ON FLOW POLYTOPES, ORDER POLYTOPES, AND CERTAIN FACES OF THE ALTERNATING SIGN MATRIX POLYTOPE

KAROLA MÉSZÁROS, ALEJANDRO H. MORALES, JESSICA STRIKER

Abstract. In this paper we study an alternating sign matrix analogue of the Chan-Robbins-Yuen polytope, which we call the ASM-CRY polytope. We show that this polytope has Catalan many vertices and its volume is equal to the number of standard Young tableaux of staircase shape; we also determine its Ehrhart polynomial. We achieve the previous by proving that the members of a family of faces of the alternating sign matrix polytope which includes ASM-CRY are both order and flow polytopes. Inspired by the above results, we relate three established triangulations of order and flow polytopes, namely Stanley's triangulation of order polytopes, the Postnikov-Stanley triangulation of flow polytopes and the Danilov-Karzanov-Koshevoy triangulation of flow polytopes. We show that when a graph G is a planar graph, in which case the flow polytope $\mathcal{F}_G$ is also an order polytope, Stanley's triangulation of this order polytope is one of the Danilov-Karzanov-Koshevoy triangulations of $\mathcal{F}_G$. Moreover, for a general graph G we show that the set of Danilov-Karzanov-Koshevoy triangulations of $\mathcal{F}_G$ is a subset of the set of Postnikov-Stanley triangulations of $\mathcal{F}_G$. We also describe explicit bijections between the combinatorial objects labeling the simplices in the above triangulations.

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1. Introduction

In this paper, we study a family of faces of the alternating sign matrix polytope inspired by an intriguing face of the Birkhoff polytope: the Chan-Robbins-Yuen (CRY) polytope [7]. We call this family of faces the ASM-CRY family of polytopes. Interest in the CRY polytope centers around its volume formula as a product of consecutive Catalan numbers; this has been proved [24], but the problem of finding a *combinatorial proof* remains open. We prove that the polytopes in the ASM-CRY family are *order polytopes* and use Stanley's theory of order polytopes [20] to give a *combinatorial proof* of formulas for their volumes and Ehrhart polynomials. We also show that these polytopes, and all order polytopes of strongly planar posets, are *flow polytopes*. This

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observation brings us to the general question of relating the different known triangulations of flow and order polytopes. We show that when G is a planar graph, in which case the flow polytope of G is also an order polytope, then Stanley's canonical triangulation of this order polytope [20] is one of the *Danilov-Karzanov-Koshevoy triangulations* of the flow polytope of G [8], a statement first observed by Postnikov [15]. Moreover, for general G we show that the set of Danilov-Karzanov-Koshevoy triangulations of the flow polytope of G is a subset of the set of *framed Postnikov-Stanley triangulations* of the flow polytope of G [15, 19]. We also describe explicit bijections between the combinatorial objects labeling the simplices in the above triangulations, answering a question posed by Postnikov [15].

We highlight the main results of the paper in the following theorems. While we define some of the notation here, some only appears in later sections to which we give pointers after the relevant statements.

In Definition 4.1, we define the *ASM-CRY family* $\mathcal{F}(ASM)(n)$ of polytopes $\mathcal{P}_\lambda(n)$ indexed by partitions $\lambda \subseteq \delta_n$ where $\delta_n := (n-1, n-2, \ldots, 1)$. In Theorem 4.3, we prove that the polytopes in this family are *faces of the alternating sign matrix polytope* $\mathcal{A}(n)$ defined in [4, 22]. In the case when $\lambda = \varnothing$ we obtain an analogue of the Chan-Robbins-Yuen (CRY) polytope, which we call the ASM-CRY polytope, denoted by $\mathcal{ASMCRY}(n)$. Our main theorem about this family of polytopes is the following. For the necessary definitions, see Sections 3.3 and 4.

Theorem 1.1. *The polytopes in the family $\mathcal{F}(ASM)(n)$ are affinely equivalent to flow and order polytopes. In particular, $\mathcal{P}_\lambda(n)$ is affinely equivalent to the order polytope of the poset $(\delta_n \setminus \lambda)^*$ and the flow polytope $\mathcal{F}_{G_{(\delta_n\setminus\lambda)^*}}$.*

By Stanley's theory of order polytopes [20] it follows that the volume of the polytope $P_\lambda(n)$ for any $P_\lambda(n) \in \mathcal{F}(ASM)(n)$ is given by the number of *linear extensions* of the poset $(\delta_n \setminus \lambda)^*$ (the number of *Standard Young Tableaux* of skew shape δ_n/λ), and its Ehrhart polynomial is given by the *order polynomial* of the poset (counting weak plane partitions of skew shape δ_n/λ with bounded parts). See Corollary 4.7 for the general statement. We give the application to $\mathcal{ASMCRY}(n)$ in the corollary below. For further examples of polytopes in $\mathcal{F}(ASM)(n)$, see Figure 7.

Corollary 1.2. *$\mathcal{ASMCRY}(n)$ is affinely equivalent to the order polytope of the poset δ_n^*. Thus, $\mathcal{ASMCRY}(n)$ has $\mathrm{Cat}(n) = \frac{1}{n+1}\binom{2n}{n}$ vertices, its normalized volume is given by*

$$\mathrm{vol}(\mathcal{ASMCRY}(n)) = \#SYT(\delta_n),$$

and its Ehrhart polynomial is

$$L_{\mathcal{ASMCRY}(n)}(t) = \Omega_{\delta_n^*}(t+1) = \prod_{1\le i<j\le n} \frac{2t+i+j-1}{i+j-1}.$$

In Theorems 3.8 and 3.10, we make explicit the relationship between flow and order polytopes, showing that they correspond under certain planarity conditions. For an introduction to flow and order polytopes, see Section 3, and for the definitions of $(\delta_n \setminus \lambda)^*$ and $G_{(\delta_n\setminus\lambda)^*}$, see Definition 4.4 and the discussion before Theorem 3.10, respectively.

As mentioned earlier, a canonical triangulation of order polytopes was given by Stanley [20], and two families of triangulations of flow polytopes were constructed by Postnikov and Stanley [15, 19] as well as Danilov, Karzanov and Koshevoy [8]. It is natural to try to understand the relation among these triangulations, and we prove the following results, the first of which was first observed by Postnikov [15]. For the necessary definitions, see Sections 5 and 6.

Theorem 1.3 (Postnikov [15])**.** *Given a planar graph G, the canonical triangulation of the order polytope $\mathcal{O}(P_G)$ is equal to the Danilov-Karzanov-Koshevoy triangulation of the flow polytope $\mathcal{F}_G$ coming from the planar framing.*

Theorem 1.4. *Given a framed graph G, the set of Danilov-Karzanov-Koshevoy triangulations of the flow polytope $\mathcal{F}_G$ is a subset of the set of framed Postnikov-Stanley triangulations of $\mathcal{F}_G$.*

All three of the above-mentioned triangulations are indexed by natural sets of combinatorial objects and we give explicit bijections between these sets in Sections 5 and 6.

The outline of the paper is as follows. In Section 2, we discuss the Birkhoff and alternating sign matrix polytopes, as well as some of their faces. In Section 3, we give background information on flow and order polytopes and show that flow polytopes of planar graphs are order polytopes and that order polytopes of strongly planar posets are flow polytopes. In Section 4, we study a family of faces of the alternating sign matrix polytopes and show that they are affinely equivalent to both flow and order polytopes and calculate their volumes and Ehrhart polynomials in particularly nice cases. In Section 5, we study triangulations of flow polytopes of planar graphs (which include the polytopes of Section 4) and show that their canonical triangulations defined by Stanley [20] are also Danilov-Karzanov-Koshevoy triangulations [8]. Finally, in Section 6, we study triangulations of flow polytopes of an arbitrary graph, that is, the Danilov-Karzanov-Koshevoy triangulations and the framed Postnikov-Stanley triangulations. We show that the former is a subset of the latter. We also exhibit explicit bijections between the combinatorial objects indexing the various triangulations, answering a question raised by Postnikov [15].

2. Faces of the Birkhoff and alternating sign matrix polytopes

In this section, we explain the motivation for our study of certain faces of the alternating sign matrix polytope. If $\mathcal{P}$ is an *integral polytope*, its **Ehrhart polynomial** $L_{\mathcal{P}}(t)$ is the polynomial that counts the number of lattice points of the dilated polytope $t \cdot \mathcal{P}$. In this case the **relative volume** of $\mathcal{P}$ is the leading term of $L_{\mathcal{P}}(t)$ and its **normalized volume** $\mathrm{vol}(\mathcal{P}) \in \mathbb{N}$ is the product of its relative volume and $\dim(\mathcal{P})!$. We start by defining the Birkhoff and Chan-Robbins-Yuen polytopes; we then define the alternating sign matrix counterparts.

Definition 2.1. The **Birkhoff polytope**, $\mathcal{B}(n)$, is defined as

$$\mathcal{B}(n) := \Big\{(b_{ij})_{i,j=1}^n \in \mathbb{R}^{n^2} \mid b_{ij} \geq 0,\ \sum_i b_{ij} = 1,\ \sum_j b_{ij} = 1\Big\}.$$

Matrices in $\mathcal{B}(n)$ are called **doubly-stochastic matrices**. A well-known theorem of Birkhoff [5] and Von Neumann [23] states that $\mathcal{B}(n)$, as defined above, equals the convex hull of the $n \times n$ permutation matrices. Note that $\mathcal{B}(n)$ has n^2 facets and dimension $(n-1)^2$, its vertices are the permutation matrices, and its volume has been calculated up to $n = 10$ by Beck and Pixton [3]. De Loera, Liu and Yoshida [9] gave a closed summation formula for the volume of $\mathcal{B}(n)$, which, while of interest on its own right, does not lend itself to easy computation. Shortly after, Canfield and McKay [6] gave an asymptotic formula for the volume.

A special face of the Birkhoff polytope, first studied by Chan-Robbins-Yuen [7], is as follows.

Definition 2.2. The **Chan-Robbins-Yuen polytope**, $\mathcal{CRY}(n)$, is defined as

$$\mathcal{CRY}(n) := \left\{(b_{ij})_{i,j=1}^n \in \mathcal{B}(n) \mid b_{ij} = 0 \text{ for } i - j \geq 2\right\}.$$

$\mathcal{CRY}(n)$ has dimension $\binom{n}{2}$ and 2^{n-1} vertices. This polytope was introduced by Chan-Robbins-Yuen [7] and in [24] Zeilberger calculated its normalized volume as the following product of *Catalan numbers.*

Theorem 2.3 (Zeilberger [24])**.**

$$\mathrm{vol}(\mathcal{CRY}(n)) = \prod_{i=1}^{n-2} \mathrm{Cat}(i)$$

where $\mathrm{Cat}(i) = \frac{1}{i+1}\binom{2i}{i}$.

The proof in [24] used a relation (see Theorem 3.3) expressing the volume as a value of the *Kostant partition function* (see Definition 3.4) and a reformulation of the Morris constant term identity [14] to calculate this value. No combinatorial proof is known.

Next we give an analogue of the Birkhoff polytope in terms of alternating sign matrices. Recall that **alternating sign matrices** (ASMs) [13] are square matrices with the following properties:

- entries $\in \{0, 1, -1\}$,
- the entries in each row/column sum to 1, and
- the nonzero entries along each row/column alternate in sign.

The ASMs with no negative entries are the permutation matrices. See Figure 1 for an example.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

FIGURE 1. All the 3×3 alternating sign matrices.

Definition 2.4 (Behrend-Knight [4], Striker [22])**.** The **alternating sign matrix polytope**, $\mathcal{A}(n)$, is defined as follows:

$$\mathcal{A}(n) := \Big\{(a_{ij})_{i,j=1}^n \in \mathbb{R}^{n^2} \mid 0 \le \sum_{i=1}^{i'} a_{ij} \le 1,\ 0 \le \sum_{j=1}^{j'} a_{ij} \le 1,\ \sum_{i=1}^{n} a_{ij} = 1,\ \sum_{j=1}^{n} a_{ij} = 1\Big\},$$

where we have the first sum for any $1 \le i', j \le n$, the second sum for any $1 \le j', i \le n$, the third sum for any $1 \le j \le n$, and the fourth sum for any $1 \le i \le n$.

Behrend and Knight [4], and independently Striker [22], defined $\mathcal{A}(n)$. The alternating sign matrix polytope can be seen as an analogue of the Birkhoff polytope, since the former is the convex hull of all alternating sign matrices (which include all permutation matrices) while the latter is the convex hull of all permutation matrices. The polytope $\mathcal{A}(n)$ has $4((n-2)^2+1)$ facets (for $n \ge 3$) [22], its dimension is $(n-1)^2$, and its vertices are the $n \times n$ alternating sign matrices [4, 22]. The Ehrhart polynomial has been calculated up to $n = 5$ [4]. Its normalized volume for $n = 1, \ldots, 5$ is calculated to be

$$1, 1, 4, 1376, 201675688,$$

and no asymptotic formula for its volume is known.

In analogy with $\mathcal{CRY}(n)$, we study a special face of the ASM polytope we call the ASM-CRY polytope (and show, in Theorem 4.3, it is indeed a face of $\mathcal{A}(n)$).

Definition 2.5. The **ASM-CRY polytope** is defined as follows.

$$\mathcal{ASMCRY}(n) := \left\{(a_{ij})_{i,j=1}^n \in \mathcal{A}(n) \mid a_{ij} = 0 \text{ for } i - j \ge 2\right\}.$$

Since the $\mathcal{CRY}(n)$ polytope has a nice product formula for its normalized volume, it is then natural to wonder if the volume of the alternating sign matrix analogue of $\mathcal{CRY}(n)$, which we denote by $\mathcal{ASMCRY}(n)$, is similarly nice. In Theorem 1.1 and Corollary 1.2, we show that $\mathcal{ASMCRY}(n)$ is both a flow and order polytope, and using the theory established for the latter, we give the volume formula and the Ehrhart polynomial of $\mathcal{ASMCRY}(n)$. Just like in the $\mathcal{CRY}(n)$ case, all formulas obtained are combinatorial. Unlike in the $\mathcal{CRY}(n)$ case, all the proofs involved are combinatorial. In Theorem 1.1, we extend these results to a family of faces $\mathcal{F}(ASM)(n)$ of the ASM polytope, of which $\mathcal{ASMCRY}(n)$ is a member; see Section 4.

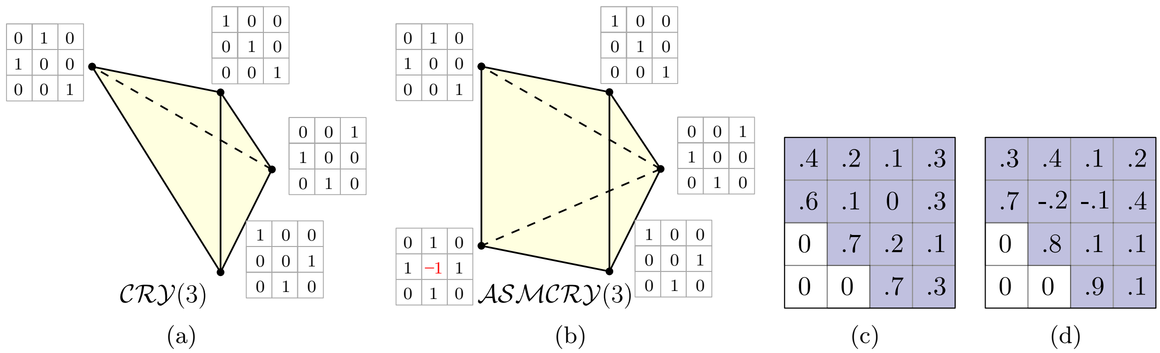


FIGURE 2. (a) The polytope $\mathcal{CRY}(3)$ in $\mathbb{R}^3$, (b) the polytope $\mathcal{ASMCRY}(3)$ in $\mathbb{R}^3$, (c) a doubly-stochastic matrix in $\mathcal{CRY}(4)$, (d) a matrix in $\mathcal{ASMCRY}(4)$.

3. Flow and order polytopes

In order to state and prove Theorem 1.1 in Section 4, we need to discuss flow and order polytopes. In Section 3.1, we define flow and order polytopes and also explain how to see $\mathcal{CRY}(n)$ as the flow polytope of the complete graph. In Sections 3.2 and 3.3, we prove that the flow polytope of a planar graph is the order polytope of a related poset, and vice versa.

3.1. Background and definitions.

Let G be a connected graph on the vertex set $[n] := \{1, 2, \ldots, n\}$ with edges directed from the smallest to largest vertex. We assume that each vertex $v \in \{2, 3, \ldots, n-1\}$ has both incoming and outgoing edges. Denote by $\mathrm{in}(e)$ the smallest (initial) vertex of edge e and $\mathrm{fin}(e)$ the biggest (final) vertex of edge e.

Definition 3.1. A **flow** fl of size one on G is a function $fl : E(G) \to \mathbb{R}_{\geq 0}$ such that

$$1 = \sum_{e \in E, \mathrm{in}(e)=1} fl(e) = \sum_{e \in E, \mathrm{fin}(e)=n} fl(e),$$

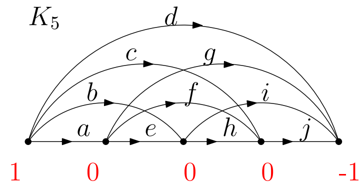
and for $2 \leq i \leq n-1$

$$\sum_{e \in E, \mathrm{fin}(e)=i} fl(e) = \sum_{e \in E, \mathrm{in}(e)=i} fl(e).$$

The **flow polytope** $\mathcal{F}_G$ associated to the graph G is the set of all flows $fl : E(G) \to \mathbb{R}_{\geq 0}$ of size one on G.

Remark 3.2. Note that the restriction that at each vertex $v \in [2, n-1]$ of G there are both incoming and outgoing edges is not a serious one. If there is a vertex $v \in [2, n-1]$ with only incoming or outgoing edges, then in $\mathcal{F}_G$ the flow on all these edges must be zero, and thus, up to removing such vertices, any flow polytope $\mathcal{F}_G$ is equivalent to a flow polytope defined as above.

The polytope $\mathcal{F}_G$ is a convex polytope in the Euclidean space $\mathbb{R}^{\#E(G)}$ and its dimension is $\dim(\mathcal{F}_G) = \#E(G) - \#V(G) + 1$ (e.g. see [1]). The vertices of $\mathcal{F}_G$ are given by unit flows along maximal directed paths or **routes** of G from the source (1) to the sink (n) [17, §13]. Figure 3 shows the equations of $\mathcal{F}_{K_5}$ and explains why this polytope is equivalent to $\mathcal{CRY}(4)$. The same correspondence shows that $\mathcal{F}_{K_{n+1}}$ and $\mathcal{CRY}(n)$ coincide. The following theorem connects volumes of flow polytopes and Kostant partition functions.



$$1 = a + b + c + d$$
$$0 = e + f + g - a$$
$$0 = h + i - b - e$$
$$0 = j - c - f - h$$

a	b	c	d
$\bullet$	e	f	g
0	$\bullet$	h	i
0	0	$\bullet$	j

FIGURE 3. Graph K_5 with edges directed from smaller to bigger vertex. The flow variables on the edges are $a, b, c, d, e, f, g, h, i, j$, the net flows in the vertices are $1, 0, 0, 0, -1$. The equations defining the flow polytope corresponding to K_5 are in the middle. Note that these same equations define $\mathcal{CRY}(4)$ as can be seen from the matrix on the left, where we denoted by $\bullet$ entries that are determined by the variables $a, b, \ldots, j$.

Theorem 3.3 (Postnikov-Stanley [15, 19], Baldoni-Vergne [1])**.** *For a loopless graph G on the vertex set $\{1, 2 \ldots, n\}$, with $d_i =$ (indegree of i) $- 1$,*

$$\mathrm{vol}\,(\mathcal{F}_G) = K_G(0, d_2, \ldots, d_{n-1}, -\sum_{i=2}^{n-1} d_i),$$

where $K_G(\mathbf{a})$ is the Kostant partition function and vol *is normalized volume.*

Recall the definition of the Kostant partition function.

Definition 3.4. The **Kostant partition function** $K_G(\mathrm{v})$ is the number of ways to write the vector v as a nonnegative linear combination of the positive type A_{n-1} roots corresponding to the edges of G, without regard to order. The edge (i, j), $i < j$, of G corresponds to the vector $e_i - e_j$, where e_i is the i^{th} standard basis vector in $\mathbb{R}^n$.

It is easy to see by definition that the Ehrhart polynomial of $\mathcal{F}_G$ in variable t is equal to $K_G(t, 0, \ldots, 0, -t)$.

Now we are ready to define order polytopes and relate them to flow polytopes.

Definition 3.5 (Stanley [20])**.** The **order polytope**, $\mathcal{O}(P)$, of a poset P with elements $\{t_1, t_2, \ldots, t_n\}$ is the set of points $(x_1, x_2, \ldots, x_n)$ in $\mathbb{R}^n$ with $0 \le x_i \le 1$ and if $t_i \le_P t_j$ then $x_i \le x_j$. We identify each point $(x_1, x_2, \ldots, x_n)$ of $\mathcal{O}(P)$ with the function $f : P \to \mathbb{R}$ with $f(t_i) = x_i$.

In general, computing or finding a combinatorial interpretation for the volume of a polytope is a hard problem. Order polytopes are an especially nice class of polytopes whose volume has a combinatorial interpretation.

Theorem 3.6 (Stanley [20])**.** *Given a poset P we have that*

(i) the vertices of $\mathcal{O}(P)$ are in bijection with the order ideals of P,
(ii) the normalized volume of $\mathcal{O}(P)$ is $e(P)$, where $e(P)$ is the number of linear extensions of P,
(iii) the Ehrhart polynomial $L_{\mathcal{O}(P)}(m)$ of $\mathcal{O}(P)$ equals the order polynomial $\Omega(P, m+1)$ of P.

Definition 3.7. Given a poset P and a positive integer m, the **order polynomial** $\Omega(P, m)$ is the number of order preserving maps $\eta : P \to \{1, 2, \ldots, m\}$.

3.2. **Flow polytopes of planar graphs are order polytopes.** The following theorem, which states that a flow polytope of a planar graph is an order polytope, is a result communicated to us by Postnikov [15]. We need the following conventions. Given a connected graph G on the vertex set $[n]$, we draw it in the plane so that the vertices $1, 2, \ldots, n$ are on a horizontal line in this order.

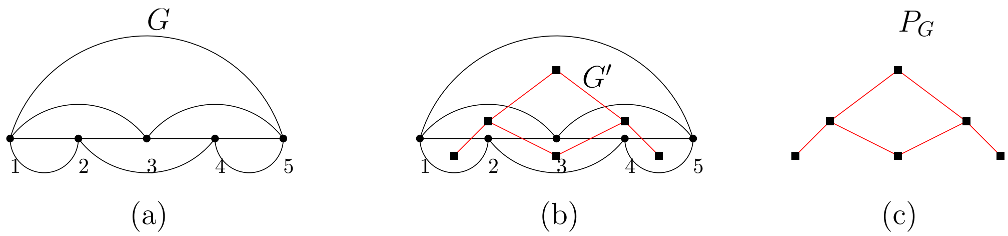


FIGURE 4. (a) A planar graph G, (b) the truncated dual graph G' shown in red, (c) the Hasse diagram of P_G.

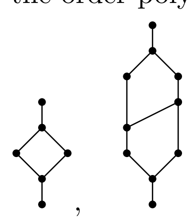
We say that G is **planar** if it has a planar embedding with $1, 2, \ldots, n$ fixed on a horizontal line. See Figure 4. Given such a planar embedding of G, we draw the **truncated dual** graph of G, denoted G', which is the dual graph with the vertex corresponding to the infinite region deleted together with its incident edges. Note that since the vertices $1, 2, \ldots, n$ are drawn on a horizontal line, we can naturally orient the edges of G' from "lower" to "higher" (see Figure 4 (b)). The poset P_G is then obtained by considering G' as its Hasse diagram (see Figure 4 (c)). Note that by Euler's formula, $\#P_G = \#E(G) - \#V(G) + 1$ which equals $\dim(\mathcal{F}_G)$.

Theorem 3.8 (Postnikov [15])**.** *Let G be a planar graph on the vertex set $[n]$ such that at each vertex $v \in [2, n-1]$ there are both incoming and outgoing edges. Fix a planar drawing of G. Then there is a linear map from the flow polytope $\mathcal{F}_G$ to the order polytope $\mathcal{O}(P_G)$ which preserves relative volume.*

Proof. For an element x of P_G let $f(x) = \sum_e fl(e)$, where the sum is taken over the edges e that are intersected by a fixed path from the element x to the "low point" in the dual graph of G. The "low point" of the dual graph is the vertex corresponding to the infinite face of G and we draw it below the graph as shown on Figure 5 (a). It is easy to see that due to flow conservation this map is well-defined, that is it does not depend on the path we choose.

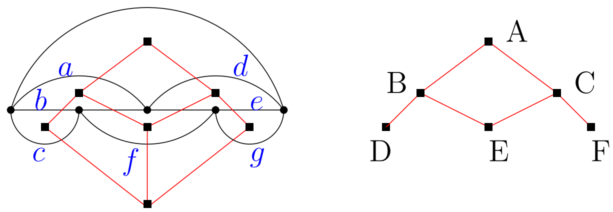
In addition, we have that $0 \leq f(x)$ since $fl(e) \geq 0$ for all edges e. Also, $f(x) \leq 1$ since the set of edges whose sum of flows equals $f(x)$ can always be extended to a cut of the graph G whose flow is the total flow 1 present in the graph. Next, if x' covers x in P_G then there is an edge e' in G separating the graph faces x and x'. Thus $f(x') = fl(e') + f(x) \geq f(x)$. Hence the linear map mentioned in the theorem takes a point $(fl(e))_{e \in E(G)}$ of $\mathcal{F}_G$ to the point $(f(x))_{x \in P_G}$ of the order polytope $\mathcal{O}(P_G)$. This map preserves the integer points in the affine spans of these polytopes, thereby preserving relative volume. □

Remark 3.9. By Theorem 3.8, if G is a planar graph then $\mathcal{F}_G$ is equivalent to an order polytope. This raises the question of whether this relation holds for non-planar graphs: for instance for the polytope $\mathcal{CRY}(n) \cong \mathcal{F}_{K_{n+1}}$ for $n \geq 4$. We can use a similar construction to that in Theorem 3.8 to show that $\mathcal{F}_{K_5}$ and $\mathcal{F}_{K_6}$ are equivalent to the order polytopes of the posets:



We leave it as a question whether $\mathcal{F}_{K_7}$ (dimension 15, 32 vertices, volume 140) is an order polytope.

3.3. **Order polytopes of strongly planar posets are flow polytopes.** We now give a converse of Theorem 3.8, showing that the order polytope of a strongly planar poset is a flow polytope. A



(a) (b)

FIGURE 5. (a) The "low point" of the dual (red) graph is the lowest point on the picture; the labels a, b, c, d, e, f, g are the names of the corresponding edges (b) our map prescribes that $f(A) = fl(a) + fl(b) + fl(c), f(B) = fl(b) + fl(c), f(C) = fl(e) + fl(g), f(D) = fl(c), f(E) = fl(f), f(F) = fl(g)$.

poset P is **strongly planar** if the Hasse diagram of $\widehat{P} := P \sqcup \{\hat{0}, \hat{1}\}$ has a planar embedding with all edges directed upward in the plane. Define the graph $\widehat{G}$ from $\widehat{P}$ by taking the Hasse diagram of $\widehat{P}$ and drawing in two additional edges from $\widehat{0}$ to $\widehat{1}$, one which goes to the left of all the poset elements and another to the right. We can then define the graph G_P to be the truncated dual of $\widehat{G}$, provided that $\widehat{G}$ is planar. $\widehat{G}$ will be planar whenever $\widehat{P}$ is planar, which in turn is when P is strongly planar. The orientation of G_P is inherited from the poset in the following way: if in the construction of the truncated dual, the edge e of G_P crosses the edge $x \to y$ where $x < y$ in P, then y is on the left and x is on the right as you traverse e.

Theorem 3.10. *If P is a strongly planar poset, then there is a linear map from the order polytope $\mathcal{O}(P)$ to the flow polytope $\mathcal{F}_{G_P}$ which preserves relative volume.*

Proof. For an edge e in G_P, let $fl(e) = f(y) - f(x)$, where in the dual construction, e crosses the Hasse diagram edge $x \to y$. It is easy to see that this map is well-defined and it maps $\mathcal{O}(P)$ to $\mathcal{F}_{G_P}$ by mapping $(f(x))_{x \in P}$ to $(fl(e))_{e \in E(G_P)}$, where $fl(e)$ is as prescribed above. This map preserves the integer points in the affine spans of these polytopes, thereby preserving relative volume. □

4. $\mathcal{ASMCRY}(n)$ AND THE FAMILY OF POLYTOPES $\mathcal{F}(ASM)$

In this section, we introduce the ASM-CRY family of polytopes $\mathcal{F}(ASM)$, which includes $\mathcal{ASMCRY}(n)$, and show that each of these polytopes is a face of the ASM polytope. We, furthermore, show that each polytope in this family is both an order and a flow polytope. Then, using the theory of order and flow polytopes as discussed in Section 3.1, we write down their volumes and Ehrhart polynomials.

Definition 4.1. Let $\delta_n = (n-1, n-2, \ldots, 2, 1)$ be the staircase partition considered as the positions (i,j) of an $n \times n$ matrix given by $\{(i,j) \mid j - i \geq 1\}$. Let the partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \subseteq \delta_n$ denote matrix positions $\{(i,j) \mid 1 \leq i \leq k,\ n - \lambda_i + 1 \leq j \leq n\}$.

We define the ASM-CRY family

$$\mathcal{F}(ASM)(n) := \{\mathcal{P}_\lambda(n) \mid \lambda \subseteq \delta_n\},$$

where

$$\mathcal{P}_\lambda(n) := \left\{(a_{ij})_{i,j=1}^n \in \mathcal{A}(n) \mid a_{ij} = 0 \text{ for } i - j \geq 2 \text{ and for } (i,j) \in \lambda\right\}.$$

Note that $\mathcal{P}_\varnothing(n) = \mathcal{ASMCRY}(n)$.

In the following proposition we give a convex hull description of the polytopes in this family.

Proposition 4.2. *The polytope $\mathcal{P}_\lambda(n) \in \mathcal{F}(ASM)(n)$ is the convex hull of the $n \times n$ alternating sign matrices $(A_{ij})_{i,j=1}^n$ with $A_{ij} = 0$ for $i - j \geq 2$ and for $(i,j) \in \lambda$.*

Proof. Let $\mathcal{Q}_\lambda(n)$ denote the convex hull of the $n \times n$ alternating sign matrices $(A_{ij})_{i,j=1}^n$ with $A_{ij} = 0$ for $i - j \geq 2$ and for $(i,j) \in \lambda$. It is easy to see that $\mathcal{Q}_\lambda(n)$ is contained in $\mathcal{P}_\lambda(n)$, since matrices in both polytopes have the same prescribed zeros and satisfy the inequality description of the full ASM polytope $\mathcal{A}(n)$.

It remains to prove that $\mathcal{P}_\lambda(n)$ is contained in $\mathcal{Q}_\lambda(n)$. Suppose there exists a matrix $b = (b_{ij})_{i,j=1}^n \in \mathcal{P}_\lambda(n)$ such that $b \notin \mathcal{Q}_\lambda(n)$. We know that b is in the convex hull of all $n \times n$ ASMs. So $b = \mu_1 A^1 + \mu_2 A^2 + \cdots + \mu_k A^k$, where $A^1, \ldots A^k$ are distinct $n \times n$ alternating sign matrices and $\mu_1, \ldots, \mu_k > 0$. At least one of these ASMs, say A^1 must have a nonzero entry A^1_{ij} for some (i,j) satisfying either $i - j \geq 2$ or $(i,j) \in \lambda$. Suppose $i - j \geq 2$; the argument follows similarly in the case $(i,j) \in \lambda$. Now since $b_{ij} = 0$ and $A^1_{ij} \neq 0$, there must be another ASM, say A^2 such that A^2_{ij} is nonzero of opposite sign. Say $A^1_{ij} = 1$ and $A^2_{ij} = -1$. Then by the definition of an alternating sign matrix, there must be $j' < j$ such that $A^2_{ij'} = 1$. But $b_{ij'} = 0$ as well, so there must be an A^3 with $A^3_{ij'} = -1$ and $j'' < j'$ such that $A^3_{ij''} = 1$. Eventually, we will reach the border of the matrix and reach a contradiction. Thus, $\mathcal{P}_\lambda(n) = \mathcal{Q}_\lambda(n)$. □

We show in Theorem 4.3 below that the polytopes in $\mathcal{F}(ASM)(n)$ are faces of $\mathcal{A}(n)$. First, we need some terminology from [22]. Consider $n^2 + 4n$ vertices on a square grid: n^2 'internal' vertices (i,j) and $4n$ 'boundary' vertices $(i,0)$, $(0,j)$, $(i,n+1)$, and $(n+1,j)$, where $1 \leq i,j \leq n$. Fix the orientation of this grid so that the first coordinate increases from top to bottom and the second coordinate increases from left to right, as in a matrix. The **complete flow grid** C_n is defined as the directed graph on these vertices with directed edges pointing in both directions between neighboring internal vertices within the grid, and also directed edges from internal vertices to neighboring border vertices. That is, C_n has edge set $\{((i,j),(i,j\pm 1)),((i,j),(i\pm 1,j)) \mid i,j = 1,\ldots,n\}$. A **simple flow grid** of order n is a subgraph of C_n consisting of all the vertices of C_n, and in which four edges are incident to each internal vertex: either four edges directed inward, four edges directed outward, or two horizontal edges pointing in the same direction and two vertical edges pointing in the same direction. An **elementary flow grid** is a subgraph of C_n whose edge set is the union of the edge sets of simple flow grids. See Figure 6.

Theorem 4.3. *The polytope $\mathcal{P}_\lambda(n) \in \mathcal{F}(ASM)(n)$ is a face of $\mathcal{A}(n)$, of dimension $\binom{n}{2} - |\lambda|$. In particular, $\mathcal{P}_\varnothing(n) = \mathcal{ASMCRY}(n)$ is a face of $\mathcal{A}(n)$, of dimension $\binom{n}{2}$.*

Proof. In Proposition 4.2 of [22], it was shown that the simple flow grids of order n are in bijection with the $n \times n$ alternating sign matrices. In this bijection, the internal vertices of the simple flow grid correspond to the ASM entries; the sources correspond to the ones of the ASM, the sinks correspond to the negative ones, and all other vertex configurations correspond to zeros. In Theorem 4.3 of [22], it was shown that the faces of $\mathcal{A}(n)$ are in bijection with $n \times n$ elementary flow grids, with the complete flow grid C_n in bijection with the full ASM polytope $\mathcal{A}(n)$. This bijection was given by noting that the convex hull of the ASMs in bijection with all the simple flow grids contained in an elementary flow grid is, in fact, an intersection of facets of the ASM polytope $\mathcal{A}(n)$, and is thus a face of $\mathcal{A}(n)$. Since, by Proposition 4.2, $\mathcal{P}_\lambda(n)$ equals the convex hull of the ASMs in it, we need only show there exists an elementary flow grid whose contained simple flow grids correspond exactly to these ASMs.

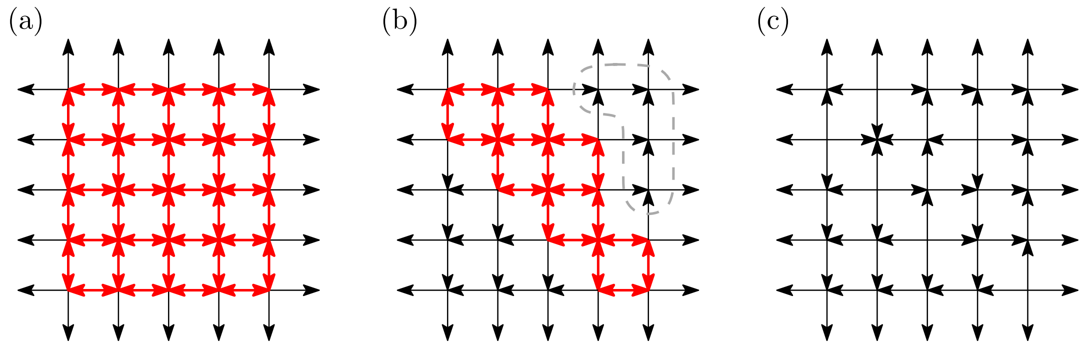


FIGURE 6. (a) The complete flow grid C_5, which corresponds to the full ASM polytope $\mathcal{A}(5)$. (b) The elementary flow grid corresponding to $\mathcal{P}_\lambda(5)$ with $\lambda = (2,1,1)$. Note there are six doubly directed regions, thus $\mathcal{P}_\lambda(5)$ is a face of $\mathcal{A}(5)$ of dimension six. (c) A simple flow grid which corresponds to a 5×5 ASM and is contained in the elementary flow grid of (b).

We can give this elementary flow grid explicitly. We claim that the edge set $\bigcup_{(i,j)} S_{i,j}$ where

$$S_{i,j} := \begin{cases} ((i,j+1),(i,j)),((i-1,j),(i,j)) & \text{if } i-j \geq 2 \\ ((i,j-1),(i,j)),((i+1,j),(i,j)) & \text{if } (i,j) \in \lambda \\ ((i,j),(i,j\pm 1)),((i,j),(i\pm 1,j)) & \text{otherwise.} \end{cases}$$

is the union of the edge sets all the simple flow grids in bijection with ASMs in $\mathcal{P}_\lambda(n)$, thus the digraph with this edge set is an elementary flow grid. Furthermore, no other simple flow grid can be constructed from directed edges in this set, since such a simple flow grid would have to include an edge pointing in the wrong direction in either the region $i-j \geq 2$ or $(i,j) \in \lambda$. Thus, $\mathcal{P}_\lambda(n)$ is a face of $\mathcal{A}(n)$.

To calculate the dimension of $\mathcal{P}_\lambda(n)$, we use the following notion from [22]. A **doubly directed region** of an elementary flow grid is a connected collection of cells in the grid completely bounded by double directed edges but containing no double directed edges in the interior. Theorem 4.5 of [22] states that the dimension of a face of $\mathcal{A}(n)$ equals the number of doubly directed regions in the corresponding elementary flow grid. The number of doubly directed regions in the elementary flow grid corresponding to $\mathcal{P}_\lambda(n)$ equals $(n-1)^2 - \left(\binom{n-1}{2} + |\lambda|\right) = \binom{n}{2} - |\lambda|$. See Figure 6. ☐

Our main result regarding $\mathcal{F}(ASM)(n)$ is Theorem 1.1, which we repeat here for convenience. It requires the following definition and see Section 3.3 for the definition of G_P.

Definition 4.4. Let δ_n and $\lambda \subseteq \delta_n$ be as in Definition 4.1. Let $(\delta_n \setminus \lambda)^*$ be the poset with elements p_{ij} corresponding to the positions $(i,j) \in \delta_n \setminus \lambda$ with partial order $p_{ij} \leq p_{i'j'}$ if $i \geq i'$ and $j \leq j'$.

Theorem 1.1. *The polytopes in the family $\mathcal{F}(ASM)(n)$ are affinely equivalent to flow and order polytopes. In particular, $\mathcal{P}_\lambda(n)$ is affinely equivalent to the order polytope of the poset $(\delta_n \setminus \lambda)^*$ and the flow polytope $\mathcal{F}_{G_{(\delta_n \setminus \lambda)^*}}$.*

We prove Theorem 1.1 by first using two lemmas to show that $\mathcal{P}_\lambda(n)$ is affinely equivalent to the order polytope of the poset $(\delta_n \setminus \lambda)^*$. Then since this poset is planar, by Theorem 3.10 its order polytope is affinely equivalent to the flow polytope $\mathcal{F}_{G_{(\delta_n \setminus \lambda)^*}}$.

Given a matrix $(m_{i,j})_{i,j=1}^n \in \mathcal{P}_\lambda(n)$, define the corner sum matrix $(c_{i,j})_{i,j=1}^n$ by

$$c_{i,j} = \sum_{\substack{1 \le i' \le i,\\ j \le j' \le n}} m_{i',j'}.$$

For $S \subseteq \mathbb{R}$, let $\mathcal{A}(\delta_n \setminus \lambda, S)$ be the set of functions $g : \delta_n \setminus \lambda \to S$. We view the order polytope of $(\delta_n \setminus \lambda)^*$ as a subset of $\mathcal{A}(\delta_n \setminus \lambda, [0,1])$. Define $\Phi : \mathcal{P}_\lambda(n) \to \mathcal{A}(\delta_n \setminus \lambda, \mathbb{R})$ by $m \mapsto g_m$ where $g_m(i,j) = 1 - c_{i,j}$. See the second map in Figure 8.

Lemma 4.5. *The image of Φ is in $\mathcal{A}(\delta_n \setminus \lambda, [0,1])$, i.e. if $m \mapsto g_m$ then $g_m(i,j) = 1 - c_{i,j} \in [0,1]$.*

Proof. We first show that $c_{i,j} \ge 0$ for all i and j. By the defining inequalities of the ASM polytope $\mathcal{A}(n)$ (see Definition 2.4), we have that the partial column sums satisfy $\sum_{1 \le i' \le i} m_{i',j} \ge 0$ for any $m \in \mathcal{A}(n)$ and $1 \le j \le n$. So since $c_{i,j}$ is a sum of partial column sums, then $c_{i,j} \ge 0$ as desired.

Next we show that $c_{i,j} \le 1$ for all $j > i \ge 1$. Note that it is not true that $c_{i,j} \le 1$ for all matrices in $\mathcal{A}(n)$ (for example the permutation matrix of 321 has $c_{2,2} = 2$). But we show $c_{i,j} \le 1$ for all $m \in \mathcal{P}_\lambda(n)$ as follows. The forced zeros $m_{ij} = 0$ for $i - j \ge 2$ and the requirement that each column sums to one imply that $\sum_{i=1}^{j+1} m_{i,j} = 1$, since the rest of the column entries equal zero. Thus we also have $\sum_{i=1}^{k} m_{i,j} = 1$ for any $k > j$. If $i = 1$ then $c_{i,j} \le 1$, since each $m_{1,j} \ge 0$ and $\sum_{j=1}^n m_{i,j} = 1$.

Now let $i \ge 2$. Note that the sum of the first i rows satisfies $c_{i,1} = i$. Also, by the discussion of the previous paragraph,

$$\sum_{j'=1}^{i-1} \sum_{i'=1}^{i} m_{i',j'} = \sum_{j'=1}^{i-1} 1 = i - 1 \ge 1,$$

since $i \ge 2$. Finally,

$$\sum_{j'=i}^{j-1} \sum_{i'=1}^{i} m_{i',j'} \ge 0,$$

since this is a sum of partial column sums. So we have

$$c_{i,j} = \sum_{j'=j}^{n} \sum_{i'=1}^{i} m_{i',j'} \le 1.$$

Thus $0 \le c_{i,j} \le 1$ so that $0 \le g_m(i,j) \le 1$.

□

Lemma 4.6. *The image of Φ is in the order polytope $\mathcal{O}\left((\delta_n \setminus \lambda)^*\right)$.*

Proof. By Lemma 4.5 we know that the image of Φ is in $\mathcal{A}(\delta_n \setminus \lambda, [0,1])$. Note that if $i' \le i$ and $j' \ge j$, then $c_{i,j} \ge c_{i',j'}$, thus we have that $g_m(i,j) \le g_m(i',j')$ if and only if $(i,j) \le (i',j')$ in $(\delta_n \setminus \lambda)^*$. So g_m is in the order polytope $\mathcal{O}\left((\delta_n \setminus \lambda)^*\right)$. □

Proof of Theorem 1.1. By Lemmas 4.5 and 4.6 we have that the map Φ is an affine map from $\mathcal{P}_\lambda(n)$ to $\mathcal{O}\left((\delta_n \setminus \lambda)^*\right)$ with homogeneous part $-A$ where A is a $0,1$-upper unitriangular matrix. Thus Φ is volume preserving ($\det(A) = 1$) and when restricted to a lattice $\mathbb{Z}^{|\delta_n \setminus \lambda|}/t$ is a bijection between the lattice points of $t \cdot \mathcal{P}_\lambda(n)$ and $t \cdot \mathcal{O}\left((\delta_n \setminus \lambda)^*\right)$, $t \in \mathbb{N}$. Thus, f is a bijective affine map from $\mathcal{P}_\lambda(n)$ to $\mathcal{O}\left((\delta_n \setminus \lambda)^*\right)$, showing that they are affinely equivalent (and thus combinatorially equivalent).

Finally since the poset $(\delta_n \setminus \lambda)^*$ is planar, by Theorem 3.10 $\mathcal{P}_\lambda(n)$ is also affinely equivalent to the flow polytope $\mathcal{F}_{G_{(\delta_n \setminus \lambda)^*}}$. □

By Stanley's theory of order polytopes [20] (see Theorem 3.6) we express the volume and Ehrhart polynomial of the polytopes in this family in terms of their associated posets.

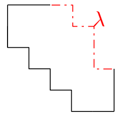
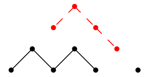
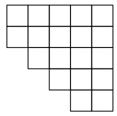
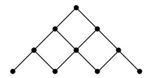
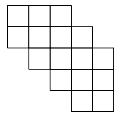
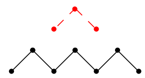
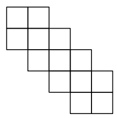
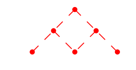
Polytope	Shape	Poset	# Vertices	Volume
$\mathcal{P}_{\lambda}(n)$			# order ideals of the poset $(\delta_n/\lambda)^*$	$\#SYT(\delta_n/\lambda)$ = # linear extensions of the poset $(\delta_n/\lambda)^*$
$\mathcal{ASMCRY}(n)$ (i.e. $\mathcal{P}_{\varnothing}(n)$)			$\frac{1}{n+1}\binom{2n}{n}$	$\frac{\binom{n}{2}!}{1^{n-1}3^{n-2}\cdots(2n-3)^1}$
$\mathcal{P}_{\delta_{n-2}}(n)$			F_{2n-1}	E_{2n-3}
$\mathcal{P}_{\delta_{n-1}}(n)$			2^{n-1}	$(n-1)!$

FIGURE 7. Some polytopes in the family $\mathcal{F}(ASM)(n)$ and their corresponding numbers of vertices and volumes; see Theorem 1.1 and Corollaries 1.2, 4.7, 4.8, and 4.9. 'Shape' refers to the entries in the matrix not fixed to be zero. All diagrams are drawn in the case $n = 5$.

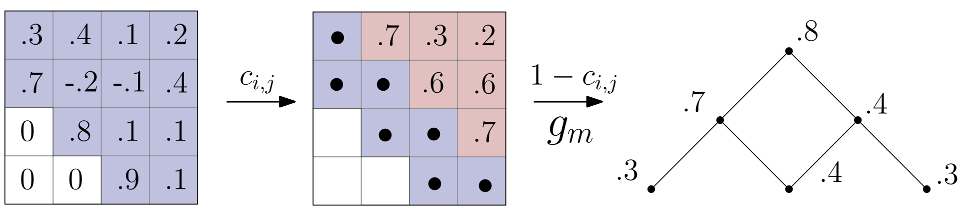


FIGURE 8. A map from a point in $\mathcal{ASMCRY}(4)$ to a point in the order polytope. First, take the northwest corner sum of each entry above the main diagonal. Then subtract that value from 1.

Corollary 4.7 ([20])**.** *For $\mathcal{P}_\lambda(n)$ in $\mathcal{F}(ASM)(n)$ we have that its normalized volume is*

$$\mathrm{vol}(\mathcal{P}_\lambda(n)) = e\left((\delta_n \backslash \lambda)^*\right),$$

and its Ehrhart polynomial is

$$L_{\mathcal{P}_\lambda(n)}(t) = \Omega_{(\delta_n\backslash\lambda)^*}(t+1).$$

Note that using Theorem 3.3 and the discussion below it, we can express the volume and Ehrhart polynomial of any flow polytope as a Kostant partition function. Thus, Theorem 1.1 gives us several Kostant partition function identities. In particular, Corollaries 1.2, 4.9 and 4.8 compute the volumes and Ehrhart polynomials of three subfamilies of polytopes in $\mathcal{F}(ASM)(n)$ that are associated to posets with a nice number of linear extensions and vertices. This includes the ASM-CRY polytope. See Figure 7.

Corollary 1.2. *$\mathcal{ASMCRY}(n)$ is affinely equivalent to the order polytope of the poset δ_n^* and the flow polytope $\mathcal{F}_{G_{\delta_n^*}}$. Thus, $\mathcal{ASMCRY}(n)$ has $\mathrm{Cat}(n) = \frac{1}{n+1}\binom{2n}{n}$ vertices, its normalized volume is* *given by*

$$\mathrm{vol}(\mathcal{ASMCRY}(n)) = \#SYT(\delta_n),$$

and its Ehrhart polynomial is

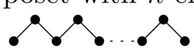
$$L_{\mathcal{ASMCRY}(n)}(t) = \Omega_{\delta_n^*}(t+1) = \prod_{1\leq i<j\leq n} \frac{2t+i+j-1}{i+j-1}. \tag{4.1}$$

Proof. When $\lambda = \varnothing$, $\mathcal{P}_\varnothing(n)$ is isomorphic to the order polytope $\mathcal{O}_{\delta_n^*}$ of the poset δ_n^* (that is, the type A_{n-1} positive root lattice). The number of linear extension of this poset is the number of standard Young tableaux (SYT) of shape $(n-1, n-2, \ldots, 2, 1)$,

$$\mathrm{vol}\mathcal{P}_\varnothing(n) = \#SYT_{(n-1,n-2,\ldots,2,1)} = \frac{\binom{n}{2}!}{1^{n-1}3^{n-2}\cdots(2n-3)^1}.$$

By Stanley's theory of order polytopes (see Theorem 3.6) $L_{\mathcal{P}_\varnothing(n)}(t) = \Omega_{\delta_{n-1}^*}(t+1)$. When t is an integer, $\Omega_{\delta_{n-1}^*}(t+1)$ counts the the number of plane partitions of shape $(n-1, n-2, \ldots, 2, 1)$ with largest part $\leq t$. By an unpublished result of Proctor [16] (see also [10]) this number is given by the product formula in the RHS of (4.1). □

We give a few other examples of polytopes in the family $\mathcal{F}(ASM)(n)$ that have known nice formulas for the volume, namely, in the cases $\lambda = \delta_{n-k}$ for $k \geq 1$. See Figure 7.

Let $[n]$ be the poset with n elements and no relations and z_{2n-1} denote the **zigzag poset** with $2n-1$ elements: .

Corollary 4.8. *$\mathcal{P}_{\delta_{n-1}}(n)$ is isomorphic to the order polytope $\mathcal{O}_{[n-1]}$ of $[n-1]$. $\mathcal{P}_{\delta_{n-2}}(n)$ has 2^{n-1} vertices and its normalized volume equals $(n-1)!$.*

Proof. Since the poset $[n-1]$ is an antichain, there are no relations, so the number of order ideals is 2^{n-1} and the number of linear extensions is $(n-1)!$. Thus, the result follows from Theorem 1.1. □

Corollary 4.9. *$\mathcal{P}_{\delta_{n-2}}(n)$ is affinely equivalent to the order polytope $\mathcal{O}_{z_{2n-1}}$ of z_{2n-1}. $\mathcal{P}_{\delta_{n-2}}(n)$ has number of vertices given by the Fibonacci number F_{2n-1} and normalized volume given by the Euler number E_{2n-3}.*

Proof. The number of order ideals of z_{2n+1} is given by the Fibonacci number F_{2n-1}. The number of linear extensions of this poset is the number of SYT of skew shape δ_n/δ_{n-2} which is given by the Euler number E_{2n-3}. Thus, the result follows from Theorem 1.1. □

Remark 4.10. For the case $\lambda = \delta_{n-k}$, the polytope $\mathcal{P}_{\delta_{n-k}}(n)$ is isomorphic to the order polytope of the poset $(\delta_n \setminus \delta_{n-k})^*$. The number of vertices of the polytope (order ideals of the poset) is given by the number of Dyck paths with height at most k [18, A211216], [11, §3.1]. The volume of the polytope is given by the number of skew SYT of shape δ_n/δ_{n-k}. There are formulas for this number of SYT as determinants of Euler numbers (e..g see Baryshnikov-Romik [2]).

We now turn from our investigation of the family of polytopes $\mathcal{F}(ASM)(n)$ to triangulations of flow and order polytopes.

5. Triangulations of flow polytopes of planar graphs

As we have seen in Section 3, flow polytopes of planar graphs are also order polytopes. Stanley [20] gave a canonical way of triangulating an order polytope $\mathcal{O}(P)$. For a linear extension $(a_1, a_2, \ldots, a_n)$ of the poset P on elements $\{a_1, a_2, \ldots, a_n\}$, define the simplex

$$\Delta_{a_1,a_2,\ldots,a_n} := \{(x_1, \ldots, x_n) \in [0,1]^n \mid x_{a_1} \leq x_{a_2} \leq \cdots \leq x_{a_n}\}. \tag{5.1}$$

Note that the $n+1$ vertices of this simplex are of the form $(0^m, 1^{n-m})$ for $m = 0, 1, \ldots, n$. The simplices corresponding to all linear extensions of P are top dimensional simplices in a triangulation of $\mathcal{O}(P)$, which we refer to as the **canonical triangulation of** $\mathcal{O}(P)$. There are also two known combinatorial ways of triangulating flow polytopes: one given by Postnikov and Stanley [15, 19], and one by Danilov, Karzanov and Koshevoy [8]. In this section, we show that given a planar graph G, the canonical triangulation of $\mathcal{O}(P_G)$ is also a triangulation obtained by the Danilov-Karzanov-Koshevoy method for $\mathcal{F}_G$. This result was first observed by Postnikov [15]. We also construct a direct bijection between linear extensions of P_G and maximal cliques of G, which index the Danilov-Karzanov-Koshevoy triangulation of $\mathcal{F}_G$. In Section 6, we will prove for a general graph G that the Danilov-Karzanov-Koshevoy triangulations of $\mathcal{F}_G$ can also be obtained by a framed Postnikov-Stanley method. Thus, in particular, the canonical triangulation of $\mathcal{O}(P_G)$ for a planar graph G is also in the set of the framed Postnikov-Stanley triangulations of $\mathcal{F}_G$.

In the following subsection, we first review the results of Danilov, Karzanov and Koshevoy [8] and then prove Theorem 1.3.

5.1. **The canonical triangulation of $\mathcal{O}(P_G)$ is a Danilov-Karzanov-Koshevoy triangulation of $\mathcal{F}_G$.** Given a connected graph G on the vertex set $[n]$ with edges oriented from smaller to bigger vertices, the vertices of the flow polytope $\mathcal{F}_G$ correspond to integer flows of size one on maximal directed paths from the source (1) to the sink (n). Following [8] we call such maximal paths **routes**. The following definitions also follow [8]. Let v be an **inner** vertex of G whenever v is neither a source nor a sink. Fix a **framing** at each inner vertex v, that is, a linear ordering $\prec_{in(v)}$ on the set of incoming edges $in(v)$ to v and the linear ordering $\prec_{out(v)}$ on the set of outgoing edges $out(v)$ from v. A **framed graph** is a graph with a framing at each inner vertex. For a framed graph G and an inner vertex v, we denote by $In(v)$ and by $Out(v)$ the set of maximal paths ending in v and the set of maximal paths starting at v, respectively. We define the order $\prec_{In(v)}$ on the paths in $In(v)$ as follows. If $P, Q \in In(v)$, then let w be the largest vertex after which P and Q coincide and before which they differ. Let e_P be the edge of P entering w and e_Q be the edge of Q entering w. Then $P \prec_{In(v)} Q$ if and only if $e_P \prec_{in(w)} e_Q$. The linear order $\prec_{Out(v)}$ on $Out(v)$ is defined analogously.

Given a route P with an inner vertex v, denote by Pv the maximal subpath of P ending at v and by vP the maximal subpath of P starting at v. We say that the routes P and Q are **coherent at a vertex** v which is an inner vertex of both P and Q if the paths Pv, Qv are ordered the same way as vP, vQ; that is, if $Pv \prec_{In(v)} Qv$ and $vP \prec_{Out(v)} vQ$. We say that routes P and Q are **coherent** if they are coherent at each common inner vertex. We call a set of mutually coherent routes a **clique**. The following theorem is a special case of [8, Theorems 1 & 2].

Theorem 5.1. [8, Theorems 1 & 2] *Given a framed graph G, the top dimensional simplices in a regular triangulation of $\mathcal{F}_G$ are obtained by taking the convex hulls of the vertices corresponding to the routes in maximal cliques. Moreover, lower dimensional simplices of this triangulation are obtained as convex hulls of the vertices corresponding to the routes in (not maximal) cliques.*

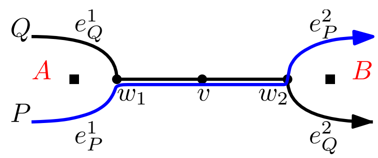
Theorem 5.1 uses the fact that the vertices of $\mathcal{F}_G$ are given by unit flows along the routes of G [17, §13].

We call the triangulations appearing in Theorem 5.1 the **Danilov-Karzanov-Koshevoy triangulations of** $\mathcal{F}_G$. Each such triangulation comes from a particular framing of the graph. We are now ready to prove that the canonical triangulation of $\mathcal{O}(P_G)$ is a Danilov-Karzanov-Koshevoy triangulations of $\mathcal{F}_G$. We now define the framing needed for this result. Consider a planar graph G on the vertex set $[n]$ with a particular planar embedding, where $1, 2, \ldots, n$ are drawn horizontally on a line. At each vertex $v \in [2, n-1]$ of G there is a natural order on the edges coming from the planar drawing of the graph: order the incoming edges as well as the outgoing edges top to bottom. We call this framing the **planar framing** of G, to emphasize that this framing comes from a particular planar embedding of the graph G.

Theorem 1.3. *Given a planar graph G, the canonical triangulation of $\mathcal{O}(P_G)$ is equal to the Danilov-Karzanov-Koshevoy triangulation of $\mathcal{F}_G$ coming from the planar framing.*

Proof. Suppose that to the contrary, there are two vertices of a simplex $\Delta_{a_1,\ldots,a_n}$ in the canonical triangulation of $\mathcal{O}(P_G)$, which correspond to non-coherent routes P and Q in the above framing of G. Suppose that P and Q are not coherent at the common inner vertex v. Suppose that the smallest vertex after which Pv and Qv agree is w_1 and the largest vertex before which vP and vQ agree is w_2. Let the edges incoming to w_1 be e_P^1 and e_Q^1 for P and Q, respectively, and let the edges outgoing from w_2 be e_P^2 and e_Q^2 for P and Q, respectively. Since P and Q are not coherent at v, this implies that either $e_P^1 \prec_{in(w_1)} e_Q^1$ and $e_Q^2 \prec_{out(w_2)} e_P^2$; or $e_Q^1 \prec_{in(w_1)} e_P^1$ and $e_P^2 \prec_{out(w_2)} e_Q^2$. We also have that the segments of P and Q between w_1 and w_2 coincide.

Consider the case where $e_P^1 \prec_{in(w_1)} e_Q^1$ and $e_Q^2 \prec_{out(w_2)} e_P^2$. Let A be the element of P_G corresponding to the region bordered from above by e_Q^1 and below by e_P^1 and let B be the element of P_G corresponding to the region bordered from above by the edge e_P^2 and below by e_Q^2:



Then in a linear extension α of P_G either $f(A) \geq f(B)$ or $f(A) \leq f(B)$. The former makes it impossible for P to be a vertex of the simplex corresponding to α since this would force $f(A) = 0$ and $f(B) = 1$. The latter makes it impossible for Q to be a vertex of the simplex corresponding to α since this would force $f(A) = 1$ and $f(B) = 0$. The case when $e_Q^1 \prec_{in(w_1)} e_P^1$ and $e_P^2 \prec_{out(w_2)} e_Q^2$ can be ruled out similarly.

We conclude that the vertices of a simplex in the canonical triangulation of $\mathcal{O}(P_G)$ correspond to a maximal clique. □

5.2. **A bijection between linear extensions of P_G and maximal cliques in G.** In this subsection, we construct an explicit bijection b between linear extensions of P_G and maximal cliques in a planar graph G framed so that at every vertex both the incoming and outgoing edges are ordered top to bottom. Recall from Section 3 that the elements of P_G correspond to bounded regions defined by G. Drawing the vertices of G on a horizontal line with vertices $1, 2, \ldots, n$ in this order, we can talk of the upper boundary of such a region. Given a linear extension $a_1 \cdots a_m$ of P_G, where $a_1, \ldots, a_m$ also denote the corresponding regions of G, map the linear extension $a_1 \cdots a_m$ to the upper boundaries of the union of regions given by the prefixes of $a_1 \cdots a_m$. See Figure 9 for an example.

Theorem 5.2. *Given a planar graph G, the map b defined above is a bijection between linear extensions of P_G and maximal cliques in G in the planar framing.*

The proof of Theorem 5.2 is in the same spirit as that of Theorem 1.3 and is left to the reader.

Note that in Theorem 5.2 we give a bijection from linear extensions of P_G to maximal cliques of G in the planar framing. In Section 6.3, we will see that given any two framings of G there is a natural bijection between their sets of maximal cliques. Therefore, combining the bijection from Theorem 5.2 and the one just mentioned in Section 6.3 we obtain a bijection between linear extensions of P_G and maximal cliques in G in any framing of a planar graph G.

Figure 9. On the left is the planar graph G and the poset P_G on elements A, B, C, D. On the right are all prefixes of the linear extension $CABD$ of P_G and the routes they correspond to under the bijection b. Note that the five resulting routes form a maximal clique in G with respect to the planar framing that orders both the incoming and outgoing edges top to bottom.

Corollary 5.3. *Given a planar graph G, the number of linear extensions of P_G equals the number of maximal cliques in G in the planar framing.*

6. Triangulations of flow polytopes

In this section, we show that the set of Danilov-Karzanov-Koshevoy triangulations of a flow polytope $\mathcal{F}_G$ is a subset of the framed Postnikov-Stanley triangulations of $\mathcal{F}_G$, which we define in this section. As a consequence of our proof, we obtain a bijection between the objects indexing the Postnikov-Stanley triangulation of a flow polytope $\mathcal{F}_G$, namely, nonnegative integer flows on the graph G with netflow $(0, d_2, \ldots, d_{n-1}, -\sum_{i=2}^{n-1} d_i)$, where $d_i = indeg_G(i) - 1$, and the objects indexing the Danilov-Karzanov-Koshevoy triangulation of a flow polytope $\mathcal{F}_G$, namely, maximal cliques of a fixed framing of G. This answers Postnikov's question [15] about a bijection between the sets indexing maximal simplices of the Postnikov-Stanley triangulations and the sets indexing maximal simplices of the Danilov-Karzanov-Koshevoy triangulation. We also obtain a natural bijection between the sets of maximal cliques of G in different framings.

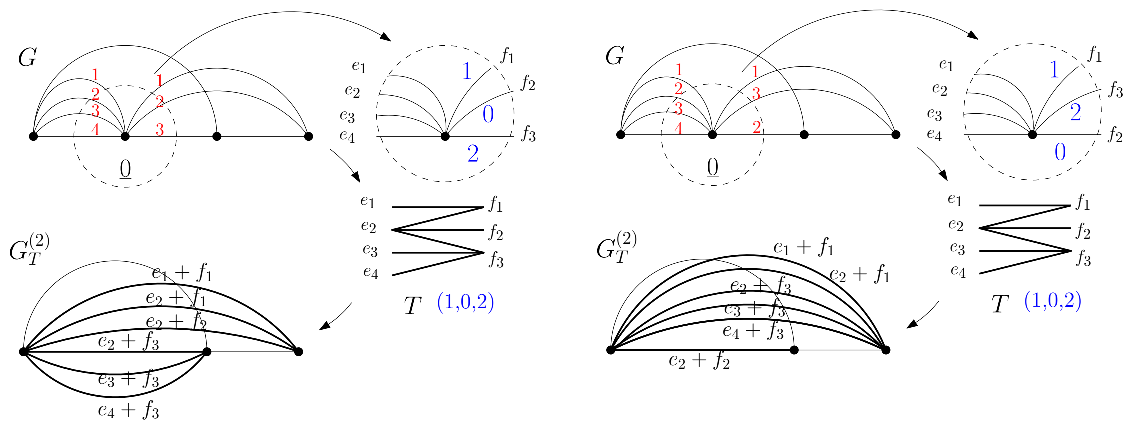
6.1. Framed Postnikov-Stanley triangulations.

We now define framed Postnikov-Stanley triangulations. These triangulations were used in [12], though they were not described explicitly there.

A **bipartite noncrossing tree** is a tree with left vertices $x_1, \ldots, x_\ell$ and right vertices $x_{\ell+1}, \ldots, x_{\ell+r}$ with no pair of edges $(x_p, x_{\ell+q}), (x_t, x_{\ell+u})$ where $p < t$ and $q > u$. We denote by $\mathcal{T}_{\mathcal{I},\mathcal{O}}$ the set of bipartite noncrossing trees where $\mathcal{I}$ and $\mathcal{O}$ are the ordered sets $(x_1, \ldots, x_\ell)$ and $(x_{\ell+1}, \ldots, x_{\ell+r})$ respectively. We have that $\#\mathcal{T}_{\mathcal{I},\mathcal{O}} = \binom{\ell+r-2}{\ell-1}$, since the elements of $\mathcal{T}_{\mathcal{I},\mathcal{O}}$ are in bijection with weak compositions of $\ell - 1$ into r parts. A tree T in $\mathcal{T}_{\mathcal{I},\mathcal{O}}$ corresponds to the composition $(b_1, \ldots, b_r)$ of (indegrees -1), where b_i denotes the number of edges incident to the right vertex $x_{\ell+i}$ in T minus 1.

Example 6.1. The bipartite tree in Figure 11 corresponds to the composition $(1, 0, 2)$.

We now define what we mean by a reduction at vertex i of a framed graph G on the vertex set $[n]$. Let $\mathcal{I}_i$ denote the multiset of incoming and $\mathcal{O}_i$ the multiset of outgoing edges of i. In addition, we assume that $\mathcal{I}_i$ and $\mathcal{O}_i$ are linearly ordered according to the framing of G. A reduction performed at i of G results in several new graphs indexed by bipartite noncrossing trees on the left vertex set I_i and right vertex set $\mathcal{O}_i$. We define these new graphs precisely below.



(a) (b)

Figure 10. Replacing the incident edges of vertex 2 in a graph H, by a noncrossing tree T encoded by the composition $(1,0,2)$ of $3 = indeg_H(2) - 1$ using two different framings (indicated by the blue numbers incident to vertex 2 in G): (a) the framing is "top to bottom", (b) different framing.

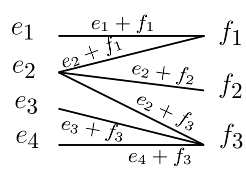


Figure 11. Based on the noncrossing tree: $S(f_1) = \{e_1+f_1, e_2+f_1\}$, $S(f_2) = \{e_2+f_2\}$, and $S(f_1) = \{e_2+f_3, e_3+f_3, e_4+f_3\}$. The local orderings of these edges at the vertices to which they are incoming are $e_1+f_1 < e_2+f_1$ and $e_2+f_3 < e_3+f_3 < e_4+f_3$.

Consider a tree $T \in \mathcal{T}_{\mathcal{I}_i,\mathcal{O}_i}$. For each tree-edge (e_1, e_2) of T where $e_1 = (r, i) \in \mathcal{I}_i$ and $e_2 = (i, s) \in \mathcal{O}_i$, let $e_1 + e_2$ be the following edge:

$$e_1 + e_2 = (r, s). \tag{6.1}$$

We call the edge $e_1 + e_2$ the **sum of edges** e_1 and e_2. Inductively, we can also define the sum of more than two consecutive edges.

The graph $G_T^{(i)}$, $T \in \mathcal{T}_{\mathcal{I}_i,\mathcal{O}_i}$, is defined as the graph obtained from G by removing the vertex i and all the edges of G incident to i and adding the multiset of edges $\{\{e_1 + e_2 \mid (e_1, e_2) \in E(T)\}\}$. See Figures 10 and 13 for examples of $G_T^{(i)}$. In these figures we also use the observation that a tree T in $\mathcal{T}_{\mathcal{I}_i,\mathcal{O}_i}$ bijects to the weak composition $(b_1, \dots, b_r)$ of (indegrees -1), where b_i denotes the number of edges incident to the ith right vertex of T minus 1. We record this composition by labeling the edges e in $\mathcal{O}_i$ of G with the corresponding part b_e. We can view this labeling as assigning a flow $b(e) = b_e$ to edges e of G in $\mathcal{O}_i$.

A **reduction of** G **at the vertex** i replaces G by the graphs in $\{G_T^{(i)}\}_{T \in \mathcal{T}_{\mathcal{I}_i,\mathcal{O}_i}}$. It also remembers which sum of the edges of G is each edge of the new graphs.

We now define an **inheritance framing** of $G_T^{(i)}$, $T \in \mathcal{T}_{\mathcal{I}_i,\mathcal{O}_i}$, which it inherits from the framing of G. We order the edges incident to vertices smaller than i arbitrarily. For each vertex j greater than i we look at the incoming and outgoing multisets of edges $\mathcal{I}_j(G_T^{(i)})$ and $\mathcal{O}_j(G_T^{(i)})$. The multiset $\mathcal{O}_j(G_T^{(i)})$ equals $\mathcal{O}_j(G)$ and is ordered the same way. If $\mathcal{I}_j(G) = \{m_1, \dots, m_k\}$, then the multiset

$\mathcal{I}_j(G_T^{(i)})$ consists of edges that are all sums of edges of G with one edge of $\{m_1, \ldots, m_k\}$. Denote by $S(m_l)$, $l \in [k]$, the edges in $\mathcal{I}_j(G_T^{(i)})$ which are sums of edges of G with m_l. Let $m_1 < \cdots < m_k$ in the framing of G. Draw T with the left and right sets of vertices ordered vertically as in Figure 11, so that we can read off its edges top to bottom. In the inheritance framing of $G_T^{(i)}$ within each set $S(m_l)$, $l \in [k]$, order the edges top to bottom when viewed in the noncrossing bipartite tree: any edge in $S(m_p)$ is less than any edge in $S(m_q)$ for $p < q$.

Given a framed graph G on the vertex set $[n]$, construct a **framed Postnikov-Stanley triangulation** of $\mathcal{F}_G$ as follows. If a vertex $v \in [2, n-1]$ has only incoming or outgoing edges, then by flow conservation those edges carry zero flow. Thus, we delete these vertices and the incident edges (without changing $\mathcal{F}_G$). Assume from now on that for every vertex $v \in [2, n-1]$ there is both an incoming and an outgoing edge.

Proceed from vertex 2 to $n-1$ performing the reduction defined above at each vertex. First we do a reduction at vertex 2. The sets $\mathcal{I}_2$ and $\mathcal{O}_2$ are ordered according to the framing of G. See Figure 10 for an example of how to eliminate the vertex 2 from a graph with two different framings. Next we need to do reductions at vertex 3 in all resulting graphs. Use inheritance framing for these graphs, which they inherit from the framing of G. Then, in all new graphs do reductions at vertex 4, and so forth. See Figure 13 for a full example.

The Subdivision Lemma formalizes that doing one reduction on G is dissecting the polytope $\mathcal{F}_G$ into smaller polytopes.

Lemma 6.2 (Subdivision Lemma [12])**.** *Let G be a connected graph with no loops on the vertex set $[n]$ and $\mathcal{F}_G$ be its flow polytope. For a fixed $i \in [2, n-1]$, the flow polytope subdivides as:*

$$\mathcal{F}_G = \bigcup_{T \in \mathcal{T}_{\mathcal{I}_i, \mathcal{O}_i}} \mathcal{F}_{G_T^{(i)}}. \tag{6.2}$$

Iterating the Subdivision Lemma, we can get a correspondence between integer flows on G with netflow vector $(0, d_2, \ldots, d_{n-1}, -\sum_{i=2}^{n-1} d_i)$, $d_i = (\text{indegree of } i) - 1$, and simplices in a triangulation of $\mathcal{F}_G$. We follow the exposition of [12] for this explanation. For G a connected loopless graph on the vertex set $[n]$, apply the Subdivision Lemma successively to vertices $2, 3, \ldots, n$. At the end we obtain the subdivision:

$$\mathcal{F}_G = \bigcup_{T_{n-1}} \cdots \bigcup_{T_3} \bigcup_{T_2} \mathcal{F}_{(\cdots (G_{T_2}^{(2)})_{T_3}^{(3)} \cdots)_{T_{n-1}}^{(n-1)}}, \tag{6.3}$$

where T_i are noncrossing trees. See Figure 12 for an example of an outcome of a subdivision of $\mathcal{F}_G$. The graph $G_{n-1} := (\cdots (G_{T_2}^{(2)})_{T_3}^{(3)} \cdots)_{T_{n-1}}^{(n-1)}$ consists of two vertices, 1 and n, with $\#E(G) - n + 2$ edges between them. Thus $\mathcal{F}_{G_{n-1}}$ is an $(\#E(G) - n + 1)$–dimensional simplex with normalized unit volume. Therefore, $\mathrm{vol}(\mathcal{F}_G)$ is the number of choices of bipartite noncrossing trees $T_2, \ldots, T_{n-1}$ where T_{i+1} encodes a composition of $\#\mathcal{I}_{i+1}(G_i) - 1$ with $\#\mathcal{O}_{i+1}(G_i)$ parts. Theorem 3.3 by Postnikov and Stanley [15, 19] shows that this number of tuples of trees is the number of integer flows on G with netflow vector $(0, d_2, \ldots, d_{n-1}, -\sum_{i=2}^{n-1} d_i)$, $d_i = (\text{indegree of } i) - 1$. For examples, see Figures 12 and 13.

6.2. **Danilov-Karzanov-Koshevoy triangulations are also framed Postnikov-Stanley triangulations.** Finally we are ready to prove that any Danilov-Karzanov-Koshevoy triangulation of $\mathcal{F}_G$ can also be constructed as a framed Postnikov-Stanley triangulation of $\mathcal{F}_G$.

Proposition 6.3. *Given a framed graph G on the vertex set $[n]$, the routes of G which give the vertices of the simplex $\mathcal{F}_{(\cdots (G_{T_2}^{(2)})_{T_3}^{(3)} \cdots)_{T_{n-1}}^{(n-1)}}$ form a maximal clique with respect to the coherence*

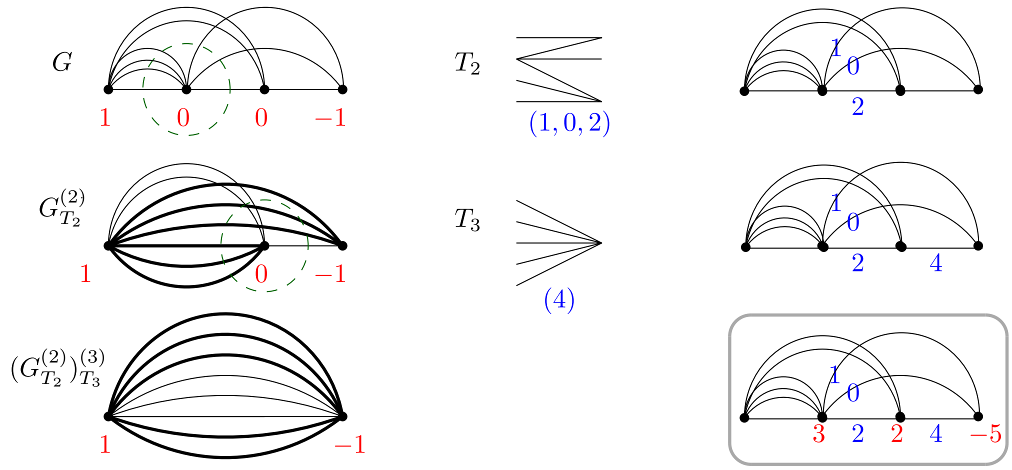


FIGURE 12. Example of the subdivision to find the volume of $\mathcal{F}_G$. The subdivision is encoded by noncrossing trees T_{i+1} that are equivalent to compositions $(b_1, \ldots, b_r)$ of $\#\mathcal{I}_{i+1}(G_i) - 1$ with $\#\mathcal{O}_{i+1}(G_i)$ parts. These trees or compositions are recorded by the integer flow on $G\backslash\{1\}$ in the box with netflow $(d_2, d_3, -d_2 - d_3) = (3, 2, -5)$ where $d_i = indeg_i(G) - 1$. The framing used is top to bottom.

relation in G. The framing of $(\cdots(G^{(2)}_{T_2})^{(3)}_{T_3}\cdots)^{(i)}_{T_i}$ is the inheritance framing obtained from the framing of $(\cdots(G^{(2)}_{T_2})^{(3)}_{T_3}\cdots)^{(i-1)}_{T_{i-1}}$.

Proof. Suppose that to the contrary, there are two vertices of a simplex $\mathcal{F}_{(\cdots(G^{(2)}_{T_2})^{(3)}_{T_3}\cdots)^{(n-1)}_{T_{n-1}}}$, which correspond to non-coherent routes P and Q in G. Suppose that P and Q are not coherent at the common inner vertex v. Suppose that the smallest vertex after which Pv and Qv agree is w_1 and the largest vertex before which vP and vQ agree is w_2. Let the edges incoming to w_1 be e^1_P and e^1_Q for P and Q, respectively, and let the edges outgoing from w_2 be e^2_P and e^2_Q for P and Q, respectively. Since P and Q are not coherent at v, this implies that either $e^1_P \prec_{in(w_1)} e^1_Q$ and $e^2_Q \prec_{out(w_2)} e^2_P$ or $e^1_Q \prec_{in(w_1)} e^1_P$ and $e^2_P \prec_{out(w_2)} e^2_Q$. We also have that the segments of P and Q between w_1 and w_2 coincide.

Denote by p the sum of edges between w_1 and w_2 on P. Denote by $*(e^1_Z + p)$, for $Z \in \{P, Q\}$, the sum of edges left of w_2 that are edges in Z (including e^1_Z in particular). After a certain number of reductions executed according to the framing, we are about to perform the reduction at vertex w_2. This reduction involves deleting w_2 and the edges incident to it, and adding the edges obtained from the noncrossing tree T we constructed based on the ordering of the incoming and outgoing edges at w_2. In such a noncrossing tree, the vertex corresponding to the edge stemming from $*(e^1_Z + p)$, $Z \in \{P, Q\}$, is above the vertex $*(e^1_{\overline{Z}} + p)$, where $\overline{Z}$ is the complement of Z in $\{P, Q\}$, in the left partition of the vertices of T. On the other hand, the vertex corresponding to $e^2_{\overline{Z}}$ is above the vertex corresponding to e^2_Z in the right partition of the vertices of T. Thus, it is impossible to obtain both routes P and Q as vertices of $\mathcal{F}_{(\cdots(G^{(2)}_{T_2})^{(3)}_{T_3}\cdots)^{(n-1)}_{T_{n-1}}}$ since that would force connecting $*(e^1_Z + p)$ and e^2_Z as well as $*(e^1_{\overline{Z}} + p)$ and $e^2_{\overline{Z}}$ in T. This would make a crossing in the noncrossing tree T, a contradiction. □

An immediate corollary of Proposition 6.3 is:

Corollary 6.4. *For a graph G, the set of Danilov-Karzanov-Koshevoy triangulations of $\mathcal{F}_G$ is a subset of the framed Postnikov-Stanley triangulations of $\mathcal{F}_G$.*

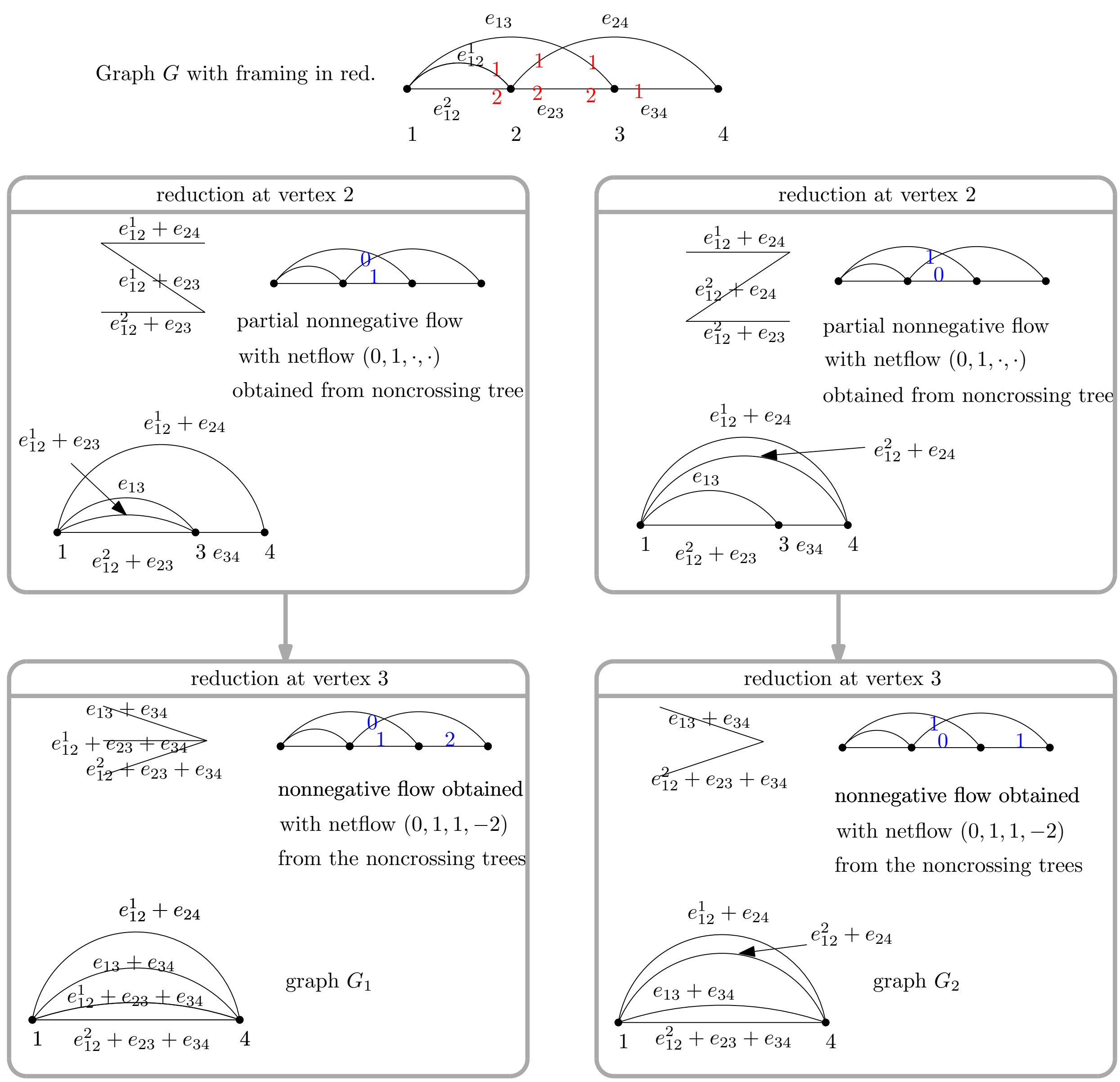


FIGURE 13. Reductions executed at vertex 2 and 3 of the framed graph G. Noncrossing trees encoding the reduction are displayed with all edges labeled. The nonnegative flow on G with netflow $(0, 1, 1, -2)$ is built. The flow polytope $\mathcal{F}_G$ is dissected into two simplices corresponding to G_1 and G_2.

6.3. **The bijection between certain nonnegative integer flows and maximal cliques.** We can use Proposition 6.3 to define a bijection b_G, where G is a framed graph on the vertex set $[n]$, from the set of nonnegative integer flows on the graph G with netflow $(0, d_2, \ldots, d_{n-1}, -\sum_{i=2}^{n-1} d_i)$ to the set of maximal cliques of G with respect to its framing. Indeed, we can use the proof of [12, Theorem 6.1] to biject each simplex in the framed Postnikov-Stanley triangulation with a nonnegative integer flow on the graph G with netflow $(0, d_2, \ldots, d_{n-1}, -\sum_{i=2}^{n-1} d_i)$, as shown for the top to bottom order for a particular simplex in Figure 12 and also in Figure 13. We can use Proposition 6.3 to biject each simplex in the framed Postnikov-Stanley triangulation with a maximal

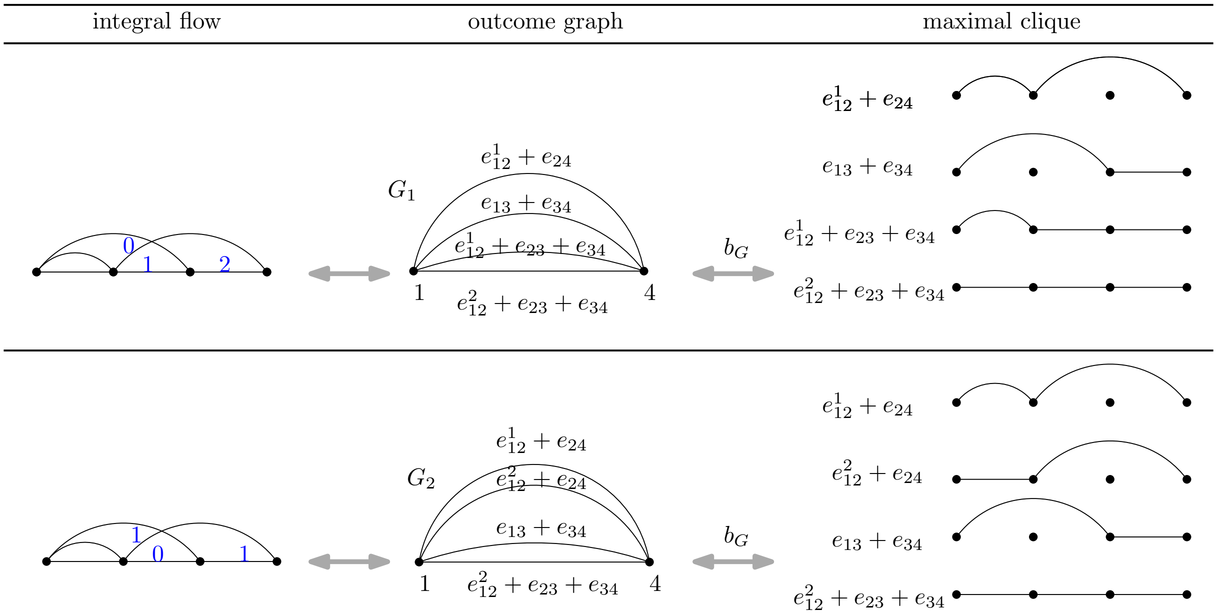


Figure 14. The four paths on the top correspond to the vertices of the simplices given by G_1 (as in Figure 13). The four paths on the bottom correspond to the vertices of the simplices given by G_2 (as in Figure 13). Both sets of paths are coherent in the framing of G given in Figure 13.

clique, simply by taking the routes corresponding to the vertices of the simplex; see Figures 13 and 14. In effect, then, we have a bijection b_G between nonnegative integer flows on the graph G with netflow $(0, d_2, \dots, d_{n-1}, -\sum_{i=2}^{n-1} d_i)$ and maximal cliques in G.

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Department of Mathematics, Cornell University, Ithaca, NY 14853

Department of Mathematics, UCLA, Los Angeles, CA 90095

Department of Mathematics, North Dakota State University, Fargo, ND 58102
E-mail address: `karola@math.cornell.edu, ahmorales@math.ucla.edu, jessica.striker@ndsu.edu`