# Surprising Relations Between Sums-Of-Squares of Characters of the Symmetric Group Over Two-Rowed Shapes and Over Hook Shapes

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**Abstract**: In a recent article, we noted (and proved) that the sum of the squares of the characters of the symmetric group, $\chi^\lambda(\mu)$, over all shapes $\lambda$ with two rows and $n$ cells and $\mu = 31^{n-3}$, equals, surprisingly, to $1/2$ of that sum-of-squares taken over all hook shapes with $n+2$ cells and with $\mu = 321^{n-3}$. In the present note, we show that this is only the tip of a huge iceberg! We will prove that if $\mu$ consists of odd parts and (a possibly empty) string of *consecutive* powers of 2, namely $2, 4, \dots, 2^{t-1}$ for $t \geq 1$, then the the sum of $\chi^\lambda(\mu)^2$ over all two-rowed shapes $\lambda$ with $n$ cells, equals exactly $\frac{1}{2}$ times the analogous sum of $\chi^\lambda(\mu')^2$ over all shapes $\lambda$ of *hook shape* with $n+2$ cells, and where $\mu'$ is the partition obtained from $\mu$ by retaining all odd parts, but replacing the string $2, 4, \dots, 2^{t-1}$ by $2^t$.

Recall that the *Constant Term* of a *Laurent polynomial* in $(x_1, \dots, x_m)$ is the free term, i.e. the coefficient of $x_1^0 \cdots x_m^0$. For example

$$CT_{x_1,x_2}(x_1^{-3}x_2 + x_1x_2^{-2} + 5) = 5 \quad .$$

Recall that a *partition* (alias *shape*) of an integer $n$, with $m$ *parts* (alias *rows*), is a non-increasing sequence of positive integers

$$\lambda = (\lambda_1, \dots, \lambda_m) \quad ,$$

where $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_m > 0$, and $\lambda_1 + \ldots + \lambda_m = n$.

If $\lambda = (\lambda_1, \dots, \lambda_m)$ and $\mu = (\mu_1, \dots, \mu_r)$ are partitions of $n$ with $m$ and $r$ parts, respectively, then it easily follows from (7.8) (p. 114) in [M], that the *characters*, $\chi^\lambda(\mu)$, of the *symmetric group*, $S_n$, may be obtained via the *constant term* expression

$$\chi^\lambda(\mu) \;=\; CT_{x_1,\dots,x_m} \frac{\prod_{1 \leq i < j \leq m}(1 - \frac{x_j}{x_i}) \prod_{j=1}^{r} \left(\sum_{i=1}^{m} x_i^{\mu_j}\right)}{\prod_{i=1}^{m} x_i^{\lambda_i}} \quad . \qquad (Chi)$$

As usual, for any partition $\mu$, $|\mu|$ denotes the sum of its parts, in other words, the integer that is being partitioned.

In [RRZ] we considered two quantities. Let $\mu_0$ be any partition with smallest part $\geq 2$. The first quantity, that we will call henceforth $A(\mu_0)(n)$, is the following sum-of-squares over two-rowed shapes $\lambda$:

$$A(\mu_0)(n) := \sum_{j=0}^{\lfloor n/2 \rfloor} \chi^{(n-j,j)}(\mu_0 1^{n-|\mu_0|})^2 \quad .$$

[Note that in [RRZ] this quantity was denoted by $\psi^{(2)}(\mu_0 1^{n-|\mu_0|})$. ]

The second quantity was the sum-of-squares over *hook-shapes*

$$B(\mu_0)(n) := \sum_{j=1}^{n} \chi^{(j,1^{n-j})}(\mu_0 1^{n-|\mu_0|})^2 \quad .$$

[Note that in [RRZ] this quantity was denoted by $\phi^{(2)}(\mu_0 1^{n-|\mu_0|})$.]

In [RRZ] we developed algorithms for discovering (and then proving) closed-form expressions for these quantities, for any *given* (specific) finite partition $\mu_0$ with smallest part larger than one. In fact we proved that each such expression is *always* a multiple of $\binom{2n}{n}$ by a certain rational function of $n$ that depends on $\mu_0$.

Unless $\mu_0$ is very small, these rational functions turn out to be very complicated, but, inspired by the OEIS([S]), Alon Regev noted (and then it was proved in [RRZ]) the *remarkable* identity

$$A(3)(n) = \frac{1}{2} B(3,2)(n+2) \quad .$$

This lead to the following natural question:

Are there other partitions, $\mu_0$, such that there exists a partition, $\mu_0'$ with $|\mu_0'| = |\mu_0| + 2$, such that the ratio $A(\mu_0)(n)/B(\mu_0')(n+2)$ is a constant?

This lead us to write a new procedure in the Maple package

`http://www.math.rutgers.edu/~zeilberg/tokhniot/Sn.txt` , that accompanies [RRZ],

called `SeferNisim(K,N0)`, that searched for such pairs $[\mu_0, \mu_0']$. We then used our *human* ability for *pattern recognition* to notice that all the successful pairs (we went up to $|\mu_0| \leq 20$) turned out to be such that $\mu_0$ either consisted of only odd parts, and then $\mu_0'$ was $\mu_0$ with 2 appended, or, more generally $\mu_0$ consisted of odd parts together with a string of *consecutive* powers of 2 (starting with 2), and $\mu_0'$ was obtained from $\mu_0$ by retaining all the odd parts but replacing the string of powers of 2 by a single power of 2, one higher then the highest in $\mu_0$. In symbols, we conjectured, (and later proved [see below], *alas*, by purely human means) the following:

**Theorem:** Let $\mu_0$ be any partition of the form

$$\mu_0 = Sort([a_1, \dots, a_s, 2, 2^2, \dots, 2^{t-1}]) \quad ,$$

where

$$a_1 \geq a_2 \geq \dots \geq a_s \geq 3 \quad ,$$

are all **odd**, and $t \geq 1$ (if $t = 1$ then $\mu_0$ only consists of odd parts). Define

$$\mu_0' = Sort([a_1, \dots, a_s, 2^t]) \quad .$$

Then, for every $n \geq |\mu_0|$, we have

$$A(\mu_0)(n) = \frac{1}{2} B(\mu_0')(n+2) \quad .$$

(For any sequence of integers, S, Sort(S) denotes that sequence sorted in non-increasing order.)

In order to prove our theorem we need to first recall, from [RRZ], the following **constant-term** expression for $B(\mu_0)(n)$.

**Lemma 1**: Let $\mu_0 = (a_1, \dots, a_r)$

$$B(\mu_0)(n) = Coeff_{x^0} \left[ \frac{(1+x)^{2n-2-2(a_1+\ldots+a_r)}}{x^{n-1}} \cdot \prod_{i=1}^{r} (x^{a_i} - (-1)^{a_i})(1 - (-1)^{a_i} x^{a_i}) \right] \quad .$$

We need an analogous constant-term expression for $A(\mu_0)(n)$. To that end, let's first spell-out Equation ($Chi$) for the two-rowed case, $m = 2$, so that we can write $\lambda = (n-j, j)$. We have, writing $\mu_0 = (a_1, \dots, a_r)$,

$$\chi^{(n-j,j)}(\mu_0 1^{n-|\mu_0|}) \;=\; CT_{x_1,x_2} \frac{(1 - \frac{x_2}{x_1})(x_1+x_2)^{n-a_1-\ldots-a_r} \prod_{i=1}^{r} (x_1^{a_i} + x_2^{a_i})}{x_1^{n-j} x_2^{j}} \quad . \qquad (Chi2)$$

This can be rewritten as

$$\chi^{(n-j,j)}(\mu_0 1^{n-|\mu_0|}) \;=\; CT_{x_1,x_2} \frac{(1 - \frac{x_2}{x_1})(1 + \frac{x_2}{x_1})^{n-a_1-\ldots-a_r} \prod_{i=1}^{r} \left(1 + (\frac{x_2}{x_1})^{a_j}\right)}{(\frac{x_2}{x_1})^{j}} \quad . \qquad (Chi2')$$

Since the *constant-termand* is of the form $P(\frac{x_2}{x_1})/(\frac{x_2}{x_1})^j$ for some *single-variable* polynomial $P(x)$, the above can be rewritten, as

$$\chi^{(n-j,j)}(\mu_0 1^{n-|\mu_0|}) \;=\; Coeff_{x^0} \frac{(1-x)(1+x)^{n-a_1-\ldots-a_r} \prod_{i=1}^{r} (1 + x^{a_i})}{x^j} \quad . \qquad (Chi2'')$$

Note that the left side is *utter nonsense* if $j > \frac{n}{2}$, but the right side makes perfect sense. It is easy to see that defining $\chi^{(n-j,j)}(\mu_0 1^{n-|\mu_0|})$ by the right side for $j > \frac{n}{2}$, we get

$$\chi^{(n-j,j)}(\mu_0 1^{n-|\mu_0|}) \;=\; -\chi^{(j,n-j)}(\mu_0 1^{n-|\mu_0|}) \quad .$$

Let's denote the numerator of the constant-termand of ($Chi''$), namely

$$(1-x)(1+x)^{n-a_1-\ldots-a_r} \prod_{i=1}^{r} (1 + x^{a_i}) \quad ,$$

by $P(x)$, then equation ($Chi2''$) can be also rewritten as a *generating function.*

$$P(x) = \sum_{j=0}^{n} \chi^{(n-j,j)}(\mu_0 1^{n-|\mu_0|})\, x^j \quad .$$

Since for any polynomial of a single variable, $P(x)=\sum_{j=0}^{n} c_j x^j$, we have

$$\sum_{j=0}^{n} c_j^2 = Coeff_{x^0}\left[P(x)P(x^{-1})\right] \quad ,$$

we get

$$\sum_{j=0}^{n} \chi^{(n-j,j)}(\mu_0 1^{n-|\mu_0|})^2 =$$

$$Coeff_{x^0}\left[\left((1-x)(1+x)^{n-a_1-\ldots-a_r}\prod_{j=1}^{r}(1+x^{a_j})\right)\cdot\left((1-x^{-1})(1+x^{-1})^{n-a_1-\ldots-a_r}\prod_{j=1}^{r}(1+x^{-a_j})\right)\right] \quad .$$

$$= -Coeff_{x^0}\left[\frac{(1-x)^2(1+x)^{2(n-a_1-\ldots-a_r)}\prod_{j=1}^{r}(1+x^{a_j})^2}{x^{n+1}}\right] \quad .$$

But since, by symmetry,

$$\sum_{j=0}^{\lfloor \frac{n}{2}\rfloor} \chi^{(n-j,j)}(\mu_0 1^{n-|\mu_0|})^2 = \frac{1}{2}\sum_{j=0}^{n} \chi^{(n-j,j)}(\mu_0 1^{n-|\mu_0|})^2 \quad ,$$

we have

**Lemma 2**: Let $\mu_0=(a_1,\ldots,a_r)$ be a partition with smallest part larger than one, then

$$A(\mu_0)(n) = -\frac{1}{2}\,Coeff_{x^0}\left[\frac{(1-x)^2(1+x)^{2(n-a_1-\ldots-a_r)}\prod_{j=1}^{r}(1+x^{a_j})^2}{x^{n+1}}\right] \quad .$$

We are now ready to prove the theorem. If $\mu_0 = Sort(a_1,\ldots,a_r,2,\ldots,2^{t-1})$ then

$$A(\mu_0)(n) = -\frac{1}{2}\,Coeff_{x^0}\left[\frac{(1-x)^2(1+x)^{2(n-a_1-\ldots-a_r-2-2^2-\ldots 2^{t-1})}\prod_{j=1}^{t-1}\left(1+x^{2^j}\right)^2\prod_{j=1}^{r}(1+x^{a_j})^2}{x^{n+1}}\right] \quad .$$

But (transferring a factor of $(1+x)^2$ from the second factor to the product, $\prod_{j=1}^{t-1}\left(1+x^{2^j}\right)^2$), we have

$$(1+x)^{2(n-a_1-\ldots-a_r-2-2^2-\ldots 2^{t-1})}\prod_{j=1}^{t-1}\left(1+x^{2^j}\right)^2 = (1+x)^{2(n-a_1-\ldots-a_r-1-2-2^2-\ldots 2^{t-1})}\prod_{j=0}^{t-1}\left(1+x^{2^j}\right)^2 \quad .$$

Hence,

$$A(\mu_0)(n) = -\frac{1}{2}\,Coeff_{x^0}\left[\frac{(1-x)^2(1+x)^{2(n-a_1-\ldots-a_r-1-2-2^2-\ldots 2^{t-1})}\prod_{j=0}^{t-1}\left(1+x^{2^j}\right)^2\prod_{j=1}^{r}(1+x^{a_j})^2}{x^{n+1}}\right] \quad .$$

By Euler's good-old $(1-x)\prod_{j=0}^{t-1}(1+x^{2^j}) = 1-x^{2^t}$. Hence

$$A(\mu_0)(n) = -\frac{1}{2}\,Coeff_{x^0}\left[\frac{(1-x^{2^t})^2(1+x)^{2(n-a_1-\ldots-a_r-1-2-2^2-\ldots 2^{t-1})}\prod_{j=1}^{r}(1+x^{a_j})^2}{x^{n+1}}\right] \quad .$$

On the other hand, since $\mu_0' = Sort(a_1,\ldots,a_r,2^t)$, and all the $a_i$'s are odd, we have

$$B(\mu_0')(n+2) = -Coeff_{x^0}\left[\frac{(1+x)^{2n+2-2(a_1+\ldots+a_r+2^t)}}{x^{n+1}}\cdot(x^{2^t}-1)^2\cdot\prod_{j=1}^{r}(x^{a_j}+1)^2\right] \quad .$$

This completes the proof, since $-(1+2+2^2+\ldots+2^{t-1}) = 1-2^t \quad . \square$

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