Problem collection from the IML programme: Graphs, Hypergraphs, and Computing

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Abstract

This is a collection of open problems and conjectures from the seminars and problem sessions of the 2014 IML programme: Graphs, Hypergraphs, and Computing.

Contents

1 Introduction

This collection of problems and conjectures is based on a subset of the open problems from the seminar series and the problem sessions of the IML programme Graphs, Hypergraphs, and Computing. Each problem contributor has provided a write up of their proposed problem and the collection has been edited by Klas Markström.

2 Seminar January 16, 2014, Jørgen Bang-Jensen

2.1 Arc-disjoint spanning strong subdigraphs and disjoint Hamilton cycles

Conjecture 2.1 (Kelly 196?). *The arc set of every regular tournament can be decomposed into Hamilton cycles.*

Theorem 2.2 (Kühn and Osthus, 2012). *The Kelly Conjecture is true for tournaments on n vertices where $n \geq M$ for some very large M.*

As every k-regular tournament is k-arc-strong, the following Conjecture implies the Kelly conjecture.

Conjecture 2.3 (Bang-Jensen and Yeo, 2004). *The arc set of every k-arc-strong tournament $T = (V, A)$ can be decomposed into k disjoint sets $A_1, \ldots, A_k$ such that each of the spanning subdigraphs $D_i = (V, A_i)$, $i = 1, 2, \ldots, k$ is strongly connected.*

Theorem 2.4 (Bang-Jensen and Yeo 2004). *Conjecture 2.3 is true in the following cases:*

- *when $k = 2$*

- *when every vertex of T has in- and out-degree at least $37k$*

- *when there exists a non-trivial (both sides of the cut have at least 2 vertices) arc-cut of size k.*

Conjecture 2.5 (Bang-Jensen and Yeo, 2004). *There exists a natural number K such that every K-arc-strong digraph $D = (V, A)$ can be decomposed into two arc-disjoint strong spanning subdigraphs $D_1 = (V, A_1)$ and $D_2 = (V, A_2)$.*

The conjecture is true for tournaments with $K = 2$ by Theorem 2.4 and for locally semicomplete digraphs[1] with $K = 3$ by a recent (non-trivial) result of Bang-Jensen and Huang (JCTB 2012).

For 2-regular digraphs (all in- and out-degrees equal to 2) the existence of arc-disjoint spanning strong subdigraphs is equivalent to the existence of arc-disjoint Hamilton cycles. Hence, by the following theorem, it is NP-complete to decide whether a digraph has a pair of arc-disjoint strong spanning subdigraphs.

Theorem 2.6 (Yeo, 2007). *It is NP-complete to decide whether a 2-regular digraph contains two arc-disjoint Hamilton cycles.*

Theorem 2.7 (Kühn, Osthus, Lapinskas and Patel, 2013). *There exists a natural number C such that every $Ck^2 \log^2 k$-strong tournament contains k arc-disjoint Hamilton cycles. This is best possible up to the log-factor.*

Conjecture 2.8 (Thomassen, 1982). *Every 3-strong[2] tournament has 2-arc-disjoint Hamilton cycles.*

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[1]a digraph is locally semicomplete if the out-neighbourhood and the in-neighbourhood of every vertex induces a semicomplete digraph. A digraph is semicomplete if it has no pair of non-adjacent vertices.

[2]A digraph D is k-strong if it has at least $k + 1$ vertices and $D - X$ is strongly connected for every subset X of $V(D)$ of size at most $k - 1$.

2.2 Decompositions into vertex disjoint pieces/digraphs

Theorem 2.9 (Kühn, Osthus and Townsend, 2014)**.** *There exists a naural number C such that the vertex set of every Ck^7t^4-strong tournament $T = (V, A)$ can be decomposed into disjoint subsets $V_1, V_2, \ldots, V_t$ such that the tournaments $T_i = T\langle V_i \rangle$ are k-strong for $i = 1, 2, \ldots, t$.*

Question 2.10. *Can we also specify t vertices $x_1, x_2, \ldots, x_t$ and find $V_1, \ldots, V_t$ as above such that $x_i \in V_i$ holds for $i = 1, 2, \ldots, t$?*

Conjecture 2.11 (Bermond and Thomassen, 1980)**.** *Every digraph with minimum out-degree $2k - 1$ contains k disjoint directed cycles.*

For $k = 2$ this was verified in 1983 by Thomassen who also proved the existence of a function $f(k)$ such that every digraph out minimum out-degree at least $f(k)$ has k disjoint cycles.

The bound $2k-1$ is best possible as seen by considering the complete digraph on $2k - 2$ vertices.

Theorem 2.12 (Bang-Jensen, Bessy and Thomassé, 2013)**.** *Conjecture 2.11 holds for tournaments*

2.3 Further open problems that were mentioned

Question 2.13. *Is there a polynomial algorithm for deciding whether the underlying graph $UG(D)$ of a digraph D contains a 2-factor $C_1, C_2, \ldots, C_k$ such that C_1 is a directed cycle in D, while C_i, $i > 1$ does not have to respect the orientations of arcs in D?*

Question 2.14. *What is the complexity of the following problem: given a 2-edge-coloured bipartite graph $B = (U, V, E)$; decide whether B has two edge-disjoint perfect matchings M_1, M_2 so that every edge of M_1 has colour 1, while M_2 may use edges of both colours?*

3 Seminar January 16, 2014, Oleg Pikhurko

Here are two open problems (as simplified as possible without losing their essence) that will be quite useful for measurable edge-colourings of graphings. –Oleg Pikhurko

3.1 An Open Question about Finite Graphs

Problem 3.1. *Estimate the minimum $f = f(d)$ such that the following holds. Let G be a (finite) graph of maximum degree at most d with at most d pendant edges pre-coloured (with no two incident pre-coloured edges having the same colour). Then this pre-colouring can be extended to a proper $(d + f)$-edge-colouring of G.*

We can show $f = O(\sqrt{d})$ suffices but it would be nice to prove that $f = O(1)$ is enough.

3.2 Towards a Measurable Local Lemma

For our purposes, it is enough to define a *graphing* $\mathcal{G}$ as a graph whose vertex set is the unit interval $I = [0, 1]$ (with the Borel σ-algebra $\mathcal{B}$ and the Lebesgue measure μ) and whose edge set E can be represented as

$$E = \{\{x, y\} \mid x, y \in I, \ x \neq y, \ \exists i \in [k] \ \phi_i(x) = y\},$$

for some (finite) family of measure-preserving invertible maps $\phi_1, \ldots, \phi_k : I \to I$. The general definition (and an excellent introduction) to graphings can be found in Lovász book [2, Chapter 18].

Problem 3.2. *Let $d \to \infty$. Prove that there is $g(d) = o(d/\log d)$ such that any graphing $\mathcal{G}$ of maximum degree at most d admits a measurable partition $I = A \cup B$ such that for* every *vertex x is (A, B)-balanced, meaning that its degrees into A and into B differ by at most $g(d)$.*

Some remarks:

1. The (finite) Local Lemma shows that the required partition $A \cup B$ exists for every finite graph with $g(d) = O(\sqrt{d \log d})$. By the Compactness Principle, this extends to all countable graphs.

2. In Problem 3.2 it is enough to find a partition such that the measure of the set X of (A, B)-unbalanced vertices x is zero. Indeed, one can show that the measure of the union Y of connectivity components of a graphing that intersect the null set X is zero too. By the previous remark, we can find a good partition of each component in Y; assuming the Axiom of Choice we can modify A, B on the null set Y to make *every* vertex (A, B)-balanced.

3. Gabor Kun [1] proved some analytic version of the Local Lemma that in particular implies for Problem 3.2 that, for every $\varepsilon > 0$, there is a measurable partition $I = A \cup B$ such that the measure of (A, B)-unbalaned vertices is at most ε. But we do need the bad set to have measure zero in our application.

References

[1] G. Kun. A measurable version of the Lovász Local Lemma. Talk at the 2-Day Combinatorics Colloquia, LSE/QMUL, London, 16 May, 2013.

[2] L. Lovász. *Large Networks and Graph Limits*. Colloquium Publications. Amer. Math. Soc, 2012.

4   Problem session Febuary 6, 2014

4.1   Peter Allen

Let $R(G, G)$ denote the 2-colour Ramsey-number for G. It is known that there exists a constant C such that if G is a planar graph on n vertices then $R(G, G) \leq Cn$. It is also known that C must be at least 4.

Question 4.1. *Is $C \leq 12$?*

4.2   Hal Kierstead

An *equitable coloring* of a graph is a partiton of its vertices into independent sets differing in size by at most one. In 1970 Hajnal and Szemerdi [1] proved that for every graph G and integer k, if $\Delta(G) < k$ then G has an equitable k-coloring. Their proof did not yield a polynomial algorithm. About 35 years later, Mydlarz and Szemerdi, and independently Kostochka and I, found such algorithms, and then joined forces to produce an $O(kn^2)$ algorithm [3].

The *maximum Ore degree* of a graph G is $\theta(G) := \max\{d(x) + d(y) : xy \in E(G)\}$. Kostochka and I [2] proved that for every graph G and integer k, if $\theta(G) < 2k$ then G has an equitable k coloring, but the proof does not yield a polynomial algorithm.

Problem 4.2. *Is there a polynomial algorithm for constructing an equitable k-coloring of any graph G with $\theta(G) < 2k$?*

References

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[2] H. Kierstead and A. Kostochka, An Ore-type Theorem on Equitable Coloring, *Journal of Combinatorial Theory, Series B* **98** (2008), 226–234.

[3] H. Kierstead, A. Kostochka, M. Mydlarz, and E. Szemerdi, A fast algorithm for equitable coloring, *Combinatorica* 30 (2010) 217–225.

4.3   Jan van den Heuvel

4.3.1   Cyclic Orderings

The following is a very special case of a much more general conjecture which appears in KAJITANI ET AL. (1988).

Conjecture 4.3.
Let T_1, T_2, T_3 be edge-disjoint spanning trees in a graph G on n vertices (so each tree has $n - 1$ edges). Then there exists a cyclic ordering of the edges in $E(T_1) \cup E(T_2) \cup E(T_3)$ such that every $n - 1$ cyclically consecutive edges in that ordering form a spanning tree.

In fact, the same question can be asked for any number of spanning trees. For two trees the result is proved in KAJITANI ET AL. (1988), who in fact prove it in the stronger form according to Conjecture 4.4 below.

Conjecture 4.3 is really a problem about matroids. The following appears in several places, including GABOV (1976), CORDOVIL & MOREIRA (1993) and WIEDEMANN (2006).

Conjecture 4.4.
Let $B = \{b_1, \ldots, b_r\}$ and $B' = \{b'_1, \ldots, b'_r\}$ be two disjoint bases of a matroid. Then there is a permutation $(b_{\pi(1)}, \ldots, b_{\pi(r)})$ of the elements of B and a permutation $(b'_{\pi'(1)}, \ldots, b'_{\pi'(r)})$ of the elements of B' such that the combined sequence $(b_{\pi(1)}, \ldots, b_{\pi(r)}, b'_{\pi'(1)}, \ldots, b'_{\pi'(r)})$ is a cyclic ordering in which every r cyclically consecutive elements form a base.

A weaker form of Conjecture 4.4 is to just ask for a cyclic for a cyclic ordering of $B_1 \cup B_2$ (so we don't require that each base appears as a consecutive part of the ordering). Even that conjecture is open for matroids in general.

The most general conjecture in this area can be found in KAJITANI ET AL. (1988); partial and related results appear in VAN DEN HEUVEL & THOMASSÉ (2012).

R. CORDOVIL AND M.L. MOREIRA *Bases-cobases graphs and polytopes of matroids.* Combinatorica **13** (1993), 157–165.

H. GABOW, *Decomposing symmetric exchanges in matroid bases.* Math. Programming **10** (1976), 271–276.

J. VAN DEN HEUVEL AND S. THOMASSÉ, *Cyclic orderings and cyclic arboricity of matroids.* J. Combin. Theory Ser. B **102** (2012), 638-646.

Y. KAJITANI, S. UENO, AND H. MIYANO, *Ordering of the elements of a matroid such that its consecutive w elements are independent.* Discrete Math. **72** (1988) 187–194.

D. WIEDEMANN, *Cyclic base orders of matroids.* Manuscript, 2006. Retrieved 23 April 2007 from `http://www.plumbyte.com/cyclic_base_orders_1984.pdf`.
Earlier version: *Cyclic ordering of matroids.* Unpublished manuscript, University of Waterloo, 1984

4.3.2 Strong Colourings of Hypergraphs

All hypergraphs in this section are allowed to have multiple edges and edges of any size. The *rank $r(H)$* of a hypergraph H is the size of the largest edge.

The *strong chromatic number $\chi_s(H)$* of a hypergraph H is the smallest number of colours needed to colour the vertices so that for every edge the vertices in that edge all receive a different colour. (So this is the same as the chromatic number of the graph obtained by replacing every edge by a clique.)

Let's call a *derived graph of a hypergraph H* a graph G on the same vertex set as H where for each edge e of H of size at least two we choose a pair $u, v \in e$ and add the edge uv to G. And let's call the following parameter the *graph chromatic number of H*:

$$\chi_d(H) = \max\{\, \chi(G) \mid G \text{ is a derived graph of } H \,\}.$$

It is obvious that $\chi_s(H) \geq \max\{r(H), \chi_d(H)\}$, and it is not so hard to prove that $\chi_s(H \leq \chi_d(H)^{\binom{r(H)}{2}}$. A little bit more thinking will give

$$\chi_s(H) \leq \chi_d(H)^{r(H)-1}.$$

The question is the find better upper bounds of $\chi_s(H)$ in terms of $\chi_d(H)$ and $r(H)$. It might even be true that there is an upper bound that is linear in $r(H)$.

Question 4.5.
Does there exist a function $f : \mathbb{N} \to \mathbb{R}_+$ so that for every hypergraph H we have

$$\chi_s(H) \leq f(\chi_d(H)) \cdot r(H) \ ?$$

A neat argument, due to my former PhD student Alexey Pokrovskiy, gives a proof that if $\chi_d(H) = 2$, then $\chi_s(H) = r(H)$.

Other special classes of hypergraphs for which Question 4.5 has a positive answer can be found in DvoŘák & Esperet (2013). That paper was also the inspiration for starting to think about this type of questions.

Z. DvoŘák and L. Esperet, *Distance-two coloring of sparse graphs.* arXiv:1303.3191 [math.CO], `http://arxiv.org/abs/1303.3191` (2013), 13 pages.

4.4 Victor Falgas-Ravry

4.4.1 Largest antichain in the independence complex of a graph

Write Q_n for the collection of all subsets of $[n] = \{1, 2, \ldots n\}$. We denote by $Q_n^{(r)}$ the rth *layer* of Q_n, that is, the collection of all subsets of $[n]$ of size r.

A family $\mathcal{A} \subseteq Q_n$ is an *antichain* if for every pair of distinct elements $A, B \in \mathcal{A}$, A is not a subset of B and B is not a subset of A. How large an antichain can we find in Q_n? Clearly each layer of Q_n forms an antichain, and a celebrated theorem of Sperner from 1928 asserts that we cannot do better than picking a largest layer:

Theorem 4.6 (Sperner [2])**.** *Let $\mathcal{A} \subseteq Q_n$ be an antichain. Then*

$$|\mathcal{A}| \leq \max_r |Q_n^{(r)}|.$$

I am interested in a generalisation of Sperner's theorem where $\mathcal{A}$ is restricted to a subset of Q_n: suppose we are given a graph G on $[n]$. The *independence complex of G* is the collection of all independent sets from $V(G) = [n]$,

$$Q(G) = \{A \subseteq [n] : \ A \text{ independent in } G\}.$$

We write $Q^{(r)}(G) = \{A \in Q(G) : \ |A| = r\}$ for the rth *layer* of $Q(G)$, and define the *width* of G, $s(G)$, to be the size of a largest antichain in $Q(G)$. Clearly we have

$$s(G) \geq \max_r |Q^{(r)}(G)|. \tag{1}$$

In general, $s(G)$ can be much larger than this: it is not hard to construct examples of graph sequences $(G_n)_{n \in \mathbb{N}}$ for which $\max_r |Q^{(r)}(G_n)| = o\left(s(G_n)\right)$. However one would expect that if G is reasonably homogeneous then (1) should be close to tight.

Question 4.7. *When do we have (almost) equality in (1) ? What conditions on G are sufficient to guarantee (almost) equality ?*

I am particularly interested in the cases where $G = C_n$, the cycle of length n, or where $G = P_n$, the path of length $n - 1$. In this setting, an analogue of the Erdős–Ko–Rado theorem in $Q(G)$ was proved by Talbot [3], using an ingenious compression argument. It is known [1] that the size of a largest antichain in a class of graphs including both C_n and P_n is of the same order as the size of a largest layer. However we really should have equality here:

Conjecture 4.8.

$$s(C_n) = \max_r |Q(C_n)^{(r)}| \ and \ s(P_n) = \max_r |Q(P_n)^{(r)}|.$$

It would also be interesting to know what happens in the case of random graphs:

Question 4.9. *Let $p = cn^{-1}$, for some constant $c > 0$. Is it true with high probability that*
$$s(G_{n,p}) = (1 + o(1)) \max_r |Q(G_{n,p})^{(r)}|?$$

References

[1] V. Falgas-Ravry. Sperner's problem for G-independent families. Accepted, *Combinatorics, Probability & Computing* (2014). Arxiv ref: 1302.6039.

[2] E. Sperner. Ein Satz über Untermengen einer endlichen Menge (in German). *Mathematische Zeitschrift*, 27(1):544–548, 1928.

[3] J. Talbot. Intersecting families of separated sets. *Journal of the London Mathematical Society*, 68(1):37–51, 2003.

5   Problem session Febuary 19, 2014

5.1   Miklós Simonovits

Problem 5.1 (Paul Erdős, via M. Simonovits). *Is it true that if G_n is a 4-chromatic n-vertex graph and deleting any edge of it we get a 3-chromatic graph, then the minimum degree of G_n is $o(n)$ (as $n \to \infty$)?*

Motivation, partial results: A graph G is called **k-color-critical** if it is k-chromatic but deleting any edge of G we get a $k - 1$-chromatic graph. (Actually, we could speak of edge-critical and vertex-critical graphs, but we stick to the edge-critical case.)

Bjarne Toft and myself, using a construction of Toft and a transformation of mine constructed (infinitely many) 4-colour-critical graphs where the minimum degree is $> c\sqrt[3]{n}$. (Simonovits, M.: On colour-critical graphs. Studia Sci. Math. Hungar. 7 (1972), 67–81, and Toft, B. Two theorems on critical 4-chromatic graphs, Studia Sci. Math. Hungar. 7 (1972), 83–89.) I do not know of anything with higher minimum degree (though I may overlook some newer results?)

A trivial construction of G. Dirac, obtained by joining two odd $n/2$-cycles completely shows that there exist 6-critical graphs with minimum degrees $n/2 + 2$. The difficulties occur for 4 and 5-critical graphs. (The 3-critical graphs are just the odd cycles.)

As I wrote, a basic ingredient of our construction was an earlier construction of Bjarne Toft, a 4-critical graph with $\approx \frac{n^2}{16}$ edges, where the vertices are in four groups of $n/4$ vertices, and (a) the first and last groups form two odd cycles, (b) the second and third groups form a complete bipartite graphs, (c) the first group is joined to the second class by a 1-factor and the third group to the last group also by a 1-factor.

5.2   Fedor Fomin

Question 5.2. *For a given n-vertex planar graph G, is it possible to find in polynomial time (or to show that this is NP-hard) an independent set of size $\lfloor n/4 \rfloor + 1$?*

5.3   Carsten Thomassen

Smith's theorem says that, for every edge e in a cubic graph, there is in an even number of Hamiltonian cycles containing e. As a consequence, every cubic Hamiltonian graph has at least 3 Hamiltonian cycles.

If e is an edge of a Hamiltonian bipartite cubic graph G, then G has en even number of Hamiltonian cycles through e. Using Smith's theorem once more, also G-e han an even number of Hamiltonian cycles. Hence the total number of Hamiltonian cycles is even, in contrast to the situation for non-bipartite graphs where it is 3 for infinitely many graphs.

Problem 5.3. *Does there exist a 3-connected cubic bipartite graph having an edge e such that there are precisely two Hamiltonian cycles containing e? (Otherwise, there will be at least 4 such cycles.)*

Problem 5.4. *Does there exist a 3-connected cubic bipartite graph having precisely 4 Hamiltonian cycles? (Otherwise, there will be at least 6 such cycles.)*

It is an old problem whether a second Hamiltonian cycle in a Hamiltonian cubic graph can be found in polynomial time. Andrew Thomason"s lollipop method is a simple algorithm producing a second Hamiltonian cycle, but it may take exponential many steps. The known examples have many edge-cuts with three edges.

Problem 5.5. *Does there exist a family of cubic, cyclically 4-edge-connected Hamiltonian graphs for which the lollipop method takes superpolynomially many steps?*

If Problem 5.5 has a negative answer, one can show that there exists a polynomially bounded algorithm for finding a second Hamiiltonian cycle in a cubic Hamiltonian graph.

5.4   Jörgen Backelin

The problem, stated briefly:

Determine the exact chromatic numbers for shift graphs with short vertices for cyclically ordered points.

Consider a finite set C, ordered cyclically. Define a shift graph by letting the vertices be all r-subsets of C, for some small r (I suggest doingt his for $r = 2$ or 3, in the first place), and by letting two vertices form an edge if their are disjoint, and intertwined in a prescribed manner.

Problem 5.6. *Determine the exact chromatic numbers for these graphs.*

Detailed explanation:

A cyclic order on a finite set C intuitively is what you think it should be, if you place the elements in a circle and follow it e. g. clockwise: It is not meaningful to say that the element a precedes b; but it is meaningful to say that starting from a we pass b before encountering c. Thus, a formal definition would have to deal with ternary rather than binary predicates (properties).

Cyclic orders was treated by P. J. Cameron 1978 (Math. Z. 148, pp. 127-139). One way to define them formally is as a property P, which holds for some triples of different elements in C, such that for any different a, b, c, d in C we have:

- Precisely one of $P(a, b, c)$ and $P(a, c, b)$ holds.

- If $P(a, b, c)$, then $P(b, c, a)$.

- If $P(a, b, c)$ and $P(a, c, d)$, then $P(a, b, d)$.

(Think of $P(a, b, c)$ as the statement "Starting from a, we pass b before arriving at c".)

"Intertwining vertices" should be done analogously as for ordinary shift graphs. E. g., for $r = 2$, there are two possibilities for two disjoint vertices $\{a, b\}$ and $\{c, d\}$: Either the pattern XXOO, which holds if either both $P(a, b, c)$ and $P(a, b, d)$ hold, or neither does, or the pattern XOXO. The first case has a trivial chromatic number for ordinary shift graphs, but not self-evidently so in the cyclic order situation.

For $r = 3$, there are essentially only three intertwinement patterns: XXXOOO, XXOXOO, and XOXOXO.

Figure 1: Sasha Kostochka receiving the award

Rationale:

I do not know if there are any direct applications of this. In general, working with cyclic orders removes a kind of lack of balance between different parts of a graph, which means that we often may construct more efficient examples of graphs with certain properties in this manner. Thus, I find this a more natural setting.

Note on generalisations: There are cyclically ordered sets of any cardinality. I do not know if there is a theory for "cyclically well-ordered sets" of large cardinalities, though.

5.5 Andrzej Ruciński

For graphs F and G and a positive integer r, we write $F \to (G)_r^v$ if every r-coloring of the vertices of F results in a monochromatic copy of G in F (not necessarily induced). E.g., $K_{r(s-1)+1} \to (K_s)_r^v$ by the Pigeon-hole Principle, where K_n is the complete graph on n vertices. Define

$$mad(F) = \max_{H \subseteq F} \frac{2|E(H)|}{|V(H)|}$$

and

$$m_{cr}(G, r) = \inf\{mad(F) \; : \; F \to (G)_r^v\}.$$

Problem 5.7. *Determine or estimate $m_{cr}(G, r)$ for every graph G and $r \geq 2$.*

It is known [1] that

$$r \max_{H \subseteq G} \delta(H) \leq m_{cr}(G, r) \leq 2r \max_{H \subseteq G} \delta(H),$$

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where $\delta(G)$ is the minimum vertex degree in G. The lower bound is attained by complete graphs, that is, $m_{cr}(K_s, r) = mad(K_{r(s-1)+1}) = r(s-1)$. On the other hand, the upper bound is asymptotically achieved by large stars as it was proved in [1] that, in particular for $r = 2$ colors,

$$4 - \frac{4}{k+1} \le m_{cr}(S_k, 2) \le 4 - \frac{2(k+1)}{k^2 + 1},$$

where S_k is the star with k edges. For $k = 2$ this reads

$$\frac{8}{3} \le m_{cr}(S_2, 2) \le \frac{14}{5}.$$

In [1] I offered 400,000 z (Polish currency in 1993, equivalent after denomination of 1995 to 40 PLN) for the determination of the exact value of $m_{cr}(S_2, 2)$. Recently, it was pointed out by A.Pokrovskiy that a result of Borodin, Kostochka, and Yancey [2] 1-improper 2-colorings of sparse graphs yields that

$$m_{cr}(S_2, 2) = \frac{14}{5}.$$

During the Open Problem session at the Mittag-Leffler Institute I handed to Sasha Kostochka an envelope with four 10 PLN bills so that he could share the award among his co-authors and A. Pokrovskiy as well.

Problem 5.8. *Determine $m_c r(S_2, 3)$ (no monetary award for that!) The current bounds, unimproved since 1994 are*

$$\frac{18}{5} \le m_c r(S_2, 3) \le \frac{22}{5}.$$

References

[1] A. Kurek and A. Rucinski, Globally sparse vertex-Ramsey graphs, *J.Graph Theory* 18 (1994) 73-81.

[2] O. V. Borodin, A. Kostochka, and M. Yancey, On 1-improper 2-colorings of sparse graphs, *Discrete Mathematics* 313 (2013) 2638-2649.

6 Seminar February 20, 2014, Allan Lo

For a graph G, we know that $\chi'(G) = \Delta(G)$ or $\chi'(G) = \Delta(G) + 1$. A subgraph H of G is *overfull* if $e(H) > \Delta(G)\lfloor|H|/2\rfloor$ (note this requires $|H|$ to be odd). Note that an overfull subgraph is a trivial obstruction for $\chi'(G) = \Delta(G)$. In 1986, Chetwynd and Hilton [1] gave the following conjecture.

Conjecture 6.1 (Overfull subgraph conjecture)**.** *A graph G on n vertices with $\Delta(G) \geq n/3$ satisfies $\chi'(G) = \Delta(G)$ if and only if G contains no overfull subgraph.*

(Some remark about regular graphs) The 1-factorization conjecture is a special case of the overfull subgraph conjecture. If G is d-regular and contains no overfull subgraph, then ($|G|$ is even) and every odd cut has size at least d edges. So G has a 1-factor. Meredith [2] showed that for all $d \geq 3$, there exists a d-regular graph G on $20d - 10$ vertices with $\chi'(G) = d + 1$, which contains no overfull subgraph.

References

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7 Problem session March 5, 2014

7.1 Jacques Verstraëte

A problem on Majority Percolation

Let G be a finite graph, and let $p \in [0,1]$. Suppose that vertices of G are randomly and independently *infected* with probability p – this is the *infection probability*. Then consider the following deterministic rule: at any stage, an uninfected vertex becomes infected if *strictly more* than half of its neighbors are infected. Let $A(G)$ be the event that in finite time every vertex of G becomes infected with associated probability measure P_p.

Problem 7.1. *Does there exist a sequence of graphs $(G_n)_{n \in \mathbb{N}}$ such that for every $p > 0$:*

$$\lim_{n \to \infty} P_p(A(G_n)) = 1 \quad ?$$

I believe the answer is no. This process is a version of a process called *bootstrap percolation*. The most studied case is the $n \times n$ grid Γ_n, with the rule that *at least* two infected neighbors of an uninfected vertex cause the vertex to become infected. In this case, a remarkable paper of Holroyd shows that for all $\varepsilon > 0$,

$$P_p(A(\Gamma_n)) \to \begin{cases} 1 & \text{if } p > (1+\varepsilon)\frac{\pi^2}{18 \log n} \\ 0 & \text{if } p < (1-\varepsilon)\frac{\pi^2}{18 \log n} \end{cases}$$

Finer control of the relationship between ε and n was obtained by Graver, Holroyd and Morris.

7.2 Klas Markström

Given a matrix $m \in \mathrm{GL}(n,2)$, i.e an invertible matrix with entries $0/1$, we let $\mathcal{D}(m)$ denote the smallest number of row-operations we can use in order to reduce m to the identity matrix. Equivalently $\mathcal{D}(m)$ is the distance from m to I in the Cayely graph $\mathrm{Cay}(\mathrm{GL}(n,2), S)$, where S is the set of elementary matrices. In [1] an algorithm was given which can row reduce a matrix m to te identity using

$$\frac{n^2}{\log_2 n} + o\left(\frac{n^2}{\log_2 n}\right)$$

row operations, and it was proven that the expected value of $\mathcal{D}(m)$ for a random matrix from $\mathrm{GL}(n,2)$ is not less than half of that. Equivalently, this shows that the diameter of $\mathrm{Cay}(\mathrm{GL}(n,2), S)$ is at most the first bound, and the average distance is at least the second.

Problem 7.2.

1. *Give an explicit (non-random) example of an matrix $m \in \mathrm{GL}(n,2)$ such that $\mathcal{D}(m) \geq 100n$*

2. *Give an explicit example of an matrix $m \in \mathrm{GL}(n,2)$ such that $\mathcal{D}(m) \geq n \log n$*

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7.3   Andrzej Ruciński

Given integers $1 \leq \ell < k$, we define an *ℓ-overlapping cycle* as a k-uniform hypergraph (or k-graph, for short) in which, for some cyclic ordering of its vertices, every edge consists of k consecutive vertices, and every two consecutive edges (in the natural ordering of the edges induced by the ordering of the vertices) share exactly ℓ vertices. If H contains an ℓ-overlapping Hamiltonian cycle then H itself is called *ℓ-Hamiltonian*.

A k-graph H is *ℓ-Hamiltonian saturated*, $1 \leq \ell \leq k-1$, if H is not ℓ-Hamiltonian but for every $e \in H^c$ the k-graph $H + e$ is such. For n divisible by $k - \ell$, let $sat(n, k, \ell)$ be the *smallest* number of edges in an ℓ-Hamiltonian saturated k-graph on n vertices. In the case of graphs, Clark and Entringer [1] proved in 1983 that $sat(n, 2, 1) = \lceil \frac{3n}{2} \rceil$ for n large enough.

A. Żak showed that for $k \geq 2$, $sat(n, k, k-1) = \Theta(n^{k-1})$ [3]. Together, we proved that for all $k \geq 3$ and $\ell = 1$, as well as for all $\frac{4}{5}k \leq \ell \leq k-1$

$$sat(n, k, \ell) = \Theta(n^{\ell}), \tag{2}$$

and conjectured that (2) holds for all k and $1 \leq \ell \leq k-1$ [2]. The smallest open case is $k = 4$, $\ell = 2$. Recently, we have got some partial results: $sat(n, k, \ell) = O(n^{(k+\ell)/2})$ and $sat(n, 4, 2) = O(n^{14/5})$.

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8 Problem session March 19, 2014

8.1 Brendan McKay

How many $n/2$-cycles can a cubic graph have?

Given a simple cubic graph with n vertices, what is a good upper bound on the number of cycles of length $n/2$ it can have?

A random cubic graph has $\Theta((4/3)n/n)$ cycles of length $n/2$. So do random cubic bipartite graphs. Also the whole cycle space has size $2^{n/2+1}$, so twice that is a (silly) upper bound.

The actual maximums for $n = 4, 6, \ldots, 24$ are: 0,2,6,12,20,20,48,48,132,118,312 (not in OEIS). All these are achieved uniquely except that for 20 vertices there are two graphs with 132 10-cycles.

Maximum automorphism group for a 3-connected cubic graph

Let $a(n)$ be the greatest order of the automorphism group of a 3-connected cubic graph with n vertices. I conjecture: for $n \geq 16$, $a(n) < n \, 2^{n/4}$.

There is a paper of Opstall and Veliche that finds the maximum over all cubic graphs, but the maximum occurs for graphs very far from being 3-connected.

When n is a multiple of 4 there is a vertex-transitive cubic graph achieving half the conjectured bound, so if true the bound is pretty sharp.

A paper of Potočnik, Spiga and Verret (arxiv.org/abs/1010.2546), together with some computation, establishes the conjecture for vertex-transitive graphs, so the remaining problem is whether one can do better for non-transitive graphs. For 20, and all odd multiples of 2 vertices from 18 to at least 998 (but not for 4–16 or 24 vertices) the graph achieving the maximum is not vertex-transitive.

Probability that a random integer matrix is positive

Let $M(n, k)$ be the set of $n \times n$ matrices of nonnegative integers such that every row and every column sums to k. Let $P(n, k)$ be the fraction of such matrices which have no zero entries, equivalently the probability that a random matrix from the uniform distribution on $M(n, k)$ has no zero entries.

Obviously $P(n, k) = 0$ for $k < n$. It seems obvious that $P(n, k)$ should be increasing as a function of k when n is fixed and $k \geq n$, but can you prove it?

We know that $P(n, k) = |M(n, k)|/|M(n, k)|$. Also, note that $M(n, k)$ is the set of integer points in the k-dilated Birkhoff polytope, and $P(n, k)$ is the fraction of such points that don't lie on the boundary. Ehrhart theory tells us that $|M(n, k)| = H_n(k)$ where H_n is a polynomial, and that $P(n, k) = (-1)^{n+1} H_n(-k)/H_n(k)$. Does it help?

8.2 Andrew Thomason (by proxy)

Suppose you are given a set E and a collection of finite sequences of elements of E. We now wish to determine if there is a graph such that E is the edges of the graph and the sequences are (nice, simple) paths in the graph.

This can be done, the graph would have at most $2|E|$ vertices so you can search all possibilities. So the question would be whether you can do it efficiently.

8.3 Bruce Richter

Hajós conjectured that if the chromatic number $\chi(G)$ of a graph G is at least r, then G contains a subdivision of K_r. Hadwiger conjectured that G has K_r

as a minor.

Albertson's Conjecture: *If $\chi(G) \geq r$, then the crossing number $cr(G) \geq cr(K_r)$.*

This is known for $r \leq 16$, with the best result being that of J. Barát and G. Tóth, Towards the Albertson conjecture, Elec. J. Combin. 2010. The interesting thing here is that the crossing number of K_r is only known when $r \leq 12$.

The conjecture is easy for large r-critical graphs. The problem occurs when $|V(G)|$ is just a little larger than r.

8.4 Alexander Kostochka

8.4.1 Problem 1

For nonnegative integers j, k, a (j, k)-coloring of graph G is a partition $V(G) = J \cup K$ such that $\Delta(G[J]) \leq j$ and $\Delta(G[K]) \leq k$. An old result of Lovász implies that every graph G with $\Delta(G) \leq j + k + 1$ has a (j, k)-coloring. The proof is short and one may wonder about possible Brooks-type refinements of the result. In seeking such possibilities, Corréa, Havet, and Sereni [1], conjectured that for sufficiently large k (say, $k > 10^6$), every planar graph G with $\Delta(G) \leq 2k + 2$ has a (k, k)-coloring.

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8.4.2 Problem 2

A graph is a *circle graph*, if it is the intersection graph of a family of chords of a circle. Circle graphs arise in many combinatorial problems ranging from sorting problems to studying planar graphs to continous fractions. In particular, for a given permutation P of $\{1, 2, \ldots, n\}$, the problem of finding the minimum number of stacks needed to obtain the permutation $\{1, 2, \ldots, n\}$ from P reduces to finding the chromatic number of a corresponding circle graph. There are polynomial algorithms for finding the clique number and the independence number of a circle graph, but finding the chromatic number of a circle graph is an NP-hard problem.

Let $f(k)$ denote the maximum chromatic number of a circle graph with clique number k. Gyárfás [3] proved that $f(k)$ is well defined and $f(k) \leq 2^k(2^k - 2)k^2$. The only known exact value is $f(2) = 5$. The best bounds known to me are

$$0.5k(\ln k - 2) \leq f(k) \leq 50 \cdot 2^k.$$

The lower bound is only barely superlinear, and the upper is very superlinear. It would be interesting to improve any of them. More info on circle graphs and their colorings could be found in [1, 2, 4].

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8.5   David Conlon

Monochromatic cycle partitions in mean colourings

A well-known result of Erdős, Gyárfás and Pyber [3] says that there exists a constant $c(r)$, depending only on r, such that if the edges of the complete graph K_n are coloured with r colours, then the vertex set of K_n may be partitioned into at most $c(r)$ disjoint monochromatic cycles, where we allow the empty set, single vertices and edges to be cycles. For $r = 2$, it is known [1] that two disjoint monochromatic cycles of different colours suffice, while the best known general bound [4] is $c(r) = O(r \log r)$.

With Maya Stein [2], we recently considered a generalisation of this monochromatic cycle partition question to graphs with locally bounded colourings. We say that an edge colouring of a graph is an *r-local colouring* if the edges incident to any vertex are coloured with at most r colours. Note that we do not restrict the total number of colours. Somewhat surprisingly, we prove that even for local colourings, a variant of the Erdős-Gyárfás-Pyber result holds.

Theorem 8.1. *The vertex set of any r-locally coloured complete graph may be partitioned into $O(r^2 \log r)$ disjoint monochromatic cycles.*

For $r = 2$, we have the following more precise theorem.

Theorem 8.2. *The vertex set of any 2-locally coloured complete graph may be partitioned into two disjoint monochromatic cycles of different colours.*

An edge colouring of a graph is said to be an *r-mean colouring* if the average number of colours incident to any vertex is at most r. We suspect that a theorem analogous to Theorem 8.1 may also hold for r-mean colourings but have been unable to resolve this question in general.

Question 8.3. *Does there exist a constant $m(r)$, depending only on r, such that the vertex set of any r-mean coloured graph may be partitioned into at most $m(r)$ cycles?*

We can show that the vertex set of any 2-mean coloured graph may be partitioned into at most two cycles of different colours but the proof uses tricks which are specific to the case $r = 2$.

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9 Problem session April 2nd, 2014

9.1 Miklós Simonovits

The problem we discuss here is informally as follows:

Is it true that the Turán number of an infinite family of forbidden (bipartite) graphs $\mathcal{L}$ can be approximated arbitrarily well in the exponent by finite subfamilies?

More precisely, here we consider ordinary simple graphs: no loops or multiple edges are allowed. $\mathbf{ext}(n, \mathcal{L})$ denotes the maximum number of edges a graph G_n on n vertices can have without containing subgraphs from $\mathcal{L}$. The problem below is motivated by the fact that if $\mathcal{C}$ is the family of all cycles then $\mathbf{ext}(n, \mathcal{C}) = n - 1$, however for any finite $\mathcal{C}^* \subset \mathcal{C}$, for some $\alpha = \alpha(\mathcal{C}^*) > 0$, and $c_1 > 0$, $\mathbf{ext}(n, \mathcal{C}^*) > c_1 n^{1+\alpha}$. This means that some kind of compactness is missing here. On the other hand, the continuity of the exponent easily follows from Bondy-Simonovits theorem:

$$\mathbf{ext}(n, C_{2k}) = O(n^{1+1/k}).$$

This means a continuity in the exponent.

Problem 9.1 (Continuity of the exponent). *Let $\mathcal{L}$ denote an infinite family of bipartite (excluded) graphs and*

$$\mathcal{L}_m := \{L \ : \ L \in \mathcal{L}, \ v(L) \leq m\}.$$

Is it true that if for some $\alpha > 0$ and $c > 0$ we have $\mathbf{ext}t(n, \mathcal{L}) = O(n^{1+\alpha})$, then for any $\varepsilon > 0$, we have

$$\mathbf{ext}(n, \mathcal{L}_m) = O(n^{1+\alpha+\varepsilon}), \quad as \quad n \to \infty,$$

if m is large enough?

9.2 Erik Aas and Brendan McKay

Let $G = (V, E)$ be a connected simple graph, k^E its edge space over the field k. We are interested in the subspace $C(G)$ spanned by the (characteristic vectors of the) cycles of G.

When k is the field with two elements, it is a classical fact that the dimension of $C(G)$ is $|E(G)| - |V(G)| + 1$. This can be proved by providing an explicit basis of $C(G)$, as follows. Pick any spanning tree T of G, and for each edge e not in T consider the unique cycle whose only edge not in T is e. These cycles are linearly independent and thus span $C(G)$ in this case.

Now, when k does not have characteristic 2, the dimension is not a simple function of $|E(G)|$ and $|V(G)|$. However, in the case G is 3-edge-connected, it is not difficult to prove that in fact the dimension of $C(G)$ is $|E(G)|$.

Question: Is there a nice explicit basis for $C(G)$ consisting of cycles indexed by E, assuming G is 3-edge-connected?

9.3 András Gyárfás

A *3-tournament* T_n^3 is the set of all triples on vertex set $[n] = \{1, 2, \ldots, n\}$ such that in each triple some vertex is designated as the *root* of the triple. A set

$X \subset [n]$ is a *dominating set* in a 3-tournament T_n^3 if for every $z \in [n] \setminus X$ there exist $x \in X, y \in [n]$ ($y \neq z, y \neq x$) such that x is the root of the triple (x, y, z). Let $dom(T_n^3)$ denote the cardinality of a smallest dominating set of T_n^3.

Conjecture 9.2. *There exists a 3-tournament T_n^3 such that $dom(T_n^3) \geq 2014$.*

Conjecture 9.3. *If any four vertices of a 3-tournament T_n^3 contain at least two triples with the same root then $dom(T_n) \leq 2014$.*

I already posed this pair of conjectures at the 2012 Prague Midsummer Combinatorial Workshop (of course with 2012 in the role of 2014).

Note that the 2-dimensional versions of the above conjectures are true: there exist tournaments T with $dom(T) \geq 2014$; if any three vertices of a tournament T contain two pairs with the same root then $dom(T) = 1$. Also, if *three triples* are required with the same root in every four vertices of a T_n^3 then $dom(T_n^3) = 1$ follows easily (a remark with Tuza).

9.4   Klas Markström

If $G = (V, E)$ is an n-vertex graph then the strong chromatic number of G, denoted $s_\chi(G)$, is the minimum k such that the following hold: Any graph which is the union of G and a set of $\lceil \frac{n}{k} \rceil$ vertex-disjoint k-cliques is k-colourable. Here we take the union of edge sets, adding isolated vertices to G if necessary to make n divisible by k.

It is easy to see that $\Delta G + 1 \leq s_\chi(G)$, and Penny Haxell [1] has proven that $s_\chi(G) \leq c\Delta(G)$ for all $c > \frac{11}{4}$ if Δ is large enough, and [2] $s_\chi(G) \leq 3\Delta(G) - 1$ in general. The folklore conjecture here is that $s_\chi(G) \leq 2\Delta(G)$. This is known to be true if $\Delta \geq \frac{n}{6}$ [3].

Let us define the *biclique number* $\omega_b(G)$ to be the maximum t such that there exists a $K_{a,b} \subset G$ with $t = a + b$.

A few years ago I made the following conjecture:

Conjecture 9.4. $\omega_b(G) \leq s_\chi(G) \leq \omega_b(G) + 1$

The lower bound is easily seen to be true so the conjecture really concerns the upper bound.

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9.5 Benny Sudakov

Question 9.5. *How many edges do we need to delete to make a K_r-free graph G of order n bipartite?*

For $r = 3, 4$ this was asked long time ago by P. Erdős. For triangle-free graphs he conjectured that deletion of $n^2/25$ edges is always enough and that extremal example is a blow-up of a 5-cycle. Sudakov answered the question for $r = 4$ and proved that the unique extremal construction in this case is a complete 3-partite graph with equal parts. This result suggests that a complete $(r-1)$-partite graph of order n with equal parts is worst example also for all remaining values of r. Therefore we believe that it is enough to delete at most $\frac{(r-2)^2}{4(r-1)^2}n^2$ edges for even $r \geq 5$ and at most $\frac{r-3}{4(r-1)}n^2$ edges for odd $r \geq 5$ to make bipartite any K_r-free graph G of order n.

10 Problem session April 16th, 2014

10.1 Dhruv Mubayi

Fix $k \geq 2$ and recall that the Ramsey number $r(G, H)$ is the minimum n such that every red/blue edge-coloring of the complete k-uniform hypergraph on n vertices yields either a red copy of G or a blue copy of H. A 3-cycle C_3 is the k-uniform hypergraph comprising three edges A, B, C such that every pair of them has intersection size 1 and no point lies in all three edges. Classical results of Ajtai-Komlós-Szemerédi and a construction by Kim show that for $k = 2$, we have $r(C_3, K_t) = \Theta(t^2 / \log t)$, where K_t is the complete graph on t vertices. Kostochka, Mubayi, and Verstraëte proved that for $k = 3$, there are positive constants a, b such that

$$at^{3/2}/(\log t)^{3/4} < r(C_3, K_t) < bt^{3/2}.$$

Conjecture 10.1. *(Kostochka-Mubayi-Verstraëte) For $k = 3$, we have $r(C_3, K_t) = o(t^{3/2})$.*

10.2 Jørgen Bang-Jensen

10.2.1 Longest (x, y)-path in a tournament

A digraph on at least $k + 1$ vertices is k-strong if it remains strongly connected after the deletion of any subset X of at most $k - 1$ vertices. An (x, y)-path is a directed path from x to y. A digraph is **hamiltonian-connected** if it contains a hamiltonian (x, y)-path for every choice of distinct vertices x, y.

Theorem 10.2. *[5] Every 4-strong tournament is hamiltonian-connected and this is best possible.*

Theorem 10.3. *[4] There exists a polynomial algorithm for deciding whether a given tournament T with specified vertices x, y has an (x, y)-hamiltonian path.*

Problem 10.4 (Conjecture 9.1)**.** *[3] What is the complexity of finding the longest (x, y)-path in a tournament?*

The algorithm of Theorem 10.3 uses a divide and conquer approach to reduce a given instance into a number of smaller instances which can either be recursively solved or for which we have a theoretical result solving the problem. Thus the approach cannot be used to solve the case where we are not looking for hamiltonian paths.

10.2.2 Hamiltonian paths in path-mergeable digraphs

A digraph D is **path-mergeable** if, for every choice of distinct vertices $x, y \in V(D)$ and internally disjoint (only end vertices in common) (x, y)-paths P_1, P_2 there is an (x, y)-path P in D such that $V(P) = V(P_1) \cup V(P_2)$. It was shown in [1] that one can recognize path-mergeable digraphs in polynomial time. A **cutvertex** in a digraph is a vertex whose removal results in a digraph whose underlying undirected graph is disconnected.

Theorem 10.5. *[1] A path-mergeable digraph has a hamiltonian cycle if and only if it is strongly connected and has no cutvertex. Furthermore, a hamiltonian cycle of each block of D can be produced in polynomial time.*

Problem 10.6. *[3] What is the complexity of the hamiltonian path problem for path-mergeable digraphs?*

Note that the problem is easy if D is not connected or has no cutvertex so the problem is easy when the block graph of D is not a path (if there is just one block the digraph has a hamiltonian cycle, by Theorem 10.5 and if the block graph is not a path there can be no hamiltonian path. However, when the block graph is a path, the fact that we have a hamiltonian cycle in each block does not help much. In fact for every internal block with connection to its sourrounding blocks through the vertices x, y, we need to check the existence of an (x, y)-hamiltonian path.

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10.3 Klas Markström

Let A be an $n \times n$ array with entries from $\{0, \ldots, n\}$, such that each non-zero x appears in at most $n - 2$ positions in A. (Each entry of A is just a single number.)

Conjecture 10.7. *For any A there exists a latin square L, using the symbols $\{1, \ldots, n\}$ such $L_{i,j} \neq A_{i,j}$*

In [2] it was proven that the conjecture holds when only two symbols appear in A, and a full characterization of unavoidable arrays with two symbols was given, and a complete list of small unavoidable arrays where each entry in A can now be a list of numbers,. In [1] the conjecture was shown to hold if $n - 2$ is replaced by $\frac{n}{5}$, and in [3] it was shown to hold if A is a partial latin square.

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11   Problem session April 29th, 2014

11.1   Jørgen Bang-Jensen

A digraph $D = (V, A)$ is k-arc-strong if $D - A'$ remains strongly connected for every subset $A' \subseteq A$ with $|A'| \leq k - 1$. We denote by $\lambda(D)$ the maximum k such that D is k-arc-strong.

Theorem 11.1. *[2] Every k arc-strong tournament T on n vertices contains a spanning k-arc-strong subdigraph with at most $nk + 136k^2$ arcs.*

Theorem 11.2. *[2] Every k arc-strong tournament $T = (V, A)$ on n vertices contains a spanning subdigraph $D' = (V, A')$ such that every vertex in D' has in- and out-degree at least k and $|A'| \leq nk + \frac{k(k-1)}{2}$ and this is best possible.*

For a given tournament T let $\alpha_k(T)$ denote the minimum number of arcs in a spanning subdigraph of T which has minimum in- and out-degree at least k. For given k-arc-strong tournament T let $\beta_k(T)$ denote the minimum number of arcs in a spanning k-arc-strong subdigraph of T.

Conjecture 11.3. *[2] For every k arc-stroing tournament T we have $\alpha_k(T) = \beta_k(T)$, in particular we have $\beta_k(T) \leq nk + \frac{k(k-1)}{2}$.*

Conjecture 11.4. *[2] There exists a polynomial algorithm for finding, in a given k-arc-strong tournament $T = (V, A)$ a minimum set of arcs A' (of size $\beta_k(T)$) such that the subdigraph induced by A' is already k-arc-strong.*

Note that the following theorem shows that a similar property as that conjectured above holds when we consider the minimum number $r_k^{arc-strong}(T)$ of arcs whose reversal results in a k-arc-strong tournament. It shows that, except when the degrees are almost right already so that some cut needs more arcs reversed, we have equality between the numbers $r_k^{arc-strong}(T)$ and $r_k^{deg}(T)$, where the later is the minimum number of arcs whose reversal in T results in a tournament T'' with minimum in- and out-degree at least k.

Theorem 11.5. *[3] For every tournament T on at least $2k + 1$ vertices the number $r_k^{arc-strong}(T)$ is equal to the maximum of the numbers $k - \lambda(T)$ and $r_k^{deg}(T)$. In particular, we always have $r_k^{arc-strong}(T) \leq \frac{k(k+1)}{2}$ (equality for transitive tournaments on at least $2k + 1$ vertices).*

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11.2 Matas Šileikis

A family $\mathcal{F}$ of subsets of $[n] = \{1, \ldots, n\}$ is called

1. *k-intersecting* if for all $A, B \in \mathcal{F}$ we have $|A \cap B| \geq k$,

2. an *antichain* if for all $A, B \in \mathcal{F}$ such that $A \neq B$ we have $A \nsubseteq B$,

In 1964 Katona [2] (see also [1, p. 98]) determined the least upper bound for the size of a k-intersecting family:

$$|\mathcal{F}| \leq \begin{cases} \sum_{j=t}^{n} \binom{n}{j}, & \text{if } k+n = 2t, \\ \sum_{j=t}^{n} \binom{n}{j} + \binom{n-1}{t-1}, & \text{if } k+n = 2t-1, \end{cases} \tag{3}$$

with equality attained by the family consisting of all sets of size at least t plus, when $k+n$ is odd, subsets of $[n-1]$ of size $t-1$.

In 1966 Kleitman [3] (see also [1, p. 102]) observed that the bound (3) remains true under a weaker condition that $\mathcal{F}$ has diameter at most $n-k$, that is, when for every $A, B \in \mathcal{F}$ we have $|A \triangle B| \leq n - k$.

In 1968 Milner [4] determined the least upper bound for the size of a k-intersecting antichain (which generalizes Sperner's Lemma, when $k = 0$):

$$|\mathcal{F}| \leq \binom{n}{t}, \qquad t = \left\lceil \frac{n+k}{2} \right\rceil. \tag{4}$$

Question. Does the bound (4) still hold for antichains satisfying the weaker condition that the diameter of $\mathcal{F}$ is at most $n - k$?

References

[1] B. Bollobás. *Combinatorics. Set systems, hypergraphs, families of vectors and combinatorial probability.* Cambridge University Press, 1986.

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[3] D. J. Kleitman. On a combinatorial conjecture of Erdős. *J. Combinatorial Theory*, 1:209–214, 1966.

[4] E. C. Milner. A combinatorial theorem on systems of sets. *J. London Math. Soc.*, 43:204–206, 1968.

12 Problem session May 7th, 2014

12.1 Imre Leader

A Ramsey Question in the Symmetric Group.

Question 12.1. *Given k and r, does there exist n such that whenever the symmetric group S_n is k-coloured there is a monochromatic copy of S_r?*

To make sense of this, it is necessary to explain what a 'copy of S_r' means. We view S_n as the set of all words of length n, on symbols $1, ..., n$, such that no symbol is repeated. Given words $x_1, ..., x_r$ on symbols $1, ..., n$, such that the sum of the lengths of the x_i is n and no symbol is repeated, a copy of S_r means the set of all possible $r!$ concatenations (in any order) of $x_1, ..., x_r$.

This is easy to check when $r = 2$, but even for $r = 3$ we do not know it. In fact, we do not even know it in the case $r = 3$ and $k = 2$.