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ON A CONJECTURE OF JOHN HOFFMAN REGARDING SUMS OF PALINDROMIC NUMBERS

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Abstract. We disprove the conjecture that every sufficiently large natural number n is the sum of three palindromic natural numbers where one of them can be chosen to be the largest or second largest palindromic natural number smaller than or equal to n.

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1. Introduction

In the following, the terms *digit* and *palindromic* refer to decimal representations. For $n \in \mathbb{N}$, its unique decimal representation is given by

$$n = \sum_{j=0}^{h(n)} n_j \cdot 10^j.$$

with minimal $h(n) \in \mathbb{N}$ and digits $n_0, \ldots, n_{h(n)} \in \{0, \ldots, 9\}$. We identify n with the digit string $n_{h(n)} \ldots n_0$.

A natural number n is called *palindromic* iff $n_j = n_{h(n)-j}$ for $0 \le j \le n(h)$.

By $\mathbb{P}$ we denote the set of palindromic natural numbers, i. e.

$$\mathbb{P} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 22, 33, \ldots, 99, 101, 111, 121, \ldots\}.$$

Until recently, it was not known whether $\mathbb{P}$ is an additive basis of $\mathbb{N}$, i. e. whether there exists $d \in \mathbb{N}$ such that $\mathbb{N} = d\,\mathbb{P}$, where $d\,\mathbb{P}$ denotes the set of sums of d elements of $\mathbb{P}$. William D. Banks has in [1] given a proof for $\mathbb{N} = 49\mathbb{P}$, which leaves still quite some distance from the commonly conjectured $\mathbb{N} = 3\mathbb{P}$. [2] mentions an even stronger conjecture of John Hofmann, claiming that every sufficiently large natural number n is the sum of three elements of $\mathbb{P}$ where one of them can be chosen to be the largest or second largest palindromic natural number $p \le n$. With the palindromic precursor and palindromic successor

$$n_* := \max_{\mathbb{P} \ni p < n} p \quad \text{and} \quad n^* := \min_{\mathbb{P} \ni p > n} p\,,$$

and $n_{**} := (n_*)_*$ for $n \in \mathbb{N}$, the question is:

Is it true that $\{n - n_*, n - n_{**}\} \cap 2\mathbb{P} \neq \emptyset$ *for every sufficiently large* $n \in \mathbb{N} \setminus \mathbb{P}$*?*

We are going to show that the answer is "no".

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2. The Counterexample

The counterexample is constructed using 'non-2$\mathbb{P}$ twins', the palindromic twins $10^a \pm 1$ for suitable $a \in \mathbb{N}$ and the fact that the distance between a palindromic number p and its successor p^* can be arbitrarily large. As 'non-2$\mathbb{P}$ twins' we use the numbers $11 \cdot 10^k + 1$ and $11 \cdot 10^k + 3$ for even k.

Proposition 1. *$11 \cdot 10^k + 1 \notin 2\mathbb{P}$ for $2 \le k \in \mathbb{N}$.*

Proof. For $t := 11 \cdot 10^k + 1$ we have $h(t) = k+1$. Suppose $t = p+q$ with $p, q \in \mathbb{P}$ and $p \le q$, so $h(p) \le h(q) \le k+1$. Because $t \notin \mathbb{P}$, we have $p > 0$.

(a) Suppose $h(q) = k+1$. Then $q_{k+1} = 1$, so $q_0 = 1$, so $p_0 = 0$, which is not possible.

(b) Suppose $h(p) = h(q) = k$. Because $t_0 = 1$ and $p_0, q_0 \ne 0$, we need $p_0 + q_0 = 11$, so 1 is carried to the tens positions, and as this must add to 10 with $p_1 + q_1$, we get $p_1 + q_1 = 9$, and a 1 is carried to the hundreds position. This goes on up to $p_{k-1} + q_{k-1} = 9$ and a carry to position k. But then $p + q \ge (p_k + q_k + 1) \cdot 10^k = (p_0 + q_0 + 1) \cdot 10^k = 12 \cdot 10^k > t$.

(c) Suppose $h(p) < h(q) = k$. Then $p + q \le (10^k - 1) + (10^{k+1} - 1) = 11 \cdot 10^k - 2 < t$.

(d) Suppose $h(p) \le h(q) < k$. Then $p + q \le (10^k - 1) + (10^k - 1) = 2 \cdot 10^k - 2 < t$. □

Proposition 2. *$11 \cdot 10^k + 3 \notin 2\mathbb{P}$ for $2 \le k \in \mathbb{N}$, k even.*

Proof. For $t := 11 \cdot 10^k + 3$ we have $h(t) = k+1$. Suppose $t = p+q$ with $p, q \in \mathbb{P}$ and $p \le q$, so $h(p) \le h(q) \le k+1$. Because $t \notin \mathbb{P}$, we have $p > 0$.

In the following, for a digit α and $m \in \mathbb{N}$, $[\alpha]_m$ denotes the concatenation of m copies of α.

(a) Suppose $h(q) = k+1$. Then $q_{k+1} = 1$, so $q_0 = 1$, so $p_0 = 2$, so $p_{h(p)} = 2$, so $h(p) < k$. A carry is needed from position $h(p)$ to position $h(p)+1$ to get $(p+q)_{h(p)} = 0$, and so on, up to a carry from position $k-1$ to position k. With this carry, we would get $p + q > t$ if $q_k > 0$, so $q_k = 0$, so $q_1 = 0$. For $h(p) > 1$ we get $p_1 = 0$. For $h(p) > 2$ we get $p_{h(p)-1} = 0$.

(aa) Suppose $k = 2$. Then $q = 1001$ and $p \in \{2, 22\}$, so $p + q \ne t$.

(ab) Suppose $k = 4$. Then $q = 10\delta\delta01$ with a digit δ and $p \in \{2, 22, 202, 2002\}$, so $p + q \ne t$.

(ac) Suppose $k \ge 6 \wedge h(p) \le 5$. Then $q = 10\delta\varepsilon\alpha\varepsilon\delta01$ with digits δ and ε and a palindromic digit string α which is empty in case of $k = 6$. To get $(p+q)_k = 1$, $\delta = 9$ is needed, so $q = 109\varepsilon\alpha\varepsilon901$ and $p + q \ne t$ for $p \in \{2, 22, 202, 2002\}$. For $p = 20\varphi02$ with some digit φ, to get $(p+q)_2 = 0$ we need $\varphi = 1$ and $\varepsilon = 9$, but then in case of $k = 6$ we get $p + q = 20102 + 10999901 = 11020003 \ne t$, while in case of $k > 6$ we need $\alpha = 7[9]_{k-8}7$, so $p + q = 20102 + 10997[9]_{k-8}79901 = 10998[0]_{k-8}00003 \ne t$. For $p = 20\varphi\varphi02$ with some digit φ, to get $(p+q)_2 = 0$ we need $\varphi = 1$ and $\varepsilon = 8$, but then in case of $k = 6$ we get $p + q = 201102 + 10988901 = 11190003 \ne t$, while in case of $k > 6$ we have

$$p + q < 10^6 + 1099 \cdot 10^{k-2} \le 10^{k-2} + 1099 \cdot 10^{k-2} = 11 \cdot 10^k < t.$$

(ad) Suppose $k \ge 8 \wedge h(p) \ge 6$. Then $q = 10\delta\varepsilon\alpha\varepsilon\delta01$ and $p = 20\varphi\beta\varphi02$ with digits $\delta, \varepsilon, \varphi$ and non-empty palindromic digit strings α, β. We will construct $p', q' \in \mathbb{P}, p' \le q'$ with $h(q') = k-1$ and $p' + q' = 11 \cdot 10^{k-2} + 3$, which gives rise to an impossible infinite descent.

(ada) Suppose $\varphi = 0$. Then $\delta = 0$, hence $q = 100\varepsilon\alpha\varepsilon001$ and $p = 200\beta002$, and we can take $q' := 10\varepsilon\alpha\varepsilon01$ and $p' := 20\beta02$.

(adb) Suppose $\varphi \neq 0$ and $h(p) = k-1$. We have $\varphi + \delta = 10$ and $\delta \neq 0$, and β must have at least two digits, i. e. $\beta = \psi\gamma\psi$ with a digit ψ and a (possibly empty) palindromic digit string γ, so $p = 20\varphi\psi\gamma\psi\varphi02$, which allows to take $q' := 10\delta\alpha\delta01$ and $p' := 20\varphi\gamma\varphi02$.

(adc) Suppose $\varphi \neq 0$ and $h(p) < k-1$. We have $\varphi + \delta = 10$, and $h(p) < k-1$ leads to $\delta = 9$ and $\varphi = 1$, so $q = 109\varepsilon\alpha\varepsilon901$ and $p = 201\beta102$.

(adca) Suppose β is more than one digit, i. e. $\beta = \psi\gamma\psi$ with a digit ψ and a (possibly empty) palindromic digit string γ, hence $p = 201\psi\gamma\psi102$. Then we take $q' := 109\alpha901$ and $p' := 201\gamma102$.

(adcb) Suppose β is a single digit. As k is even, α has an even number of digits. If α were two digits, say $\alpha = \tau\tau$ with a digit τ, so $q = 109\varepsilon\tau\tau\varepsilon901$, we would need $\tau = 8$ for the lower position, but $\tau = 9$ for the higher position of τ. If α were more than two digits, say $\alpha = \tau\varrho\tau$ with a digit τ and a palindromic digit string ϱ with 2 or more digits, so $q = 109\varepsilon\tau\varrho\tau\varepsilon901$, we would again need $\tau = 8$ for the lower position, but $\tau = 9$ for the higher position of τ. So the case (adcb) is not possible at all.

(b) Suppose $h(p) = h(q) = k$. Then $p_0 + q_0 = 3, p_k + q_k \in \{10, 11\}$, but $p_k = p_0, q_k = q_0$.

(c) Suppose $h(p) < h(q) = k$. Then $p + q \leq (10^k - 1) + (10^{k+1} - 1) = 11 \cdot 10^k - 2 < t$.

(d) Suppose $h(p) \leq h(q) < k$. Then $p + q \leq (10^k - 1) + (10^k - 1) = 2 \cdot 10^k - 2 < t$. □

Proposition 3. *There are infinitely many $n \in \mathbb{N} \setminus \mathbb{P}$ with $n - n_*, n - n_{**} \notin 2\mathbb{P}$.*

Proof. Let $1 \leq j \in \mathbb{N}$. Then for $t := 11 \cdot 10^{2j} + 1$, propositions 1 and 2 show $t, t+2 \notin 2\mathbb{P}$. Take $m \in \mathbb{N}$ with $10^m > t$ and set $p := 10^{2m} + 1 \in \mathbb{P}$. Then $p^* = 10^{2m} + 10^m + 1 = p + 10^m$ and $p_* = 10^{2m} - 1 = p - 2$. For $n := p + t$ we have $p < n < p + 10^m = p^*$, so $n \notin \mathbb{P}$ and $n_* = p$, hence $n - n_* = n - p = t \notin 2\mathbb{P}$ and $n - n_{**} = n - p_* = n - (p-2) = t + 2 \notin 2\mathbb{P}$.

In this way, for every $j \geq 1$ choose an $m(j)$ and get an $n(j)$ with the desired properties. Taking $m(j+1) > m(j)$ gives $n(j+1) > n(j)$. □

Choosing the smallest possible m with $10^m > 11 \cdot 10^{2j} + 1$, namely $m = 2j + 2$, in the proof of proposition 3 yields $n(j) = 10\,000^{j+1} + 11 \cdot 100^j + 2$.

On a related note, we would like to point out that the greedy algorithm which, given a natural number, repeatedly subtracts the largest possible palindromic number, can result in an arbitrarily large number of palindromic summands: Start with $n(1) := 1$. To get $n(j+1)$, take $m \in \mathbb{N}$ with $10^m > n(j)$ and set $n(j+1) := 10^{2m} + 1 + n(j)$. Then $n(j+1) \notin \mathbb{P}$ and $n(j+1)_* = 10^{2m} + 1$, so $n(j+1) - n(j+1)_* = n(j)$. For every $j \in \mathbb{N}$, the greedy algorithm partitions $n(j)$ into j palindromic summands. Consequently, and in confirmation of a recent presumption of Neil Sloane [3], the OEIS sequence A088601 is unbounded.

References

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