EXPONENTIALLY S-NUMBERS

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Abstract. Let $\mathbf{S}$ be the set of all finite or infinite increasing sequences of positive integers. For a sequence $S = \{s(n)\}, n \geq 1$, from $\mathbf{S}$, let us call a positive number N an exponentially S-number ($N \in E(S)$), if all exponents in its prime power factorization are in S. Let us accept that $1 \in E(S)$. We prove that, for every sequence $S \in \mathbf{S}$ with $s(1) = 1$, the exponentially S-numbers have a density $h = h(E(S))$ such that

$$\sum_{i \leq x,\ i \in E(S)} 1 = h(E(S))x + O(\sqrt{x} \log x e^{c \frac{\sqrt{\log x}}{\log \log x}}),$$

where $c = 4\sqrt{\frac{2.4}{\log 2}} = 7.4430...$ and $h(E(S)) = \prod_p (1 + \sum_{i \geq 2} \frac{u(i) - u(i-1)}{p^i})$, where $u(n)$ is the characteristic function of S.

1. Introduction

Let $\mathbf{S}$ be the set of all finite or infinite increasing sequences of positive integers. For a sequence $S = \{s(n)\}, n \geq 1$, from $\mathbf{S}$, let us call a positive number N an exponentially S-number ($N \in E(S)$), if all exponents in its prime power factorization are in S. Let us accept that $1 \in E(S)$. For example, if $S = \{1\}$, then the exponentially 1-numbers form the sequence B of square-free numbers, and, as well-known,

$$\sum_{i \leq x,\ i \in B} 1 = \frac{6}{\pi^2} x + O(x^{\frac{1}{2}}). \tag{1}$$

In case, when $S = B$, we obtain the exponentially square-free numbers (for the first time this notion was introduced by M. V. Subbarao in 1972 [6], see A209061[5]). Namely the exponentially square-free numbers were studied by many authors (for example, see [2], [6] (Theorem 6.7), [7], [8], [9]). In these papers, the authors analyzed the following asymptotic formula

$$\sum_{i \leq x,\ i \in E(B)} 1 = \prod_p (1 + \sum_{a=4}^{\infty} \frac{\mu^2(a) - \mu^2(a-1)}{p^a}) x + R(x), \tag{2}$$

where the product is over all primes, μ is the Möbius function. The best result of type $R(x) = o(x^{\frac{1}{4}})$ was obtained by Wu (1995) without using RH (more exactly see [9]). In 2007, assuming that RH is true, Tóth [8] obtained $R(x) = O(x^{\frac{1}{5}+\varepsilon})$ and in 2010, Cao and Zhai [2] more exactly found that $R(x) = Cx^{\frac{1}{5}} + O(x^{\frac{38}{193}+\varepsilon})$, where C is a computable constant. Besides, Tóth [7] studied also the exponentially k-free numbers, $k \geq 2$.

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In this paper, without using RH, we obtain a general formula with a remainder term $O(\sqrt{x}\log x e^{c\frac{\sqrt{\log x}}{\log\log x}})$ (c is a constant) not depended on $S \in \mathbf{S}$ beginning with 1. More exactly, we prove the following.

Theorem 1. *For every sequence $S \in \mathbf{S}$ the exponentially S-numbers have a density $h = h(E(S))$ such that, 1) if $s(1) > 1$, then $h = 0$, while 2) if $s(1) = 1$, then*

$$\sum_{i\le x,\ i\in E(S)} 1 = h(E(S))x + O(\sqrt{x}\log x e^{c\frac{\sqrt{\log x}}{\log\log x}}), \tag{3}$$

with $c = 4\sqrt{\frac{2.4}{\log 2}} = 7.443083...$ and

$$h(E(S)) = \prod_p (1 + \sum_{i\ge 2} \frac{u(i) - u(i-1)}{p^i}), \tag{4}$$

where $u(n)$ is the characteristic function of sequence S: $u(n) = 1$, if $n \in S$ and $u(n) = 0$ otherwise.

In particular, in case $S = B$ we obtain (2) with a less good remainder term, but which is suitable for all sequences in $\mathbf{S}$ beginning with 1.

2. Lemma

For proof Theorem 1 we need a lemma proved earlier (2007) by the author [4], pp.200-202. For a fixed square-free number r, denote by B_r the set of square-free numbers n for which $\gcd(n, r) = 1$, and put

$$b_r(x) = |B_r \cap \{1, 2, ..., x\}|.$$

In particular, $B = B_1$ is the set of all square-free numbers.

Lemma 1.

$$b_r(x) = \frac{6r}{\pi^2}\prod_{p|r}(p+1)^{-1}x + R_r(x),$$

where for every $x \ge 1$ and every $r \in B$

$$|R_r(x)| \le \begin{cases} k\sqrt{x}, & \text{if } r \le N \\ ke^{c\frac{\sqrt{\log r}}{\log\log r}}\sqrt{x}, & \text{if } r \ge N+1. \end{cases}$$

where $k = 3.5\prod_{2\le p\le 23}(1 + \frac{1}{\sqrt{p}}) = 57.682607...$ (in case $r = 1$, $k = 3.5$), $c = 4\sqrt{\frac{2.4}{\log 2}} = 7.443083...$, $N = 6469693229$.

3. Proof of Theorem 1

1) Denote by Υ the sequence $\{2, 3, 4, ...\}$ of all natural numbers without 1. Let S do not contain 1. Then, evidently, $E(S) \subseteq E(\Upsilon)$. Note that the sequence $E(\Upsilon)$ is called also powerful numbers (sequence A001694 in [5]).

Bateman and Grosswald [1] proved that

$$\sum_{i\le x,\ \ i\in E(\Upsilon)} 1=\frac{\zeta(3/2)}{\zeta(3)}x^{1/2}+\frac{\zeta(2/3)}{\zeta(2)}x^{1/3}+O(x^{1/6}). \tag{5}$$

So, $h(E(\Upsilon))=0$. Then what is more $h(E(S))=0$.

Furthermore, *denote by* $r(n)$ *the product of all distinct prime divisors of* n; set $r(1)=1$.

2) Now let $1\in S$. Note that the set $E(\Upsilon)\cap E(S)$ contains 1 and all numbers of $E(S)$ whose exponents in their prime power factorizations are more than 1. Evidently, every number $y\in E(S)$ has a unique representation as the product of some number $a\in E(\Upsilon)\cap E(S)$ and a number $m\in B_{r(a)}$. In particular, if y is square-free, then $a=1,\ m=y(\in B_1)$. For a fixed $a\in E(\Upsilon)\cap E(S)$, denote the set of $y=am\in E(S)$ by $E(S)^{(a)}$. Then $E(S)=\bigcup\limits_{a\in E(S)\cap E(\Upsilon)} E(S)^{(a)}$, where the union is disjoint. Consequently, by Lemma 1, we have

$$\sum_{i\le x,\ \ i\in E(S)} 1=b_1(x)+\sum_{4\le a\le x,\ \ a\in E(S)\cap E(\Upsilon)} b_{r(a)}\left(\frac{x}{a}\right) \tag{6}$$

$$=\frac{6}{\pi^2}\left(1+\sum_{4\le a\le x,\ \ a\in E(S)\cap E(\Upsilon)}\ \prod_{p|r(a)}\left(1-\frac{1}{p+1}\right)\frac{1}{a}\right)x+R(x),$$

where

$$|R(x)|\le 3.5\sqrt{x}+\sum_{4\le a\le x,\ \ a\in E(S)\cap E(\Upsilon)}\left|R_{r(a)}\left(\frac{x}{a}\right)\right|\le 3.5\sqrt{x}+$$

$$+\sum_{\substack{4\le a\le x: r(a)\le N\\ a\in E(S)\cap E(\Upsilon)}}\left|R_{r(a)}\left(\frac{x}{a}\right)\right|+\sum_{\substack{a\le x: r(a)\ge N+1\\ a\in E(S)\cap E(\Upsilon)}}\left|R_{r(a)}\left(\frac{x}{a}\right)\right| \tag{7}$$

with $N=6469693229$.

Let $x>N$ go to infinity. Distinguish two cases: (i) $r(a)\le N$; (ii) $r(a)>N$.

(i) $r(a)\le N$. Denote by $E(\Upsilon)(n)$ the n-th powerful number (in increasing order). According to (5), $E(\Upsilon)(n)=(\frac{\zeta(3)}{\zeta(3/2)})^2n^2(1+o(1))$. So, $\Sigma_{1\le n\le x}\frac{1}{\sqrt{E(\Upsilon)(n)}}=O(\log x)$. Hence, by (7) and Lemma 1,

$$|R(x)|\le 3.5\sqrt{x}+k\sqrt{x}\sum_{a\le x,\ \ a\in E(S)\cap E(\Upsilon)}\frac{1}{\sqrt{a}}=O(\sqrt{x}\log x).$$

(ii) $r(a)>N$. Then, by (7) and Lemma 1,

$$R(x) \le k\sqrt{x} \sum_{\substack{a\le x: r(a)\ge N+1\\ a\in E(S)\cap E(\Upsilon)}} \frac{1}{\sqrt{a}} e^{c\frac{\sqrt{\log r(a)}}{\log\log r(a)}},$$

where the last sum does not exceed

$$\sum_{N+1\le a\le x:\ \ r(a)\ge N+1} \frac{1}{\sqrt{a}} e^{c\frac{\sqrt{\log a}}{\log\log a}} \le e^{c\frac{\sqrt{\log x}}{\log\log x}} O(\log x).$$

So, $R(x) = O(\sqrt{x}\log x e^{c\frac{\sqrt{\log x}}{\log\log x}})$ and, by (6), we have

$$\sum_{i\le x,\ \ i\in E(S)} 1 =$$

$$\frac{6}{\pi^2}\left(1 + \sum_{4\le a\le x,\ \ a\in E(S)\cap E(\Upsilon)} \prod_{p|r(a)} \left(1 - \frac{1}{p+1}\right)\frac{1}{a}\right) x + O(\sqrt{x}\log x e^{c\frac{\sqrt{\log x}}{\log\log x}}).$$

Moreover, if we replace here the sum $\sum_{a\le x, a\in E(S)\cap E(\Upsilon)}$ by the sum $\sum_{a\in E(S)\cap E(\Upsilon)}$, then the error does not exceed $\frac{6x}{\pi^2}\sum\limits_{n>x}\frac{1}{E(\Upsilon)(n)} = \frac{6x}{\pi^2}O(1/x) = O(1)$, then the result does not change. So, finally,

(8) $$\sum_{i\le x,\ \ i\in E(S)} 1 =$$

$$\frac{6}{\pi^2}\left(\sum_{a\in E(S)\cap E(\Upsilon)} \prod_{p|r(a)} \left(1 - \frac{1}{p+1}\right)\frac{1}{a}\right) x + O(\sqrt{x}\log x e^{c\frac{\sqrt{\log x}}{\log\log x}}).$$

Formula (8) shows that, if $1 \in S$, then $E(S)$ has a density.

4. Completion of the proof

It remains to evaluate the sum (8). For that we follow the scheme of [4], pp.203-204. For a fixed $l \in B$, denote by $C(l)$ the set of all $E(S)\cap E(\Upsilon)$-numbers a with $r(a) = l$. Recall that $r(1) = 1$. By (8), we have

(9) $$\sum_{i\le x,\ \ i\in E(S)} 1 = \frac{6}{\pi^2}x\sum_{l\in B}\prod_{p|l}\left(1 - \frac{1}{p+1}\right)\sum_{a\in C(l)}\frac{1}{a} + R(x).$$

Consider the function $A : N \to R$ given by:

$$A(l) = \begin{cases} \sum\limits_{a\in C(l)} \frac{1}{a}, & l \in B,\\ 0, & l \notin B.\end{cases}$$

Example 1.

$$A(1) = \sum_{a\in C(1)} \frac{1}{a} = \sum_{r(a)=1}\frac{1}{a} = 1.$$

Example 2. *Let p be prime. Since $r(p)=p$, then*

$$A(p)=\sum_{a\in C(p)}\frac{1}{a}=\sum_{i\geq 2}\frac{1}{p^{s(i)}}.$$

The sum not contains $\frac{1}{p^{s(1)}}=\frac{1}{p}$ since, by the condition, $a\in E(S)\cap E(\Upsilon)$, but the sequence $E(\Upsilon)$ not contains any prime.

Example 3. *Let $p<q$ be primes. Since $r(pq)=pq$, then*

$$A(pq)=\sum_{i\geq 2, j\geq 2}\frac{1}{p^{s(i)}}\frac{1}{q^{s(j)}}.$$

It is evident that, if $l_1, l_2\in B$ and $\gcd(l_1,l_2)=1$, then

$$A(l_1l_2)=\sum_{a\in C(l_1l_2)}\frac{1}{a}=\sum_{a\in C(l_1)}\frac{1}{a}\sum_{a\in C(l_2)}\frac{1}{a}=A(l_1)A(l_2).$$

It follows that $A(l)$ is a multiplicative function. Hence the function f which is defined by

$$f(l)=\prod_{p|l}\left(1-\frac{1}{p+1}\right)A(l)$$

is also multiplicative. Evidently, by the definition of $A(n)$,

$$\sum_{n=1}^{\infty}f(n)\leq\sum_{n=1}^{\infty}A(n)\leq\sum_{a\in E(\Upsilon)}\frac{1}{a}<\infty.$$

Consequently ([3],p.103):

$$\sum_{n=1}^{\infty}f(n)=\prod_{p}(1+f(p)+f(p^2)+\ldots). \tag{10}$$

Since $f(p^k)=0$ for $k\geq 2$, then by (9):

$$\sum_{i\leq x,\ i\in E(S)}1=\frac{6}{\pi^2}x\sum_{l=1}^{\infty}f(l)+R(x)=\frac{6}{\pi^2}x\prod_{p}(1+f(p))+R(x)=$$

$$=\frac{6}{\pi^2}x\prod_{p}\left(1+\left(1-\frac{1}{p+1}\right)\left(\frac{1}{p^{s(2)}}+\frac{1}{p^{s(3)}}+\frac{1}{p^{s(4)}}+\ldots\right)\right)+R(x).$$

Now we have

$$h(E(S))=\frac{6}{\pi^2}\prod_{p}(1+(1-\frac{1}{p+1})\sum_{i\geq 2}\frac{1}{p^{s(i)}})= \tag{11}$$

$$\frac{6}{\pi^2}\prod_{p}(1+\sum_{i\geq 2}\frac{p}{(p+1)p^{s(i)}})=$$

$$\prod_p((1-\frac{1}{p^2})-(1-\frac{1}{p})\sum_{i\geq 2}\frac{u(i)}{p^i})$$

and, taking into account that $u(1)=1$, we find

$$h(E(S))=\prod_p(1-\frac{1}{p^2}-(1-\frac{1}{p})\frac{1}{p}+(1-\frac{1}{p})\sum_{i\geq 1}\frac{u(i)}{p^i})=$$

$$\prod_p((1-\frac{1}{p})+\sum_{i\geq 1}\frac{u(i)}{p^j}-\frac{1}{p}\sum_{i\geq 1}\frac{u(i)}{p^j})=$$

$$\prod_p((1-\frac{1}{p})+\frac{1}{p}+\sum_{i\geq 2}\frac{u(i)}{p^j}-\frac{1}{p}\sum_{i\geq 2}\frac{u(i-1)}{p^{i-1}})=$$

$$\prod_p(1+\sum_{i\geq 2}\frac{u(i)-u(i-1)}{p^i})$$

which gives the required evaluation of the sum in (8) and completes the proof of the theorem.

5. A question of D. Berend

Let p_n be the n-th prime. Let $\mathrm{A}=\{S_1,S_2,...\}$ be an infinite sequence of sequences $S_i\in\mathbf{S}$ beginning with 1. We say that a positive number N is an exponentially A-number ($N\in E(\mathrm{A})$), if in case that $p_n,\ n\geq 1$, divides N, then its exponent in the prime power factorization of N belongs to S_n. We accept that $1\in E(\mathrm{A})$. How will change Theorem 1 for the exponentially A-numbers?

An analysis of the proof of Theorem 1 shows that also in this more general case, for every sequence A there exists a density $h(\mathrm{A})$ of the exponentially A-numbers such that

$$\sum_{i\leq x,\ i\in E(\mathrm{A})}1=h(E(\mathrm{A}))x+R(x),\tag{12}$$

where $R(x)$ is the same as in Theorem 1 and

$$h(E(\mathrm{A}))=\prod_{n\geq 1}(1+\sum_{i\geq 2}\frac{u_n(i)-u_n(i-1)}{{p_n}^i}),\tag{13}$$

where $u_n(k)$ is the characteristic function of sequence S_n : $u_n(k)=1$, if $k\in S_n$ and $u_n(k)=0$ otherwise.

Example 4. *Let*

$$\mathrm{A}=\{S_1=\{1\},S_2=\{1,2\},...,S_n=\{1,...,n\},...\}.$$

Then, by (13),

$$h(E(A))=\prod_{n\geq 1}(1-\frac{1}{{p_n}^{n+1}})=0.7210233...\ .$$

6. A question

Let $1 \in S$. Then the density $h(E(S))$ is in the interval $[6/\pi^2, 1]$. Whether the set $\{h(E(S))\}$ is a dense set in this interval?

D. Berend (private communication) gave a negative answer. Indeed, consider the set $\mathbf{S}_1$ of sequences $\{S\}$ containing 2. Then, evidently, $h(E(S)) \geq h(E(\{1,2\}))$ such that, by Theorem 1,

$$h(E(S))|_{S \in \mathbf{S}_1} \geq \prod_p (1 - \frac{1}{p^3}). \tag{14}$$

Now consider the set $\mathbf{S}_2$ of sequences $\{S\}$ not containing 2. Then $h(E(S)) \leq h(E(\{1,3,4,5,6,...\}))$ such that, by Theorem 1,

$$h(E(S))|_{S \in \mathbf{S}_2} \leq \prod_p (1 - \frac{1}{p^2} + \frac{1}{p^3}) = \prod_p (1 - \frac{p-1}{p^3}). \tag{15}$$

Thus, by (14)-(15), we have a gap in the set $\{h(E(S))\}$ in interval

$$(\prod_p (1 - \frac{p-1}{p^3}),\ \prod_p (1 - \frac{1}{p^3})).$$

Of course, this Berend's idea has far-reaching effects.

7. Acknowledgement

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