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COMBINATORICS OF POLY-BERNOULLI NUMBERS

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Abstract

The $\mathbb{B}_n^{(k)}$ poly-Bernoulli numbers — a natural generalization of classical Bernoulli numbers ($B_n = \mathbb{B}_n^{(1)}$) — were introduced by Kaneko in 1997. When the parameter k is negative then $\mathbb{B}_n^{(k)}$ is a nonnegative number. Brewbaker was the first to give combinatorial interpretation of these numbers. He proved that $\mathbb{B}_n^{(-k)}$ counts the so called lonesum 0-1 matrices of size $n \times k$. Several other interpretations were pointed out. We survey these and give new ones. Our new interpretation, for example, gives a transparent, combinatorial explanation of Kaneko's recursive formula for poly-Bernoulli numbers.

1. Introduction

In the 17th century Faulhaber [10] listed the formulas giving the sum of the k^{th} powers of the first n positive integers when $k \leq 17$. These formulas are always polynomials. Jacob Bernoulli [4] realized the scheme in the coefficients of these polynomials. Describing the coefficients he introduced a new sequence of rational numbers. Later Euler [9] recognized the significance of this sequence (that was connected his several celebrated results). He named the elements of the sequence as Bernoulli numbers. For example Bernoulli numbers appear in the closed formula for $\zeta(2k)$ (determining $\zeta(2)$ is the famous Basel problem, that was solved by Euler).

In 1997 Kaneko ([17]) introduced the poly-Bernoulli numbers. Later Arakawa and Kaneko ([18], [2]) established strong connections to multiple zeta values (or Euler-Zagier sums, that were introduced and successfully used in different branches of mathematics earlier).

Definition 1. ([17]) *The poly-Bernoulli numbers* $\{\mathbb{B}_n^{(k)}\}_{n\in\mathbb{N},k\in Z}$ *are defined by the following exponential generating function*

$$\sum_{n=0}^{\infty} \mathbb{B}_n^{(k)} \frac{x^n}{n!} = \frac{Li_k(1-e^{-x})}{1-e^{-x}}, \qquad \textit{for all } k \in \mathbb{Z}$$

where

$$Li_k(x) = \sum_{i=1}^{\infty} \frac{x^i}{i^k}.$$

The sequence $\{\mathbb{B}_n^{(1)}\}_n$ is just the Bernoulli numbers (with $\mathbb{B}_1^{(1)} = +\frac{1}{2}$). Later Kaneko gave a recursive definition of the poly-Bernoulli numbers:

Theorem 2. ([18], quoted in [13])

$$\mathbb{B}_n^{(k)} = \frac{1}{n+1}\left(\mathbb{B}_n^{(k-1)} - \sum_{m=1}^{n-1}\binom{n}{m-1}\mathbb{B}_m^{(k)}\right),$$

or equivalently

$$\mathbb{B}_n^{(k-1)} = \mathbb{B}_n^{(k)} + \sum_{m=1}^{n}\binom{n}{m}\mathbb{B}_{n-(m-1)}^{(k)}.$$

We need a more combinatorial description of poly-Bernoulli numbers, given by Arakawa and Kaneko.

Definition 3. *Let a and b be two natural numbers. The Stirling number of the second kind is the number of partitions of $[a] := \{1, 2, \dots, a\}$ (or any set of a elements) into b classes, and is denoted by $\left\{ {a \atop b} \right\}$.*

Partitions can be described as equivalence relations ([12]).

Theorem 4. ([1]) *For any natural numbers n and k the following formula holds*

$$\mathbb{B}_n^{(-k)} = \sum_{m=0} m!\left\{ {n+1 \atop m+1} \right\} m!\left\{ {k+1 \atop m+1} \right\}.$$

This theorem exhibits the fact that $\mathbb{B}_n^{(-k)}$ numbers are natural numbers. This formula has initiated the combinatorial investigations of poly-Bernoulli numbers. There are several combinatorially described sequence of sets, such that their size is $\mathbb{B}_n^{(-k)}$ (we call them *poly-Bernoulli families*). We can consider these statements as alternative definitions of poly-Bernoulli numbers or as answers to enumeration problems.

These combinatorial definitions give us the possibility to explain previous identities — originally proven by algebraic methods — combinatorially.

The importance of the notion of poly-Bernoulli numbers is underlined by the fact that there are several drastically different combinatorial descriptions.

After reviewing the previous works we give a new poly-Bernoulli family. In our family we consider 0-1 matrices with certain forbidden submatrix, what we call Γ. Our main result is that the number of $n \times k$ 0-1 matrices that avoid the two submatrices $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ is $\mathbb{B}_n^{(k)}$. We present a bijective ([25], [26]) proof of this.

There are several papers investigating enumeration problems of matrices with some forbidden substructure/pattern (see [15], [19]). Their initial setups are different from ours. In [15] the order of rows and order of columns of matrices do not count. By permuting the rows and columns of a matrix M we obtain a matrix that includes the same patterns as M. In [19] the elements of the matrices do not play crucial role. For example exchanging the 0's and 1's in a 0-1 matrix provides a matrix with the same patterns as the initial matrix. Also their techniques are mainly analytical, that is different from our approach.

Finally some classical results are explained combinatorially. Among others a recursion for $\mathbb{B}_n^{(k)}$ is proven based on our interpretation. The proof is simple and straightforward. The recursion is not new — as we mentioned earlier — ([18]), quoted in [13]). All previous explanations of this recursive formula were analytical. It was checked by algebraic manipulation of generating functions. This underlines a point: our main result can be proven using a shortcut. The number of Γ-free matrices satisfy the mentioned recursion (see the end of our paper). We know the same is true for poly-Bernoulli numbers ([18]), hence by induction the main theorem follows.

From a combinatorialist's point of view a simple checking the validity of a formula is not satisfactory. A combinatorial, bijective proof gives a deeper understanding of the meaning of a formula. Stanley ([26]) collected several theorems, identities where no bijective proof is known and urges the combinatorialists to provide one. Our paper is written in the spirit of Stanley's ideas.

The most recent research on poly-Bernoulli numbers is mostly on extensions of the definition of poly-Bernoully numbers and the number theoretical, analytical investigations of these extensions by analytical methods ([3], [16], [23]). Our combinatorial approach is different from the methods of these papers, and it might shed light on some connections and might lead to new directions.

2. Previous poly-Bernoulli families

2.1. The obvious interpretation

Seeing the formula of Arakawa and Kaneko one can easily come up with a combinatorial problem such that the answer to it is $\mathbb{B}_n^{(-k)}$.

Let N be a set of n elements and K a set of k elements. One can think as $N = \{1, 2, \ldots, n\} =: [n]$ and $K = [k]$. Extend both sets with a special element: $\widehat{N} = N \,\dot\cup\, \{n+1\}$ and $\widehat{K} = K \,\dot\cup\, \{k+1\}$. Take $\mathcal{P}_{\widehat{N}}$ a partition of $\widehat{N}$ and $\mathcal{P}_{\widehat{K}}$ a partition of $\widehat{K}$ with the same number of classes as $\mathcal{P}_{\widehat{N}}$. Both partitions have a special class: the class of the special element. We call the other classes as *ordinary* classes. Let m denote the number of ordinary classes in $\mathcal{P}_{\widehat{N}}$ (that is the same as the number of ordinary classes in $\mathcal{P}_{\widehat{K}}$). Obviously $m \in \{0, 1, 2, \ldots, \min\{n, k\}\}$. Order the ordinary classes arbitrarily in both partitions. How many ways can we do this?

For fixed m choosing $\mathcal{P}_{\widehat{N}}$ and ordering its ordinary classes can be done $m!\left\{ {n+1 \atop m+1} \right\}$

ways. Choosing the pair of ordered partitions can be done $m!\left\{{n+1 \atop m+1}\right\}m!\left\{{k+1 \atop m+1}\right\}$ ways. The answer to our question is

$$\sum_{m=0} m!\left\{{n+1 \atop m+1}\right\}m!\left\{{k+1 \atop m+1}\right\} = \mathbb{B}_n^{(-k)}.$$

2.2. Lonesum matrices

Definition 5. ([22]) *A* 0-1 *matrix is lonesum iff it can be reconstructed from its row and column sums.*

Obviously a lonesum matrix cannot contain the

$$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \qquad \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

submatrices (a submatrix is a matrix that can be obtained by deletion of rows and columns). Indeed, in the case of the existence of one of the forbidden submatrices we can switch it to the other one. This way we obtain a different matrix with the same row and column sums. It turns out that this property is a characterization [22].

It is obvious that in a lonesum matrix for two rows 'having the same row sum' is the same relation as 'being equal'. Even more, for rows r_1 and r_2 'the row sum in r_1 is at least the row sum in r_2' is the same as 'r_1 has 1's in the positions of the 1's of r_2'. The same is true for columns. Another easy observation is that changing the order of the rows/columns does not affects the lonesum property. These two observations guarantee that a lonesum matrix can be rearranged by row/column order changes into a matrix where the 1's in each row occupy a leading block of positions and these blocks are non-increasing as we follow the natural order of rows. I.e. substituting the 1s with solid squares we obtain a rotated stairs-like picture (sequence of rectangles of height 1, starting at the same vertical line and with non-increasing width). This type of diagrams are called Young diagrams (see [25] for further information). This is also a characterization of lonesum matrices.

Look at the 1s in a lonesum matrix in the previous normal form, and consider it as a Young diagram: the number of steps (block of rows with the same width) is the number of different non-0 row sums and at the same time it is the number of different non-0 column sums. I.e. the number of different non-0 row sums is the same as the number of different non-0 column sums.

Let M be a 0-1 lonesum matrix of size $n \times k$. Add a special row and column with all 0's. Let $\widehat{M}$ be the extended $(n+1) \times (k+1)$ matrix. 'Having the same row sum' is an equivalence relation. The corresponding partition has a special class, the set of 0 rows. By the extension we ensured that the special class exists/non-empty. Let m be the number of non-special/ordinary classes. The ordinary classes are ordered by their corresponding row sums. In the same way we obtain an ordered partition of columns. It is straightforward to prove that the two ordered partitions give a coding of lonesum matrices. This gives us the following theorem of Brewbaker, first presented in his MSc thesis.

Theorem 6. ([5],[6]) *Let $\mathcal{L}_n^{(k)}$ denote the set of lonesum 0-1 matrices of size $n \times k$. Then*

$$|\mathcal{L}_n^{(k)}| = \sum_{m=0} m! \left\{ {n+1 \atop m+1} \right\} m! \left\{ {k+1 \atop m+1} \right\} = \mathbb{B}_n^{(-k)}.$$

2.3. Callan permutations and max-ascending permutations

Callan [7] considered the set $[n+k]$. We call the elements $1, 2, \ldots, n$ left-value elements (n of them) and $n+1, n+2, \ldots, n+k$ right-value elements (k of them). We extend our universe with 0, a special left-value element and with $n+k+1$, a special right-value element. Let $N = [n]$, $K = \{n{+}1, n{+}2, \ldots, n{+}k\}$, $\widehat{N} = \{0\} \dot\cup [n]$, $\widehat{K} = K \dot\cup \{n+k+1\}$, Consider

$$\pi : 0, \pi_1, \pi_2, \ldots, \pi_{n+k}, n+k+1$$

a permutation of $\widehat{N} \dot\cup \widehat{K}$ with the restriction that its first element is 0 and its last element is $n+k+1$. Consider the following equivalence relation/partition of left-values: two left-values are equivalent iff 'each element in the permutation between them is a left-value'. Similarly one can define an equivalence relation on the right-values: 'each element in the permutation between them is a right-value'. The equivalence classes are just the "blocks" of left- and right-values in permutation π. The left-right reading of π gives an ordering of left-value and right-value blocks/classes. The order starts with a left-value block (the equivalence class of 0, the special class) and ends with a right-value block (the equivalence class of $n+k+1$, the special class). Let m be the common number of ordinary left-value blocks and ordinary right-value blocks.

Callan considered permutations such that in each block the numbers are in increasing order. Let $\mathcal{C}_n^{(k)}$ denote the set of these permutations. For example

$$\mathcal{C}_2^{(2)} = \{012\mathbf{345}, 01\mathbf{3}2\mathbf{45}, 014\mathbf{23}\mathbf{5}, 01\mathbf{34}2\mathbf{5}, 02\mathbf{3}1\mathbf{45}, 0241\mathbf{35}, 02\mathbf{34}1\mathbf{5}, 0\mathbf{3}12\mathbf{45},$$

$$0\mathbf{3}14\mathbf{2}\mathbf{5}, 0\mathbf{3}2\mathbf{4}1\mathbf{5}, 0\mathbf{34}12\mathbf{5}, 041\mathbf{235}, 041\mathbf{3}2\mathbf{5}, 042\mathbf{3}1\mathbf{5}\}$$

(the right-value numbers are in boldface).

It is easy to see that describing a Callan permutation we need to give the two ordered partitions of the left-value and right-value elements. Indeed, inside the blocks the 'increasing' condition defines the order, and the ordering of the classes let us know how to merge the left-value and right-value blocks. We obtain the following theorem.

Theorem 7. ([7])

$$|\mathcal{C}_n^{(k)}| = \sum_{m=0} m! \left\{ {n+1 \atop m+1} \right\} m! \left\{ {k+1 \atop m+1} \right\} = \mathbb{B}_n^{(-k)}.$$

⋆

He, Munro and Rao [14] introduced the notion of max-ascending permutations. This is (in some sense) a "dual" of the notion of Callan permutation. We mention that [24] does not contain this description of poly-Bernoulli numbers. Now we give a slightly different version of max-ascending permutations than the one presented in [14].

Again we consider

$$\pi : 0, \pi_1, \pi_2, \ldots, \pi_{n+k}, n+k+1$$

permutations of $\widehat{N} \dot\cup \widehat{K}$ with the restriction that its first element is 0 and its last element is $n+k+1$. We call the first $n+1$ elements of the permutation left-position elements (0 will be referred to as special left-position element). Consider the following equivalence relation/partition of left-positions: two left-positions, say i and j, are equivalent iff 'any integer v between $\pi(i)$ and $\pi(j)$ occupies a left position in π'. Similarly one can define an equivalence relation on the right-positions: 'each value between the ones, that occupy the positions, is in a right-position'. The max-ascending property of a permutation is that in a class of positions our numbers must be in increasing order.

For example consider the case when $n = 4$ and $k = 2$. Let π be the permutation 621534. We extend it with a first 0 and last 7:

$$\begin{array}{cccccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \pi : & \big(0 & 6 & 2 & 1 & 5 & \mathbf{3} & \mathbf{4} & \mathbf{7}\big). \end{array}$$

The top row contains the positions, and the lower row shows the values that are permuted. The numbers in bold face are the values in right positions ($k+1$ of them). The two left positions 1 and 3 are equivalent, since the corresponding values at these positions are 0 and 2; only the value 1 is between then, and that is at a left position. The two left positions 1 and 2 are not equivalent, since the corresponding values at these positions are 0 and 6; the value 3 is between them, but it is in a right position (in the 6^{th} place). The equivalence classes of left positions are $\{1, 3, 4\}$ and $\{2, 5\}$. The corresponding values standing in the first equivalence class are $0, 1, 2$, and $5, 6$ are the values that occupy the positions of the second class. The equivalence classes of right positions are $\{6, 7\}$ and $\{8\}$. The permutation is not max-ascending permutation: since at the positions $1, 3, 4$ (they form an equivalence class) the values $0, 1, 2$ are not in increasing order. In the case of 512634 the equivalence classes are the same (we permuted the values within positions forming an equivalence class). It is a max-ascending permutation.

We give another example

$$\mathcal{A}_2^{(2)} = \{012\mathbf{345}, 013\mathbf{425}, 013\mathbf{245}, 014\mathbf{235}, 031\mathbf{425}, 031\mathbf{245}, 041\mathbf{235},$$

$$023\mathbf{415}, 023\mathbf{145}, 034\mathbf{125}, 024\mathbf{135}, 024\mathbf{315}, 042\mathbf{135}, 042\mathbf{315}\},$$

where boldface denotes the numbers at right-positions.

The definitions of Callan and max-ascending permutations are very similar. By exchanging the roles of position/value we transform one of them into the other. Specially if we consider our permutations as a bijection from $\{0, 1, \ldots, n+k, n+k+1\}$ to itself then "invert permutation" is a bijection from $\mathcal{C}_n^{(k)}$ to $\mathcal{A}_n^{(k)}$.

Theorem 8. *Let $\mathcal{A}_n^{(k)}$ denote the set of max-ascending permutations of $\{0, 1, 2, \ldots, n+k+1\}$. Then*

$$|\mathcal{A}_n^{(k)}| = \sum_{m=0} m! \left\{ {n+1 \atop m+1} \right\} m! \left\{ {k+1 \atop m+1} \right\} = \mathbb{B}_n^{(-k)}.$$

2.4. Vesztergombi permutations

Vesztergombi [27] investigated permutations of $[n+k]$ with the property that $-k < \pi(i) - i < n$. She determined a formula for their number. Lovász ([21] Exercise 4.36.) gives a combinatorial proof of this result. Launois working on quantum matrices slightly modified Vesztergombi's set and realized the connection to poly-Bernoulli numbers. Many of these results are summarized in the following theorem.

Theorem 9. *Let $\mathcal{V}_n^{(k)}$ denote the set of permutations π of $[n+k]$ such that $-k \leq \pi(i) - i \leq n$ for all i in $[n+k]$.*

$$|\mathcal{V}_n^{(k)}| = \sum_{m=0} m! \left\{ {n+1 \atop m+1} \right\} m! \left\{ {k+1 \atop m+1} \right\} = \mathbb{B}_n^{(-k)}.$$

2.5. Acyclic orientations of $K_{n,k}$

Very recently P. Cameron, C. Glass and R. Schumacher gave a new combinatorial interpretation of poly-Bernoulli numbers. (Peter Cameron published this in a note at his blog on 19th of January, 2014 [8].)

Let $K_{n,k}$ be the complete bipartite graph with color classes of size n and k. An orientation of a graph is acyclic iff it does not contain a directed cycle.

Theorem 10. ([8]) *The number of acyclic orientations of $K_{n,k}$ is $\mathbb{B}_n^{(-k)}$.*

Let $\mathcal{O}_n^k$ be the set of acyclic orientations of $K_{n,k}$. A simple graph theoretical observation gives us that in a complete bipartite graph acyclicity is equivalent to 'not having oriented C_4'. The theorem is immediate from a bijection between $\mathcal{O}_n^k$ and $\mathcal{L}_n^k$ (i.e. set of lonesum matrices). The bijection is easy and natural. Identify the two parts of nodes in $K_{n,k}$ with the rows and columns of a matrix size $n \times k$. Any edge has two possible orientations, hence we can code an actual oriented edge by a bit (0/1) according to its direction between the two color classes. The oriented graph can be coded by a 0-1 matrix of size $n \times k$. Forbidding oriented C_4's is equivalent to forbidding two submatrices of size 2×2. These matrices are the same as the ones in Ryser's characterization of lonesum matrices. So the desired bijection is just the simple coding we have described.

3. A new poly-Bernoulli family

Let M be a 0-1 matrix. We say that three 1s in M form a Γ configuration iff two of them are in the same row (one, let us say a, precedes the other) and the third is under a. I.e. the three 1s form the upper left, upper right and lower left elements of a submatrix of size 2×2. So we do not have any condition on the lower right element of the submatrix of size 2×2, containing the Γ.

We will consider matrices without Γ configuration. Let $\mathcal{G}_n^{(k)}$ denote the set of all 0-1 matrices of size $n \times k$ without Γ.

The following theorem is our main theorem.

Theorem 11.

$$|\mathcal{G}_n^{(k)}| = \mathbb{B}_n^{(-k)}.$$

The rest of the section is devoted to the combinatorial proof of this statement.

The obvious way to prove our claim is to give a bijection to one of the previous sets, where the size is known to be $\mathbb{B}_n^{(-k)}$. The obvious candidate is $\mathcal{L}_n^{(k)}$. Γ-free matrices were considered from the point of extremal combinatorics (see [11]). It is known that Γ-free matrices of size $n \times k$ contain at most $n + k - 1$ many 1s. Among Brewbaker's lonesum matrices (in contrast) there are some with many 1s (for example the all-1 matrix) and there are others with few 1s. We do not know straight, simple bijection between lonesum matrices and matrices with no Γ. Instead, we follow the obvious scheme: we code Γ-free matrices with two partitions and two orders. From this and from the previous bijections one can construct a direct bijection between the two sets of matrices but that is not appealing.

Proof. Let M be a 0-1 matrix of size $n \times k$. We say that a position/element has *height* $n-i$ iff it is in the i^{th} row. The *top*-1 of a column is its 1 element of maximal height. The *height of a column* is the height of its top-1 or 0, whenever it is a 0 column.

Let M be a matrix without Γ configuration. Let $\widehat{M}$ be the extension of it with an all 0s column and row. (We have defined the *height of all-0 columns* to be 0. In $\widehat{M}$ non-0 columns have 0 at the bottom, hence their heights are at least 1.) 'Having the same height' is an equivalence relation on the set of columns in $\widehat{M}$. The class of the additional column is the set of 0 columns (that is not empty since we work with the extended matrix). We call the class of the additional column 'the special class'. Its elements are the special columns. So special column means 'all-0 column'. The additional column in $\widehat{M}$ ensures that we have this special class. The other classes are the ordinary classes. Let m be the number of the ordinary classes. These m classes partition the set of non-0 columns. The total number of equivalence classes is $m + 1$.

In order to clarify the details after the formal description we explain the steps on a specific example. $\diamondsuit$ denote the end of example, when we return to the abstract discussion,

Example. M is a Γ-free matrix:

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \widehat{M} = \left(\begin{array}{cccccccc|c} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$\widehat{M}$ is "basically the same" as M. It only contains an additional all-0 column (the last one), and an additional all-0 row (the last one). In $\widehat{M}$ each column has a height. The height depends on the position of the top-1 of the considered column (it counts how many positions are under it). The all-0 column has height 0. We circled the top-1s and marked the height of the columns at the upper border of our matrix $\widehat{M}$:

$$\widehat{M} = \begin{matrix} 4 & 6 & 4 & 3 & 0 & 3 & 1 & 5 & 0 \\ \end{matrix}$$
$$\begin{pmatrix} 0 & \textcircled{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \textcircled{1} & 0 \\ \textcircled{1} & 0 & \textcircled{1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \textcircled{1} & 0 & \textcircled{1} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \textcircled{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

'Having the same height' is an equivalence relation among the columns. In our example there are six different heights considering all the columns: $0, 1, 3, 4, 5, 6$. Two columns have height 0 (they are the two all-0 columns). One of them is the additional column of $\widehat{M}$, the other was present in M. They are the special columns, forming the special class of our equivalence relation on columns. The other five heights define five ordinary classes. One of these classes is formed by the first and third column, they are the columns with height 4. $\diamondsuit$

Take $\mathcal{C}$, any non-special class of columns (the columns in $\mathcal{C}$ are ordered as the indices order the whole set of columns). Since our matrix does not contain Γ all columns but the last one has only one 1 (that is necessarily the top-1) of the same height. We say that the last elements/columns of non-special classes are *important columns*. Important columns in $\widehat{M}$ form a submatrix M_0 of size $(n+1) \times m$.

In M_0 the top-1s are called *important elements*. In each row without top-1 the *leading* 1 (the 1 with minimal column index) is also called *important* 1. So in all non-0 rows of M_0 there is exactly one important 1.

Each row has an 'indentation': the position of the important 1, i.e. last top-1 if the row contains a top-1, otherwise the position of the first 1 (or 0 if the row is all 0s). The row indentations determine a partition of the set of rows.

The two partitions have the same number of parts, namely, $m + 1$ where m was introduced when describing the column partition.

The last top-1s are in different rows and columns, hence determine an $m \times m$ submatrix which becomes a permutation matrix if all entries except the last top-1s are zeroed out. This permutation matrix determines an identification of the ordinary column classes and ordinary row classes.

Example. M_0 contains the last columns of the ordinary equivalence classes. In our example it has 5 columns (the upper border of our example we see the common height of the column class, and the original index of each row).

The top-1s are circled. There are two rows without top-1. The last row is all-0, the other in not all-0. Its leading 1 is in bold face. We also marked (at the left border of M_0) the indentations of rows. The indentation/label of an all-0 row is 0:

$$M_0 = \begin{array}{c} \\ 1\\5\\2\\3\\2\\4\\0 \end{array} \begin{array}{c} \begin{array}{ccccc} \overset{6}{2^{\text{nd}}} & \overset{4}{3^{\text{rd}}} & \overset{3}{6^{\text{th}}} & \overset{1}{7^{\text{th}}} & \overset{5}{8^{\text{th}}} \end{array} \\ \begin{pmatrix} ① & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & ① \\ 0 & ① & 0 & 0 & 0 \\ 1 & 0 & ① & 0 & 0 \\ 0 & \mathbf{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & ① & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}.$$

Note that since each column has a top-1 we have $1, 2, 3, 4, 5$ as labels (all-0 columns form the special column class that has no representative in M_0). The last row of M_0 is an all-0 row, hence we have the 0 label too.

The top-1s define an identification of ordinary row and column classes. We use the letters a, b, c, d, e for the identified row/column classes. s marks the special rows (in our case there is only one special row, the last row):

$$M_0 = \begin{array}{c} \\ a\\e\\b\\c\\b\\d\\s \end{array} \begin{array}{c} \begin{array}{ccccc} \overset{a}{2^{\text{nd}}} & \overset{b}{3^{\text{rd}}} & \overset{c}{6^{\text{th}}} & \overset{d}{7^{\text{th}}} & \overset{e}{8^{\text{th}}} \end{array} \\ \begin{pmatrix} ① & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & ① \\ 0 & ① & 0 & 0 & 0 \\ 1 & 0 & ① & 0 & 0 \\ 0 & \mathbf{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & ① & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}.$$

◊

A partition of columns into $m+1$ classes, and partition of rows into $m+1$ classes, and a bijection between the non-special row- and column-classes — after fixing m — leaves

$$m!\left\{ {n+1 \atop m+1} \right\}\left\{ {k+1 \atop m+1} \right\}$$

possibilities. This information (knowing the two partitions and the correspondence) codes a big part of matrix $\widehat{M}$:

We know that the columns and rows of the special classes are all 0s. A non-special column class $\mathcal{C}$ has a corresponding class of rows. The top row of the corresponding row class gives us the common height of the columns in $\mathcal{C}$. So we know each non-important columns (they have only one 1, defining its known height). We narrowed the unknown 1s of M into the non-0 rows of M_0. Easy to check that $\widehat{M}$ contains Γ iff M_0 contains one.

Example. Let us consider our example. $m = 5$ and the row/column partitions are marked at the left and upper border of our matrix. We use a, b, c, d, e, s as names for the classes in both cases (hence the classes of the two partitions are identified). s denotes the two special classes (the class of the last row and last column).

$$\widehat{M} = \begin{array}{c} \\ a \\ e \\ b \\ c \\ b \\ d \\ s \end{array} \begin{array}{c} \begin{array}{ccccccccc} b & a & b & c & s & c & d & e & s \end{array} \\ \left(\begin{array}{ccccccccc} & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \end{array} \right) \end{array}$$

We can recover a large portion of $\widehat{M}$ from this information. The class of the last column, and the class of the last row are the two special classes (named by s). We must have all 0s in these rows/columns.

The columns of M_0 are the last columns of the ordinary classes. For example the top-1s in the column class a can be decoded from the rows belonging to class a: The highest row with label a marks the common row of top-1s in columns labelled by a. This way we can determine the common heights of the ordinary column classes, hence recover the top-1s. Then we know all elements above a top-1 must have value 0. All columns with label a but the last one contains only its top-1 as non-0 element.

In our example we sum up the information gained so far (the top border contains the recovered heights and labels for the columns from M_0):

$$\widehat{M} = \begin{array}{c} b \\ f \\ a \\ c \\ a \\ e \\ d \end{array} \begin{array}{c} \begin{array}{ccccccccc} 4 & \overset{6}{M_0} & \overset{4}{M_0} & 3 & 0 & \overset{3}{M_0} & \overset{1}{M_0} & \overset{5}{M_0} & 0 \end{array} \\ \left(\begin{array}{ccccccccc} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & & 1 & 0 & 0 & 0 & 0 & & 0 \\ 0 & & & 1 & 0 & 1 & 0 & & 0 \\ 0 & & & 0 & 0 & & 0 & & 0 \\ 0 & & & 0 & 0 & & 1 & & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right) \end{array}$$

Knowing the top-1s enable us to recover the indentations (relative to M_0) belonging to the ordinary row classes. If there is a row without a top-1, then from its row label we know the position of its first 1 (hence we know that in that row at previous positions we have 0s):

$$
\widehat{M} = \begin{array}{r} \\ 1/M_0 \\ 5/M_0 \\ 2/M_0 \\ 3/M_0 \\ 2/M_0 \\ 4/M_0 \\ 0/M_0 \end{array}
\begin{array}{c} \begin{array}{ccccccccc} 4 & \overset{6}{M_0} & \overset{4}{M_0} & 3 & 0 & \overset{3}{M_0} & \overset{1}{M_0} & \overset{5}{M_0} & 0 \end{array} \\
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & & 1 & 0 & 0 & 0 & 0 & & 0 \\
0 & & & 1 & 0 & 1 & 0 & & 0 \\
0 & 0 & 1 & 0 & 0 & & 0 & & 0 \\
0 & & & 0 & 0 & & 1 & & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} \end{array}
$$

◇

Note that there are many positions where we do not know the elements of our matrix (all are located in M_0). Also when counting the possibilities we have a missing $m!$ factor. The rest of the proof shows that filling in the missing elements (resulting a Γ-free matrix) can be done $m!$ many ways.

Now on we concentrate on M_0 (that is where the unknown elements are). The positions of important ones are known. In each column of M_0 there is a lowest important 1. We call them *crucial* 1s. (Specially crucial 1s are important 1s too.) We have m many crucial 1s, one is in each column of M_0. A 1 in M_0 that is non-important is called *hiding* 1.

Lemma 12. *Consider a hiding* 1 *in* M_0*. Then exactly one of the following two possibilities holds:*

(1) there is a crucial 1 *above it and a top-*1 *to the right of it,*

(2) there is a crucial 1 *on its left side (and of course a top-*1 *above it).*

Example. The following figure exhibits the two options: $1^{(h1)}$, $1^{(h2)}$ are two hiding 1s, corresponding case (1) and case (2) respectively. Top-1s are the circled 1s. The two hiding 1s "share" the crucial 1 in the lemma.

$$
\begin{pmatrix}
 & \vdots & & & & \\
 & & \cdots & \textcircled{1} & & \\
 & \vdots & & \uparrow & & \\
\cdots & 1^{(c)} & \leftarrow & 1^{(h2)} & & \\
 & \uparrow & & \vdots & & \\
 & 1^{(h1)} & \rightarrow & & \rightarrow & \textcircled{1} \\
 & \vdots & & & &
\end{pmatrix}
$$

◇

Proof. Let h be a hiding 1 in M_0.

First, assume that the row of h does not contain a top-1. Then the first 1 in this row (f) is an important 1 (hence it differs from h). Since the matrix is Γ-free, we cannot have a 1 under f, i.e. f is a crucial 1. h is not important, so it is not a top-1. The top-1 in its column must be above it. We obtained that case (2) holds.

Second, assume that the row of h contains a top-1, t. If t is on the left of h then the forbidden Γ ensures that under t there is no other 1. Hence t is crucial and case (2) holds again. If t is on the right of h then the forbidden Γ ensures that under h there is no other 1. Hence the lowest important 1 in the column of h (a crucial 1) is above of it. Case (1) holds.

(1) and (2) cases are exclusive since if both are satisfied then h has a crucial 1 on its left and a top-1 on its right. That is impossible since the 1s in a row of a top-1 are not even important. ☐

Let h be a hiding 1. There must be a unique crucial 1 corresponding to it: If h satisfies case (1), then it is the crucial one above it. If h satisfies case (2), then it is the crucial one on the left side of it. In this case we say that this crucial c is *responsible* for h.

Take a crucial 1 in M_0, that we call c. For any top-1, t that comes in a later column and it is higher than c the position in the row of c under t we call *questionable*. Also for any top-1, t that comes in a later column and it is lower than c the position in the column of c before t we call *questionable*. In M_0 there are m many crucial 1. If c is in the i^{th} column, then there are $m-i$ column that comes later and each defines one questionable position.

The meaning of the lemma is that each hiding 1 must be in a questionable position.

Example. We continue our previous example (but only M_0 is followed on). We circled the top-1s in M_0. We added an index c to crucial 1s. (All the important 1s are identified so we are able to locate these elements.

$$\widehat{M} = \begin{pmatrix} \textcircled{1}^{(c)} & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & \textcircled{1}^{(c)} \\ & \textcircled{1} & 0 & 0 & \\ & & \textcircled{1}^{(c)} & 0 & \\ 0 & 1^{(c)} & & 0 & \\ & & & \textcircled{1}^{(c)} & \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

According to our argument, for the crucial 1 in the first column there are four top-1s in the later columns and there are four questionable positions corresponding to them. We mark them as $?_1$. Similarly, three questionable positions belongs to the crucial 1 in the second column, marked as $?_2$. (If there is hiding 1 in one of these positions then the crucial 1 of the second column would be responsible to it). We put ? to each questionable position and add an index marking the column of the crucial 1 that is connected to it:

$$M_0 = \begin{pmatrix} \textcircled{1}_c & 0 & 0 & 0 & 0 \\ ?_1 & 0 & 0 & 0 & \textcircled{1}^{(c)} \\ ?_1 & \textcircled{1} & 0 & 0 & \\ ?_1 & & \textcircled{1}^{(c)} & 0 & ?_3 \\ 0 & 1^{(c)} & ?_2 & 0 & ?_2 \\ ?_1 & ?_2 & ?_3 & \textcircled{1}^{(c)} & ?_4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Note that there are positions that are not questionable. The lemma says there can not be a hiding 1. Indeed, a 1 at these positions would create a Γ. $\diamondsuit$

First rephrase our lemma:

Corollary 13. *All hiding* 1*s are in questionable positions.*

It is obvious that we have $(m-1)+(m-2)+\ldots+2+1$ many questionable positions (to the crucial 1 in the i^{th} column there are $m-i$ many questionable position is assigned).

Easy to check that if we put the important 1s into M_0 and add a new 1 into a questionable position then we won't create a Γ configuration. The problem is that the different questionable positions are not independent.

Lemma 14. *There are* $m!$ *ways to fill the questionable positions with* 0*s and* 1*s without forming a* Γ*.*

Proof. Let c be a crucial 1. We divide the set of questionable positions that corresponds to c, depending their positions relative to c into two parts: Let R_c be the set of questionable positions in the row of c, that is right from c. Let D_c be the set of questionable positions in the column of c, that is down from c.

The following two observation is immediate:

(i) At most one of R_c and D_c contains a 1.

(ii) If D_c contains a 1 (hence R_c is empty), then it contains only one 1.

Indeed, if the two claims are not true then we can easily recognize a Γ.

For each crucial c describe the following 'piece of information': I_1: the position of the first 1 in R_c or I_2: the position of the only one 1 in D_c (this informs us that R_c contains only 0s) or I_3: say "all the positions of $R_c \cup D_c$ contain 0".

If c comes from the first column of M_0, then we have m many outcomes for this piece of information. $m-1$ many of these are such that one position of a 1 is revealed (the first 1 in R_c or the the only one 1 in D_c). Placing a 1 there doesn't harm the Γ-free property of our matrix. One possible outcome of the information is the one that reveals that there is no 1 in $R_c \cup D_c$.

Example. In our example let c be the crucial 1 of the first column, i.e. the first element of the first row:

$$M_0 = \begin{pmatrix} c = ①^{(c)} & 0 & 0 & 0 & 0 \\ ?_1 & 0 & 0 & 0 & ①^{(c)} \\ ?_1 & ① & 0 & 0 & 0 \\ ?_1 & 0 & ①^{(c)} & 0 & ?_3 \\ 0 & 1^{(c)} & ?_2 & 0 & ?_2 \\ ?_1 & ?_2 & ?_3 & ①^{(c)} & ?_4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

R_c is empty, D_c contains four positions from the first column (2^{nd}, 3^{rd}, 4^{th}, 6^{th}). So when we reveal the above mentioned information about c then we have the following 5 possibile outcomes:

$I_2(2)$: "The only 1 under c is in the second row and there is no 1 in the row of c."

$I_2(3)$: "The only 1 under c is in the third row and there is no 1 in the row of c."

$I_2(4)$: "The only 1 under c is in the fourth row and there is no 1 in the row of c."

$I_2(6)$: "The only 1 under c is in the sixth row and there is no 1 in the row of c."

I_3: "There is no 1 under and after c."

Let us assume that we get the the third possibility ($I_2(4)$) as the additional information. Then we can continue filling the missing elements of M_0:

$$M_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & ①^{(c)} \\ 0 & ① & 0 & 0 & 0 \\ 1 & 0 & ①^{(c)} & 0 & ?_3 \\ 0 & c = 1^{(c)} & ?_2 & 0 & ?_2 \\ 0 & ?_2 & ?_3 & ①^{(c)} & ?_4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Now let c the crucial 1 in the second column. R_c contains two positions (3^{rd} and 5^{th} column), D_c contains one position from the column of the actual c (the one in the 6^{th} row). So when we reveal the above mentioned information about c then we have the following 4 possibilities:

$I_1(3)$: "The first 1 after c is in the third column and there is no 1 in D_c."

$I_1(5)$: "The first 1 after c is in the fifth column and there is no 1 in D_c."

$I_2(6)$: "The only 1 under c is in the sixth row and there is no 1 in the row of c."

I_3: "There is no 1 under and after c."

Let us assume that we get the the first possibility ($I_1(3)$) as an additional information. The hidden 1 that is revealed is under the crucial 1 (c') of its column. The Γ-free property of M (and hence M_0) guarantees that cannot be a hiding 1 after c'. Again we summarize the information gained:

$$M_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & ①^{(c)} \\ 0 & ① & 0 & 0 & 0 \\ 1 & 0 & c' = ①^{(c)} & 0 & 0 \\ 0 & 1^{(c)} & 1 & 0 & ?_2 \\ 0 & 0 & ?_3 & ①^{(c)} & ?_4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

In the third column the old crucial 1 (c') will be replaced by the 1, (c) revealed by the previous information.

$$M_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & ①^{(c)} \\ 0 & ① & 0 & 0 & 0 \\ 1 & 0 & c' = ① & 0 & 0 \\ 0 & 1^{(c)} & c = 1^{(c)} & 0 & ?_3 \\ 0 & 0 & ?_3 & ①^{(c)} & ?_4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\diamondsuit$

If we get I_2 or I_3 then we know all the elements at the questionable positions corresponding to c. In this case we can inductively continue and finish the description of M. If the information, we obtain is I_1 then our knowledge about the 1s at the questionable positions corresponding to c is not complete. But we can deduce many additional information.

Assume that I_1 says that on the right of c the first 1 in questionable position is in the j^{th} column. Let $\widetilde{c}_j$ be the position of this 1. The position c_j is above of it. We know that R_i doesn't contain a 1 (indeed, that would form a Γ with the 1s at c_j and at $\widetilde{c}_j$). For similar reasons also we cannot have a 1 at a questionable position between c_j and $\widetilde{c}_j$.

This knowledge guarantees that we can substitute c_j with $\widetilde{c}_j$ ($\widetilde{c}_j$ will be a crucial 1 substituting c_j). The corresponding questionable positions will be the questionable positions that are down and right from it. We still encounter all the hiding ones (there must be at the questionable positions corresponding to the crucial 1s, we didn't confronted yet). So we can induct.

The above argument proves that any element of $\{1, 2, \ldots, m\} \times \{1, 2, \ldots, m-1\} \times \{1, 2\} \times \{1\}$ codes the outcome of the information revealing process, hence a Γ-free completion of our previous knowledge. The i^{th} component of the code says that in the i^{th} column of M_0 which information on the actual crucial 1 is true. Our previous argument just describe how to do the first few steps of the decoding and how to recursively continue it. □

The lemma finishes the enumeration of Γ-free 0-1 matrices of size $n \times k$. Also finishes a description of a constructive bijection from $\mathcal{G}_n^{(k)}$ to the obvious poly-Bernoulli set. Our main theorem is proven a bijective way.

4. Combinatorial proofs

Theorem 15.

$$\mathbb{B}_n^{(-k)} = \mathbb{B}_k^{(-n)}.$$

The relation originally was proven by Kaneko. It is obvious from any of the combinatorial definitions. Arakawa–Kaneko formula also exhibits this symmetry an algebraic way.

Theorem 16.

$$\mathbb{B}_n^{(-k)} = \mathbb{B}_n^{(-(k-1))} + \sum_{j=1}^{n} \binom{n}{j} \mathbb{B}_{n-(j-1)}^{(-(k-1))}.$$

Proof. Our main theorem gives that $\mathbb{B}_n^{(-k)}$ counts the Γ-free matrices of size $n \times k$.

Each row of a Γ-free matrix

A. starts with a 0, or

B. starts with a 1, followed only by 0s, or

C. starts with a 1, and contains at least one more 1.

Let j denote the number of rows of type B/C.

If $j = 0$, then the first column is all-0 column, and it has $\mathbb{B}_n^{(-(k-1))}$ many extensions as Γ-free matrix.

If $j \geq 1$, then we must choose the j many rows of type B/C. Our decision describes the first column of our matrix. The first $j - 1$ many chosen rows cannot contain any other 1, since a Γ would appear. I.e. they are type B, and completely described.

The further elements (a submatrix of size $(n - j + 1) \times (k - 1)$) can be filled with an arbitrary Γ-free matrix. The recursion is proven. □

We can state the theorem (without a reference to the main theorem) as a recursion for $|\mathcal{G}_n^{(k)}|$. Since the same recursion is known for $\mathbb{B}_n^{(-k)}$, an easy induction proves the main theorem. Our first proof, the main part of this paper is purely combinatorial and explains a previously known recursion without algebraic manipulations of generating functions.

Theorem 17.

$$\sum_{n,k\in\mathbb{N}:n+k=N} (-1)^n \mathbb{B}_n^{(-k)} = 0.$$

Proof. We use Callan's description of poly-Bernoulli numbers. We consider Callan permutations of N objects (the extended base set has size $N+2$). We underline that to speak about Callan permutations we must divide the N objects into left and right value category. For this we need to write N as a term sum: $n+k$.

For technical reasons we change the base set of our permutations. The extended left values remain $0, 1, 2, \ldots, n$, the extended right values will be $\mathbf{1}, \mathbf{2}, \ldots, \mathbf{k}, \mathbf{k+1}$. We note that the calligraphic distinction between the left and right values allows us to use any $n+1$ numbers for the extended left values, and the same is true for the right values.

The combinatorial content of the claim is that if we consider Callan permutation of N objects (with all possible $n+k$ partitions), then those where the number of left values is even has the same cardinality as those where the number of left values is odd.

$\mathbb{B}_n^{(-k)}$ is the size of $\mathcal{C}_n^{(k)}$. We divide it into two subsets according to the type of the element following the leading 0. Let $\mathcal{C}_n^{(k)}(\ell)$ be the set of those elements from $\mathcal{C}_n^{(k)}$, where the leading 0 is followed by a left value element. Let $\mathcal{C}_n^{(k)}(r)$ be the set of those elements from $\mathcal{C}_n^{(k)}$, where the leading 0 is followed by a right value element.

We give two examples:

$$\mathcal{C}_3^{(1)}(\ell) = \{01\mathbf{1}23\mathbf{2}, 02\mathbf{1}13\mathbf{2}, 03\mathbf{1}12\mathbf{2}, 012\mathbf{1}3\mathbf{2}, 013\mathbf{1}2\mathbf{2}, 023\mathbf{1}1\mathbf{2}, 0123\mathbf{12}\},$$

$$\mathcal{C}_2^{(2)}(r) = \{0\mathbf{1}1\mathbf{2}2\mathbf{3}, 0\mathbf{12}21\mathbf{3}, 0\mathbf{1}12\mathbf{23}0\mathbf{12}12\mathbf{3}, 0\mathbf{2}1\mathbf{1}2\mathbf{3}, 0\mathbf{2}2\mathbf{1}1\mathbf{3}, 0\mathbf{2}12\mathbf{13}, \}.$$

We will describe a $\varphi : \mathcal{C}_n^{(k)}(\ell) \to \mathcal{C}_{n-1}^{(k+1)}(r)$ bijection. (Hence we will have a $\varphi : \mathcal{C}_n^{k}(r) \to \mathcal{C}_{n+1}^{(k-1)}(\ell)$ bijection too.) Our map reverses the parity of the number of left values and completes the proof.

The bijection goes as follows: Take a permutation from $\mathcal{C}_n^{(k)}(\ell)$. Find '1' in the permutation. It follows the leading 0 or it will be the first element of a block of left values (that is preceded by a block of right value, say R). In the first case we substitute 1 by $\mathbf{0}$. In the second case we also substitute 1 by $\mathbf{0}$ but additionally we move the R block right after the leading 0.

We warn the reader that the image permutations have extended left values $0, 2, \ldots, n$ and extended right values $\mathbf{0}, \mathbf{1}, \ldots, \mathbf{k+1}$. This change does not effect the essence.

An example helps to digest the technicalities:

$$012\mathbf{1}3\mathbf{2} \to 0\mathbf{0}2\mathbf{1}3\mathbf{2} \equiv 0\mathbf{1}1\mathbf{2}2\mathbf{3},$$

$$03\mathbf{1}12\mathbf{2} \to 0\mathbf{1}3\mathbf{0}2\mathbf{2} \equiv 0\mathbf{2}2\mathbf{1}1\mathbf{3}.$$

The inverse of our map can be easily constructed. It must be based on $\mathbf{1}$. The details are left to the reader. $\square$

5. Conclusion

We presented a summary of previous descriptions of poly-Bernoulli numbers, including a new one. Our list isn't as respectful as Stanley's list for Catalan numbers, but suggests that poly-Bernoulli numbers are natural and central. The last two proofs are the only combinatorial explanations (as far we know) for basic relationships for poly-Bernoulli numbers. We expect further combinatorial definitions and proofs, enriching the understanding poly-Bernoulli numbers.

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