in table 5.

Now, let us consider the set  $\tilde{\Pi}_n^*$  of non-overlapping partitions without singletons. Since the corresponding expressions  $\varphi$  do not contain  $d_0$ , hence the first occurrence of  $u$  is not differentiated when computing the monomial  $p(\varphi)$ , and therefore  $p(\varphi)$  is divisible by  $u$ . So, let us assign to these partitions the polynomial

$$\tilde{f}_n(u, \dots, u_n) = \frac{1}{u} \sum_{\varphi \in \tilde{\Phi}_n^*} p(\varphi).$$

In particular, one finds easily from table 4

$$\tilde{f}_0 = 1, \quad \tilde{f}_1 = 0, \quad \tilde{f}_2 = u, \quad \tilde{f}_3 = u_1, \quad \tilde{f}_4 = u_2 + 2u^2.$$

It turns out that these polynomials are very simply related with the polynomials for all non-overlapping partitions.

**Statement 11.** *The generating function for  $\tilde{f}_n$  is equal to*

$$\tilde{f} = \tilde{f}_0 z^{-1} + \tilde{f}_1 z^{-2} + \cdots + \tilde{f}_{n-1} z^{-n} + \cdots = \frac{1}{z/2 - f}. \quad (30)$$

*Proof.* By repeating the arguments in the previous proof, we obtain that  $\tilde{f} = \tilde{f}_0 z^{-1} + \tilde{f}_1 z^{-2} + \cdots$  satisfies the equation  $D(\tilde{f}) + u\tilde{f}^2 - z\tilde{f} + 1 = 0$ . From here, all  $\tilde{f}_n$  are determined uniquely through recurrence relations which can be easily written down. The change  $f = z/2 - 1/\tilde{f}$  brings to equation (29), and since the leading term of  $f$  is equal to  $z/2 - 1/(1/z) = -z/2$ , hence  $f$  coincides with the series (27).  $\square$

Let us pass to the simplified statistics, as in the previous sections, by merging the terms of the same degree in  $f_n$  under the substitution  $u_i = u$ . Since  $\deg p(\varphi)$  is one more than the number of multipliants in the corresponding partition  $\varphi(\emptyset)$ , hence the coefficient of  $u^{k+1}$  is equal to the cardinality of the set  $\Phi_{n,k}^*$ , or, what is the same, of the set  $\Pi_{n,k}^*$ .

$$\begin{aligned}
f_0 &= u \\
f_1 &= u_1 \\
f_2 &= u_2 + u^2 \\
f_3 &= u_3 + 4uu_1 \\
f_4 &= u_4 + (6uu_2 + 5u_1^2) + 2u^3 \\
f_5 &= u_5 + (8uu_3 + 18u_1u_2) + 16u^2u_1 \\
f_6 &= u_6 + (10uu_4 + 28u_1u_3 + 19u_2^2) + (30u^2u_2 + 50uu_1^2) + 5u^4
\end{aligned}$$

$n \setminus k$	0	1	2	3	4	5	$B_n^*$	$\tilde{B}_n^*$
0	1						1	1
1	1						1	0
2	1	1					2	1
3	1	4					5	1
4	1	11	2				14	3
5	1	26	16				43	7
6	1	57	80	5			143	20
7	1	120	324	64			509	60
8	1	247	1170	490	14		1922	195
9	1	502	3948	2944	256		7651	675
10	1	1013	12776	15403	2730	42	31965	2480

**Table 5.** Polynomials  $f_n$ , homogeneity  $w(u_j) = j + 2$ . Number triangle  $|\Pi_{n,k}^*|$  for non-overlapping partitions of  $[n]$  with  $k$  multiplets. The Bessel numbers  $B_n^* = |\Pi_n^*|$  are equal to the sums over the rows;  $\tilde{B}_n^* = |\tilde{\Pi}_n^*|$ .

**Corollary 12.** *The number  $|\Pi_{n,k}^*|$  of non-overlapping partitions of  $[n]$  with  $k$  multiplets is equal to the coefficient of  $u^{k+1}$  in the polynomial in one variable  $F_n(u)$ , defined by recurrence relations*

$$F_0 = u, \quad F_{n+1} = uF_n' + \sum_{s=0}^{n-1} F_s F_{n-1-s}, \quad n = 0, 1, 2, \dots \quad (31)$$

The generating function  $F = -z/2 + \sum_{n \geq 1} F_{n-1} z^{-n}$  satisfies the equation

$$uF'(u) + F(u)^2 = z^2/4 - u. \quad (32)$$

**Corollary 13.** *The number  $|\tilde{\Pi}_{n,k}^*|$  of non-overlapping partitions  $[n]$  with  $k$  multiplets and without singletons is equal to the coefficient of  $u^k$  in the polynomial  $\tilde{F}_n(u)$ , where*

$$\sum_{n \geq 1} \tilde{F}_{n-1} z^{-n} = \frac{1}{z - \sum_{n \geq 1} F_{n-1} z^{-n}}.$$

The substitution  $u = \frac{y^2}{4}$ ,  $F(u) = \frac{y\psi'(y)}{2\psi(y)}$  reduces (32) to the Bessel equation

$$y^2\psi''(y) + y\psi'(y) + (y^2 - z^2)\psi(y) = 0$$

with the linearly independent solutions  $\psi = J_{\pm z}(y)$ , where  $J_z$  is the Bessel function

$$J_z(y) = \sum_{k=0}^{\infty} \frac{(-1)^k (y/2)^{z+2k}}{k! \Gamma(z+k+1)}.$$

This function satisfies the recurrence relations

$$J_{z-1}(y) = J'_z(y) + \frac{z}{y} J_z(y), \quad J_{z+1}(y) - \frac{2z}{y} J_z(y) + J_{z-1}(y) = 0.$$

Generating functions  $f, F, \tilde{f}, \tilde{F}$  satisfy the corresponding Riccati equations, as formal asymptotic series with respect to the parameter  $z$ . Taking into account that  $F(u) = -z/2 + 0z^0 + \dots$ , we obtain the asymptotic expansion

$$\frac{yJ'_z(y)}{2J_z(y)} = -\frac{z}{2} + \frac{yJ_{z-1}(y)}{2J_z(y)} \sim F\left(\frac{y^2}{4}, -z\right), \quad \operatorname{Re} z \rightarrow +\infty. \quad (33)$$

*Bessel numbers*  $B_n^* = |\Pi_n^*| = F_n(1)$  counting the non-overlapping partitions of  $[n]$  are equal to the total sums of the coefficients of  $f_n$ . Analogously, *2-associated Bessel numbers*  $\tilde{B}_n^* = |\tilde{\Pi}_n^*| = \tilde{F}_n(1)$  count the non-overlapping partitions without singletons. For the first time, both sequences were defined in [14], where an asymptotic expansion was obtained which coincides with (33) under the substitution  $y = 2$ . However, its derivation was based on a different combinatorial technique: instead of the above generating operations which bring to the Riccati equation, the so-called path diagrams were used [13] which lead to a representation of  $F$  as a continued fraction, which is equivalent to the three-term recurrence relation for  $J_z$ . From here, a functional equation for the generating function  $F(1, z)$  can be derived, which implies a recurrence relation intermediately for the numbers  $B_n^*$  [20, eq.(22)], see also [9], [31, A006789] and [31, A099950] (sequence  $\tilde{B}_n^* + \tilde{B}_{n+1}^*$ ). This is equivalent to the following statement after elimination of  $\tilde{B}_n^*$ .

**Statement 14.** *Sequences  $B_n^*, \tilde{B}_n^*$  satisfy the recurrence relations*

$$\tilde{B}_0^* = 1, \quad B_n^* = \sum_{j=0}^n \binom{n}{j} \tilde{B}_{n-j}^*, \quad \tilde{B}_{n+1}^* = \sum_{j=0}^{n-1} \tilde{B}_j^* B_{n-1-j}^*, \quad n \geq 0. \quad (34)$$

*Proof.* The equation for  $\tilde{B}_{n+1}^*$  follows from the Corollary 13, the equation for  $B_n^*$  appears if we enlarge the partitions without singletons until the necessary size, by inserting singletons in all possible ways.  $\square$

The diagonal entries of the number triangle in table 5, that is, the coefficients of “dispersionless terms”  $u^{k+1}$  in  $f_{2k}$ , form the sequence of Catalan numbers [31, A000108]. These monomials correspond to partitions builded only by operation  $P$ , that is, consisting only from non-overlapping doublets, which can be easily identified with the strings of balanced parentheses. Correspondingly, the replacement of the initial condition in equation (28) by 1 leads to the recurrence for the Catalan numbers interpolated with zeroes [31, A126120]:

$$c_0 = 1, \quad c_{n+1} = \sum_{s=0}^{n-1} c_s c_{n-1-s} \rightarrow 1, 0, 1, 0, 2, 0, 5, 0, \dots$$

### 5.3 Korteweg–de Vries hierarchy

Recall that the Riccati equation (29) plays the key role in the theory of KdV equation

$$u_{,t_3} = u_3 + 6uu_1,$$

as a tool for computing both conservation laws and higher symmetries [15]. The KdV hierarchy is defined by the compatibility conditions for the Schrödinger equation

$$D^2(\psi) = (z^2/4 - u)\psi \quad (35)$$

and equations of the form  $\psi_{,t_{2n+1}} = G_{2n}D(\psi) - \frac{1}{2}D(G_{2n})\psi = 0$ , where  $G_{2n} = z^{2n} + 2g_0z^{2n-2} + \dots + 2g_{2n-2}$ . A straightforward computation brings to equations  $u_{,t_{2n+1}} = D(g_{2n})$  and the recurrence relations

$$g_0 = u_0, \quad D(g_{2n+2}) = D^3(g_{2n}) + 4uD(g_{2n}) + 2u_1g_{2n}, \quad n \geq 0. \quad (36)$$

This implies the equation for the generating function  $g = 1 + 2 \sum_{n \geq 0} g_{2n}z^{-2n-2}$

$$2gD^2(g) - D(g)^2 + (4u - z^2)g^2 = -z^2, \quad (37)$$

which uniquely defines all  $g_{2n}$  as polynomials, homogeneous with respect to the weight  $w(u_i) = i+2$  (the right hand side of (37) can be replaced with arbitrary constant series in powers of  $z^{-2}$ , then the homogeneous flows are replaced by their linear combinations). It is easy to check that solution of equation (37) is expressed through solutions of equation (29):

$$g = \frac{z}{f(-z) - f(z)}, \quad (38)$$

which is equivalent to equations

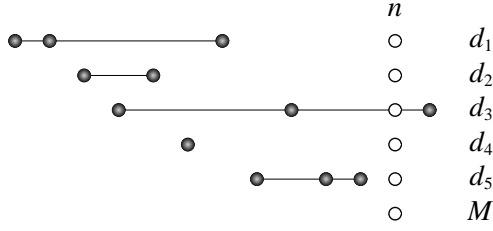
$$g_0 = f_0 = u, \quad g_{2n} = f_{2n} + 2 \sum_{s=0}^{n-1} g_{2s}f_{2n-2-2s}, \quad n = 1, 2, \dots,$$

complementary to recurrence relations (28). Equation (29) is brought to (35) by substitution  $f = D(\psi)/\psi$  and, therefore, equation (38) is equivalent to relation  $g = \text{const} \psi(z)\psi(-z)$ . The equalities

$$f_{,t_{2n+1}} = D(G_{2n}f) - \frac{1}{2}D^2(G_{2n}), \quad f(z) + f(-z) = -D(\log g)$$

demonstrate that  $f$  is a generating function for the densities of conservation laws, and the densities  $f_{2n+1}$  are trivial.

Although the relation between the generating functions  $f$  and  $g$  is very simple and looks like equation (30), no combinatorial interpretation is known for the polynomials  $g_{2n}$  or recurrence relations (36), in terms of non-overlapping partitions or any other combinatorial objects. Nevertheless, there exist explicit, although rather complicated, expressions for the coefficients of these polynomials. One of them was obtained already in [15], it defines the coefficient of a given monomial as a certain multiple integral. The structure of expressions from [30, 2, 28] is more combinatorial.



**Figure 7.** Operations  $d_j$ ,  $M$  for atomic partitions. Empty circles mark the vacancies for new element.

## 6 Atomic partitions

### 6.1 Generating operations

Atomic partitions were introduced in the study of the algebra of symmetric polynomials in noncommutative variables (these set partitions enumerate a certain basis of generators for this algebra) [5]. Like in the case of non-overlapping partitions, the definition makes use of the ordering of the underlying set.

**Definition 3.** A partition  $\pi = \{\pi_1, \dots, \pi_k\} \vdash [n]$  is called *atomic*, if no subset of its blocks forms a partition of  $[m]$  for  $1 \leq m < n$ .

Equivalently, no non-empty partitions  $\rho, \sigma$  exist such that  $\pi = \rho | \sigma$ , where  $|$  is the concatenation (11). Yet another equivalent definition reads: the set partition is atomic if the supports of its blocks cover the whole interval of the partition,

$$\bigcup_{j=1}^k \text{supp } \pi_j = [1, n]_{\mathbb{R}}. \quad (39)$$

Let  $\Pi_n^a$  and  $\Pi_{n,k}^a$  denote the sets of all atomic partitions of  $[n]$  and those which consist of  $k$  blocks,  $k = 1$  at  $n = 1$  and  $1 \leq k \leq n - 1$  at  $n > 1$ .

Construction of atomic partitions employs the operations

$$\begin{aligned} d_j : \Pi_{n,k}^a &\rightarrow \Pi_{n+1,k}^a, & 1 \leq j \leq k, & & M : \Pi_{n,k}^a &\rightarrow \Pi_{n+1,k+1}^a, & n \geq 2, \\ P : \Pi_{n,k}^a \times \Pi_{m,l}^a &\rightarrow \Pi_{m+n,k+l}^a, & m, n \geq 2. & \end{aligned}$$

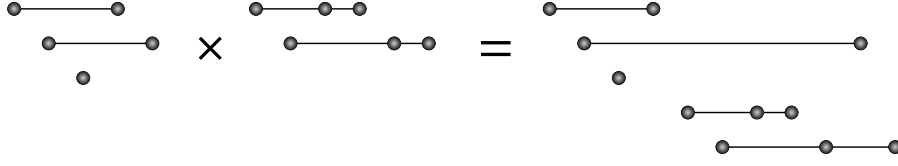
Two first operations are similar to (2), but the difference is that the new element is added at the  $n$ -th position, displacing the old element from this position to  $n + 1$ -st position, cf. fig. 1 and 7; moreover, operation  $M$  is applied only for  $n > 1$ . In more details, we first replace the element  $n$  by  $n + 1$  in the partition  $\pi = \{\pi_1, \dots, \pi_k\} \vdash [n]$ :

$$\pi' = \{\pi'_1, \dots, \pi'_k\} = \pi|_{n \rightarrow n+1},$$

next, operation  $d_j$  adds the element  $n$  to the block  $\pi'_j$  (in particular, if  $\pi_j$  contains  $n$  then this operation just adds  $n + 1$  to this block); operation  $M$  adds  $n$  as a new singleton:

$$d_j \pi = \{\pi'_1, \dots, \pi'_j \cup \{n\}, \dots, \pi'_k\}; \quad M \pi = \{\pi'_1, \dots, \pi'_k, \{n\}\}, \quad n \geq 2. \quad (40)$$





**Figure 8.** Operation  $P$  for atomic partitions.

Operation  $P$  is a modification of concatenation, with displacing of the last elements of both partitions (cf. with (24)). Let  $\rho \in \Pi_n^a$ ,  $\sigma \in \Pi_m^a$ ,  $m, n \geq 2$ , then

$$P(\rho, \sigma) = \rho|_{n \rightarrow m+n-1} \cup (\sigma|_{m \rightarrow m+1+n-1}). \quad (41)$$

In other words, we first concatenate the partitions with the maximal elements removed, then we place these elements back to their blocks, in the last but one position for the block from the first partition and in the last position for the block from the second one, see fig. 8.

**Statement 15.** Operations (40), (41) generate all atomic partitions by application to the seed partition  $\{\{1\}\}$  in all possible ways:

$$\begin{aligned} \Pi_1^a &= \{\{\{1\}\}\}, & \Pi_2^a &= d_1 \Pi_1^a = \{\{\{1, 2\}\}\}, \\ \Pi_{n+1}^a &= \bigcup_{k=1}^{n-1} \bigcup_{j=1}^k d_j \Pi_{n,k}^a \cup M \Pi_n^a \cup \bigcup_{j=2}^{n-1} P(\Pi_j^a, \Pi_{n+1-j}^a), & n &\geq 2, \end{aligned} \quad (42)$$

moreover, any partition appear only once.

*Proof.* Let us prove that our operations preserve the atomicity property. Notice, that in partition  $\pi \in \Pi_n^a$ , the block  $\pi_j$  containing  $n$  is not a singleton if  $n > 1$ , since otherwise the partition is of the form  $\pi = \rho|_{\{1\}}$ . Therefore, the passage to the partition  $\pi' = \pi|_{n \rightarrow n+1}$  enlarges the support of this block:  $\text{supp } \pi'_j = \text{supp } \pi_j \cup [n, n+1]_{\mathbb{R}}$ . The further adding of the element  $n$  does not reduce the blocks supports and it follows by (39) that partitions  $d_j \pi$ ,  $M \pi$  are atomic as well. The exceptional case is  $n = 1$ , for which  $M\{\{1\}\} = \{\{1\}, \{2\}\}$ , this is why the definition of this operation includes the restriction  $n \geq 2$ .

In (41), analogously, the blocks of partitions  $\rho|_{m \rightarrow m+n-1}$  and  $\sigma|_{n \rightarrow n+1+m-1}$  cover, respectively, the intervals  $[1, m+n-1]_{\mathbb{R}}$  and  $[m, m+n]_{\mathbb{R}}$  (again, take into account that if  $m, n \geq 2$  then the blocks with maximal elements are not singletons), and their union covers  $[1, m+n]_{\mathbb{R}}$ .

Now let us prove that any partition  $\pi \in \Pi_{n+1}^a$  can be represented, in a unique way, as  $\pi = M\tau$  or  $\pi = d_j \tau$  or  $\pi = P(\rho, \sigma)$ , where  $\rho, \sigma, \tau$  are atomic partitions. Let  $n$  (last to the end element of  $\pi$ ) belongs to the block  $\pi_j$ . Construct the partition  $\tau \vdash [n]$  by deleting this element and moving  $n+1$  at the vacant position. The following cases are possible.

1)  $\pi_j$  is a singleton. Then  $\tau \in \Pi_n^a$  and  $\pi = M\tau$ ; moreover,  $\pi$  cannot be obtained in other way, because the operations  $d_j$ ,  $P$  add the element  $n$  to a block which already contains some elements.

2)  $\pi_j$  is not a singleton and  $\tau \in \Pi_n^a$ . Then  $\pi = d_j\tau$ ; the uniqueness follows from the fact that if  $\pi = P(\rho, \sigma)$  then  $\tau \notin \Pi_n^a$  and  $j$  is uniquely determined as the index of a block containing the element  $n$ .

3)  $\pi_j$  is not a singleton and  $\tau \notin \Pi_n^a$ . Then  $\tau$  is uniquely represented as  $\tau = \tilde{\rho}|_s$ , where  $\sigma \in \Pi_s^a$ ,  $s > 1$ . The partition  $\sigma$  is characterized by the property that the union of the supports of  $\sigma + |\tilde{\rho}|$  covers the interval to the right of the rightmost gap in the union of the supports of  $\tau$ . Moreover, the block  $\pi_j \setminus \{n\}$  belongs to the partition  $\tilde{\rho}$  and appending of  $|\tilde{\rho}| + 1$  to this block gives us an atomic partition  $\rho$  such that  $\pi = P(\rho, \sigma)$ .  $\square$

## 6.2 Statistics

1) *Back to the Bell polynomials.* We will consider two statistics for the atomic partitions. First, let us copy the definition of the Bell polynomials  $Y_n$  (4):

$$p(\pi) = \prod_{j=1}^{|\pi|} v_{|\pi_j|}, \quad \tilde{Y}_n(v_1, \dots, v_n) = \sum_{\pi \in \Pi_n^a} p(\pi). \quad (43)$$

It turns out that polynomials  $\tilde{Y}_n$  and  $Y_n$  are closely related (cf. with Statement 11). Pay attention that equation (44) involves the ordinary generating functions, in contrast to the exponential one in (7). The differential equations for this functions can be solved in quadratures, but these formulae are of little value for us, since  $Y$  and  $\tilde{Y}$  serve just as asymptotic expansions for solutions, like the series in section 5.2.

**Statement 16.** *Generating functions  $\tilde{Y} = \tilde{Y}_1/z + \tilde{Y}_2/z^2 + \dots$  and  $Y = Y_0 + Y_1/z + Y_2/z^2 + \dots$  are related by equation*

$$\tilde{Y} = 1 - 1/Y. \quad (44)$$

*Proof.* Let  $v_i = D^i(v)$ ,  $D = \partial_x$ . It is easy to prove that the generating operations (40), (41) satisfy the property

$$\sum_{j=1}^{|\pi|} p(d_j\pi) = D(p(\pi)), \quad p(M\pi) = v_1 p(\pi), \quad p(P(\rho, \sigma)) = p(\rho)p(\sigma).$$

Statement 15 implies the recurrence relations

$$\tilde{Y}_1 = v_1, \quad \tilde{Y}_2 = v_2, \quad \tilde{Y}_{n+1} = D(\tilde{Y}_n) + v_1 \tilde{Y}_n + \sum_{s=2}^{n-1} \tilde{Y}_s \tilde{Y}_{n+1-s}, \quad n \geq 2, \quad (45)$$

which are equivalent to the Riccati equation

$$D(\tilde{Y}) + (\tilde{Y} - 1)(z\tilde{Y} - v_1) = 0. \quad (46)$$

The change (44) brings to equation  $D(Y) = (z - v_1)Y - z$  which is equivalent to the recurrence relations (6) for the Bell polynomials.  $\square$

$$\begin{aligned}
r_2 &= u \\
r_3 &= u_1 + uv \\
r_4 &= u_2 + (2u_1v + uv_1 + u^2) + uv^2 \\
r_5 &= u_3 + (3u_2v + 4uu_1 + 3u_1v_1 + uv_2) + (3u_1v^2 + 3uvv_1 + 3u^2v) + uv^3 \\
r_6 &= u_4 + (4u_3v + 6uu_2 + 6u_2v_1 + 5u_1^2 + 4u_1v_2 + uv_3) \\
&\quad + (6u_2v^2 + 16uu_1v + 12u_1vv_1 + 2u^3 + 5u^2v_1 + 4uvv_2 + 3uv_1^2) \\
&\quad + (4u_1v^3 + 6u^2v^2 + 6uv^2v_1) + uv^4
\end{aligned}$$

$n \setminus k$	1	2	3	4	5	6	7	8	$a_n$
1	1								1
2	1								1
3	1	1							2
4	1	4	1						6
5	1	11	9	1					22
6	1	26	48	16	1				92
7	1	57	202	140	25	1			426
8	1	120	747	916	325	36	1		2146
9	1	247	2559	5071	3045	651	49	1	11624

**Table 6.** Polynomials  $r_n$ . The numbers  $a_{n,k}$  of atomic partitions  $[n]$  with  $k$  blocks; sums over rows give the total numbers of atomic partitions  $a_n$ .

Let  $a_{n,k} = |\Pi_{n,k}^a|$  denote the number of atomic partitions  $[n]$  with  $k$  blocks [31, A087903] and  $a_n = |\Pi_n^a|$  be the total number of atomic partitions [31, A074664], see table 6. The substitution of  $v_j = v$  and  $v_j = 1$  into (44) establishes the connection of these numbers with the Stirling numbers of the 2nd kind and the Bell numbers:

$$\sum_{n \geq 1} \sum_{k=1}^n a_{n,k} \frac{v^k}{z^n} = 1 - \left( \sum_{n \geq 0} \sum_{k=0}^n \left\{ \begin{matrix} n \\ k \end{matrix} \right\} \frac{v^k}{z^n} \right)^{-1}, \quad \sum_{n \geq 1} \frac{a_n}{z^n} = 1 - \left( \sum_{n \geq 0} \frac{B_n}{z^n} \right)^{-1}.$$

For the first time, these relations were found in [5, eq.(1),(2)].

*Remark 4.* The so-called unsplitable partitions with a close statistics were studied in [4, 7]. The total number of such partitions of size  $n$  coincides with the number of atomic partitions, but another number triangle appears instead of  $a_{n,k}$ . A rather complicated bijection between these two classes of partitions was established in [8]. It is not clear at the moment, whether this correspondence is described by some transformation of generating functions or variables  $v_i$ .

2) *Polynomials in two variables.* Let us consider expressions  $\varphi(u)$ , builded from the variable  $u$  by use of operations  $P(a, b)$ ,  $Ma$  and  $d_j a$ ,  $1 \leq j \leq \deg a$ , where  $\deg a$  is equal to the common number of occurrences of  $u$  and  $M$  in  $a$ . Let the set  $\Phi_n^a$ ,  $n \geq 2$ , consists of all such expressions containing  $n-1$  characters  $u, d, M, P$ , and the set  $\Phi_{n,k}^a$  includes those with the degree equal to  $k$ . According to Statement 15, the map  $\varphi(u) \mapsto \varphi(\{\{1, 2\}\})$

$n$	$\varphi(u)$	$\varphi(\{\{1, 2\}\})$	$q(\varphi)$
2	$u$	12	$u$
3	$d_1u$	123	$u_1$
	$Mu$	13 2	$uv$
4	$d_1d_1u$	1234	$u_2$
	$d_1Mu$	134 2	$u_1v$
	$d_2Mu$	14 23	$uv_1$
	$Md_1u$	124 3	$u_1v$
	$MMu$	14 2 3	$uv^2$
	$P(u, u)$	13 24	$u^2$

**Table 7.** The sets  $\Phi_n^a$ ,  $\Pi_n^a$  and monomials  $q(\varphi)$  for  $n = 2, 3, 4$ .

defines a bijection between  $\Phi_{n,k}^a$  and  $\Pi_{n,k}^a$ , for  $n \geq 2$  (notice, that the partition  $\{\{1\}\}$  which makes up  $\Pi_1^a$ , is actually not involved into (42)).

In addition to the variable  $u = u_0$ , we introduce the variable  $v = v_0$  and define the monomial  $q(\varphi)$ , as the value of expression  $\varphi$  computed according to the following rules, applied in the given order:

- (i) all operations  $Ma$  are replaced with  $P(a, v)$ ;
  - (ii) the operation  $d_ja$  adds 1 to the subscript of the  $j$ -th occurrence of  $u$  or  $v$  into  $a$ , counting from the left;
  - (iii) all operations  $P(a, b)$  are replaced with  $ab$ .
- Summation over all expressions defines the polynomials

$$r_n(u, \dots, u_{n-2}, v, \dots, v_{n-3}) = \sum_{\varphi \in \Phi_n^a} q(\varphi), \quad (47)$$

see table 7, where the list of expressions  $\Phi_n^a$  is given for  $n = 2, 3, 4$ , along with the respective partitions and monomials, and table 6 containing few first polynomials  $r_n$ .

*Remark 5.* A comparison of tables 4 and 7 demonstrates that non-overlapping partitions  $\Pi_n^*$  are in one-to-one correspondence with the atomic partitions  $\Pi_{n+2}^a$  constructed without the use of operation  $M$ .

Notice, that in partition  $P(\rho, \sigma)$  the blocks of  $\rho$  and  $\sigma$  are enumerated sequentially, and operation  $M$  adds singleton as the last block. On the step (i), the variable  $v$  is introduced as the second argument, hence after this step the enumeration of variables  $u, v$  from left to right is consistent with the enumeration of blocks in respective partition.

Therefore, after the step (ii), a block with  $l$  elements corresponds to variable  $u_{l-2}$  or  $v_{l-1}$ , depending on whether this block was generated by operation  $P$  or  $M$ . This implies the relation between the monomials  $p$  and  $q$ , and therefore, between polynomials  $\tilde{Y}_n$  and  $r_n$ :

$$p(\varphi(\{\{1, 2\}\})) = q(\varphi)|_{u_l=v_{l+2}, v_l=v_{l+1}}, \quad \tilde{Y}_n = r_n|_{u_l=v_{l+2}, v_l=v_{l+1}}$$

(except for  $\tilde{Y}_1 = v_1$ , because the polynomial  $r_1$  is not defined).

**Statement 17.** *The polynomials  $r_n$  of the variables  $u_i = D^i(u)$ ,  $v_i = D^i(v)$ ,  $D = \partial_x$ , satisfy the recurrence relations*

$$r_2 = u, \quad r_{n+1} = D(r_n) + vr_n + \sum_{s=2}^{n-1} r_s r_{n+1-s}, \quad n \geq 2. \quad (48)$$

The generating function  $r = r_2/z + r_3/z^2 + \dots$  satisfies the relation

$$D(r) + r^2 + vr + u = zr. \quad (49)$$

*Proof.* Equation (48) follows from (42) and identities

$$\sum_{j=1}^{\deg \varphi} q(d_j \varphi) = D(q(\varphi)), \quad q(M\varphi) = vq(\varphi), \quad q(P(\varphi, \phi)) = q(\varphi)q(\phi).$$

The two last identities are obvious. In order to prove the first, let us write the variables  $u_i, v_i$  in the monomial in the same order as they stay after applying the operations  $M$  and  $d_j$  according to the rules (i), (ii). Then the difference between the monomials  $q(d_j \varphi)$  and  $q(\varphi)$  is that the subscript of  $j$ -th factor is increased by 1, and we only have to take the sum over  $j$ .  $\square$

### 6.3 Kaup–Broer hierarchy

Linearization of the Riccati equation (49) brings to equation

$$D^2(\psi) + (v - z)D(\psi) + u\psi = 0. \quad (50)$$

The associated hierarchy of nonlinear equations appears from the compatibility conditions of (50) with equation  $\psi_{,t} = GD(\psi) + H\psi = 0$ , where  $G$  and  $H$  are polynomials in  $z$ . Straightforward computations brings to equations

$$G = (z^{n-1}g)_{\geq 0}, \quad H = \frac{1}{2}(z^{n-1}((v - z)g - D(g)))_{\geq 0},$$

where the subscript  $\geq 0$  denotes the deleting of negative powers of  $z$ , and to equation for the generating function  $g = 1 + g_1/z + g_2/z^2 + \dots$

$$2gD^2(g) - D(g)^2 + (4u - 2v_1 - (v - z)^2)g^2 = -z^2, \quad (51)$$

which uniquely determines all  $g_j$ . Moreover, the compatibility conditions are equivalent to the system

$$u_{,t_n} = \frac{1}{2}D(D(g_n) - vg_n + g_{n+1}), \quad v_{,t_n}(v) = D(g_n).$$

The choice of the right hand side in (51) provides the homogeneity with respect to the weight  $w(u_i) = i + 2$ ,  $w(v_i) = i + 1$ . The basic equations of the hierarchy are of the form

$$u_{,t_2} = D(u_1 + 2uv), \quad v_{,t_2} = D(-v_1 + v^2 + 2u).$$

This is the Kaup–Broer system [22], one of gauge equivalent forms of the nonlinear Schrödinger equation (the chain of substitution can be found in [26, eq. (A)]). Let us write down also the 3-rd order symmetry:

$$u_{,t_3} = D(u_2 + 3u^2 + 3u_1v + 3uv^2), \quad v_{,t_3} = D(v_2 - 3vv_1 + v^3 + 6uv).$$

A comparison of the Riccati equations (49) and (46) proves that all flows admit the reductions  $u = 0$  and  $u = v_1$ , both leading to the Burgers hierarchy; a comparison with the Riccati equation (29) proves that all odd flows admit the reduction  $v = 0$  to the KdV hierarchy.

The generating functions  $g$  and  $r = r(u, v, z)$  from Statement 17 are related by the formula  $g = z/(\bar{r} - r)$ , where the series

$$\bar{r} = -\frac{u}{r(u, -u_1/u - v, -z)} = z - v + \frac{v_1 - u}{z} + \dots$$

is another solution of the Riccati equation (49). Like in the case of equation (37), an intermediate combinatorial interpretation of polynomials  $g_j$  is unknown.

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