ON COMPOSITION POLYNOMIALS

JEAN-CHRISTOPHE NOVELLI AND JEAN-YVES THIBON

Abstract. We provide a combinatorial interpretation of the reduced composition polynomials of Ardila and Doker [Adv. Appl. Math. **50** (2013), 607], and relate them to the $(1-q)$-transform of noncommutative symmetric functions.

1. Composition polynomials and noncommutative symmetric functions

Let $I = (i_1, \ldots, i_r)$ be a composition of n. The *composition polynomial* $g_I(q)$ is defined in [1] as

$$g_I(q) = \int_q^1 \int_q^{t_r} \cdots \int_q^{t_2} t_1^{i_1-1} t_2^{i_2-1} \cdots t_r^{i_r-1} dt_1 dt_2 \cdots dt_r. \tag{1}$$

This polynomial is divisible by $(1-q)^r$, and $f_I(q) = g_I(q)/(1-q)^r$ is called the *reduced composition polynomial*. It is proved in [1] that its coefficients are nonnegative. Our aim is to give a combinatorial interpretation of the coefficients of $n! f_I(q)$, which turn out to be integers. Actually, these polynomials have a natural interpretation in the context of noncommutative symmetric functions [5, 6].

Let $S_n(A)$ denote the noncommutative complete symmetric functions of an alphabet A, and $\Psi_n(A)$ the noncommutative power-sums of the first kind.

Their generating series

$$\sigma_t(A) := \sum_{n\geq 0} S_n(A) t^n \quad \text{and} \quad \psi_t(A) := \sum_{n\geq 1} \Psi_n t^{n-1} \tag{2}$$

are related by the differential equation

$$\frac{d}{dt}\sigma_t(A) = \sigma_t(A)\psi_t(A) \tag{3}$$

with the initial condition $\sigma_0(A) = 1$. This can be recast as an integral equation

$$\sigma_t(A) = 1 + \int_0^t \sigma_u(A)\psi_u(A) du \tag{4}$$

which may be solved by iterated integrals[1].

In [6], an analogue of the classical $(1-q)$-transform of ordinary symmetric functions (sending the power sums p_n to $(1-q^n)p_n$) is defined:

$$\sigma_t((1-q)A) := \lambda_{-qt}(A)\sigma_t(A) \quad (\text{where } \lambda_t(A) = \sigma_t(A)^{-1}). \tag{5}$$

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[1]The reader unfamiliar with noncommutative symmetric function can assume that σ_t is a generic noncommutative power series and take this as a definition of ψ_t.

Proposition 1.1. *The coefficient of* Ψ^I *in* $S_n((1-q)A)$ *is* $g_I(q)$.

Proof – The series $X(t) = \sigma_1((t-q)A) := \lambda_{-q}(A)\sigma_t(A)$ satisfies the differential equation

$$\frac{dX}{dt} = \lambda_{-q}(A)\frac{d}{dt}\sigma_t(A) = \lambda_{-q}(A)\sigma_t(A)\psi_t(A) = X(t)\psi_t(A) \tag{6}$$

with the initial condition

$$X(q) = 1, \tag{7}$$

so that

$$X(t) = 1 + \int_q^t X(t_1)\psi_{t_1} dt_1 \tag{8}$$

whose solution is

$$X(t) = 1 + \sum_{r\geq 1} \int_q^t \int_q^{t_1} \cdots \int_q^{t_{r-1}} \psi_{t_r}\cdots\psi_{t_2}\psi_{t_1} dt_r\cdots dt_2 dt_1. \tag{9}$$

∎

Example 1.2. For $n = 4$,

$$\begin{aligned} 4!S_4((1-q)A) = {} & 6\,(1-q)\,(q+1)\left(q^2+1\right)\Psi^4 + 2\left(3\,q^2+2\,q+1\right)(1-q)^2\,\Psi^{31} \\ & + 3\,(1-q)^2\,(q+1)^2\,\Psi^{22} + (3\,q+1)\,(1-q)^3\,\Psi^{211} \\ & + 2\left(q^2+2\,q+3\right)(1-q)^2\,\Psi^{13} + 2\,(q+1)\,(1-q)^3\,\Psi^{121} \\ & + (q+3)\,(1-q)^3\,\Psi^{112} + (1-q)^4\,\Psi^{1111}. \end{aligned} \tag{10}$$

Definition 1.3. *For a composition* $I = (i_1, \dots, i_r)$ *of* n, *we set*

$$P_I(q) = n!f_I(q) = n!\frac{g_I(q)}{(1-q)^r}. \tag{11}$$

Example 1.4. From Example 1.2, we have

$$\begin{array}{lcl} P_4 & = & 6q^3+6q^2+6q+6, \\ P_{31} & = & 6q^2+4q+2, \\ P_{22} & = & 3q^2+6q+3, \\ P_{211} & = & 3q+1, \\ P_{13} & = & 2q^2+4q+6, \\ P_{121} & = & 2q+2, \\ P_{112} & = & q+3, \\ P_{1111} & = & 1. \end{array} \tag{12}$$

Note that the sequence of the sums of all P_I of a given weight at $q = 1$ begins with

$$1, 1, 3, 13, 73 \tag{13}$$

which suggests a combinatorial interpretation using sets of lists.

2. Combinatorial interpretation of the reduced polynomials

A *set of lists* of size n is a set partition π of $[n]$ endowed with a total order inside each block [7, A000262]. Since this object is intermediate between a segmented permutation and a set partition, we propose to call it a *permutation partition*, or for short, a *permutition*.

Let $\mathfrak{P}_n$ be the set of permutitions of size n. We define a statistic sinv (special inversions) on $\mathfrak{P}_n$ as follows: $\operatorname{sinv}(\pi)$ is the number of letters in π greater than the last letter of their block.

We also introduce an order among the blocks, according to the value of the last letter of each block, and define the composition $c(\pi)$ as the sequence of lengths of the blocks ordered in this way.

For example, with $\pi = \{53, 4612, 978\}$, $\operatorname{sinv}(\pi) = 1+2+1 = 4$, and the canonical ordering of π is the segmented permutation $\sigma = (4612|53|978)$, so that $c(\pi) = (4,2,3)$.

Theorem 2.1. *The reduced composition polynomial P_I is the generating function of* sinv *on permutitions of composition I:*

$$P_I(q) = \sum_{c(\pi)=I} q^{\operatorname{sinv}(\pi)}. \tag{14}$$

Example 2.2. With $I = (1,3)$, one gets 12 permutitions of shape I:

$$(1|234),(1|324),(1|243),(1|423),(1|342),(1|432),$$
$$(2|134),(2|314),(2|143),(2|413),$$
$$(3|124),(3|214)$$

whose numbers of special inversions are respectively $0,0,1,1,2,2,0,0,1,1,0,0$, yielding the polynomial $6+4q+2q^2$ which is indeed P_{13}, see Example 12.

Proof – Ardila and Doker [1] give the induction formula

$$\frac{1}{i_1} g_{(i_1+i_2,i_3,\dots,i_r)}(q) = g_I(q) - \frac{q^{i_1}}{i_1} g_{(i_2,\dots,i_r)}(q) \tag{15}$$

which translates into

$$q i_1 P_I(q) + P_{(i_1+i_2,i_3,\dots,i_r)}(q) = i_1 P_I(q) + \frac{n!}{(n-i_1)!} q^{i_1} P_{(i_2,\dots,i_r)}(q), \tag{16}$$

which, together with the initial conditions $P_{1^n}(q) = 1$, completely determine all the P_I. To prove that the polynomials defined by (14) satisfy this induction, we interpret both sides of (16) as q-enumerations of disjoint unions of sets.

Let us first describe the combinatorial interpretation of each of the four terms.

First, both terms $i_1 P_I(q)$ are seen as the set E_1 of pairs (τ, x) where τ satisfies $c(\tau) = I$ and x is an element of $\{\tau_1, \dots, \tau_{i_1}\}$.

Following Ardila-Doker, denote by I^1 the composition $(i_1+i_2, i_3, \dots, i_r)$. The term P_{I^1} is interpreted as the set E_2 of permutitions τ satisfying $c(\tau) = I^1$.

Finally, the term $\frac{n!}{(n-i_1)!}P_{(i_2,\dots,i_r)}$ is translated as the set E_3 of pairs (τ,w) where τ satisfies $c(\tau)=(i_2,\dots,i_r)$ and where $w=w_1\dots w_{i_1}$ is a word of size i_1 with distinct values between 1 and n.

We shall now proceed as follows: first define a map from E_1 to $E_1\cup E_3$, characterize its image set, show that it is bijective and show that it behaves as desired with respect to the q-enumeration constraint. Then, define a map from E_2 to $E_1\cup E_3$, show the same properties as before and also show that the images of both partition the set $E_1\cup E_3$.

Let us first define our map ϕ_1 from E_1. Consider an element (τ,x) of E_1 and set $S=\{\tau_1,\dots,\tau_{i_1}\}$.

- If τ_{i_1} is the smallest value of S, define w as the word obtained from $\tau_1\dots\tau_{i_1}$ by exchanging x with τ_{i_1}. Then

$$\phi_1((\tau,x))=(\mathrm{std}(\tau_{i_1+1}\dots\tau_n),w), \tag{17}$$

  where std denotes the usual standardization of words over an ordered alphabet.
- Otherwise, let m be the maximal value of S strictly smaller than τ_{i_1}. Then let τ' be obtained from τ by exchanging τ_{i_1} with m and define

$$\phi_1((\tau,x))=(\tau',x). \tag{18}$$

By construction, in both cases, each element in the image of ϕ_1 has a unique preimage, so that ϕ_1 is a bijection from E_1 to its image set. Indeed, one can retrieve τ in the first case, since the values $\tau_{i_1+1}\dots\tau_n$ are obtained from their standardized word by the unique increasing morphism from $[n-i_1]$ to the set of values in $[n]$ not occuring in w. Then, $\tau_1\dots\tau_{i_1}$ is obtained from w by exchanging the smallest value of w with its last one, and x is w_{i_1}. The second case is straightforward, as x plays no rôle in the definition of τ', from which τ can be easily recovered.

Let us now describe the image set $\phi_1(E_1)$. The intersection of E_1 and $\phi_1(E_1)$ is the set of elements where x can take any value and such that τ_{i_1} is not the greatest value of $\{\tau_1,\dots,\tau_{i_1}\}$ smaller than $\tau_{i_1+i_2}$. Indeed, such a pair clearly has a preimage as described above and the other elements of E do not. The intersection of E_3 after applying the increasing morphism to the standard word and $\phi_1(E_1)$ is the set of elements where the smallest value of w is smaller than $\tau_{i_1+i_2}$. Indeed, the condition on the smallest value of w is necessary and sufficient to have a permutition as preimage.

Finally, with respect to the q-enumeration according to the number of special inversions of the permutitions on both sides, ϕ_1 behaves as expected. Any element $(\tau,x)\in E_1$ sent to $(\pi,x)\in E_1$ satisfies $\mathrm{sinv}(\tau)=\mathrm{sinv}(\pi)-1$, since there will be one more inversion in π than in τ among their first i_1 values, that is, before the first bar. Now, for any $(\tau,x)\in E_1$ sent to $(\pi,w)\in E_3$, $\mathrm{sinv}(\tau)=\mathrm{sinv}(\pi)+i_1-1$ since there will be i_1-1 less inversions in π than in τ: indeed, τ_{i_1} is the smallest value before its first bar, so that there are i_1-1 inversions in τ before its first bar.

Let us now define a second map ϕ_2 from $E_2=\{\tau|c(\tau)=I^1\}$ to $E_1\cup E_3$.

- If $\tau_\alpha > \tau_{i_1+i_2}$ for all $\alpha \le i_1$, then

$$\phi_2(\tau) = (\mathrm{std}(\tau_{i_1+1}\dots\tau_{i_1+i_2}|\dots), \tau_1\dots\tau_{i_1}). \tag{19}$$

- Otherwise, let k be the greatest element of $\{\tau_1,\dots,\tau_{i_1}\}$ smaller than $\tau_{i_1+i_2}$ and let τ' be obtained from $\tau_1\dots\tau_{i_1}$ by exchanging k with τ_{i_1}, and define

$$\phi_2(\tau) = ((\tau'|\tau_{i_1+1}\dots\tau_{i_1+i_2}|\dots), \tau_{i_1}). \tag{20}$$

As with ϕ_1, ϕ_2 is a bijection from E_2 to its image set.

Let us now describe the image set $\phi_2(E_2)$. The intersection of E_1 and $\phi_2(E_2)$ is the set of elements such that x can take any value and such that τ_{i_1} is the greatest element of $\{\tau_1,\dots,\tau_{i_1}\}$ smaller than $\tau_{i_1+i_2}$. The intersection of E_3 and $\phi_2(E_2)$ is the set of elements such that the smallest value of w is greater than $\tau_{i_1+i_2}$. Hence, both sets constitute the complement of the image of ϕ_1, so that the map ϕ defined by $\phi|_{E_1} = \phi_1$ and $\phi|_{E_2} = \phi_2$ is a bijection between $E_1 \cup E_2$ and $E_1 \cup E_3$.

Again, ϕ_2 behaves as expected with respect to the statistic sinv. Any element $\tau \in E_2$ sent to $(\pi, x) \in E_1$ satisfies $\mathrm{sinv}(\tau) = \mathrm{sinv}(\pi)$, since the inversions between τ_k with $k \le i_1$ and $\tau_{i_1+i_2}$ become all inversions between π_i with $i \le i_1$ and π_{i_1}. Moreover, for any $\tau \in E_2$ sent to $(\pi, w) \in E_3$, $\mathrm{sinv}(\tau) = \mathrm{sinv}(\pi) + i_1$, since there are inversions between all τ_k with $k \le i_1$ and $\tau_{i_1+i_2}$ that disappear in π. Therefore, ϕ is a bijection between $E_1 \sqcup E_2$ and $E_1 \sqcup E_3$, behaving as claimed with respect to sinv. ∎

2.1. **Examples.**

2.1.1. Let us first give four examples of ϕ_1 and ϕ_2 and their inverse maps, covering all possible cases.

We have

$$\begin{aligned}
\phi_1((361|74|258), 3) &= ((42|135), 163)\\
\phi_1((163|74|258), 1) &= ((361|74|258), 1)\\
\phi_2((26371|458)) &= ((41|235), 263)\\
\phi_2((36174|258)) &= ((163|74|258), 1)
\end{aligned} \tag{21}$$

Now, from $((42|135), 163)$, one easily recovers the suffix of the permutition τ of its preimage. It is $(74|258)$, from which one can see that 163 are not all greater than 4. Hence, it belongs to the image of ϕ_1, and then easily gives back the first permutition of the example.

In the case of $((41|235), 263)$, one recovers as suffix the expression $(71|458)$, and since all values of 263 are greater than 1, it belongs to the image of ϕ_2. Finally, on the other two examples $((361|74|258), 1)$ and $((163|74|258), 1)$, since 3 is the greatest possible value of the prefix considering that the second expression between bars ends with a 4, these elements respectively belong to the image of ϕ_1 and ϕ_2.

2.1.2. Let us now illustrate the complete bijection on the example of the composition $I = (2,1,1)$. The following table shows the images of $E_1 = \{(\tau, x)\}$ satisfying $c(\tau) = (2,1,1)$ and $x \in \{1,2\}$ with the corresponding q-statistics on the right:

$$
\begin{array}{lr}
(12|3|4,1) \mapsto (21|3|4,1) & (1,q) \\
(12|3|4,2) \mapsto (21|3|4,2) & (1,q) \\
(21|3|4,1) \mapsto (1|2,21) & (q,1) \\
(21|3|4,2) \mapsto (1|2,12) & (q,1) \\
(31|3|4,1) \mapsto (1|2,31) & (q,1) \\
(31|3|4,3) \mapsto (1|2,13) & (q,1) \\
(41|3|4,1) \mapsto (1|2,41) & (q,1) \\
(41|3|4,4) \mapsto (1|2,14) & (q,1)
\end{array}
\tag{22}
$$

The next table shows the images of $E_2 = \{\tau\}$ satisfying $c(\tau) = (3,1)$ with the corresponding q-statistics on the right:

$$
\begin{array}{lr}
(123|4) \mapsto (12|3|4,2) & (1,1) \\
(213|4) \mapsto (12|3|4,1) & (1,1) \\
(132|4) \mapsto (31|2|4,3) & (q,q) \\
(312|4) \mapsto (31|2|4,1) & (q,q) \\
(142|3) \mapsto (41|2|3,4) & (q,q) \\
(412|3) \mapsto (41|2|3,1) & (q,q) \\
(231|4) \mapsto (1|2,23) & (q^2,1) \\
(321|4) \mapsto (1|2,32) & (q^2,1) \\
(241|3) \mapsto (1|2,24) & (q^2,1) \\
(421|3) \mapsto (1|2,42) & (q^2,1) \\
(341|2) \mapsto (1|2,34) & (q^2,1) \\
(431|2) \mapsto (1|2,43) & (q^2,1)
\end{array}
\tag{23}
$$

Now, summing up the q-statistics on both side of (22) and (23), one gets that its left-hand side is

$$q(6q+2) + (6q^2+4q+2) = 12q^2+6q+2, \tag{24}$$

and its right-hand side is

$$6q+2+q^2 \times 12, \tag{25}$$

so that they coincide.

3. Miscellaneous comments

3.1. Noncommutative symmetric functions of degree n can be intepreted as elements of the descent algebra Σ_n of $\mathfrak{S}_n$. In this context, Ψ_n is the Dynkin symmetrizer (iterated bracketing): as a linear combination of permutations,

$$\Psi_n = [\cdots[[1,2],3],\cdots,n] \tag{26}$$

and $S_n((1-q)A$ is the iterated q-bracketing

$$S_n((1-q)A) = (1-q)[\cdots[[1,2]_q,3]_q,\cdots,n]_q. \tag{27}$$

By [6, Lemma 5.11], writing

$$S_n((1-q)A) = S_n((1-q)A) * S_n(A) \tag{28}$$

(internal product) and inserting the expansion

$$S_n(A) = \sum_{r=1}^{n} \sum_{\substack{I \vDash n \\ \ell(I)=r}} \frac{\Psi^I}{i_1(i_1+i_2)\cdots(i_1+\cdots i_r)} \tag{29}$$

we obtain

$$S_n((1-q)A) = \sum_{r=1}^{n} \sum_{\substack{I \vDash n \\ \ell(I)=r}} (1-q^{i_1}) \frac{[\cdots[[\Psi_{i_1},\Psi_{i_2}]_{q^{i_2}},\Psi_{i_3}]_{q^{i_3}},\cdots,\Psi_{i_r}]_{q^{i_r}}}{i_1(i_1+i_2)\cdots(i_1+\cdots i_r)} \tag{30}$$

3.2. There exists a combinatorial Hopf algebra based on permutitions [2]. It plays with respect to the Hopf algebra of set partitions (symmetric functions in noncommuting variables) a rôle symmetrical to that of **WQSym** (quasi-symmetric functions in noncommuting variables).

The number of permutitions of length n is equal to the number of stalactic classes of parking functions of the same length, [4], on which a combinatorial Hopf algebra structure can also be defined. There is also a bijection between permutitions and noncrossing set compositions [3]. The bijection goes through Dyck paths with labelled peaks, which are easily identified with the canonical representatives of stalactic classes of parking functions, obtained by permuting among themselves the blocks of identical letters in a nondecreasing parking function. Composing these bijections, on can obtain a bijection between permutitions and stalactic classes of parking functions.

References

[1] F. Ardila and J. Doker, *Lifted generalized permutahedra and composition polynomials*, Adv. Appl. Math. **50** (2013), 607–633.

[2] J.-P. Bultel, A. Chouria, J.-G. Luque, and O. Mallet, *Word symmetric functions and the Redfield-Pólya theorem*, 25th International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2013), 563–574, Discrete Math. Theor. Comput. Sci. Proc., AS, Assoc. Discrete Math. Theor. Comput. Sci., Nancy, 2013.

[3] D. Callan, *Sets, lists and noncrossing partitions*, J. Integer Seq. **11** (2008), no. 1, Article 08.1.3, 7 pp.

[4] F. Hivert, J.-C. Novelli, and J.-Y. Thibon, *Commutative combinatorial Hopf algebras*, J. Algebraic Combin. **28** (2008), no. 1, 65–95.

[5] I. M. Gelfand, D. Krob, A. Lascoux, B. Leclerc, V. S. Retakh, and J.-Y. Thibon. *Noncommutative symmetric functions*, Adv. Math. **112**, 1995, 218–348.

[6] Krob D., Leclerc B., and Thibon J.-Y., *Noncommutative symmetric functions II: Transformations of alphabets*, Intern. J. Alg. Comput., **7**, (2), (1997), 181–264.

[7] *The On-Line Encyclopedia of Integer Sequences*, published electronically at http://oeis.org, 2010.

[Novelli, Thibon] Laboratoire d'informatique Gaspard-Monge, Université Paris-Est Marne-la-Vallée, 5, Boulevard Descartes, Champs-sur-Marne, 77454 Marne-la-Vallée cedex 2, France

E-mail address, Jean-Christophe Novelli: `novelli@univ-mlv.fr`

E-mail address, Jean-Yves Thibon: `jyt@univ-mlv.fr`