A Fibonacci analogue of Stirling numbers

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Abstract

In recent years, there has been renewed interest in the Fibonomials $\binom{n}{k}_F$. That is, we define the Fibonacci numbers by setting $F_1 = 1 = F_2$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$. We let $n_F! = F_1 \cdots F_n$ and $\binom{n}{k}_F = \frac{n_F!}{k_F!(n-k)_F!}$. One can easily prove that $\binom{n}{k}_F$ is an integer and a combinatorial interpretation of $\binom{n}{k}_F$ was given in [9].

The goal of this paper is to find similar analogues for the Stirling numbers of the first and second kind and the Lah numbers. That is, we let $(x)_{\downarrow_0} = (x)_{\uparrow_0} = 1$ and for $k \geq 1$, $(x)_{\downarrow_k} = x(x-1)\cdots(x-k+1)$ and $(x)_{\uparrow_k} = x(x+1)\cdots(x+k-1)$. Then the Stirling numbers of the first and second kind are the connections coefficients between the usual power basis $\{x^n : n \geq 0\}$ and the falling factorial basis $\{(x)_{\downarrow_n} : n \geq 0\}$ in the polynomial ring $\mathbb{Q}[x]$ and the Lah numbers are the connections coefficients between the rising factorial basis $\{(x)_{\uparrow_n} : n \geq 0\}$ and the falling factorial basis $\{(x)_{\downarrow_n} : n \geq 0\}$ in the polynomial ring $\mathbb{Q}[x]$. In this paper, we will focus the Fibonacci analogues of the Stirling numbers. Our idea is to replace the falling factorial basis and the rising factorial basis by the Fibo-falling factorial basis $\{(x)_{\downarrow_{F,n}} : n \geq 0\}$ and the Fibo-rising factorial basis $\{(x)_{\uparrow_{F,n}} : n \geq 0\}$ where $(x)_{\downarrow_{F,0}} = (x)_{\uparrow_{F,0}} = 1$ and for $k \geq 1$, $(x)_{\downarrow_{F,k}} = x(x - F_1)\cdots(x - F_{k-1})$ and $(x)_{\uparrow_{F,k}} = x(x + F_1)\cdots(x + F_{k-1})$. Then we study the combinatorics of the connection coefficients between the usual power basis, the Fibo-falling factorial basis, and the Fibo-rising factorial basis. In each case, we can give a rook theory model for the connections coefficients and show how this rook theory model can give combinatorial explanations for many of the properties of these coefficients.

1 Introduction

In 1915, Fontené in [3] suggested a generalization $n_A!$ and $\binom{n}{k}_A$ of $n!$ and the binomial coefficient $\binom{n}{k}$ depending on any sequence $A = \{A_n : n \geq 0\}$ of real or complex numbers such that $A_0 = 0$ and $A_n \neq 0$ for all $n \geq 1$ by defining $0_A! = 1$, $n_A! = A_1 A_2 \cdots A_n$ for $n \geq 1$, and $\binom{n}{k}_A = \frac{n_A!}{k_A!(n-k)_A!}$ for $0 \leq k \leq n$. If we let $F = \{F_n : n \geq 0\}$ be the sequence of Fibonacci numbers defined by

$F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$, then the $\binom{n}{k}_F$'s are known as the Fibonomials. The Fibonomial $\binom{n}{k}_F$ are integers since Gould [4] proved that

$$\binom{n}{k}_F = F_{k+1}\binom{n-1}{k}_F + F_{n-k+1}\binom{n-1}{k-1}_F.$$

There have been a series of papers looking at the properties of the Fibonomials [1,2,4–6,9–11]. There is a nice combinatorial model for the Fibonomial coefficients given in [9] and the papers [1,2] have developed further properties and generalizations of the Fibonomials using this model.

Our goal in this paper is to define Fibinomials type analogues for the Stirling numbers of the first and second kind and to study their properties. Let $\mathbb{Q}$ denote the rational numbers and $\mathbb{Q}[x]$ denote the ring of polynomials over $\mathbb{Q}$. There are three very natural bases for $\mathbb{Q}[x]$. The usual power basis $\{x^n : n \geq 0\}$, the falling factorial basis $\{(x)_{\downarrow_n} : n \geq 0\}$, and the rising factorial basis $\{(x)_{\uparrow_n} : n \geq 0\}$. Here we let $(x)_{\downarrow_0} = (x)_{\uparrow_0} = 1$ and for $k \geq 1$, $(x)_{\downarrow_k} = x(x-1)\cdots(x-k+1)$ and $(x)_{\uparrow_k} = x(x+1)\cdots(x+k-1)$. Then the Stirling numbers of the first kind $s_{n,k}$, the Stirling numbers of the second kind $S_{n,k}$ and the Lah numbers $L_{n,k}$ are defined by specifying that for all $n \geq 0$,

$$(x)_{\downarrow_n} = \sum_{k=1}^{n} s_{n,k}\ x^k, \quad x^n = \sum_{k=1}^{n} S_{n,k}\ (x)_{\downarrow_k}, \text{ and } \ (x)_{\uparrow_n} = \sum_{k=1}^{n} L_{n,k}\ (x)_{\downarrow_k}.$$

The signless Stirling numbers of the first kind are defined by setting $c_{n,k} = (-1)^{n-k} s_{n,k}$. Then it is well known that $c_{n,k}$, $S_{n,k}$, and $L_{n,k}$ can also be defined by the recursions that $c_{0,0} = S_{0,0} = L_{0,0} = 1$, $c_{n,k} = S_{n,k} = L_{n,k} = 0$ if either $n < k$ or $k < 0$, and

$$c_{n+1,k} = c_{n,k-1} + nc_{n,k}, \quad S_{n+1,k} = S_{n,k-1} + kS_{n,k}, \text{ and } \ L_{n+1,k} = L_{n,k-1} + (n+k)L_{n,k}$$

for all $n, k \geq 0$. There are well known combinatorial interpretations of these connection coefficients. That is, $S_{n,k}$ is the number of set partitions of $[n] = \{1, \ldots, n\}$ into k parts, $c_{n,k}$ is the number of permutations in the symmetric group S_n with k cycles, and $L_{n,k}$ is the number of ways to place n labeled balls into k unlabeled tubes with at least one ball in each tube.

We start with the tiling model of the F_n of [9]. That is, let $\mathcal{FT}_n$ denote the set of tilings a column of height n with tiles of height 1 or 2 such that bottom most tile is of height 1. For example, possible tiling configurations for $\mathcal{FT}_i$ for $i \leq 4$ are shown in

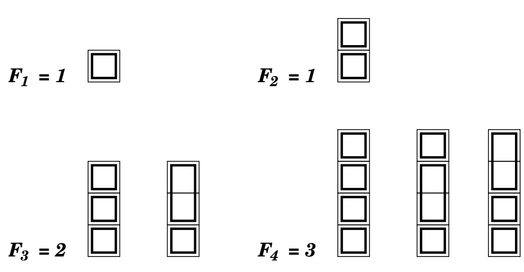

Figure 1: The tilings counted by F_i for $1 \leq i \leq 4$.

For each tiling $T \in \mathcal{FT}_n$, we let $\text{one}(T)$ is the number of tiles of height 1 in T and $\text{two}(T)$ is the number of tiles of height 2 in T and define

$$F_n(p, q) = \sum_{T \in \mathcal{FT}_n} q^{\text{one}(T)} p^{\text{two}(T)}. \tag{1}$$

It is easy to see that $F_1(p,q) = q$, $F_2(p,q) = q^2$ and $F_n(p,q) = qF_{n-1}(p,q)+pF_{n-2}(p,q)$ for $n \geq 2$ so that $F_n(1,1) = F_n$. We then define the p,q-Fibo-falling factorial basis $\{(x)_{\downarrow_{F,p,q,n}} : n \geq 0\}$ and the p,q-Fibo-rising factorial basis $\{(x)_{\uparrow_{F,p,q,n}} : n \geq 0\}$ by setting $(x)_{\downarrow_{F,p,q,0}} = (x)_{\uparrow_{F,p,q,0}} = 1$ and setting

$$(x)_{\downarrow_{F,p,q,k}} = x(x - F_1(p,q)) \cdots (x - F_{k-1}(p,q)) \text{ and} \tag{2}$$

$$(x)_{\uparrow_{F,p,q,k}} = x(x + F_1(p,q)) \cdots (x + F_{k-1}(p,q)) \tag{3}$$

for $k \geq 1$.

Our idea to define p,q-Fibonacci analogues of the Stirling numbers of the first kind, $\mathbf{sf}_{n,k}(p,q)$, the Stirling numbers of the second kind, $\mathbf{Sf}_{n,k}(p,q)$, and the Lah numbers, $\mathbf{Lf}_{n,k}(p,q)$, is to define them to be the connection coefficients between the usual power basis $\{x^n : n \geq 0\}$ and the p,q-Fibo-rising factorial and p,q-Fibo-falling factorial bases. That is, we define $\mathbf{sf}_{n,k}(p,q)$, $\mathbf{Sf}_{n,k}(p,q)$, and $\mathbf{Lf}_{n,k}(p,q)$ by the equations

$$(x)_{\downarrow_{F,p,q,n}} = \sum_{k=1}^{n} \mathbf{sf}_{n,k}(p,q)\ x^k, \tag{4}$$

$$x^n = \sum_{k=1}^{n} \mathbf{Sf}_{n,k}(p,q)\ (x)_{\downarrow_{F,p,q,k}}, \tag{5}$$

and

$$(x)_{\uparrow_{F,p,q,n}} = \sum_{k=1}^{n} \mathbf{Lf}_{n,k}(p,q)\ (x)_{\downarrow_{F,p,q,k}} \tag{6}$$

for all $n \geq 0$.

It is easy to see that these equations imply simple recursions for the connection coefficients $\mathbf{sf}_{n,k}(p,q)$s, $\mathbf{Sf}_{n,k}(p,q)$s, and $\mathbf{Lf}_{n,k}(p,q)$s. That is, the $\mathbf{sf}_{n,k}(p,q)$s can be defined by the recursions

$$\mathbf{sf}_{n+1,k}(p,q) = \mathbf{sf}_{n,k-1}(p,q) - F_n(p,q)\mathbf{sf}_{n,k}(p,q) \tag{7}$$

plus the boundary conditions $\mathbf{sf}_{0,0}(p,q) = 1$ and $\mathbf{sf}_{n,k}(p,q) = 0$ if $k > n$ or $k < 0$. The $\mathbf{Sf}_{n,k}(p,q)$s can be defined by the recursions

$$\mathbf{Sf}_{n+1,k}(p,q) = \mathbf{Sf}_{n,k-1}(p,q) + F_k(p,q)\mathbf{Sf}_{n,k}(p,q) \tag{8}$$

plus the boundary conditions $\mathbf{Sf}_{0,0}(p,q) = 1$ and $\mathbf{Sf}_{n,k}(p,q) = 0$ if $k > n$ or $k < 0$. The $\mathbf{Lf}_{n,k}(p,q)$s can be defined by the recursions

$$\mathbf{Lf}_{n+1,k}(p,q) = \mathbf{Lf}_{n,k-1}(p,q) + (F_k(p,q) + F_n(p,q))\mathbf{Lf}_{n,k}(p,q) \tag{9}$$

plus the boundary conditions $\mathbf{Lf}_{0,0}(p,q) = 1$ and $\mathbf{Lf}_{n,k}(p,q) = 0$ if $k > n$ or $k < 0$. If we define $\mathbf{cf}_{n,k}(p,q) := (-1)^{n-k}\mathbf{sf}_{n,k}(p,q)$, then $\mathbf{cf}_{n,k}(p,q)$s can be defined by the recursions

$$\mathbf{cf}_{n+1,k}(p,q) = \mathbf{cf}_{n,k-1}(p,q) + F_n(p,q)\mathbf{cf}_{n,k}(p,q) \tag{10}$$

plus the boundary conditions $\mathbf{cf}_{0,0}(p,q) = 1$ and $\mathbf{cf}_{n,k}(p,q) = 0$ if $k > n$ or $k < 0$. It also follows that

$$(x)_{\uparrow_{F,p,q,n}} = \sum_{k=1}^{n} \mathbf{cf}_{n,k}(p,q)\ x^k. \tag{11}$$

The goal of this paper is to develop a combinatorial model for the Fibo-Stirling numbers $\mathbf{Sf}_{n,k}$ and $\mathbf{cf}_{n,k}$. Our combinatorial model is a modification of the rook theory model for $S_{n,k}$ and $c_{n,k}$ except that we replace rooks by Fibonacci tilings. We will show that we can use this model to give combinatorial proofs of the recursions (10) and (8) and defining equations (11) and (5) as well as a combinatorial proof of the fact that the infinite matrices $||\mathbf{Sf}_{n,k}||_{n,k\geq 0}$ and $||\mathbf{sf}_{n,k}||_{n,k\geq 0}$ are inverses of each other. There is also a rook theory model for the $\mathbf{Lf}_{n,k}(p,q)$s, but it is significantly different from the rook theory model for the $\mathbf{Sf}_{n,k}(p,q)$s and the $\mathbf{cf}_{n,k}(p,q)$s. Thus we will give such a model in a subsequent paper. We should also note that there is a more general rook theory model which can be used to give combinatorial interpretations for the coefficients $\mathbf{sf}_{n,k}(1,1)$, $\mathbf{Sf}_{n,k}(1,1)$, $\mathbf{Lf}_{n,k}(1,1)$, and $\mathbf{cf}_{n,k}(1,1)$ due to Miceli and the third author [7]. However, that model does not easily adapt to give a rook theory model for the coefficients $\mathbf{sf}_{n,k}(p,q)$, $\mathbf{Sf}_{n,k}(p,q)$, $\mathbf{Lf}_{n,k}(p,q)$, and $\mathbf{cf}_{n,k}(p,q)$.

The outline of this paper is as follows. In Section 2, we shall give a general rook theory model of tiling placements on Ferrers boards B. In the special case where B is the board whose column heights are $0,1,\ldots,n-1$, reading from left to right, our rook theory model will give us combinatorial interpretations for the $\mathbf{Sf}_{n,k}(p,q)$s and the $\mathbf{cf}_{n,k}(p,q)$s. We shall develop general recursions for the analogue of file and rook numbers in this model which will specialize to give combinatorial proofs of the recursions (10) and (8). Similarly, we shall give combinatorial proofs of two general product formulas in this model which will specialize to give combinatorial proofs of (11) and (5). In Section 3, we shall give a combinatorial proof that the infinite matrices $||\mathbf{Sf}_{n,k}(p,q)||_{n,k\geq 0}$ and $||\mathbf{sf}_{n,k}(p,q)||_{n,k\geq 0}$ are inverses of each other. In Section 4, we shall give various generating functions and identities for the $\mathbf{Sf}_{n,k}(p,q)$s and the $\mathbf{cf}_{n,k}(p,q)$s.

2 A rook theory model for the $\mathbf{Sf}_{n,k}(p,q)$s and the $\mathbf{cf}_{n,k}(p,q)$s.

In this section, we shall develop a new type of rook theory model which is appropriate to interpret the $\mathbf{Sf}_{n,k}(p,q)$s and the $\mathbf{cf}_{n,k}(p,q)$s. A Ferrers board $B=F(b_1,\ldots,b_n)$ is a board whose column heights are $b_1,\ldots,b_n$, reading from left to right, such that $0\leq b_1\leq b_2\leq\cdots\leq b_n$. We shall let B_n denote the Ferrers board $F(0,1,\ldots,n-1)$. For example, the Ferrers board $B=F(2,2,3,5)$ is pictured on the left of Figure 2 and the Ferrers board B_4 is pictured on the right of Figure 2.

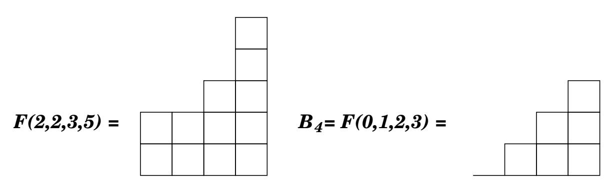

Figure 2: Ferrers boards.

Classically, there are two type of rook placements that we consider on a Ferrers board B. First we let $\mathcal{N}_k(B)$ be the set of all placements of k rooks in B such that no two rooks lie in the same row or column. We shall call an element of $\mathcal{N}_k(B)$ a placement of k non-attacking rooks in B or just a rook placement for short. We let $\mathcal{F}_k(B)$ be the set of all placements of k rooks in B such that no two rooks lie in the same column. We shall call an element of $\mathcal{F}_k(B)$ a file placement of k rooks in B. Thus file placements differ from rook placements in that file placements allow two rooks to be in the same row. For example, we exhibit a placement of 3 non-attacking rooks in $F(2,2,3,5)$ on the left in Figure 3 and a file placement of 3 rooks on the right in Figure 3.

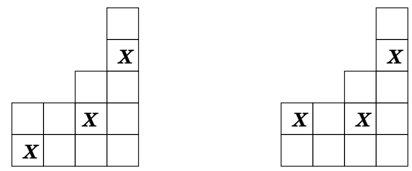

Figure 3: Examples of rook and file placements.

Given a Ferrers board $B = F(b_1,\ldots,b_n)$, we define the k-th rook number of B to be $r_k(B) = |\mathcal{N}_k(B)|$ and the k-th file number of B to be $f_k(B) = |\mathcal{F}_k(B)|$. Then the rook theory interpretation of the classical Stirling numbers is

$$\begin{aligned} S_{n,k} &= r_{n-k}(B_n) \text{ for all } 1 \leq k \leq n \text{ and} \\ c_{n,k} &= f_{n-k}(B_n) \text{ for all } 1 \leq k \leq n. \end{aligned}$$

Our idea is to modify the sets $\mathcal{N}_k(B)$ and $\mathcal{F}_k(B)$ to replace rooks with Fibonacci tilings. The analogue of file placements is very straightforward. That is, if $B = F(b_1,\ldots,b_n)$, then we let $\mathcal{FT}_k(B)$ denote the set of all configurations such that there are k columns $(i_1,\ldots,i_k)$ of B where $1 \leq i_1 < \cdots < i_k \leq n$ such that in each column i_j, we have placed one of the tilings $T_{i,j}$ for the Fibonacci number $F_{b_{i_j}}$. We shall call such a configuration a Fibonacci file placement and denote it by

$$P = ((c_{i_1}, T_{i_1}), \ldots, (c_{i_k}, T_{i_k})).$$

Let one(P) denote the number of tiles of height 1 that appear in P and two(P) denote the number of tiles of height 2 that appear in P. We then define the weight of P, $WF(P,p,q)$, to be $q^{\mathrm{one}(P)}p^{\mathrm{two}(P)}$. For example, we have pictured an element P of $\mathcal{FT}_3(F(2,3,4,4,5))$ in Figure 4 whose weight is q^7p^2.

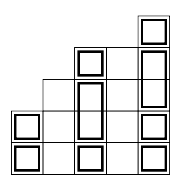

Figure 4: A Fibonacci file placement.

We define the k-th p,q-Fibonacci file polynomial of B, $\mathbf{fT}_k(B,p,q)$, by setting

$$\mathbf{fT}_k(B,p,q) = \sum_{P \in \mathcal{FT}_k(B)} WF(P,p,q). \tag{12}$$

If $k = 0$, then the only element of $\mathcal{FT}_k(B)$ is the empty placement whose weight by definition is 1.

Then we have the following two theorems concerning Fibonacci file placements in Ferrers boards.

Theorem 1. *Let $B = F(b_1, \ldots, b_n)$ be a Ferrers board where $0 \leq b_1 \leq \cdots \leq b_n$ and $b_n > 0$. Let $B^- = F(b_1, \ldots, b_{n-1})$. Then for all $1 \leq k \leq n$,*

$$\mathbf{fT}_k(B, p, q) = \mathbf{fT}_k(B^-, p, q) + F_{b_n}(p, q)\mathbf{fT}_{k-1}(B^-, p, q). \tag{13}$$

Proof. It is easy to see that the right-hand side (13) is just the result of classifying the Fibonacci file placements P in $\mathcal{FT}_k(B)$ by whether there is a tiling in the last column. If there is no tiling in the last column of P, then removing the last column of P produces an element of $\mathcal{FT}_k(B^-)$. Thus such placements contribute $\mathbf{fT}_k(B^-, p, q)$ to $\mathbf{fT}_k(B, p, q)$. If there is a tiling in the last column, then the Fibonacci file placement that results by removing the last column is an element of $\mathcal{FT}_{k-1}(B^-)$ and the sum of the weights of the possible Fibonacci tilings of height b_n for the last column is $F_{b_n}(p, q)$. Hence such placements contribute $F_{b_n}(p, q)\mathbf{fT}_{k-1}(B^-, p, q)$ to $\mathbf{fT}_k(B, p, q)$. □

If $B = F(b_1, \ldots, b_n)$ is a Ferrers board, then we let B_x denote the board that results by adding x rows of length n below B. We label these rows from top to bottom with the numbers $1, 2, \ldots, x$. We shall call the line that separates B from these x rows the *bar*. A mixed file placement P on the board B_x consists of picking for each column b_i either (i) a Fibonacci tiling T_i of height b_i above the bar or (ii) picking a row j below the bar to place a rook in the cell in row j and column i. Let $\mathcal{M}_n(B_x)$ denote set of all mixed rook placements on B. For any $P \in \mathcal{M}_n(B_x)$, we let $\text{one}(P)$ denote the number of tiles of height 1 that appear in P and $\text{two}(P)$ denote the set tiles of height 2 that appear in P. We then define the weight of P, $WF(P, p, q)$, to be $q^{\text{one}(P)}p^{\text{two}(P)}$. For example, Figure 5 pictures a mixed placement P in B_x where $B = F(2, 3, 4, 4, 5, 5)$ and x is 9 such that $WF(P, p, q) = q^7p^2$.

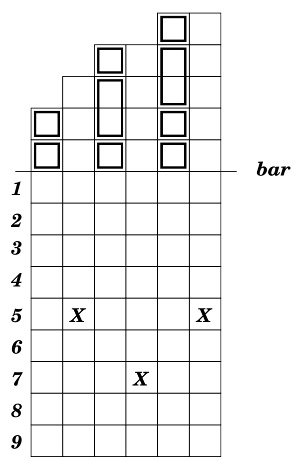

Figure 5: A mixed file placement.

Our next theorem results from counting $\sum_{P \in \mathcal{M}_n(B_x)} WF(P, p, q)$ in two different ways.

Theorem 2. *Let $B = F(b_1, \ldots, b_n)$ be a Ferrers board where $0 \leq b_1 \leq \cdots \leq b_n$.*

$$(x + F_{b_1}(p, q))(x + F_{b_2}(p, q)) \cdots (x + F_{b_n}(p, q)) = \sum_{k=0}^{n} \mathbf{fT}_k(B, p, q)x^{n-k}. \tag{14}$$

Proof. Since both sides of (14) are polynomials of degree n, it is enough to show that there are $n+1$ different values of x for which the two sides are equal. In fact, we will show that the two sides are equal for any positive integer x.

Thus fix x to be a positive integer and consider the sum $S = \sum_{P \in \mathcal{M}_n(B_x)} WF(P,p,q)$. It is clear that each column of b_i of B contributes a factor of $x + F_{b_i}(p,q)$ to S so that

$$S = \prod_{i=1}^{n} (x + F_{b_i}(p,q)).$$

On the other hand, suppose that we fix a Fibonacci file placement $P \in \mathcal{FT}_k(B)$. Then we want to compute $S_P = \sum_{Q \in \mathcal{M}_n(B), Q \cap B = P} WF(Q,p,q)$ which is the sum of $WF(Q,p,q)$ over all mixed placements Q such that Q intersect B equals P. It it easy to see that such a Q arises by choosing a rook to be placed below the bar for each column that does not contain a tiling. Since there are x^{n-k} ways to do this, it follows that $S_P = WF(P,p,q)x^{n-k}$. Hence it follows that

$$\begin{aligned} S &= \sum_{k=0}^{n} \sum_{P \in \mathcal{FT}_k(B)} S_P \\ &= \sum_{k=0}^{n} x^{n-k} \sum_{P \in \mathcal{FT}_k(B)} WF(P,p,q) \\ &= \sum_{k=0}^{n} \mathbf{fT}_k(B,p,q)\ x^{n-k}. \end{aligned}$$

□

We should note that neither the proof of Theorem 1 nor 2 depended on the fact that $b_1 \leq b_2 \leq \ldots \leq b_n$. Thus they hold for arbitrary sequences of non-negative integers $(b_1, \ldots, b_n)$.

Now consider the special case of the previous two theorems when $B_n = F(0,1,2,\ldots,n-1)$. Then (13) implies that

$$\mathbf{fT}_{n+1-k}(B_{n+1},p,q) = \mathbf{fT}_{n+1-k}(B_n,p,q) + F_n(p,q)\mathbf{fT}_{n-k}(B_n,p,q).$$

It then easily follows that for all $0 \leq k \leq n$,

$$\mathbf{cf}_{n,k}(p,q) = \mathbf{fT}_{n-k}(B_n,p,q). \tag{15}$$

Note that $\mathbf{cf}_{n,0}(p,q) = 0$ for all $n \geq 1$ since there are no Fibonacci file placements in $\mathcal{FT}_n(B_n)$ since there are only $n-1$ non-zero columns. Moreover such a situation, we see that (13) implies that

$$x(x + F_1(p,q))(x + F_2(p,q)) \cdots (x + F_{n-1}(p,q)) = \sum_{k=1}^{n} \mathbf{cf}_{n,k}(p,q)\ x^k.$$

Thus we have given a combinatorial proof of (11).

Our Fibonacci analogue of rook placements is a slight variation of Fibonacci file placements. The main difference is that each tiling will cancel some of the top most cells in each column to its right that has not been canceled by a tiling which is further to the left. Our goal is to ensure that if we start with a Ferrers board $B = F(b_1, \ldots, b_n)$, our cancellation will ensure that the number of uncanceled cells in the empty columns are $b_1, \ldots, b_{n-k}$, reading from left to right. That is, if $B = F(b_1, \ldots, b_n)$, then we let $\mathcal{NT}_k(B)$ denote the set of all configurations such that that there are k columns $(i_1, \ldots, i_k)$ of B where $1 \leq i_1 < \cdots < i_k \leq n$ such that the following conditions hold.

1. In column c_{i_1}, we place a Fibonacci tiling $T_{i,1}$ of height b_{i_1} and for each $j > i_1$, this tiling cancels the top $b_j - b_{j-1}$ cells at the top of column j. This cancellation has the effect of ensuring that the number of uncanceled cells in the columns without tilings at this point is $b_1, \ldots, b_{n-1}$, reading from left to right.

2. In column c_{i_2}, our cancellation due to the tiling in column i_1 ensures that there are b_{i_2-1} uncanceled cells in column i_2. Then we place a Fibonacci tiling $T_{i,2}$ of height b_{i_2-1} and for each $j > i_2$, we cancel the top $b_{j-1} - b_{j-2}$ cells in column j that has not been canceled by the tiling in column i_1. This cancellation has the effect of ensuring that the number of uncanceled cells in the columns without tilings at this point is $b_1, \ldots, b_{n-2}$, reading from left to right.

3. In general, when we reach column i_s, we assume that the cancellation due to the tilings in columns $i_1, \ldots, i_{j-1}$ ensure that the number of uncanceled cells in the columns without tilings is $b_1, \ldots, b_{n-(s-1)}$, reading from left to right. Thus there will be $b_{i_s-(s-1)}$ uncanceled cells in column i_s at this point. Then we place a Fibonacci tiling $T_{i,s}$ of height $b_{i_s-(s-1)}$ and for each $j > i_s$, this tiling will cancel the top $b_{j-(s-1)} - b_{j-s}$ cells in column j that has not been canceled by the tilings in columns $i_1, \ldots, i_{s-1}$. This cancellation has the effect of ensuring that the number of uncanceled cells in columns without tilings at this point is $b_1, \ldots, b_{n-s}$, reading from left to right.

We shall call such a configuration a Fibonacci rook placement and denote it by

$$P = ((c_{i_1}, T_{i_1}), \ldots, (c_{i_k}, T_{i_k})).$$

Let one(P) denote the number of tiles of height 1 that appear in P and two(P) denote the number of tiles of height 2 that appear in P. We then define the weight of P, $WF(P,p,q)$, to be $q^{\mathrm{one}(P)}p^{\mathrm{two}(P)}$. For example, on the left in Figure 6, we have pictured an element P of $\mathcal{NT}_3(F(2,3,4,4,6,6))$ whose weight is q^4p^2. In Figure 6, we have indicated the cells canceled by the tiling in column i by placing an i in the cell. We note in the special case where $B = F(0,k,2k,\ldots,(n-1)k)$, then our cancellation scheme is quite simple. That is, each tiling just cancels the top k cells in each column to its right which has not been canceled by tilings to its left. For example, on the right in Figure 6, we have pictured an element P of $\mathcal{NT}_3(F(0,1,2,3,4,5))$ whose weight is q^6p. Again, we have indicated the canceled cells by the tiling in column i by placing an i in the cell.

We define the k-th p,q-Fibonacci rook polynomial of B, $\mathbf{rT}_k(B,p,q)$, by setting

$$\mathbf{rT}_k(B,p,q) = \sum_{P \in \mathcal{NT}_k(B)} WF(P,p,q). \tag{16}$$

If $k = 0$, then the only element of $\mathcal{FT}_k(B)$ is the empty placement whose weight by definition is 1.

Then we have the following two theorems concerning Fibonacci rook placements in Ferrers boards.

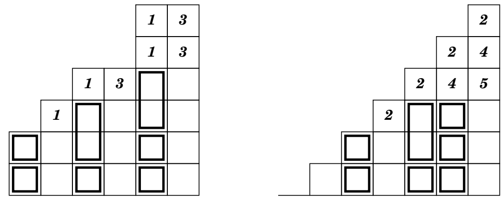

Figure 6: A Fibonacci rook placement.

Theorem 3. *Let $B = F(b_1, \ldots, b_n)$ be a Ferrers board where $0 \leq b_1 \leq \cdots \leq b_n$ and $b_n > 0$. Let $B^- = F(b_1, \ldots, b_{n-1})$. Then for all $1 \leq k \leq n$,*

$$\mathbf{rT}_k(B, p, q) = \mathbf{rT}_k(B^-, p, q) + F_{b_{n-(k-1)}}(p, q)\mathbf{rT}_{k-1}(B^-, p, q). \tag{17}$$

Proof. It is easy to see that the right-hand side (17) is just the result of classifying the Fibonacci rook placements P in $\mathcal{NT}_k(B)$ by whether there is a tiling in the last column. If there is no tiling in the last column of P, then removing the last column of P gives an element of $\mathcal{NT}_k(B^-)$. Thus such placements contribute $\mathbf{rT}_k(B^-, p, q)$ to $\mathbf{rT}_k(B, p, q)$. If there is a tiling in the last column, then the Fibonacci rook placement that results by removing the last column is an element of $\mathcal{NT}_{k-1}(B^-)$ and these tilings cancel the top $b_n - b_{n-(k-1)}$ cells of the last column. Then the weights of the possible Fibonacci tilings of height $b_{n-(k-1)}$ for the last column is $F_{b_{n-(k-1)}}(p, q)$. Hence such placements contribute $F_{b_{n-(k-1)}}(p, q)\mathbf{rT}_{k-1}(B^-, p, q)$ to $\mathbf{rT}_k(B, p, q)$. $\square$

We also have a product formula for Fibonacci rook placements in B_n. In this case, we have to use ideas from the proof of an even more general product formula due to Miceli and third author in [7].

Let $B = F(b_1, \ldots, b_n)$ be a Ferrers board and x be a positive integer. Then we let $AugB_x$ denote the board where we start with B_x and add the flip of the board B about its baseline below the board. We shall call the the line that separates B from these x rows the *upper bar* and the line that separates the x rows from the flip of B added below the x rows the *lower bar*. We shall call the flipped version of B added below B_x the board $\overline{B}$. For example, if $B = F(2, 3, 4, 4, 5, 5)$, then the board $AugB_7$ is pictured in Figure 7.

The analogue of mixed placements in $AugB_x$ are more complex than the mixed placements for B_x. We process the columns from left to right. If we are in column 1, then we can do one of the following three things.

i. We can put a Fibonacci tiling in cells in column b_1 in B. Then we must cancel the top-most cells in each of the columns in B to its right so that the number of uncanceled cells in the columns to its right are $b_1, b_2, \ldots, b_{n-1}$, respectively, as we read from left to right. This means that we will cancel $b_i - b_{i-1}$ at the top of column i in B for $i = 2, \ldots, n$. We also cancel the same number of cells at the bottom of the corresponding columns of $\overline{B}$.

ii. We can place a rook in any row of column 1 that lies between the upper bar and lower bar. This rook will not cancel anything.

iii. We can put a flip of Fibonacci tiling in column b_1 of $\overline{B}$. This tiling will not cancel anything.

Next assume that when we get to column j, the number of uncanceled cells in the columns that have no tilings in B and $\overline{B}$ are $b_1, \ldots, b_k$ for some k as we read from left to right. Suppose there are b_i uncanceled cells in B in column j. Then we can do one of three things.

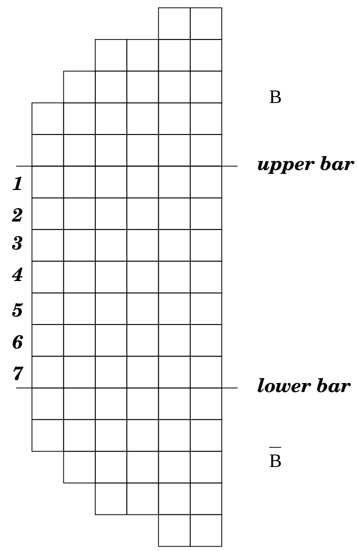

Figure 7: An example of an augmented board $AugB_x$..

i. We can put a Fibonacci tiling of height b_i in the uncanceled cells in column j in B. Then we must cancel top-most cells of the columns in B to its right so that the number of uncanceled cells in the columns which have no tilings up to this point are $b_1, b_2, \ldots, b_{k-1}$, We also cancel the same number of cells at the bottom of the corresponding columns of $\overline{B}$

ii. We can place a rook in any row of column j that lies between the upper bar and lower bar. This rook will not cancel anything.

iii. We can put a flip of Fibonacci tiling in the b_i uncanceled cells in column b_j of $\overline{B}$. This tiling will not cancel anything

We let $\mathcal{M}_n(AugB_x)$ denote set of all mixed rook placements on $AugB_x$. For any $P \in \mathcal{M}_n(AugB_x)$, we let $\text{one}_B(P)$ denote the number of tiles of height 1 that appear in P that lie in B, $\text{two}_B(P)$ denote the number of tiles of height 2 that appear in P that lie in B, $\text{one}_{\overline{B}}(P)$ denote the number of tiles of height 1 that appear in P that lie in $\overline{B}$, and $\text{two}_{\overline{B}}(P)$ denote the number of tiles of height 2 that appear in P that lie in $\overline{B}$. We then define the weight of P, $\overline{WF}(P,p,q)$ to be $q^{\text{one}_B(P)} p^{\text{two}_B(P)} - q^{\text{one}_{\overline{B}}(P)} p^{\text{two}_{\overline{B}}(P)}$. For example, Figure 8 pictures a mixed placement P in $AugB_x$ where $B = F(2,3,4,4,5,5)$ and x is 7 such that $\overline{WF}(P,p,q) = q^3p^2 - q^2$. In this case we have put 2s in the cells that are canceled by the tiling in B in column 2 and 4s in the cells that are canceled by the tiling in B in column 4. Note that if we process the columns from left to right, after we have placed the tiling in column 2, the number of uncanceled cells in the columns which do not have tiling above the upper bar are 2,3,4,4,5 as we read from left to right in both B and $\overline{B}$ and after we have placed the tiling in column 4, the number of uncanceled cells in the columns which do not have tilings above the upper bar are 2,3,4,4 as we read from left to right in both B and $\overline{B}$.

Our next theorem results from counting $\sum_{P \in \mathcal{M}_n(AugB_x)} \overline{WF}(P,p,q)$ in two different ways.

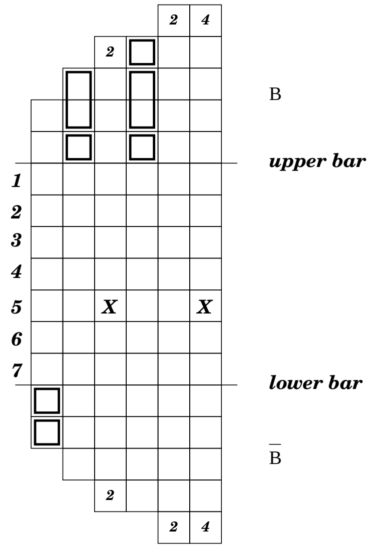

Figure 8: A mixed rook placement.

Theorem 4. *Let $B = F(b_1, \ldots, b_n)$ be a Ferrers board where $0 \leq b_1 \leq \cdots \leq b_n$.*

$$x^n = \sum_{k=0}^{n} \mathbf{rT}_{n-k}(B,p,q)(x - F_{b_1}(p,q))(x - F_{b_2}(p,q)) \cdots (x - F_{b_k}(p,q)). \tag{18}$$

Proof. Since both sides of (18) are polynomials of degree n, it is enough to show that there are $n+1$ different values of x for which the two sides are equal. In fact, we will show that the two sides are equal for any positive integer x.

Thus, fix x to be a positive integer and consider the sum $S = \sum_{P \in \mathcal{M}_n(AugB_x)} \overline{WF}(P,p,q)$. First we consider the contribution of each column as we proceed from left to right. Given our three choices in column 1, the contribution of our choices of the tilings of height b_1 that we can place in column 1 of B is $F_{b_1}(p,q)$, the contribution of our choices of placing a rook in between the upper bar and the lower is x, and the contribution of our choices of the tilings of height b_1 that we can place in column 1 of $\overline{B}$ is $-F_{b_1}(p,q)$. Thus the contribution of our choices in column 1 to S is $F_{b_1}(p,q) + x - F_{b_1}(p,q) = x$.

In general, after we have processed our choices in the first j columns, our cancellation scheme ensures that the number of uncanceled cells in B and $\overline{B}$ in the j-th column is b_i for some $i \leq j$. Thus given our three choices in column j, the contribution of our choices of the tilings of height b_i that we can place in column j of B is $F_{b_i}(p,q)$, our choices of placing a rook in between the upper bar and the lower is x, and the contribution of our choices of the tilings of height b_i that we can place in column j of $\overline{B}$ is $-F_{b_i}(p,q)$. Thus the contribution of our choices in column j to S is $F_{b_i}(p,q) + x - F_{b_i}(p,q) = x$. It follows that $S = x^n$.

On the other hand, suppose that we fix a Fibonacci rook placement $P \in \mathcal{NT}_{n-k}(B)$. Then we want to compute the $S_P = \sum_{Q \in \mathcal{M}_n(AugB_x), Q \cap B = P} \overline{WF}(Q,p,q)$ which is the sum of $\overline{WF}(Q,p,q)$ over all mixed placements Q such that Q intersect B equals P. Our cancellation scheme ensures that the number of uncanceled cells in B and $\overline{B}$ in the k columns that do not contain tilings in P is $b_1, \ldots, b_k$ as we read from right to left. For each such $1 \leq i \leq k$, the factor that arises from either choosing a rook to be placed in between the upper bar and lower bar or a flipped

Fibonacci tiling of height b_i in $\overline{B}$ is $x - F_{b_i}(p,q)$. It follows that

$$S_P = WF(P,p,q)\prod_{i=1}^{k}(x - F_{b_i}(p,q)).$$

Hence it follows that

$$\begin{aligned} S &= \sum_{k=0}^{n} \sum_{P \in \mathcal{NT}_{n-k}(B)} S_P \\ &= \sum_{k=0}^{n} \left(\prod_{i=1}^{k}(x - F_{b_i}(p,q))\right) \sum_{P \in \mathcal{NT}_k(B)} WF(P,p,q) \\ &= \sum_{k=0}^{n} \mathbf{rT}_{n-k}(B,p,q)\left(\prod_{i=1}^{k}(x - F_{b_i}(p,q))\right). \end{aligned}$$

□

Now consider the special case of the previous two theorems when $B_n = F(0,1,2,\ldots,n-1)$. Then (17) implies that

$$\mathbf{rT}_{n+1-k}(B_{n+1},p,q) = \mathbf{rT}_{n+1-k}(B_n,p,q) + F_k(p,q)\mathbf{rT}_{n-k}(B_n,p,q).$$

It then easily follows that for all $0 \le k \le n$,

$$\mathbf{Sf}_{n,k}(p,q) = \mathbf{rT}_{n-k}(B_n,p,q). \tag{19}$$

Note that $\mathbf{Sf}_{n,0}(p,q) = 0$ for all $n \geq 1$ since there are no Fibonacci rook placements in $\mathcal{NT}_n(B_n)$ since there are only $n-1$ non-zero columns. Moreover such a situation, we see that (18) implies that

$$x^n = \sum_{k=1}^{n} \mathbf{Sf}_{n,k}(p,q)x(x - F_1(p,q))(x - F_2(p,q))\cdots(x - F_{k-1}(p,q))$$

Thus we have given a combinatorial proof of (5).

3 A combinatorial proof that $||\mathbf{Sf}_{n,k}(p,q)||^{-1} = ||\mathbf{sf}_{n,k}(p,q)||$.

In this section, we shall give a combinatorial proof that the infinite matrices $||\mathbf{Sf}_{n,k}(p,q)||_{n,k\geq 0}$ and $||\mathbf{sf}_{n,k}(p,q)||_{n,k\geq 0}$ are inverses of each other.

Since the matrices $||\mathbf{Sf}_{n,k}(p,q)||_{n,k\geq 0}$ and $||\mathbf{sf}_{n,k}(p,q)||_{n,k\geq 0}$ are lower triangular, we must prove that for any $0 \le k \le n$,

$$\sum_{j=k}^{n} \mathbf{Sf}_{n,j}(p,q)\mathbf{sf}_{j,k}(p,q) = \chi(n=k) \tag{20}$$

where, for a statement A, we let $\chi(A) = 1$ if A is true and $\chi(A) = 0$ if A is false. Given our combinatorial interpretation of $\mathbf{Sf}_{n,j}(p,q)$ and $\mathbf{sf}_{j,k}(p,q)$, we must show that

$$\sum_{j=k}^{n} \sum_{(P,Q) \in \mathcal{NT}_{n-j}(B_n) \times \mathcal{FT}_{j-k}(B_j)} (-1)^{j-k} WF(P,p,q)WF(Q,p,q) = \chi(n=k). \tag{21}$$

Note that if $(P,Q) \in \mathcal{NT}_{n-j}(B_n) \times \mathcal{FT}_{j-k}(B_j)$, then the sign associated with (P,Q) is just $(-1)^{\text{no. of rooks in } Q}$ so that we define $sgn(P,Q) = (-1)^{\text{no. of rooks in } Q}$.

In the case when $n = k$, (21) reduces to the fact that

$$1 = \sum_{(P,Q) \in \mathcal{NT}_{n-n}(B_n) \times \mathcal{FT}_{n-n}(B_n)} (-1)^{n-n} WF(P,p,q) WF(Q,p,q).$$

This is clear since the only P in $\mathcal{NT}_{n-n}(B_n)$ is the empty configuration and $WF(P,p,q) = 1$ and the only Q in $\mathcal{FT}_{n-n}(B_n)$ is the empty configuration and $WF(Q,p,q) = 1$. Moreover in such a case $sgn(P,Q) = 1$.

If $n \geq 1$ and $k = 0$, the result is also immediate. In that case our sum becomes

$$\sum_{j=0}^{n} \sum_{(P,Q) \in \mathcal{NT}_{n-j}(B_n) \times \mathcal{FT}_{j-0}(B_j)} (-1)^{n-n} WF(P,p,q) WF(Q,p,q).$$

However, for all $j \geq 1$, $\mathcal{FT}_j(B_j)$ is empty because you can place at most $j-1$ file tilings on B_j. In the case when $j = 0$, then $\mathcal{NT}_{n-0}(B_n)$ is empty so that the entire sum is empty. Hence for all $n \geq 1$,

$$\sum_{j=0}^{n} \mathbf{Sf}_{n,j}(p,q) \mathbf{sf}_{j,0}(p,q) = 0.$$

Thus we can assume that $n > k \geq 1$. Our goal is to define an involution

$$I_{n,k} : \bigcup_{j=k}^{n} \mathcal{NT}_{n-j}(B_n) \times \mathcal{FT}_{j-k}(B_j) \to \bigcup_{j=k}^{n} \mathcal{NT}_{n-j}(B_n) \times \mathcal{FT}_{j-k}(B_j)$$

such that for all for $(P,Q) \in \bigcup_{j=k}^{n} \mathcal{NT}_{n-j}(B_n) \times \mathcal{FT}_{j-k}(B_j)$, $I_{n,k}(P,Q) = (P',Q') \neq (P,Q)$, $sgn(P,Q) = -sgn(P',Q')$, and $WF(P,p,q)WF(Q,p,q) = WF(P',p,q)WF(Q',p,q)$.

We shall proceed by induction on n. The base case of our induction is $n = 2$ and $k = 1$. In that case (21) becomes

$$\sum_{j=1}^{2} \sum_{(P,Q) \in \mathcal{NT}_{2-j}(B_2) \times \mathcal{FT}_{j-1}(B_j)} (-1)^{j-1} WF(P,p,q) WF(Q,p,q).$$

However, in the case $j = 2$, there is a single pair in $\mathcal{NT}_{2-1}(B_2) \times \mathcal{FT}_{1-1}(B_1)$ which is pictured on the left in Figure 9 and there is a single pair in $\mathcal{NT}_{2-2}(B_2) \times \mathcal{FT}_{2-1}(B_2)$ which is pictured on the right in Figure 9. These two pairs each have weight q but have opposite signs so that our involution $I_{2,1}$ just maps each pair to the other pair.

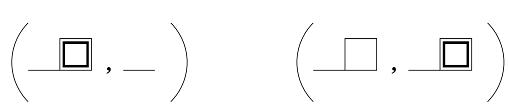

Figure 9: $I_{2,1}$.

Thus assume that $n > 2$ and $n > k \geq 1$. We define $I_{n,k}$ via 3 cases.

Case 1. $(P,Q) \in \mathcal{NT}_{n-j}(B_n) \times \mathcal{FT}_{j-k}(B_j)$ and there is a tiling in the last column of P.

In this case, there are $n-j-1$ tilings in the first $n-1$ columns of P so that there are $n-1-(n-j-1) = j$ uncanceled cells in the last column of P. Note that the last column of Q is of height $j-1$. Then we let $I_{n,k}(P,Q) = (P',Q')$ where P' arises from P by removing the tiling in the last column of P and Q' results from Q by taking the tiling in the last column of P and placing it at the end of Q. It will then be the case that $(P',Q') \in \mathcal{NT}_{n-(j+1)}(B_n) \times \mathcal{FT}_{j+1-k}(B_{j+1})$. Note that $sgn(P,Q) = (-1)^{j-k}$ and $sgn(P',Q') = (-1)^{j+1-k}$. In addition, since we did not change the total number of tiles of size 1 and 2, we have that $WF(P,p,q)WF(Q,p,q) = WF(P',p,q)WF(Q',p,q)$.

Case 2. $(P,Q) \in \mathcal{NT}_{n-j}(B_n) \times \mathcal{FT}_{j-k}(B_j)$ and there is no tiling in the last column of P but there is a tiling in the last column of Q.

In this case, there are $n-j$ tilings in the first $n-1$ columns of P so that there are $n-1-(n-j) = j-1$ uncanceled cells in the last column of P. Note that the last column of Q is of height $j-1$ in this case. Then we let $I_{n,k}(P,Q) = (P',Q')$ where P' arises from P by taking the tiling in the last column of Q and placing it in the $j-1$ uncanceled cells of the last column of P and Q' results from Q removing its last column.

It will then be the case that $(P',Q') \in \mathcal{NT}_{n-(j-1)}(B_n) \times \mathcal{FT}_{j-1-k}(B_{j-1})$. Note that $sgn(P,Q) = (-1)^{j-k}$ and $sgn(P',Q') = (-1)^{j-1-k}$. Since we did not change the total number of tiles of size 1 and 2, we have that $WF(P,p,q)WF(Q,p,q) = WF(P',p,q)WF(Q',p,q)$.

It is easy to see that if (P,Q) is in Case 1, then $I_{n,k}(P,Q)$ is in Case 2 and if (P,Q) is in Case 2, then $I_{n,k}(P,Q)$ is in Case 1. An example of these two cases is given in Figure 10 where the pair (P,Q) pictured at the top is in $\mathcal{NT}_{6-3}(B_6) \times \mathcal{FT}_{3-2}(B_3)$ and satisfies the conditions of Case 1 and the pair (P',Q') pictured at the bottom is in $\mathcal{NT}_{6-4}(B_6) \times \mathcal{FT}_{4-2}(B_4)$ and satisfies the conditions of Case 2.

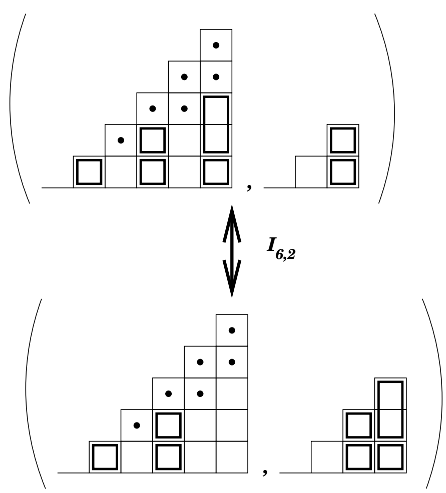

Figure 10: The involution $I_{n,k}$.

It follows that sum of $sgn(P,Q)WF(P,p,q)WF(Q,p,q)$ over all (P,Q) satisfying the conditions of Case 1 or Case 2 for some j is 0. Thus we have one last case to consider.

Case 3. $(P,Q) \in \mathcal{NT}_{n-j}(B_n) \times \mathcal{FT}_{j-k}(B_j)$ and there is no tiling in the last column of P and there is no tiling in the last column of Q.

In this case, we let (P',Q') be the result of removing the last column of both P and Q. It follows that $(P',Q') \in \mathcal{NT}_{n-1-(j-1)}(B_{n-1}) \times \mathcal{FT}_{(j-1)-(k-1)}(B_{j-1})$. It is easy to see that the map θ which sends $(P,Q) \to (P',Q')$ is a sign preserving and weight preserving bijection from the set of all $(P,Q) \in \bigcup_{j=k}^{n} \mathcal{NT}_{n-j}(B_n) \times \mathcal{FT}_{j-k}(B_j)$ which are in case 3 onto the set $\bigcup_{i=k-1}^{n-1} \mathcal{NT}_{n-1-i}(B_{n-1}) \times \mathcal{FT}_{i-(k-1)}(B_i)$. An example of the θ maps is given in Figure 11. But then we know by induction that

$$\sum_{(P',Q')\in\bigcup_{i=k-1}^{n-1} \mathcal{NT}_{n-1-i}(B_{n-1})\times\mathcal{FT}_{i-(k-1)}(B_i)} sgn(P',Q')WF(P',p,q)WF(Q',p,q) = 0.$$

This shows that (21) holds which is what we wanted to prove.

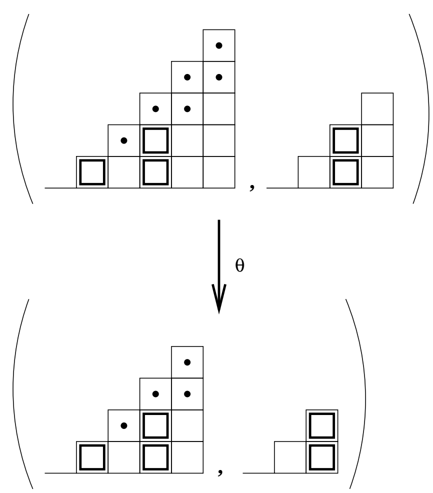

Figure 11: An example of the θ map.

Remark. It is to see that the proofs in Sections 2 and 3 did not use any particular properties of the Fibonacci tilings. Thus for example, let $\mathcal{PT}_n$ denote the set of all tilings of a column of height n with tiles of height 1, 2, or 3 such that the bottom tile is a size 1.

we could consider numbers defined $P_1 = P_2 = 1$, $P_3 = 2$ and $P_n = P_{n-1} + P_{n-2} + P_{n-3}$ for $n \geq 4$. Then it then easy to see that P_n equals the number of tilings of height n using tiles of size 1, 2, and 3 such that the bottom tile is of size 1. We will call such tilings P-tilings. Given tiling $T \in \mathcal{PT}_n$, we let one(T) denote the number of tiles of height 1 in T, two(T) denote the number of tiles of height 2 in T, and three(T) denote the number of tiles of height 3 in T. Then we let

$$P_n(p,q,r) = \sum_{T\in\mathcal{PT}_n} q^{\mathrm{one}(T)} p^{\mathrm{two}(T)} r^{\mathrm{three}(T)}.$$

For example, Figure 12 gives the set of tiles for $\mathcal{PT}_1, \ldots, \mathcal{PT}_5$.

Given any such P tiling T, we let one(T) denote the number of tiles of height 1 in T, two(T) denote the number of tiles of height 2 in T, and three(T) denote the number of tiles of height

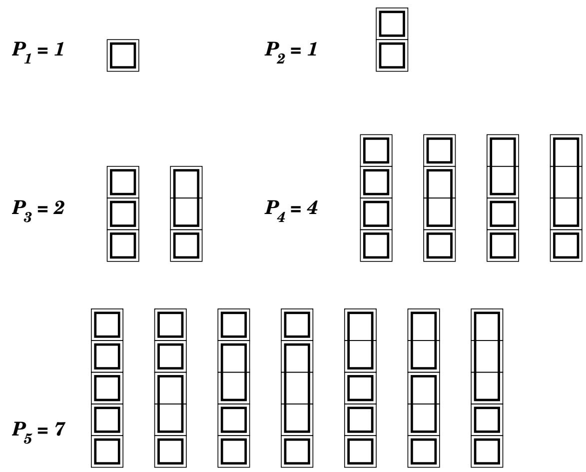

Figure 12: P-tilings for $\mathcal{PT}_1, \ldots, \mathcal{PT}_5$.

3 in T. The we can define $P_n(q,p,r)$ as the sum of weights $WP(T)$ of all P-tilings T of height n where $WP(T) = q^{\mathrm{one}(T)} p^{\mathrm{two}(T)} r^{\mathrm{three}(T)}$ over all tilings for P_n. It is then easy to see that $P_1(q,p,r) = q$, $P_2(q,p,r) = q^2$, $P_3(q,p,r) = q^3 + qp$, and

$$P_n(q,p,r) = qP_{n-1}(q,p,r) + pP_{n-2}(q,p,r) + rP_{n-3}(q,p,r)$$

for $n \geq 4$.

Then for any Ferrers board B, we can define P-analogues $\mathbf{rPT}_k(B,p,q,r)$ of the rook numbers $\mathbf{rT}_k(B,p,q)$ and P-analogues $\mathbf{fPT}_k(B,p,q,r)$ of the file numbers $\mathbf{fT}_k(B,p,q)$ exactly as before except that we replace Fibonacci tilings by P-tilings and we keep track of the number of tiles of size 1, 2, and 3 instead of keeping track of the tiles of size 1 and 2. Then we have the following analogues of Theorems 1, 2, 3, and 4 with basically the same proofs.

Theorem 5. *Let $B = F(b_1, \ldots, b_n)$ be a Ferrers board where $0 \leq b_1 \leq \cdots \leq b_n$ and $b_n > 0$. Let $B^- = F(b_1, \ldots, b_{n-1})$. Then for all $1 \leq k \leq n$,*

$$\mathbf{fPT}_k(B,p,q,r) = \mathbf{fPT}_k(B^-,p,q,r) + P_{b_n}(p,q,r)\mathbf{fPT}_{k-1}(B^-,p,q,r). \tag{22}$$

Theorem 6. *Let $B = F(b_1, \ldots, b_n)$ be a Ferrers board where $0 \leq b_1 \leq \cdots \leq b_n$.*

$$(x + P_{b_1}(p,q,r))(x + P_{b_2}(p,q,r)) \cdots (x + P_{b_n}(p,q,r)) = \sum_{k=0}^{n} \mathbf{fPT}_k(B,p,q,r)x^{n-k}. \tag{23}$$

Theorem 7. *Let $B = F(b_1, \ldots, b_n)$ be a Ferrers board where $0 \leq b_1 \leq \cdots \leq b_n$ and $b_n > 0$. Let $B^- = F(b_1, \ldots, b_{n-1})$. Then for all $1 \leq k \leq n$,*

$$\mathbf{rPT}_k(B,p,q,r) = \mathbf{rPT}_k(B^-,p,q,r) + P_{b_{n-(k-1)}}(p,q,r)\mathbf{rPT}_{k-1}(B^-,p,q,r). \tag{24}$$

Theorem 8. *Let $B = F(b_1, \ldots, b_n)$ be a Ferrers board where $0 \leq b_1 \leq \cdots \leq b_n$.*

$$x^n = \sum_{k=0}^{n} \mathbf{rPT}_{n-k}(B,p,q,r)(x - P_{b_1}(p,q,r))(x - P_{b_2}(p,q,r)) \cdots (x - P_{b_k}(p,q,r)). \tag{25}$$

In particular, if we let $\mathbf{cp}_{n,k}(p,q,r) = \mathbf{fPT}_{n-k}(B_n,p,q,r)$, then we will have that $\mathbf{cp}_{n,n}(p,q,r) = 1$ for all $n \geq 0$, $\mathbf{cp}_{n,0}(p,q,r) = 0$ for all $n \geq 1$, and

$$\mathbf{cp}_{n+1,k}(p,q,r) = \mathbf{cp}_{n,k-1}(p,q,r) + P_n(p,q,r)\mathbf{cp}_{n,k}(p,q,r)$$

for $1 \leq k \leq n+1$ and

$$x(x+P_1(p,q,r))\cdots(x+P_{n-1}(p,q,r)) = \sum_{k=1}^{n} \mathbf{cp}_{n,k}(p,q,r)\ x^k.$$

Similarly, if we let $\mathbf{Sp}_{n,k}(p,q,r) = \mathbf{rPT}_{n-k}(B_n,p,q,r)$, then we will have that $\mathbf{Sp}_{n,n}(p,q,r) = 1$ for all $n \geq 0$, $\mathbf{Sp}_{n,0}(p,q,r) = 0$ for all $n \geq 1$, and

$$\mathbf{Sp}_{n+1,k}(p,q,r) = \mathbf{Sp}_{n,k-1}(p,q,r) + P_k(p,q,r)\mathbf{Sp}_{n,k}(p,q,r)$$

for $1 \leq k \leq n+1$ and

$$x^n = \sum_{k=1}^{n} \mathbf{Sp}_{n,k}(p,q,r)x(x-P_1(p,q,r))\cdots(x-P_{k-1}(p,q,r)).$$

Finally, if we let $\mathbf{sp}_{n,k}(p,q,r) = (-1)^{n-k}\mathbf{cp}_{n,k}(p,q,r)$, then essentially the same proof that we used in this section, we will give a combinatorial proof of the fact that the matrices $||\mathbf{Sp}_{n,k}(p,q)||$ and $||\mathbf{sp}_{n,k}(p,q)||$ are inverses of each other.

4 Identities for $\mathbf{Sf}_{n,k}(p,q)$ and $\mathbf{cf}_{n,k}(p,q)$

In this section, we shall derive various identities for the Fibonacci analogues of the Stirling numbers $\mathbf{Sf}_{n,k}(p,q)$ and $\mathbf{cf}_{n,k}(p,q)$. We let $[0]_q = 1$ and, for any positive integer n, let $[n]_q = 1+q+\cdots+q^{n-1}$. Then the usual q-analogues of $n!$ and $\binom{n}{k}$ are defined by

$$\begin{aligned} [n]_q! &= [n]_q[n-1]_q\cdots[2]_q[1]_q \text{ and} \\ \begin{bmatrix} n \\ k \end{bmatrix}_q &= \frac{[n]_q!}{[k]_q![n-k]_q!}. \end{aligned}$$

A partition $\lambda = (\lambda_1,\ldots,\lambda_k)$ of n is a weakly increasing sequence of positive integers such that $\sum_{i=1}^k \lambda_i = n$. We let $|\lambda| = n$ denote the size of λ and $\ell(\lambda) = k$ denote the number of parts of λ. For this paper, we will draw the Ferrers diagram of a partition consistent with the convention for Ferrers boards. That is, the Ferrers diagram of $\lambda = (\lambda_1 \leq \cdots \leq \lambda_k)$ is the Ferrers board $F(\lambda_1,\ldots,\lambda_k)$. A standard combinatorial interpretation of the q-binomial coefficient $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is that $\begin{bmatrix} n \\ k \end{bmatrix}_q$ equals the sum of $q^{|\lambda|}$ over all partitions whose Ferrers diagram are contained in the $(n-k)\times k$ rectangle.

We have already seen that $\mathbf{cf}_{n,n}(p,q) = \mathbf{Sf}_{n,n}(p,q) = 1$ since these correspond to the empty placements in B_n. Then we have the following simple theorem.

Theorem 9. *For all $n \geq 1$,*

$$\mathbf{Sf}_{n,1}(p,q) = q^{n-1} \text{ and } \mathbf{cf}_{n,1}(p,q) = \prod_{i=1}^{n-1} F_i(p,q).$$

For all $n \geq 2$,

$$\begin{aligned}\mathbf{Sf}_{n,2}(p,q) &= q^{n-2}[n-1]_q \text{ and} \\ \mathbf{cf}_{n,n-1}(p,q) &= \mathbf{Sf}_{n,n-1}(p,q) = \sum_{i=1}^{n-1} F_i(p,q).\end{aligned}$$

Proof. For $\mathbf{Sf}_{n,1}(p,q)$, we must count the weights of the Fibonacci rook tilings of B_n in which every column has a tiling. For each column $i \geq 2$ in B_n, our cancellation scheme ensures that all but one square in column i is canceled by the tilings to its left. Thus there is only one Fibonacci rook tiling that contributes to $\mathbf{Sf}_{n,1}(p,q)$ which is the tiling where every column has exactly one tile of height 1. For example, Figure 13 picture such a tiling for $\mathbf{Sf}_{5,1}(p,q)$. Hence $\mathbf{Sf}_{n,1}(p,q) = q^{n-1}$.

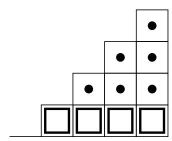

Figure 13: The tiling for $\mathbf{Sf}_{5,1}(p,q)$.

For $\mathbf{cf}_{n,1}(p,q)$, we must count the weights of the Fibonacci file tilings of B_n in which every column has a tiling. Since the sum of the weights of the tilings in column i is $F_{i-1}(p,q)$ for $i = 2,\ldots,n$, it follows that $\mathbf{cf}_{n,1}(p,q) = \prod_{i=1}^{n-1} F_i(p,q)$.

For $\mathbf{Sf}_{n,2}(p,q)$, we know that $\mathbf{Sf}_{2,2}(p,q) = 1$ so that our formula holds for $n = 2$. For $n \geq 3$, we must count the weights of all the Fibonacci rook tilings such that there is exactly one empty column. It is easy to see that if the empty column is at the end, then by our argument of $\mathbf{Sf}_{n,1}(p,q)$, there is exactly one tile in each columns $2,\ldots,n-1$ so that the weight of such a tiling is q^{n-2}. Then as the empty column moves right to left, we see that we replace a column with one tile with a column with two tiles. This process is pictured in Figure 14 for $n = 6$. It follows that for $n \geq 3$,

$$\mathbf{Sf}_{n,2}(p,q) = q^{n-2} + q^{n-1} \cdots + q^{2(n-2)} = q^{n-2}(1 + q + \cdots + q^{n-2}) = q^{n-2}[n-1]_q.$$

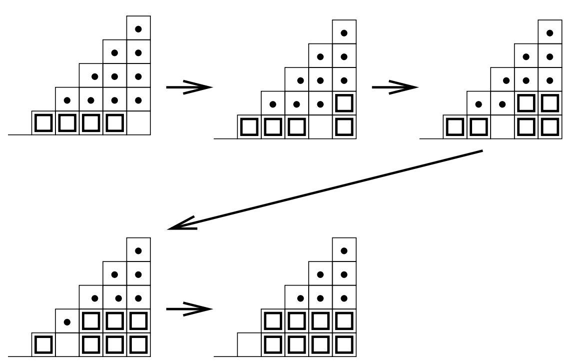

Figure 14: The tilings for $\mathbf{Sf}_{6,2}(p,q)$.

For $\mathbf{Sf}_{n,n-1}(p,q)$ and $\mathbf{cf}_{n,n-1}(p,q)$, we must count the tilings of B_n in which exactly one column is tiled. In this case, the rook tilings and the file tilings are the same. Hence $\mathbf{cf}_{n,n-1}(p,q) = \mathbf{Sf}_{n,n-1}(p,q) = \sum_{i=1}^{n-1} F_i(p,q)$. $\square$

Next we define

$$\mathbb{SF}_k(p,q,t) := \sum_{n\geq k} \mathbf{Sf}_{n,k}(p,q)t^n$$

for $k \geq 1$ It follows from Theorem 9 that

$$\mathbb{SF}_1(p,q,t) = \sum_{n\geq 1} q^{n-1}t^n = \frac{t}{(1-qt)} = \frac{t}{(1-F_1(p,q)t)}. \tag{26}$$

Then for $k > 1$,

$$\begin{aligned}
\mathbb{SF}_k(p,q,t) &= \sum_{n\geq k} \mathbf{Sf}_{n,k}(p,q)t^n \\
&= t^k + \sum_{n>k} \mathbf{Sf}_{n,k}(p,q)t^n \\
&= t^k + t\sum_{n>k} \left(\mathbf{Sf}_{n-1,k-1}(p,q) + F_k(p,q)\mathbf{Sf}_{n-1,k-1}(p,q)\right) t^{n-1} \\
&= t^k + t\left(\sum_{n>k} \mathbf{Sf}_{n-1,k-1}(p,q)t^{n-1}\right) + F_k(p,q)t\left(\sum_{n>k} \mathbf{Sf}_{n-1,k}(p,q)t^{n-1}\right) \\
&= t^k + t(\mathbb{SF}_{k-1}(p,q,t) - t^{k-1}) + F_k(p,q)t\mathbb{SF}_k(p,q,t).
\end{aligned}$$

It follows that

$$\mathbb{SF}_k(p,q,t) = \frac{t}{(1-F_k(p,q)t)}\mathbb{SF}_{k-1}(p,q,t). \tag{27}$$

The following theorem easily follows from (26) and (27).

Theorem 10. *For all $k \geq 1$,*

$$\mathbb{SF}_k(p,q,t) = \frac{t^k}{(1-F_1(p,q)t)(1-F_2(p,q)t)\cdots(1-F_k(p,q)t)}.$$

For any formal power series in $f(x) = \sum_{n\geq 0} f_n x^n$, we let $f(x)|_{x^n} = f_n$ denote the coefficient of x^n in $f(x)$.

Our next result will give formulas for $\mathbf{Sf}_{n,k}(p,q)|_{p^0}$ and $\mathbf{Sf}_{n,k}(p,q)|_p$. Note that we have already shown that $\mathbf{Sf}_{n,2}(p,q) = q^{n-2}[n-1]_q$ so that $\mathbf{Sf}_{n,2}(p,q)|_p = 0$.

Theorem 11. *For all $n \geq k \geq 1$,*

$$\mathbf{Sf}_{n,k}(p,q)|_{p^0} = q^{n-k}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q \tag{28}$$

and for all $n > k \geq 3$,

$$\mathbf{Sf}_{n,k}(p,q)|_p = q^{n-k}\sum_{s=1}^{k-2} sq^{s-1} \sum_{i=0}^{n-k-1} q^{i(s+1)} \begin{bmatrix} i+k-s-2 \\ i \end{bmatrix}_q \begin{bmatrix} s+n-k-i \\ s+1 \end{bmatrix}_q. \tag{29}$$

Proof. There are two proofs that we can give for (28). The first uses the generating function

$$\mathbb{SF}_k(p,q,t) = \sum_{n\geq k} \mathbf{Sf}_{n,k}(p,q)t^n = \frac{t^k}{(1-F_1(p,q)t)(1-F_2(p,q)t)\cdots(1-F_k(p,q)t)}. \tag{30}$$

Clearly, for all $n \geq 1$, $F_n(p,q)|_{p^0} = q^n$ because the only Fibonacci tiling $T \in \mathcal{FT}_n$ which has no tiles of height 2 is the tiling which consists of n tiles of height 1. Taking the coefficient of p^0 on both sides of (30), we see that

$$\begin{aligned}\sum_{n\geq k} \mathbf{Sf}_{n,k}(p,q)|_{p^0}t^n &= \frac{t^k}{(1-F_1(p,q)|_{p^0}t)(1-F_2(p,q)|_{p^0}t)\cdots(1-F_k(p,q)|_{p^0}t)} \\ &= \frac{t^k}{(1-qt)(1-q^2t)\cdots(1-q^kt)}. \end{aligned} \tag{31}$$

Taking the coefficient of t^n on both sides of (31), we see that $\mathbf{Sf}_{n,k}(p,q)|_{p^0}$ equals the sum of $q^{|\lambda|}$ over all partitions λ with $n-k$ parts whose parts are from $\{1,\ldots,k\}$. If we subtract 1 from each part of λ, we will end up with a partition contained in the $(n-k)\times(k-1)$. Since the sum of $q^{|\pi|}$ over all partitions π whose Ferrers diagram is contained in $(n-k)\times(k-1)$ rectangle is $\left[\begin{smallmatrix} n-1 \\ k-1 \end{smallmatrix}\right]_q$, it follows that $\mathbf{Sf}_{n,k}(p,q)|_{p^0} = q^{n-k}\left[\begin{smallmatrix} n-1 \\ k-1 \end{smallmatrix}\right]_q$.

In fact, this result can be seen directly from our rook theory interpretation for $\mathbf{Sf}_{n,k}(p,q)$. That is, $\mathbf{Sf}_{n,k}(p,q)|_{p^0}$ is the sum of $q^{\mathrm{one}(T)}$ over all Fibonacci rook tilings T of B_n with k empty columns that only use tiles of height 1. It is easy to see that that every time we traverse an empty column, the number of tiles that we can put in a column goes up by one. Since the first column is empty, this means that we can start with tilings of height 1 and as we traverse the $k-1$ remaining empty column, the maximum number of tiles of height one that we can put in any column is k. It follows that if we remove the tiles of height 1 at the bottom of any Fibonacci rook tiling T of B_n with k empty columns that use only tiles of height 1, we will be left with a Ferrers diagram of a partition which is contained in the $(n-k)\times(k-1)$ rectangle. This process is pictured in Figure 15 in the case where $n=11$ and $k=4$.

We can also reverse this correspondence. That is, if we are given the Ferrers diagram of partition μ contained the $(n-k)\times(k-1)$ rectangle, we can reconstruct the tiling $P \in \mathcal{NT}_{n_k}(B_n)$ which gave rise to μ. That is, we first add tiles of height 1 at the bottom of μ which will give us the Ferrers diagram of partition $\lambda = (\lambda_1,\ldots,\lambda_{n-k})$ with $n-k$ parts with parts from $\{1,\ldots,k\}$. Thus $1 \leq \lambda_1 \leq \cdots \leq \lambda_{n-k} \leq k$. Then if $\lambda_1 = 1$, we put a tiling of height 1 in column 2. If $\lambda_1 = j > 1$, then we start with j empty columns and place a tiling of height j in column $j+1$. Then assuming that we have placed the tilings corresponding to $\lambda_1,\ldots,\lambda_i$, we place the tiling for λ_{i+1} next to the tiling for λ_i if $\lambda_i = \lambda_{i+1}$ and we put $\lambda_{i+1}-\lambda_i$ consecutive empty columns next to the column that contains the tiling for λ_i and place the tiling for λ_{i+1} in the next column if $\lambda_{i+1}-\lambda_i > 0$. This process is pictured in Figure 16 in the case where $n=11$ and $k=5$. That is, we start with the partition $(0,0,0,2,3,3)$ contained in the 6×4 rectangle. Then we add one to each part of the partition to get the partition $\lambda = (1,1,1,3,4,4)$. Then our process says we start putting a tiling of height 1 in column 2 and add two more tiling of height 1 columns 3 and 4. Next, since $\lambda_4 - \lambda_3 = 2$, we put two empty columns followed by a column of height 3 in column 7. Next, since $\lambda_5 - \lambda_4 = 1$, we put an empty column followed by a column of height 4 in column 9. Finally since $\lambda_5 = \lambda_6$, we add another column of height 4 in column 10.

Next we consider (29). In this case, the Fibonacci rook tilings $P \in \mathcal{NT}_{n-k}(B_n)$ that we must consider are those tilings which have exactly one tile of height 2 and the rest of tiles must be of

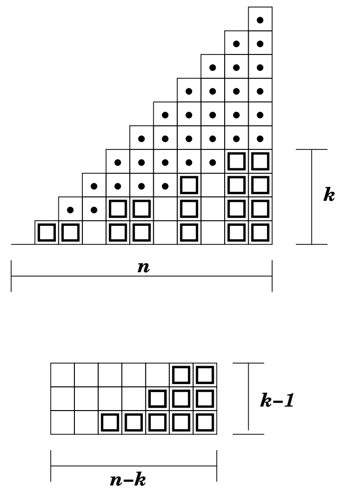

Figure 15: The correspondence between tilings in $\mathcal{NT}_{n-k}(B_n)$ using only tiles of height 1 and partitions contained in the $(n-k)\times(k-1)$ rectangle.

height 1. We let c denote the column which contains the tile of height 2. We shall classify such tilings by the number s of tiles of height 1 that are in column c. Since the maximum number of non-canceled cells in any column is k and every column which is tiled has a tile of height 1 at the bottom of the column, s can vary from 1 to $k-2$. We shall think of the factor q^{n-k} that sits outside of the first sum in (29) as the contribution from the tiles of height 1 at the bottom of the $n-k$ columns that have tilings. The factor sq^{s-1} accounts for the factor that comes from the column c. That is, there are $s-1$ tiles of height 1 in c other than the tile of height 1 at the bottom of column c and the number of tiles of height 1 that can lie below the tile of height of 2 in c can vary from 1 to s.

We interpret the i in the inner sum as the number of columns to the right of c which have tiles. There are $(s+1)i$ tiles of height 1 in rows 2 through $s+2$ in each of these i columns which account for the factor $q^{i(s+1)}$. Next we consider the set partition β induced by that tiles above row $s+2$ that lie in these i columns. β must be contained in the $i\times(k-s-2)$ since the maximum number of uncanceled cells in any column is k. As β varies over all possible partitions contained in $i\times(k-s-2)$, we get a factor of $\begin{bmatrix} i+k-s-2 \\ i \end{bmatrix}_q$. Finally we let α be the partition induced by the tilings in the columns to the left of column c minus the tiles of height 1 at the bottom of these columns. There are $n-k-i-1$ such columns and the maximum number of tiles in any such column is $s+1$. As α varies over all possible partitions contained in $(n-k-i-1)\times(s+1)$, we get a factor of $\begin{bmatrix} s+n-k-i \\ s+1 \end{bmatrix}_q$.

The decomposition of a $P\in\mathcal{NT}_{n-k}(B_n)$ where $n=17$, $s=2$, $k=7$, and $i=3$ into the partitions α, β, and the $i\times(s+1)$ rectangle is pictured in Figure 17.

We can use the same argument that we did in the proof of (28) to prove that we can reconstruct a $P\in\mathcal{NT}_{n-k}(B_n)$ from s, i, and the partitions α and β.

Thus we have proved that

$$\mathbf{Sf}_{n,k}(p,q)|_{p^1} = q^{n-k}\sum_{s=1}^{k-2} sq^{s-1}\sum_{i=0}^{n-k-1} q^{i(s+1)}\begin{bmatrix} i+k-s-2 \\ i \end{bmatrix}_q \begin{bmatrix} s+n-k-i \\ s+1 \end{bmatrix}_q.$$

$\square$

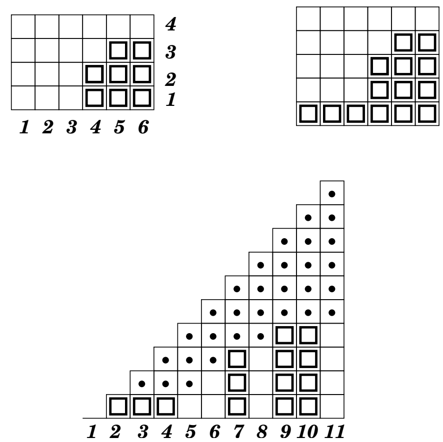

Figure 16: Reconstructing of an element $P \in \mathcal{NT}_{n-k}(B_n)$ from a partitions contained in the $(n-k) \times (k-1)$ rectangle.

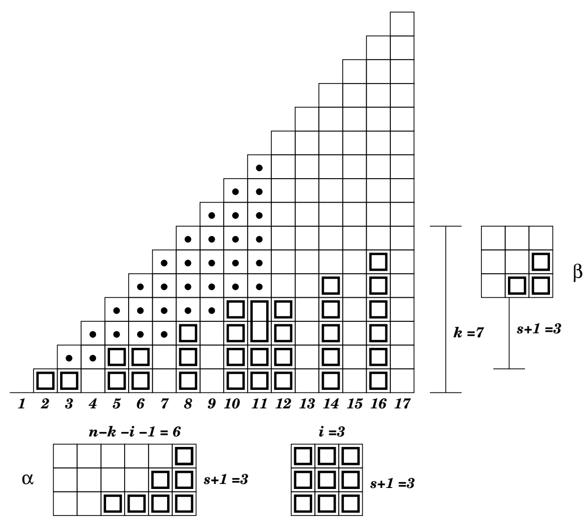

Figure 17: The decomposition of $P \in \mathcal{NT}_{n-k}(B_n)$.

We claim that (29) is just a q-analogue of $\binom{k-1}{2}\binom{n-1}{k}$. That is, we have the following theorem.

Theorem 12. *For all $n \geq k \geq 3$,*

$$\mathbf{Sf}_{n,k}(p,1)|_{p^1} = \binom{k-1}{2}\binom{n-1}{k} \tag{32}$$

Proof. We can easily prove (32) by induction on k and then by induction on n. First observe $\mathbf{Sf}_{k,k}(p,1)|_{p^1} = 0$ since $\mathbf{Sf}_{k,k}(p,q) = 1$. Thus for each k, the base case of the induction on n holds.

Next observe that for $k \geq 3$, $F_k(p,1)|_{p^1} = k-2$ and $F_k(p,1)|_{p^0} = 1$. Hence

$$\begin{aligned}
\mathbf{Sf}_{n+1,k}(p,1)|_{p^1} &= \mathbf{Sf}_{n,k-1}(p,1)|_{p^1} + \left(F_k(p,1)\mathbf{Sf}_{n,k}(p,1)\right)|_{p^1} \\
&= \mathbf{Sf}_{n,k-1}(p,1)|_{p^1} + F_k(p,1)|_{p^1}\mathbf{Sf}_{n,k}(p,1)|_{p^0} + F_k(p,1)|_{p^0}\mathbf{Sf}_{n,k}(p,1)|_{p^1} \\
&= \mathbf{Sf}_{n,k-1}(p,1)|_{p^1} + (k-2)\binom{n-1}{k-1} + \mathbf{Sf}_{n,k}(p,1)|_{p^1}. \qquad (33)
\end{aligned}$$

Now suppose that $k = 3$. Then we need to show that $\mathbf{Sf}_{n,3} = \binom{n-1}{3}$. Note that for all $n \geq 2$, $\mathbf{Sf}_{n,2}(p,q) = q^{n-2}[n-1]_q$ so that $\mathbf{Sf}_{n,2}(p,1)|_{p^1} = 0$. Thus using (33) and induction, we see that

$$\mathbf{Sf}_{n+1,3}(p,1)|_{p^1} = \binom{n-1}{2} + \binom{n-1}{3} = \binom{n}{3}.$$

This establish (32) in the case $k = 3$.

For $k > 3$, assume by induction that $\mathbf{Sf}_{n,k-1} = \binom{k-1}{2}\binom{n-1}{k-1}$. Then using (33) and induction, we see that

$$\begin{aligned}
\mathbf{Sf}_{n+1,3}(p,1)|_{p^1} &= \binom{k-1}{2}\binom{n-1}{k-1} + (k-2)\binom{n-1}{k-1} + \binom{k-1}{2}\binom{n-1}{k} \\
&\binom{k-1}{2}\binom{n-1}{k-1} + \binom{k-1}{2}\binom{n-1}{k} \\
&= \binom{k-1}{2}\binom{n}{k}.
\end{aligned}$$

$\square$

A sequence of real numbers $a_0, \ldots, a_n$ is is said to be *unimodal* if there is a $0 \leq j \leq n$ such that $a_0 \leq \cdots \leq a_j \geq a_{j+1} \geq \cdots \geq a_n$ and is said to be *log concave* if for $0 \leq i \leq n$, $a_i^2 - a_{i-1}a_{i+1} \geq 0$ where we set $a_{-1} = a_{n+1} = 0$. If a sequence is log concave, then it is unimodal. A polynomial $P(x) = \sum_{k=0}^n a_k x^k$ is said to be *unimodal* if $a_0, \ldots, a_n$ is a unimodal sequence and is said to be log concave if $a_0, \ldots, a_n$ is log concave. Computational evidence suggests that the polynomials $\mathbf{Sf}_{n,k}(p,1)$ are log concave for all $n \geq k$. We can prove this for $k \leq 4$. Clearly, this is true for $k = 1$ and $k = 2$ because by Theorem 9 both $S_{n,1}(p,1)$ and $S_{n,2}(p,1)$ are just constants. For $k = 3$ and $k = 4$, we have the following theorem.

Theorem 13. *For all $n \geq 3$ and $s \geq 0$,*

$$\mathbf{Sf}_{n,3}(p,1)|_{p^s} = \binom{n-1}{s+2}. \qquad (34)$$

and for all $n \geq 4$ and $s \geq 0$,

$$\mathbf{Sf}_{n,4}(p,1)|_{p^s} = (2^{s+1}-1)\binom{n-1}{s+3}. \qquad (35)$$

Proof. By Theorem 11, we know that $\mathbf{Sf}_{n,3}(p,1)|_{p^0} = \binom{n-1}{2}$ and $\mathbf{Sf}_{n,4}(p,1)|_{p^0} = \binom{n-1}{3}$. Thus our formulas hold for $s = 0$.

We then proceed first by induction on n and then by induction on s. We know by Theorem 9 that for $s \geq 1$, $\mathbf{Sf}_{n,2}|_{p^s} = 0$. Thus for $s \geq 1$,

$$\begin{aligned}
\mathbf{Sf}_{n,3}(p,1)|_{p^s} &= \mathbf{Sf}_{n-1,2}(p,1)|_{p^s} + (F_3(p,1)\mathbf{Sf}_{n-1,3}(p,1))|_{p^s} \\
&= ((1+p)\mathbf{Sf}_{n-1,3}(p,1))|_{p^s} \\
&= \mathbf{Sf}_{n-1,3}(p,1))|_{p^s} + \mathbf{Sf}_{n-1,3}(p,1))|_{p^{s-1}} \\
&= \binom{n-2}{s+2} + \binom{n-2}{s+1} = \binom{n-1}{s+2}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathbf{Sf}_{n,4}(p,1)|_{p^s} &= \mathbf{Sf}_{n-1,3}(p,1)|_{p^s} + (F_4(p,1)\mathbf{Sf}_{n-1,4}(p,1))|_{p^s} \\
&= \mathbf{Sf}_{n-1,3}(p,1)|_{p^s} + ((1+2p)\mathbf{Sf}_{n-1,4}(p,1))|_{p^s} \\
&= \mathbf{Sf}_{n-1,3}(p,1)|_{p^s} + \mathbf{Sf}_{n-1,4}(p,1))|_{p^s} + 2\mathbf{Sf}_{n-1,4}(p,1))|_{p^{s-1}} \\
&= \binom{n-2}{s+2} + (2^{s+1}-1)\binom{n-2}{s+3} + 2(2^s-1)\binom{n-2}{s+2} \\
&= (2^{s+1}-1)\left(\binom{n-2}{s+2} + \binom{n-1}{s+3}\right) \\
&= (2^{s+1}-1)\binom{n-1}{s+3}.
\end{aligned}$$

□

It is then easy to prove by direct calculation that the sequences $\{\binom{n-1}{s+2}\}_{s>0}$ and $\{(2^s-1)\binom{n-1}{s+3}\}_{s>0}$ are log concave. It is not obvious how this direct approach can be extended to prove that the polynomials $\mathbf{Sf}_{n,k}(p,1)$ are log concave for $k \geq 5$ because the formulas for $\mathbf{Sf}_{n,k}(p,1)|_{p^s}$ become more complicated. For example, the following formulas are straightforward to prove by induction:

$$\begin{aligned}
\mathbf{Sf}_{n,5}(p,1)|_{p} &= 6\binom{n-1}{5}, \\
\mathbf{Sf}_{n,5}(p,1)|_{p^2} &= 25\binom{n-1}{6} + \binom{n-1}{5}, \\
\mathbf{Sf}_{n,5}(p,1)|_{p^3} &= 90\binom{n-1}{7} + 9\binom{n-1}{6}, \text{ and} \\
\mathbf{Sf}_{n,5}(p,1)|_{p^4} &= 301\binom{n-1}{8} + 52\binom{n-1}{7} + \binom{n-1}{6}.
\end{aligned}$$

Our next two theorems concern some results on $\mathbf{cf}_{n,1}(p,q)|_{p^i}$ and $\mathbf{cf}_{n,2}(p,q)|_{p^i}$ for small values of i. We start by considering $\mathbf{cf}_{n,1}(p,q)|_{p^i}$ for $i = 0,1,2,3$.

Theorem 14.

$$\mathbf{cf}_{n,1}(p,q)|_{p^0} = q^{\binom{n}{2}} \text{ for all } n \geq 1. \tag{36}$$

$$\mathbf{cf}_{n,1}(p,q)|_{p^1} = \begin{cases} 0 & \text{for } n = 1,2,3 \\ \binom{n-2}{2} q^{\binom{n}{2}-2} & \text{for all } n \geq 4. \end{cases} \tag{37}$$

$$\mathbf{cf}_{n,1}(p,q)|_{p^2} = \begin{cases} 0 & \textit{for } n = 1,2,3,4 \\ \left(3\binom{n-1}{4} - \binom{n-3}{2}\right) q^{\binom{n}{2}-4} & \textit{for all } n \geq 5. \end{cases} \tag{38}$$

$$\mathbf{cf}_{n,1}(p,q)|_{p^3} = \begin{cases} 0 & \textit{for } n = 1,2,3,4,5 \\ \left(15\binom{n}{6} - 6\binom{n-2}{4} + \binom{n-4}{4}\right) q^{\binom{n}{2}-6} & \textit{for all } n \geq 5. \end{cases} \tag{39}$$

Proof. Equation (36) follows from the fact that $\mathbf{cf}_{n,1}(p,q)|_{p^0}$ counts the Fibonacci file tiling of B_n where all but the first column are filled with tiles of size 1. There are clearly $1+2+\cdots+(n-1) = \binom{n}{2}$ such tiles.

$\mathbf{cf}_{n,1}(p,q)|_{p^1}$ in (37) counts the (p,q)-Fibonacci file tilings of B_n where all but the first column are filled with tiles of size 1 except for one column $c \in \{4,\ldots,n\}$ which is tiled with $c-3$ tiles of height 1 and one tile of height 2. Thus the total number of tiles of height 1 in any such tiling is $q^{\binom{n}{2}-2}$ and the number of ways to tile column c is $c-3$ depending on how many tiles of height 1 in c lies below the tile of height 2 in c. Thus if $n \geq 4$, there are $\sum_{c=4}^{n}(c-3) = \binom{n-2}{2}$ such file tilings. Thus $\mathbf{cf}_{n,1}(p,q)|_{p^1} = 0$ if $n \leq 3$ and $\mathbf{cf}_{n,1}(p,q)|_{p^1} = \binom{n-2}{2} q^{\binom{n}{2}-2}$ if $n \geq 4$.

$\mathbf{cf}_{n,1}(p,q)|_{p^2}$ in (38) counts the (p,q)-Fibonacci file tilings of B_n where there are tilings in columns $2,\ldots,n$ and we use exactly two tiles of height 2. Clearly, there are no such tilings for $n = 1,2,3,4$. For $n = 5$, there are two such tilings which are pictured in Figure 18. Thus our formula holds for $n = 5$.

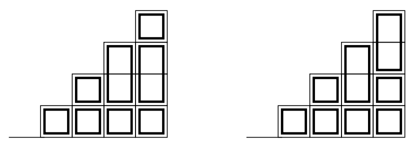

Figure 18: The two file tiling for $\mathbf{cf}_{5,1}|_{p^2}$.

For $n > 5$, we proceed by induction. Note that $F_n(p,q)|_{p^2} = \binom{n-3}{2}$. That is, for $F_n(p,q)|_{p^2}$, we are considering Fibonacci tilings of height n where we have $n-4$ tiles of height 1 and two tiles of height 2. Since we must start with a tile of height 1, the number of such tilings is the number of rearrangement of $1^{n-5}2^2$ which is $\binom{n-3}{2}$. It is also easy to see that $F_n(p,q)|_p = n-2$. Then

$$\begin{aligned}
\mathbf{cf}_{n,1}|_{p^2} &= \mathbf{cf}_{n-1,0}|_{p^2} + \left(F_{n-1}(p,q)\mathbf{cf}_{n-1,0}\right)|_{p^2} \\
&= \left(F_{n-1}(p,q)|_{p^2}\right)\left(\mathbf{cf}_{n-1,1}|_{p^0}\right) + \left(F_{n-1}(p,q)|_{p^1}\right)\left(\mathbf{cf}_{n-1,1}|_{p^1}\right) + \\
&\quad \left(F_{n-1}(p,q)|_{p^0}\right)\left(\mathbf{cf}_{n-1,1}|_{p^2}\right) \\
&= \left(\binom{n-4}{2}\right)\left(q^{\binom{n-1}{2}}\right) + \left((n-3)q^{n-3}\right)\left(\binom{n-3}{2}q^{\binom{n-1}{2}-2}\right) + \\
&\quad \left(q^{n-1}\right)\left(q^{\binom{n-1}{2}-4}\left(3\binom{n-2}{4} - \binom{n-4}{2}\right)\right) \\
&= q^{\binom{n}{2}-4}\left(\binom{n-4}{2} + (n-3)\binom{n-3}{2} + 3\binom{n-2}{4} - \binom{n-4}{2}\right) \\
&= q^{\binom{n}{2}-4}\left(3\binom{n-2}{3} - \binom{n-3}{2} + 3\binom{n-2}{4}\right) \\
&= q^{\binom{n}{2}-4}\left(3\binom{n-1}{4} - \binom{n-3}{2}\right).
\end{aligned}$$

$\mathbf{cf}_{n,1}(p,q)|_{p^3}$ in (39) counts the Fibonacci file tilings of B_n where there are tilings in columns $2,\ldots,n$ and we use exactly three tiles of height 2. In general, the number of tiles of height 1 in such tiling is $\binom{n-2}{2}-6$. It is easy to check that there are no such tilings for $n=1,2,3,4,5$. For $n=6$, there are nine such tilings which are pictured in Figure 19. Thus our formula holds for $n=6$.

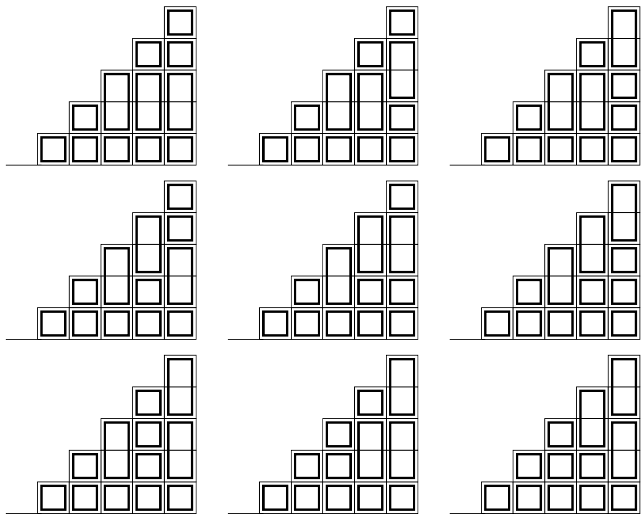

Figure 19: The 9 file tiling for $\mathbf{cf}_{6,1}|_{p^3}$.

For $n>6$, we proceed by induction. Note that $F_n(p,q)|_{p^3}=\binom{n-4}{3}$. That is, for $F_n(p,q)|_{p^3}$, we are considering Fibonacci tilings of height n where we have $n-6$ tiles of height 1 and three tiles of height 2. Since we must start with a tile of height 1, the number of such tilings is the number of rearrangement of $1^{n-7}2^3$ which is $\binom{n-4}{3}$. Then

$$\begin{aligned}
\mathbf{cf}_{n,1}|_{p^3} &= \mathbf{cf}_{n-1,0}|_{p^3}+\left(F_{n-1}(p,q)\mathbf{cf}_{n-1,1}\right)|_{p^3}\\
&= \left(F_{n-1}(p,q)|_{p^3}\right)\left(\mathbf{cf}_{n-1,1}|_{p^0}\right)+\left(F_{n-1}(p,q)|_{p^2}\right)\left(\mathbf{cf}_{n-1,1}|_{p^1}\right)+\\
&\quad \left(F_{n-1}(p,q)|_{p^1}\right)\left(\mathbf{cf}_{n-1,1}|_{p^2}\right)+\left(F_{n-1}(p,q)|_{p^0}\right)\left(\mathbf{cf}_{n-1,1}|_{p^3}\right)\\
&= \left(\binom{n-5}{3}q^{n-7}\right)\left(q^{\binom{n-1}{2}}\right)+\left(\binom{n-4}{2}q^{n-5}\right)\left(\binom{n-3}{2}q^{\binom{n-1}{2}-2}\right)+\\
&\quad \left((n-3)q^{n-3}\right)\left(q^{\binom{n-1}{2}-4}\left(3\binom{n-2}{4}-\binom{n-4}{2}\right)\right)+\left(q^{n-1}\right)\mathbf{cf}_{n-1,1}|_{p^3}\\
&= q^{n-1}\mathbf{cf}_{n-1,1}|_{p^3}+\\
&\quad q^{\binom{n}{2}-6}\left(\binom{n-5}{3}+\binom{n-4}{2}\binom{n-3}{2}+(n-3)\left(3\binom{n-2}{2}-\binom{n-4}{2}\right)\right).
\end{aligned}$$

It follows that we have the recursion

$$\begin{aligned}
&\left(\mathbf{cf}_{n,1}|_{p^3}\right)|_{q^{\binom{n}{2}-6}}=\\
&\left(\mathbf{cf}_{n-1,1}|_{p^3}\right)|_{q^{\binom{n-1}{2}-6}}+\binom{n-5}{3}+\binom{n-4}{2}\binom{n-3}{2}+(n-3)\left(3\binom{n-2}{2}-\binom{n-4}{2}\right). \quad (40)
\end{aligned}$$

Iterating (40), it follows that for $n \geq 7$,

$$\mathbf{cf}_{n,1}|_{p^3} =$$
$$q^{\binom{n}{2}-6}\left(9+\sum_{k=7}^{n}\binom{k-5}{3}+\binom{k-4}{2}\binom{k-3}{2}+(k-3)\left(3\binom{k-2}{2}-\binom{k-4}{2}\right)\right) =$$
$$\binom{n-4}{2}\left(\frac{12+28n+n^2-6n^3+n^4}{24}\right) = 15\binom{n}{6}-6\binom{n-2}{4}+\binom{n-4}{4}$$

where we have used Mathematica to verify the last two equalities. ☐

We note that the sequence $\{3\binom{n-1}{4}-\binom{n-3}{2}\}_{n\geq 5}$ starts out

$$2, 12, 39, 95, 195, 357, 602, 954, \ldots.$$

This is sequence A086602 in the OEIS [8]. This sequence does not have a combinatorial interpretation so that we have now given a combinatorial interpretation of this sequence.

The sequence $\{15\binom{n}{6}-6\binom{n-2}{4}+\binom{n-4}{4}\}_{n\geq 6}$ starts out

$$9, 75, 331, 1055, 2745, 6209, 12670, 23886, 42285, 71115, \ldots.$$

This sequence does not appear in the OEIS.

Next we consider $\mathbf{cf}_{n,2}(p,q)|_{p^i}$ for $i = 0, 1$.

Theorem 15. *For $n \geq 2$,*

$$\mathbf{cf}_{n,2}(p,q)|_{p^0} = q^{\binom{n-1}{2}}[n-1]_q. \tag{41}$$

$$\mathbf{cf}_{n,2}(p,q)|_{p^1} = \begin{cases} q^2+q^3 & \text{for } n=4 \\ \binom{n-2}{2}q^{\binom{n}{2}-3}+q^{\binom{n-1}{2}-2}\sum_{i=0}^{n-3}\left(\binom{n-3}{2}+i\right)q^i & \text{for } n\geq 5. \end{cases} \tag{42}$$

Proof. For $\mathbf{cf}_{n,2}(p,q)|_{p^0}$, we know that $\mathbf{cf}_{2,2}(p,q) = 1$ so that our formula holds for $n = 2$. For $n \geq 3$, we must count the weights of all the Fibonacci file tilings where we use no tiles of height 2 such that there is exactly one empty column. It is easy to see that if the empty column is at the end, the number of tiles of size 1 is $1+2+\cdots+(n-2) = \binom{n-1}{2}$. Then as the empty column moves right to left, we see that we replace a column with i tiles of height 1 by a column with $i+1$ tiles of height 1. This process is pictured in Figure 20 for $n = 6$. It follows that for $n \geq 3$,

$$\mathbf{cf}_{n,2}(p,q) = q^{\binom{n-1}{2}}+q^{\binom{n-1}{2}+1}\cdots+q^{\binom{n-1}{2}+(n-2)} = q^{\binom{n-1}{2}}(1+q+\cdots+q^{n-2}) = q^{\binom{n-1}{2}}[n-1]_q.$$

For $\mathbf{cf}_{4,2}(p,q)|_p$, there are only two Fibonacci file tilings which have one tile of height 2. These are pictured in Figure 21. Thus $\mathbf{cf}_{4,2}(p,q)|_p = q^2+q^3$.

Note that by (37) $\mathbf{cf}_{4,1}(p,q)|_p = q^4$. Hence

$$\begin{aligned}
\mathbf{cf}_{5,2}(p,q)|_p &= \mathbf{cf}_{4,1}(p,q)|_p + \left(F_4(p,q)\mathbf{cf}_{4,2}(p,q)\right)|_p \\
&= q^4 + \left(F_4(p,q)|_p\right)\left(\mathbf{cf}_{4,2}(p,q)|_{p^0}\right) + \left(F_4(p,q)|_{p^0}\right)\left(\mathbf{cf}_{4,2}(p,q)|_p\right) \\
&= q^4 + (2q^2)\left(\binom{3}{2}[3]_q\right) + q^4(q^2+q^3) \\
&= q^4 + 2q^5 + 3q^6 + 3q^7.
\end{aligned}$$

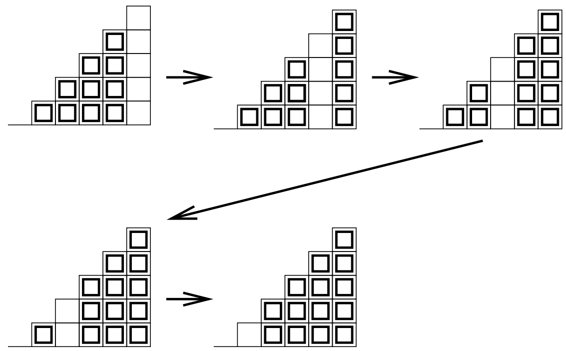

Figure 20: The tilings for $\mathbf{cf}_{6,2}(p,q)$.

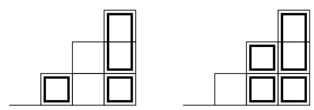

Figure 21: The tilings for $\mathbf{cf}_{4,2}(p,q)|_p$.

This verifies our formula for $n=5$.

For $n>5$, we proceed by induction. That is,

$$
\begin{aligned}
\mathbf{cf}_{n,2}(p,q)|_p &= \mathbf{cf}_{n-1,1}(p,q)|_p + \left(F_{n-1}(p,q)\mathbf{cf}_{n-1,2}(p,q)\right)|_p \\
&= \binom{n-3}{2} q^{\binom{n-1}{2}-2} + \\
&\quad \left(F_{n-1}(p,q)|_p\right)\left(\mathbf{cf}_{n-1,2}(p,q)|_{p^0}\right) + \left(F_{n-1}(p,q)|_{p^0}\right)\left(\mathbf{cf}_{n-1,2}(p,q)|_p\right) \\
&= \binom{n-3}{2} q^{\binom{n-1}{2}-2} + \left((n-3)q^{n-3}\right)\left(q^{\binom{n-2}{2}}[n-2]_q\right) + \\
&\quad \left(q^{n-1}\right)\left(\binom{n-3}{2} q^{\binom{n-1}{2}-3} + q^{\binom{n-2}{2}-2}\sum_{i=0}^{n-4}\left(\binom{n-4}{2}+i\right)q^i\right) \\
&= \binom{n-3}{2} q^{\binom{n-1}{2}-2} + \\
&\quad q^{\binom{n-1}{2}-1}\left((n-3)q^{n-3} + \sum_{i=0}^{n-4}(n-3)q^i\right) + \\
&\quad \left(\binom{n-3}{2} q^{\binom{n}{2}-3} + q^{\binom{n-1}{2}-1}\sum_{i=0}^{n-4}\left(\binom{n-4}{2}+i\right)q^i\right) \\
&= \binom{n-3}{2} q^{\binom{n-1}{2}-2} + q^{\binom{n}{2}-3}\left((n-3)+\binom{n-3}{2}\right) + \\
&\quad q^{\binom{n-1}{2}-1}\sum_{i=0}^{n-4}\left((n-3)+\binom{n-4}{2}+i\right)q^i \\
&= \binom{n-2}{2} q^{\binom{n}{2}-3} + q^{\binom{n-1}{2}-2}\sum_{i=0}^{n-3}\left(\binom{n-3}{2}+i\right)q^i.
\end{aligned}
$$

□

It is easy to see from our formula for $\mathbf{cf}_{n,2}(p,q)|_p$ that

$$\begin{aligned}
\mathbf{cf}_{n,2}(p,1)|_p &= \binom{n-2}{2} + (n-2)\binom{n-3}{2} + \binom{n-2}{2} \\
&= \binom{n-2}{2} + (n-4)\binom{n-2}{2} + \binom{n-2}{2} \\
&= (n-2)\binom{n-2}{2}.
\end{aligned}$$

We note that the sequence $\{\mathbf{cf}_{n,2}(p,q)|_p\}_{n\geq 4}$ starts out

$$2, 9, 24, 90, 147, 224, 324, 450, 605, 792, 1014, \ldots.$$

This is sequence A006002 in the OEIS which does not have a combinatorial interpretation. Thus we have given a combinatorial interpretation to this sequence.

Our computational evidence suggests that the polynomials $\mathbf{cf}_{n,k}(p,1)$ are also log concave. We can prove this in the case $k = 1$. In that case, $\mathbf{cf}_{n,k}(p,1) = \prod_{i=1}^{n-1} F_i(p,1)$ and one can prove that the polynomials $F_n(p,1)$ have real roots. Thus $\mathbf{cf}_{n,k}(p,1)$ has real roots and, hence, is log-concave.

5 Conclusions

In this paper, we studied Fibonacci analogues of the Stirling numbers of the first and second kind. That is, we have studied the connection coefficients defined by the equations

$$x(x+F_1(p,q))\cdots(x+F_{n-1}(p,q)) = \sum_{k=1}^{n} \mathbf{cf}_{n,k}(p,q)x^k$$

and

$$x^n = \sum_{k=1}^{n} \mathbf{Sf}_{n,k}(p,q)x(x-F_1(p,q))\cdots(x-F_{k-1}(p,q)).$$

We also have given a rook theory model for the $\mathbf{cf}_{n,k}(p,q)$s and $\mathbf{Sf}_{n,k}(p,q)$s.

There are other natural q-analogues for Fibonacci analogues of the Stirling numbers of the first and second kind. For example, we could study the connection coefficients defined by the equations

$$[x]_q[x+F_1]_q\cdots[x+F_{n-1}]_q = \sum_{k=1}^{n} \mathbf{cF}_{n,k}(q)[x]_q^k$$

and

$$[x]_q^n = \sum_{k=1}^{n} \mathbf{SF}_{n,k}(q)[x]_q[x-F_1]_q\cdots[x-F_{k-1}]_q.$$

It turns out that our basic rook theory model can also be used to give a combinatorial interpretation to $\mathbf{cF}_{n,k}(q)$'s and $\mathbf{SF}_{n,k}(q)$'s. In this case, we have to weight the Fibonacci tilings of height n in a different way. The basic idea is that there is a natural tree associated with the

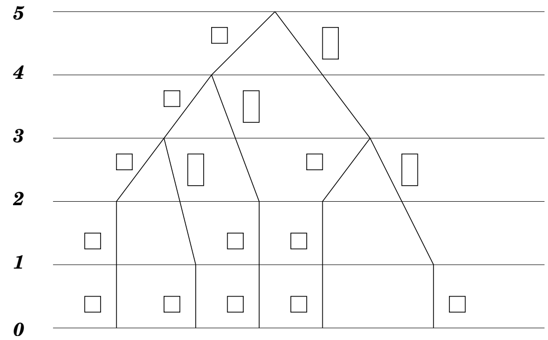

Figure 22: The tree for F_5

Fibonacci tilings of height n. That is, we start from the top of a Fibonacci tiling and branch left if we see a tile of height 1 and branch right if we see a tiling of height 2. We shall call such a tree, the Fibonacci tree for F_n. For example, the Fibonacci tree for F_5 is pictured in Figure 22.

Then the paths in the tree correspond to Fibonacci tilings and we define the rank of a Fibonacci tiling T of height n, $\mathbf{rank}(T)$, to be the number of paths in the Fibonacci tree for F_n which lie to left of the path that corresponds to T. In this way, the ranks of that Fibonacci tiling range from 0 to $F_n - 1$ and, hence

$$\sum_{T \text{ is a Fibonacci tiling of height } n} q^{\mathbf{rank}(T)} = 1 + q + \cdots + q^{F_n - 1} = [F_n]_q.$$

We shall show in a subsequent paper that by weighting a Fibonacci rook tilings or a Fibonacci file tilings of B_n,

$$P = ((c_{i_1}, T_{i_1}), \ldots, (c_{i_k}, T_{i_k}))$$

by $q^{s(P)} q^{\sum_{j=1}^{k} \mathbf{rank}(T_{i_j})}$ for some appropriate statistic $s(P)$, we can give a combinatorial interpretation to $\mathbf{cF}_{n,k}(q)$s and $\mathbf{SF}_{n,k}(q)$s.

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