

SQUARE LATTICE WALKS AVOIDING A QUADRANT

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ABSTRACT. In the past decade, a lot of attention has been devoted to the enumeration of walks with prescribed steps confined to a convex cone. In two dimensions, this means counting walks in the first quadrant of the plane (possibly after a linear transformation).

But what about walks in non-convex cones? We investigate the two most natural cases: first, square lattice walks avoiding the negative quadrant $\mathcal{Q}_1 = \{(i, j) : i < 0 \text{ and } j < 0\}$, and then, square lattice walks avoiding the West quadrant $\mathcal{Q}_2 = \{(i, j) : i < j \text{ and } i < -j\}$. In both cases, the generating function that counts walks starting from the origin is found to differ from a simple D -finite series by an algebraic one. We also obtain closed form expressions for the number of n -step walks ending at certain prescribed endpoints, as a sum of three hypergeometric terms.

One of these terms already appears in the enumeration of square lattice walks confined to the cone $\{(i, j) : i + j \geq 0 \text{ and } j \geq 0\}$, known as Gessel's walks. In fact, the enumeration of Gessel's walks follows, by the reflection principle, from the enumeration of walks starting from $(-1, 0)$ and avoiding \mathcal{Q}_1 . Their generating function turns out to be purely algebraic (as the generating function of Gessel's walks).

Another approach to Gessel's walks consists in counting walks that start from $(-1, 1)$ and avoid the West quadrant \mathcal{Q}_2 . The associated generating function is D -finite but transcendental.

1. INTRODUCTION

In recent years, the enumeration of lattice walks confined to convex cones has attracted a lot of attention. In two dimensions, this means counting walks in the intersection of two half-spaces, which we can always assume (Figure 1) to form the first quadrant $\mathcal{Q} = \{(i, j) : i \geq 0 \text{ and } j \geq 0\}$. The problem is then completely specified by prescribing a starting point and a set of allowed steps. The two most natural examples are walks on the square lattice (with steps $\rightarrow, \uparrow, \leftarrow, \downarrow$), and walks on the *diagonal* square lattice (with steps $\nearrow, \searrow, \swarrow, \nwarrow$). Both cases can be solved via the classical reflection principle [15, 16]. The enumeration usually records the length n of the walk (with a variable t), and the coordinates (i, j) of its endpoint (with variables x and y). For instance, the generating function of square lattice walks starting from $(0, 0)$ and confined to \mathcal{Q} is [16, 10]:

$$Q(x, y) = \sum_{i, j, n \geq 0} \frac{(i+1)(j+1)}{(n+1)(n+2)} \binom{n+2}{n-i-j} \binom{n+2}{\frac{n+i-j+2}{2}} x^i y^j t^n, \quad (1)$$

where the sum is restricted to integers i, j, n such that n and $i + j$ have the same parity. (To lighten notation, we ignore the dependence in t of this series.) This series is D -finite [21]: this means that it satisfies a linear differential equation in each of its variables t, x and y , with coefficients in the field $\mathbb{Q}(t, x, y)$ of rational functions in t, x and y .

In the past decade, a systematic study of quadrant walks with *small steps* (that is, steps in $\{-1, 0, 1\}^2$) has been carried out, and a complete classification is now

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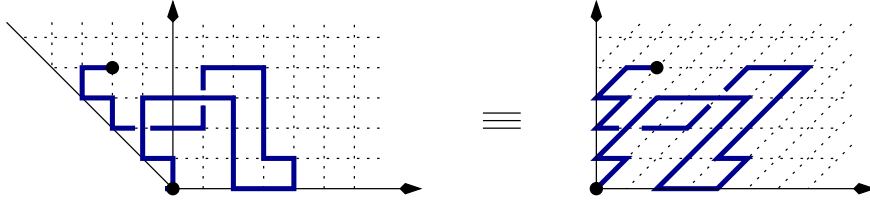


FIGURE 1. Square lattice walks staying in a 135° wedge are equivalent to quadrant walks with steps $\rightarrow, \nearrow, \leftarrow, \swarrow$.

available. For walks starting at $(0, 0)$, the generating function is D-finite if and only if a certain group of birational transformations is finite. The proof combines an attractive combination of approaches: algebraic [7, 10, 14, 15, 23, 26], computer-algebraic [3, 18, 19], analytic [4, 20, 29], asymptotic [5, 11, 22, 24].

The most intriguing D-finite case is probably Gessel's model, illustrated in Figure 1. Around 2000, Ira Gessel conjectured that the number of $2n$ -step walks of this type starting and ending at $(0, 0)$ was

$$g_{0,0}(2n) = 16^n \frac{(1/2)_n (5/6)_n}{(2)_n (5/3)_n}, \quad (2)$$

where $(a)_n = a(a+1)\cdots(a+n-1)$ is the ascending factorial. A computer-aided proof of this conjecture was finally found in 2009 by Kauers, Koutschan and Zeilberger [18]. A year later, Bostan and Kauers [2] proved, using again intensive computer algebra, that the three-variate generating function of Gessel's walks starting at $(0, 0)$ and ending anywhere in the quadrant is not only D-finite, but even algebraic: this means that it satisfies a polynomial equation over $\mathbb{Q}(t, x, y)$. Three other proofs have now been given [1, 4, 6], but none of them explains combinatorially the simplicity of the numbers, nor the algebraicity of the series.

The primary objective of this paper is to initiate a parallel study for walks confined to non-convex cones. In two dimensions, this means that walks live in the *union* of two half-spaces, which we can assume to form the three-quadrant cone

$$\mathcal{C} := \{(i, j) : i \geq 0 \text{ or } j \geq 0\}.$$

In other words, these walks avoid the negative quadrant. We solve here the two most natural cases (and possibly the simplest): the square lattice, and the diagonal square lattice (Figure 2). By a simple rotation, the latter model is equivalent to

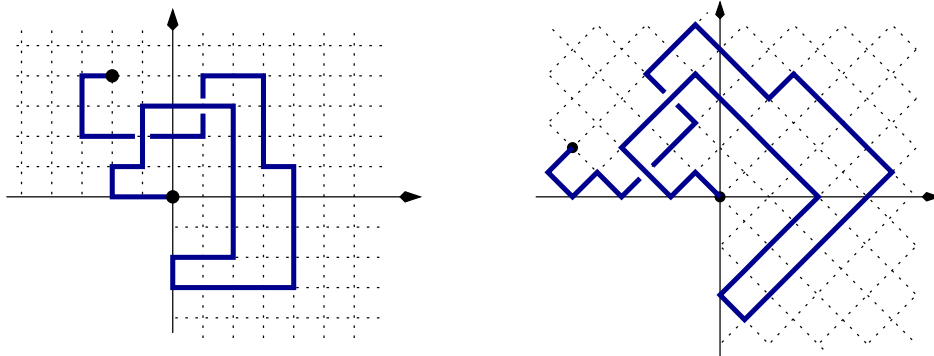


FIGURE 2. Walks confined to the non-convex cone \mathcal{C} : on the square lattice (left), and on the diagonal square lattice (right).

square lattice walks avoiding the West quadrant $\{(i, j) : i < -|j|\}$, as described in the abstract. The first problem was raised in 2001 by David W. Wilson in entry A060898 of the OEIS [17].

These two problems are far from being as simple as their quadrant counterparts. Their solutions exhibit, as in Gessel’s model, a combination of algebraicity phenomena and hypergeometric numbers. For instance, the generating function $C(x, y)$ of square lattice walks starting at $(0, 0)$ and confined to \mathcal{C} differs from the D-finite series

$$\frac{1}{3} \left(Q(x, y) - \bar{x}^2 Q(\bar{x}, y) - \bar{y}^2 Q(x, \bar{y}) \right)$$

by an *algebraic* one (we have written \bar{x} for $1/x$ and \bar{y} for $1/y$, and the series Q is given by (1)). This holds as well on the diagonal square lattice, if $Q(x, y)$ now counts quadrant walks with diagonal steps. In terms of numbers, we find for instance that the number of walks of length $2n$ starting and ending at $(0, 0)$ on the diagonal square lattice is

$$c_{0,0}(2n) = \frac{16^n}{9} \left(3 \frac{(1/2)_n^2}{(2)_n^2} + 8 \frac{(1/2)_n(7/6)_n}{(2)_n(4/3)_n} - 2 \frac{(1/2)_n(5/6)_n}{(2)_n(5/3)_n} \right).$$

Two ingredients of this formula are familiar: $16^n \frac{(1/2)_n^2}{(2)_n^2}$ counts walks confined to the first quadrant, while $16^n \frac{(1/2)_n(5/6)_n}{(2)_n(5/3)_n}$ counts Gessel’s walks. Asymptotically, these two terms are dominated by the central one, and

$$c_{0,0}(2n) \sim \frac{2^5}{3^2} \frac{\Gamma(2/3)}{\pi} \frac{4^{2n}}{(2n)^{5/3}}.$$

We obtain a similar, slightly more complicated formula for the square lattice (see (10)).

That Gessel’s numbers are involved in this problem should not be too surprising. Indeed, halving three quadrants gives a 135° cone as in Figure 1, and the solution of Gessel’s problem can be recovered from the reflection principle if we count square lattice walks starting from $(-1, 0)$ and confined to \mathcal{C} (Figure 3). An alternative approach is to count walks on the *diagonal* square lattice starting from $(-2, 0)$ and confined to \mathcal{C} (Figure 4). This connection between three-quadrant problems and Gessel’s walks was in fact another motivation of our study, and we solve these two problems with shifted starting point. We do not claim to have explained combinatorially Gessel’s ex-conjecture: as all proofs of this conjecture, our approach to the three-quadrant problem consists in solving a functional equation satisfied by the generating function $C(x, y)$. The tools involved in the solution consist of elementary power series manipulations, coefficient extractions, polynomial elimination. We have at the moment no combinatorial understanding of our results.

We hope that this work will be the starting point of a systematic study of walks avoiding a quadrant, analogous to what has been done so far for walks confined to a quadrant. The difficulty of the “simple” square lattice case suggests that this study may turn out to be even more challenging.

The paper is organized as follows. In Section 2 we count square lattice walks starting from $(0, 0)$ and confined to \mathcal{C} . Analogous results are proved in Section 3 for walks on the diagonal square lattice. In Section 4 we go back to the square lattice, but change the starting point to $(-1, 0)$. The x/y symmetry is lost, but we still obtain a complete solution, in fact simpler than in the original case:

the generating function of square lattice walks that start from $(-1, 0)$ and avoid the negative quadrant is algebraic.

In Section 5 we count similarly walks starting from $(-2, 0)$ on the diagonal square lattice. The generating function is not algebraic, but differs from a simple D-finite

series by an algebraic one. In Section 6 we derive from these results a new solution of Gessel's problem. Some perspectives and open questions are discussed in the final section. The paper is accompanied by two Maple sessions (one for the square lattice, one for the diagonal square lattice) available on the author's [webpage](#).

Notation. For a ring R , we denote by $R[t]$ (resp. $R[[t]]$) the ring of polynomials (resp. formal power series) in t with coefficients in R . If R is a field, then $R(t)$ stands for the field of rational functions in t . This notation is generalized to several variables. For instance, the generating function $Q(x, y)$ that counts quadrant walks is a series of $\mathbb{Q}[x, y][[t]]$, while the generating function $C(x, y)$ of walks confined to \mathcal{C} belongs to $\mathbb{Q}[x, \bar{x}, y, \bar{y}][[t]]$, where $\bar{x} = 1/x$ and $\bar{y} = 1/y$.

If $G(x)$ is a power series in t with coefficients in $\mathbb{Q}[x, \bar{x}]$, written as

$$G(x) = \sum_{i \in \mathbb{Z}, n \geq 0} g_i(n) t^n x^i,$$

we denote by G_i the coefficient of x^i in G :

$$G_i = [x^i]G(x) := \sum_{n \geq 0} g_i(n) t^n. \quad (3)$$

We denote by $[x^>]G(x)$ the *positive part of G in x* :

$$[x^>]G(x) := \sum_{i > 0, n \geq 0} g_i(n) t^n x^i.$$

We similarly define the non-positive, negative, and non-negative parts of $G(x)$. Finally, for a series $G(x, y)$ in t with coefficients in $\mathbb{Q}[x, \bar{x}, y, \bar{y}]$, we denote by $G_{i,j}$ the coefficient of $x^i y^j$, which is a series in t .

We refer to [21] for properties of D-finite series.

2. THE SQUARE LATTICE

The aim of this section is to determine the generating function of square lattice walks starting from $(0, 0)$ and confined to the three-quadrant cone \mathcal{C} . It reads

$$C(x, y) = \sum_{(i,j) \in \mathcal{C}} \sum_{n \geq 0} c_{i,j}(n) x^i y^j t^n = 1 + t(x + \bar{x} + y + \bar{y}) + O(t^2),$$

where $c_{i,j}(n)$ counts n -step walks going from $(0, 0)$ to (i, j) . For walks confined to the first quadrant \mathcal{Q} , we define the series $Q(x, y)$ and its coefficients $q_{i,j}(n)$ similarly. As recalled in the introduction, these coefficients have a simple hypergeometric form (see (1)).

Our first result (Theorem 1) states that the generating function $C(x, y)$ differs from the simple D-finite series

$$\frac{1}{3} \left(Q(x, y) - \bar{x}^2 Q(\bar{x}, y) - \bar{y}^2 Q(x, \bar{y}) \right)$$

by an algebraic series, which we describe explicitly. From there, we can express the generating function $C_{i,j}$ of walks ending at a prescribed point (i, j) (Corollary 2), and in some cases, obtain closed form expressions for its coefficients $c_{i,j}(n)$.

Theorem 1. *The generating function of square lattice walks starting at $(0, 0)$, confined to \mathcal{C} and ending in the first quadrant (resp. at a negative abscissa) is*

$$\frac{1}{3} Q(x, y) + P(x, y), \quad \left(\text{resp.} \quad -\frac{1}{3} \bar{x}^2 Q(\bar{x}, y) + \bar{x} M(\bar{x}, y) \right), \quad (4)$$

where $M(x, y)$ and $P(x, y)$ are algebraic of degree 72 over $\mathbb{Q}(x, y, t)$.

More precisely, P can be expressed in terms of M by:

$$P(x, y) = \bar{x}(M(x, y) - M(0, y)) + \bar{y}(M(y, x) - M(0, x)), \quad (5)$$

M satisfies the functional equation

$$(1 - t(x + \bar{x} + y + \bar{y}))(2M(x, y) - M(0, y)) = 2x/3 - 2t\bar{y}M(x, 0) + t(x - \bar{x})M(0, y) + t\bar{y}M(y, 0), \quad (6)$$

and the specializations $M(x, 0)$ and $M(0, y)$ have respective degrees 24 and 12 over $\mathbb{Q}(t, x)$ and $\mathbb{Q}(t, y)$.

Moreover, these algebraic series admit rational parametrizations. Let T be the unique series in t with constant term 1 satisfying:

$$T = 1 + 256t^2 \frac{T^3}{(T+3)^3}, \quad (7)$$

and let $Z = \sqrt{T}$. Let U be the only power series in t with constant term 1 satisfying

$$16T^2(U^2 - T) = x(U + UT - 2T)(U^2 - 9T + 8TU + T^2 - TU^2). \quad (8)$$

Then the series $tM(xt, 0)$ and $tM(0, xt)$ (both even series in t) admit rational expressions in terms of Z and U , given in Appendix A.1.

Remarks

1. Equation (4) gives the generating functions of walks ending in two quadrants of \mathcal{C} . By symmetry, the generating function of walks ending in the third quadrant, that is, at a negative ordinate, is $-\bar{y}^2 Q(\bar{y}, x)/3 + \bar{y}M(\bar{y}, x)$.
2. The parametrization by T , Z and U already appears in van Hoeij's parametrization for Gessel's walks in the quadrant [3, 6]. As explained later in Section 6, this is no coincidence. A fourth series is involved in the parametrization of Gessel's problem, and we will find it when counting walks confined to \mathcal{C} on the diagonal square lattice (Theorem 4).
3. A more compact statement of Theorem 1 reads as follows:

$$C(x, y) = A(x, y) + \frac{1}{3} \left(Q(x, y) - \bar{x}^2 Q(\bar{x}, y) - \bar{y}^2 Q(x, \bar{y}) \right),$$

where $A(x, y)$ satisfies

$$(1 - t(x + \bar{x} + y + \bar{y}))A(x, y) = (2 + \bar{x}^2 + \bar{y}^2)/3 - t\bar{y}A_-(\bar{x}) - t\bar{x}A_-(\bar{y}),$$

and $A_-(x)$ is a series in t with coefficients in $\mathbb{Q}[x]$, algebraic of degree 24. It equals the series $xM(x, 0)$ given in Appendix A.1.

Of course the algebraicity of $A(x, y) = P(x, y) + \bar{x}M(\bar{x}, y) + \bar{y}M(\bar{y}, x)$ follows from this statement, but it hides the fact that the series $P(x, y)$ and $M(x, y)$ are algebraic themselves (there is no reason why extracting say, negative powers of x in an algebraic series with coefficients in $\mathbb{Q}[x, \bar{x}, y, \bar{y}]$ should yield an algebraic series).

4. The four series $Q(x, y)$, $Q(\bar{x}, y)$, $Q(x, \bar{y})$ and $Q(\bar{x}, \bar{y})$ are related by a simple identity (see (16)), which allows us to write:

$$C(x, y) = A(x, y) - \frac{1}{3} \bar{x}^2 \bar{y}^2 Q(\bar{x}, \bar{y}) + \frac{(x - \bar{x})(y - \bar{y})}{3xy(1 - t(x + \bar{x} + y + \bar{y}))}.$$

This implies that $C(x, y)$, as $Q(x, y)$ itself, is D-finite but transcendental.

5. By combining the above results and *singularity analysis* of algebraic (and D-finite) series [13], one can derive from the above theorem that the number of n -step walks confined to \mathcal{C} is

$$[t^n]C(1, 1) \sim \frac{2^5 \sqrt{3}}{3^3 \Gamma(2/3)} \frac{4^n}{n^{1/3}}.$$

We now focus on walks ending at a prescribed point. For $(i, j) \in \mathcal{C}$, let $C_{i,j}$ denote the length generating function of walks going from $(0, 0)$ to (i, j) in \mathcal{C} . Define similarly $Q_{i,j}$, for $i, j \geq 0$. According to (1), the latter series has a simple form:

$$Q_{i,j} = \sum_{n \geq 0} \frac{(i+1)(j+1)}{(n+1)(n+2)} \binom{n+2}{\frac{n-i-j}{2}} \binom{n+2}{\frac{n+i-j+2}{2}} t^n.$$

The following corollary clarifies the nature of the series $C_{i,j}$.

Corollary 2 (Walks ending at a prescribed position). *Let T be the unique series in t with constant term 1 satisfying (7), and let $Z = \sqrt{T}$.*

For $j \geq 0$, the series $C_{-1,j}$ belongs to $t^{j+1}\mathbb{Q}(Z)$, and is thus algebraic. More generally, for $i \geq 1$ and $j \geq 0$, the series $C_{-i,j}$ is D-finite, of the form

$$-\frac{1}{3}Q_{i-2,j} + t^{i+j} \text{Rat}(Z)$$

for some rational function Rat. It is transcendental as soon as $i \geq 2$.

Finally, for $i \geq 0$ and $j \geq 0$, the series $C_{i,j}$ is of the form

$$\frac{1}{3}Q_{i,j} + t^{i+j} \text{Rat}(Z).$$

It is D-finite and transcendental.

The series $C_{i,j}$ can be effectively computed. For instance,

$$tC_{-1,0} = \frac{(Z^2 - 1)(11 + 6Z^2 - Z^4)}{(Z^2 + 3)^3}, \quad (9)$$

$$C_{-1,1} = 1024 \frac{Z^3 (Z^2 + 1)^2 (Z - 1)(1 + 2Z - Z^2)}{(Z^2 + 3)^6 (Z + 1)},$$

$$C_{-2,0} = -\frac{1}{3}Q_{0,0} + \frac{256Z^3(4 + 4Z - 4Z^2 + 23Z^3 - 9Z^4 + 18Z^5 - 6Z^6 + 3Z^7 - Z^8)}{3(Z^2 + 3)^6(Z + 1)},$$

$$C_{0,0} = \frac{1}{3}Q_{0,0} + \frac{512Z^3(4 + 4Z - 4Z^2 + 23Z^3 - 9Z^4 + 18Z^5 - 6Z^6 + 3Z^7 - Z^8)}{3(Z^2 + 3)^6(Z + 1)}.$$

The similarity between the last two expressions comes from (5), which tells us that $P_{0,0} = 2M_{1,0}$, while $C_{0,0} = Q_{0,0}/3 + P_{0,0}$ and $C_{-2,0} = -Q_{0,0}/3 + M_{1,0}$ by (4).

Starting from the expression of $C_{i,j}$ (more precisely, of its algebraic part $P_{i,j}$ or $M_{-i-1,j}$, depending on the sign of i), one can decide if the coefficients $c_{i,j}(n)$ have an expression as a finite sum of hypergeometric terms: one first computes a linear recurrence relation with polynomial coefficients satisfied by the coefficients (for instance using the Maple commands `algeqtodiffeq` and `diffeqtorec`) and then applies the Hyper algorithm from [27], which determines all hypergeometric solutions of such a recurrence relation. Using its Maple incarnation `hypergeomso1s`, we thus obtain:

$$\begin{aligned} c_{0,0}(2n) &= \frac{4 \cdot 16^n}{3^5} \left(3^4 \frac{(1/2)_n (1/2)_{n+1}}{(2)_n (2)_{n+1}} + 4(24n^2 + 60n + 29) \frac{(1/2)_n (7/6)_n}{(2)_{n+1} (4/3)_{n+1}} \right. \\ &\quad \left. - 2(12n^2 + 30n + 5) \frac{(1/2)_n (5/6)_n}{(2)_{n+1} (5/3)_{n+1}} \right) \quad (10) \\ &\sim \frac{2^9}{3^4} \frac{\Gamma(2/3)}{\pi} \frac{4^{2n}}{(2n)^{5/3}}. \end{aligned}$$

Given the link between $C_{0,0}$ and $C_{-2,0}$, this gives a closed form expression of the same type for $c_{-2,0}(2n)$. We have found similar expressions for the endpoints $(1, 1)$ and $(-3, 1)$, but not for $(-1, 0)$, $(-3, 0)$, $(-4, 0)$, $(0, 1)$, $(-1, 1)$, nor $(-2, 1)$.

As in the systematic study of quadrant models [10], the starting point of our approach is a functional equation that translates the step by step construction of walks confined to \mathcal{C} . It reads:

$$C(x, y) = 1 + t(x + \bar{x} + y + \bar{y})C(x, y) - t\bar{y}C_-(\bar{x}) - t\bar{x}C_-(\bar{y}),$$

where

$$C_-(\bar{x}) = \sum_{i < 0, n \geq 0} c_{i,0}(n)x^i t^n \in \bar{x}\mathbb{Q}[\bar{x}][[t]] \quad (11)$$

counts walks ending on the negative x -axis. The terms $t\bar{y}C_-(\bar{x})$ and $t\bar{x}C_-(\bar{y})$ correspond to forbidden moves yielding in the negative quadrant. Equivalently,

$$K(x, y)C(x, y) = 1 - t\bar{y}C_-(\bar{x}) - t\bar{x}C_-(\bar{y}) \quad (12)$$

where

$$K(x, y) = 1 - t(x + \bar{x} + y + \bar{y})$$

is the *kernel* of the equation.

Before going further, we review a solution of the associated quadrant model, which adapts to many other quadrant models [10]. In the square lattice case which we consider here, it is essentially a power series version of the classical reflection principle [15].

2.1. WARMING UP: WALKS CONFINED TO THE POSITIVE QUADRANT

We start from the functional equation obtained by constructing quadrant walks step by step:

$$K(x, y)Q(x, y) = 1 - t\bar{y}Q_+(x) - t\bar{x}Q_+(y) \quad (13)$$

where

$$Q_+(x) = Q(x, 0) = \sum_{i \geq 0, n \geq 0} q_{i,0}(n)x^i t^n \in \mathbb{Q}[x][[t]].$$

Equivalently,

$$xyK(x, y)Q(x, y) = xy - txQ_+(x) - tyQ_+(y). \quad (14)$$

The kernel $K(x, y)$ is invariant by the transformations $x \mapsto \bar{x}$ and $y \mapsto \bar{y}$. Hence we also have:

$$\begin{aligned} \bar{x}yK(x, y)Q(\bar{x}, y) &= \bar{x}y - t\bar{x}Q_+(\bar{x}) - tyQ_+(y), \\ \bar{x}\bar{y}K(x, y)Q(\bar{x}, \bar{y}) &= \bar{x}\bar{y} - t\bar{x}Q_+(\bar{x}) - t\bar{y}Q_+(\bar{y}), \\ x\bar{y}K(x, y)Q(x, \bar{y}) &= x\bar{y} - txQ_+(x) - t\bar{y}Q_+(\bar{y}). \end{aligned}$$

The *orbit equation* is the alternating sum of the last four equations:

$$\begin{aligned} K(x, y)(xyQ(x, y) - \bar{x}yQ(\bar{x}, y) + \bar{x}\bar{y}Q(\bar{x}, \bar{y}) - x\bar{y}Q(x, \bar{y})) &= xy - \bar{x}y + \bar{x}\bar{y} - x\bar{y} \\ &= (x - \bar{x})(y - \bar{y}). \end{aligned} \quad (15)$$

Observe that the right-hand side is now explicit. We call it the *orbit sum* of this quadrant model. The above equation can be rewritten as

$$xyQ(x, y) - \bar{x}yQ(\bar{x}, y) + \bar{x}\bar{y}Q(\bar{x}, \bar{y}) - x\bar{y}Q(x, \bar{y}) = \frac{(x - \bar{x})(y - \bar{y})}{1 - t(x + \bar{x} + y + \bar{y})}. \quad (16)$$

Extracting the positive part in x and y gives the following classical result [16, 10].

Proposition 3. *The series $xyQ(x, y)$ is the positive part (in x and y) of the rational function*

$$\frac{(x - \bar{x})(y - \bar{y})}{1 - t(x + \bar{x} + y + \bar{y})}.$$

It is thus D -finite. For $i, j, m \geq 0$, the number of quadrant walks of length $n = i + j + 2m$ starting at $(0, 0)$ and ending at (i, j) is:

$$\frac{(i+1)(j+1)n!(n+2)!}{m!(m+i+j+2)!(m+i+1)!(m+j+1)!}.$$

2.2. REDUCTION TO AN EQUATION WITH ORBIT SUM ZERO

We now return to walks avoiding the negative quadrant. We first apply to the functional equation (12), written as

$$xyK(x, y)C(x, y) = xy - txC_-(\bar{x}) - tyC_-(\bar{y}),$$

the treatment that we have just applied to the quadrant equation (14). The orbit equation reads as above

$$K(x, y)(xyC(x, y) - \bar{x}yC(\bar{x}, y) + \bar{x}\bar{y}C(\bar{x}, \bar{y}) - x\bar{y}C(x, \bar{y})) = (x - \bar{x})(y - \bar{y}). \quad (17)$$

We have just seen that $Q(x, y)$ also satisfies this equation (see (15)). The same holds for $-\bar{x}^2Q(\bar{x}, y)$, for $-\bar{y}^2Q(x, \bar{y})$ and for $\bar{x}^2\bar{y}^2Q(\bar{x}, \bar{y})$ (by (15) again). Let a be a real number and write

$$C(x, y) = A(x, y) + (1 - 2a)Q(x, y) - a\bar{x}^2Q(\bar{x}, y) - a\bar{y}^2Q(x, \bar{y}), \quad (18)$$

where $A(x, y)$ is a new series. (We do not involve the fourth solution $\bar{x}^2\bar{y}^2Q(\bar{x}, \bar{y})$, since $C(x, y)$ contains no monomial that is negative in x and in y). The above observations imply that the orbit sum associated with A vanishes:

$$xyA(x, y) - \bar{x}yA(\bar{x}, y) + \bar{x}\bar{y}A(\bar{x}, \bar{y}) - x\bar{y}A(x, \bar{y}) = 0. \quad (19)$$

Moreover, we can compute from the equations (12) and (13) satisfied by C and Q a functional equation for A . We first note that

$$C_-(\bar{x}) = A_-(\bar{x}) - a\bar{x}^2Q_+(\bar{x}),$$

where, as above,

$$A_-(\bar{x}) = \sum_{i < 0, n \geq 0} a_{i,0}(n)x^i t^n \in \bar{x}Q[\bar{x}][[t]].$$

Then we derive from (12) and (13) that:

$$K(x, y)A(x, y) = a(2 + \bar{x}^2 + \bar{y}^2) - t\bar{y}A_-(\bar{x}) - t\bar{x}A_-(\bar{y}) + t\bar{y}(1 - 3a)Q_+(x) + t\bar{x}(1 - 3a)Q_+(y).$$

This suggests to choose $a = 1/3$, so that

$$K(x, y)A(x, y) = (2 + \bar{x}^2 + \bar{y}^2)/3 - t\bar{y}A_-(\bar{x}) - t\bar{x}A_-(\bar{y}). \quad (20)$$

Note that the equations (12) and (20) satisfied by C and A only differ by their constant term on the right-hand side: it is simply 1 for C , but $(2 + \bar{x}^2 + \bar{y}^2)/3$ for A . This results into a zero orbit sum for A . Observe also that (20) characterizes $A(x, y)$ uniquely as a formal power series in t . The series $3A(x, y)$ counts walks in \mathcal{C} starting from $(0, 0)$, $(-2, 0)$ or $(0, -2)$, but those starting at $(0, 0)$ get weight 2.

We will show that $A(x, y)$ is algebraic (hence the notation A), as claimed by Theorem 1. We recall that all quadrant models with small steps and orbit sum zero have an algebraic generating function too [3, 4, 6, 7, 8, 10, 14]. One main difference with these quadrant models is that $A(x, y)$ involves positive and negative powers of x and y . The next subsection takes care of this difference.

2.3. REDUCTION TO A QUADRANT-LIKE PROBLEM

We now separate in $A(x, y)$ the contribution of the three quadrants:

$$A(x, y) = P(x, y) + \bar{x}M(\bar{x}, y) + \bar{y}M(\bar{y}, x), \quad (21)$$

where $P(x, y) \in \mathbb{Q}[x, y][[t]]$ and $M(\bar{x}, y) \in \mathbb{Q}[\bar{x}, y][[t]]$. Note that we have exploited the obvious x/y symmetry, and that this identity defines P and M uniquely in terms of A . The letter P stands for *positive*, and the letter M for *mixed*. Extracting the positive part in x and y from the orbit equation (19) gives

$$xyP(x, y) = y(M(x, y) - M(0, y)) + x(M(y, x) - M(0, x)), \quad (22)$$

which is equivalent to (5). Hence it suffices to determine M . We can write the above series A in terms of M only:

$$A(x, y) = \bar{x}(M(x, y) - M(0, y)) + \bar{y}(M(y, x) - M(0, x)) + \bar{x}M(\bar{x}, y) + \bar{y}M(\bar{y}, x).$$

Note that $A_-(\bar{x}) = \bar{x}M(\bar{x}, 0)$. Plugging this in the functional equation (20) gives:

$$\begin{aligned} K(x, y)(\bar{x}M(x, y) - \bar{x}M(0, y) + \bar{y}M(y, x) - \bar{y}M(0, x) + \bar{x}M(\bar{x}, y) + \bar{y}M(\bar{y}, x)) \\ = (2 + \bar{x}^2 + \bar{y}^2)/3 - t\bar{x}\bar{y}M(\bar{x}, 0) - t\bar{x}\bar{y}M(\bar{y}, 0). \end{aligned}$$

Let us extract the negative part in x of this equation. We obtain:

$$\begin{aligned} -t\bar{x}(M_x(0, y) + \bar{y}M(y, 0) - \bar{y}M(0, 0)) + K(x, y)\bar{x}M(\bar{x}, y) + tM(0, y) - t\bar{x}\bar{y}M(\bar{y}, 0) \\ = \bar{x}^2/3 - t\bar{x}\bar{y}M(\bar{x}, 0) - t\bar{x}\bar{y}M(\bar{y}, 0), \quad (23) \end{aligned}$$

with $M_x = \partial M / \partial x$. Observe that the term $t\bar{x}\bar{y}M(\bar{y}, 0)$ occurs on both sides and thus cancels. Extracting from this the coefficient of \bar{x} gives:

$$-t(M_x(0, y) + \bar{y}M(y, 0) - \bar{y}M(0, 0)) + (1 - t(y + \bar{y}))M(0, y) - tM_x(0, y) = -t\bar{y}M(0, 0).$$

From this, we obtain an expression of $M_x(0, y)$ in terms of $M(y, 0)$ and $M(0, y)$. By plugging it in (23), we obtain

$$K(x, y)(2M(\bar{x}, y) - M(0, y)) = 2\bar{x}/3 - 2t\bar{y}M(\bar{x}, 0) + t(\bar{x} - x)M(0, y) + t\bar{y}M(y, 0).$$

Replacing x by \bar{x} gives the functional equation (6) for $M(x, y)$, which we repeat here for convenience:

$$K(x, y)(2M(x, y) - M(0, y)) = 2x/3 - 2t\bar{y}M(x, 0) + t(x - \bar{x})M(0, y) + t\bar{y}M(y, 0). \quad (24)$$

We recall that $M(x, y)$ is a series in t with polynomial coefficients in x and y . The equation it satisfies is reminiscent of the quadrant equation (13). However, its right-hand side involves the series $M(y, 0)$ in addition to the two standard specializations $M(x, 0)$ and $M(0, y)$. Still, several ingredients in the rest of the solution are borrowed from former solutions of quadrant models [7, 6, 10].

2.4. CANCELLING THE KERNEL: AN EQUATION BETWEEN $M(x, 0)$, $M(0, x)$ AND $M(0, \bar{x})$

As a polynomial in y , the kernel $K(x, y)$ has two roots. Only one of them is a power series in t (with coefficients in $\mathbb{Q}[x, \bar{x}]$). We denote it by $Y \equiv Y(x)$:

$$Y = \frac{1 - t(x + \bar{x}) - \sqrt{(1 - t(x + \bar{x}))^2 - 4t^2}}{2t} = t + (x + \bar{x})t^2 + O(t^3). \quad (25)$$

The other root is $1/Y$, and its expansion in t involves a term $1/t$.

Specializing y to Y in (24) gives a relation between the three series on the right-hand side:

$$2x/3 - 2tM(x, 0)/Y + t(x - \bar{x})M(0, Y) + tM(Y, 0)/Y = 0. \quad (26)$$

Since Y is symmetric in x and \bar{x} , we also have, upon replacing x by \bar{x} :

$$2\bar{x}/3 - 2tM(\bar{x}, 0)/Y + t(\bar{x} - x)M(0, Y) + tM(Y, 0)/Y = 0. \quad (27)$$

Since the kernel is symmetric in x and y , we can rewrite (24) as

$$K(x, y)(2M(y, x) - M(0, x)) = 2y/3 - 2t\bar{x}M(y, 0) + t(y - \bar{y})M(0, x) + t\bar{x}M(x, 0).$$

Setting $y = Y$ in this equation gives

$$2Y/3 - 2t\bar{x}M(Y, 0) + t(Y - 1/Y)M(0, x) + t\bar{x}M(x, 0) = 0. \quad (28)$$

Finally, using once more $Y(x) = Y(\bar{x})$ gives a fourth equation:

$$2Y/3 - 2txM(Y, 0) + t(Y - 1/Y)M(0, \bar{x}) + txM(\bar{x}, 0) = 0. \quad (29)$$

We have thus obtained four equations, namely (26–29), relating the six series $M(x, 0)$, $M(\bar{x}, 0)$, $M(Y, 0)$, $M(0, x)$, $M(0, \bar{x})$ and $M(0, Y)$. By eliminating $M(0, Y)$ and $M(Y, 0)$, we obtain two equations between the remaining series, which only differ by the transformation $x \mapsto \bar{x}$. Eliminating $M(\bar{x}, 0)$ between them gives:

$$(Y - 1/Y)(xM(0, x) - 2\bar{x}M(0, \bar{x})) - 2\bar{x}Y/t + 3M(x, 0) = 0. \quad (30)$$

Our next step is to eliminate $M(x, 0)$.

2.5. AN EQUATION BETWEEN $M(0, x)$ AND $M(0, \bar{x})$

Let us denote the discriminant occurring in the expression (25) of Y by

$$\Delta(x) := (1 - t(x + \bar{x}))^2 - 4t^2.$$

Denote also

$$R(x) = tM(x, 0) \quad \text{and} \quad S(x) = txM(0, x). \quad (31)$$

Then the above equation (30) reads

$$\sqrt{\Delta(x)}(S(x) - 2S(\bar{x}) - \bar{x}) + \bar{x} - t - t\bar{x}^2 = 3tR(x). \quad (32)$$

Consequently,

$$\Delta(x)(S(x) - 2S(\bar{x}) - \bar{x})^2 = (3tR(x) - \bar{x} + t + t\bar{x}^2)^2.$$

Recall that $S(x)$ is a series in t with coefficients in $x\mathbb{Q}[x]$. Extracting from the above identity the negative part in x gives (after dividing by 4):

$$\Delta(x)(S(\bar{x})^2 + \bar{x}S(\bar{x})) - [x^<](\Delta(x)S(x)S(\bar{x})) = F_0 + \bar{x}F_1 + \bar{x}^2F_2,$$

where F_0, F_1 and F_2 are series in t than can be expressed in terms of $S(x)$ and $R(x)$:

$$F_0 = t^2S_1(1 + S_1), \quad F_1 = \frac{t}{2}(tS_2 + 3tR_1 - 5S_1), \quad F_2 = t^2(1 + 2S_1), \quad (33)$$

where we have used the notation (3) (note that $R_0 = S_1$ by definition of R and S). We have thus obtained an expression for the negative part of the series $\Delta(x)S(x)S(\bar{x})$:

$$[x^<](\Delta(x)S(x)S(\bar{x})) = \Delta(x)(S(\bar{x})^2 + \bar{x}S(\bar{x})) - F_0 - \bar{x}F_1 - \bar{x}^2F_2.$$

Let us denote

$$P_0 := [x^0](\Delta(x)S(x)S(\bar{x})). \quad (34)$$

By symmetry, we can now express $\Delta(x)S(x)S(\bar{x})$ as follows:

$$\Delta(x)S(x)S(\bar{x}) = P_0 + \Delta(x)(S(x)^2 + S(\bar{x})^2 + xS(x) + \bar{x}S(\bar{x})) - 2F_0 - (x + \bar{x})F_1 - (x^2 + \bar{x}^2)F_2.$$

Equivalently:

$$\Delta(x)(S(x)^2 + S(\bar{x})^2 - S(x)S(\bar{x}) + xS(x) + \bar{x}S(\bar{x})) = 2F_0 - P_0 + (x + \bar{x})F_1 + (x^2 + \bar{x}^2)F_2. \quad (35)$$

This is the promised equation relating $S(x)$ and $S(\bar{x})$, or equivalently, $M(0, x)$ and $M(0, \bar{x})$. In the next step, we will get rid of $M(0, \bar{x})$.

Remark. One may wonder whether the right-hand side of (35), as a polynomial in x , is divisible by $\Delta(x)$, or at least by one of its two factors $(1 - t(x + \bar{x} + 2))$ and $(1 - t(x + \bar{x} - 2))$. This would simplify our calculations, but one can easily check that this is not the case.

2.6. AN EQUATION FOR $M(\mathbf{0}, x)$ ONLY

The product $S(x)S(\bar{x})$ occurring in the above equation (35) makes it difficult to extract the positive part. To eliminate this cross term, we multiply the equation by $S(x) + S(\bar{x}) + x + \bar{x}$. This “trick” (based on the identity $(a^2 + b^2 - ab)(a + b) = a^3 + b^3$) was already used in the solution [6] of Gessel’s quadrant model. This gives:

$$\begin{aligned} \Delta(x) \left(S(x)^3 + S(\bar{x})^3 + (2x + \bar{x})S(x)^2 + (2\bar{x} + x)S(\bar{x})^2 + x(x + \bar{x})S(x) + \bar{x}(x + \bar{x})S(\bar{x}) \right) \\ = \left(2F_0 - P_0 + (x + \bar{x})F_1 + (x^2 + \bar{x}^2)F_2 \right) (S(x) + S(\bar{x}) + x + \bar{x}). \end{aligned}$$

Now we can extract the non-negative part in x :

$$\begin{aligned} \Delta(x) \left(S(x)^3 + (2x + \bar{x})S(x)^2 + x(x + \bar{x})S(x) \right) = t^2(x - \bar{x})(1 + S_1)^2 + (1 + S_1)(F_1 + 2tS_1) \\ + \left(2F_0 - P_0 + (x + \bar{x})F_1 + (x^2 + \bar{x}^2)F_2 \right) (S(x) + x). \end{aligned}$$

(We have used the expression (33) of F_2 to simplify the right-hand side). Extracting the constant term in x gives $F_1 = -2tS_1$, so that the equation satisfied by $S(x) = txM(0, x)$ is:

$$\begin{aligned} \Delta(x) \left(S(x)^3 + (2x + \bar{x})S(x)^2 + x(x + \bar{x})S(x) \right) = t^2(x - \bar{x})(1 + S_1)^2 \\ + \left(2t^2S_1^2 + 2t(tx^2 + t\bar{x}^2 - x - \bar{x} + t)S_1 - P_0 + t^2(x^2 + \bar{x}^2) \right) (S(x) + x). \quad (36) \end{aligned}$$

In addition to $S(x)$, it involves two series depending on t only, namely S_1 and P_0 . Still, we will see that this equation, combined with the fact that $S(x)$ has polynomial coefficients in x and the values of the first few of these coefficients, determines uniquely P_0 and $S(x)$ (and consequently S_1).

2.7. THE GENERALIZED QUADRATIC METHOD: ALGEBRAICITY OF $A(x, y)$

General principle. We have described in [9] how to study equations of the form

$$\text{Pol}(S(x), A_1, \dots, A_k, t, x) = 0, \quad (37)$$

where $\text{Pol}(x_0, x_1, \dots, x_k, t, x)$ is a polynomial with complex coefficients, $S(x)$ is a formal power series in t with coefficients in $\mathbb{Q}[x]$, and A_1, \dots, A_k are k auxiliary series depending on t only, under the assumption that these $k + 1$ series are uniquely determined by (37). (In the above example (36), Pol is a *Laurent* polynomial in x , but this makes no difference.) The strategy of [9] instructs us to look for power series $X \in \mathbb{C}[[t]]$ satisfying

$$\frac{\partial \text{Pol}}{\partial x_0}(S(X), A_1, \dots, A_k, t, X) = 0. \quad (38)$$

Indeed, by differentiating (37) with respect to x , we see that any such series also satisfies

$$\frac{\partial \text{Pol}}{\partial x}(S(X), A_1, \dots, A_k, t, X) = 0, \quad (39)$$

and we thus obtain three polynomial equations, namely Eq. (37) written for $x = X$, Eqs. (38) and (39), that relate the $(k + 2)$ unknown series $S(X)$, A_1, \dots, A_k and X . If we can prove the existence of k distinct series X_1, \dots, X_k satisfying (38), we will have $3k$ equations between the $3k$ unknown series $S(X_1), \dots, S(X_k)$, A_1, \dots, A_k , X_1, \dots, X_k . If there is no redundancy in this system, we will have proved that each of the $3k$ unknown series is algebraic over $\mathbb{C}(t)$.

Identifying the series X_i . We apply this strategy to (36), with $A_1 = S_1$ and $A_2 = P_0$. The polynomial $\text{Pol}(x_0, x_1, x_2, t, x)$ is

$$\begin{aligned} \Delta(x) & \left(x_0^3 + (2x + \bar{x})x_0^2 + x(x + \bar{x})x_0 \right) - t^2(x - \bar{x})(1 + x_1)^2 \\ & - \left(2t^2x_1^2 + 2t(tx^2 + t\bar{x}^2 - x - \bar{x} + t)x_1 - x_2 + t^2(x^2 + \bar{x}^2) \right) (x_0 + x). \end{aligned} \quad (40)$$

Equation (38) reads:

$$\begin{aligned} \Delta(X) & \left(3S(X)^2 + 2(2X + 1/X)S(X) + X(X + 1/X) \right) \\ & = 2t^2S_1^2 + 2t(tX^2 + t/X^2 - X - 1/X + t)S_1 - P_0 + t^2(X^2 + 1/X^2). \end{aligned} \quad (41)$$

Recall the definitions (31) and (34) of S and P_0 . Once multiplied by X , the above equation has the following form:

$$X(1 + X^2) = -XP_0 + t\text{Pol}_1 \left(\frac{M(0, X) - M(0, 0)}{X}, M(0, 0), t, X \right)$$

for some polynomial Pol_1 . This already shows that there exist two solutions X_1 and X_2 in $\mathbb{C}[[t]]$ having constant terms i and $-i$ respectively: since P_0 is a multiple of t , the above equation allows one to compute their coefficients inductively, assuming $M(0, x)$ is known. But we will also use a third solution, which has constant term 0. To show its existence, let us write $X = t\hat{X}$. Then the above equation, once divided by t , reads

$$\hat{X} = 2 - \hat{X}P_0 + t\text{Pol}_2 \left(\frac{M(0, t\hat{X}) - M(0, 0)}{t\hat{X}}, M(0, 0), t, \hat{X} \right),$$

which gives a third solution X_0 , of the form $2t + O(t^2)$. Using the first few coefficients of $M(0, x)$, we obtain:

$$X_0 = 2t + 8t^3 + 64t^5 + 640t^7 + 7168t^9 + O(t^{11}), \quad (42)$$

$$X_{1,2} = \pm i + 2t^3 + 16t^5 \mp 2it^6 + 156t^7 + O(t^8). \quad (43)$$

In particular, these three series are non-zero, and are thus the three solutions of (41).

We note that

$$X_0 = \frac{1}{2t} \left((4t^2) + (4t^2)^2 + 2(4t^2)^3 + 5(4t^2)^4 + 14(4t^2)^5 + \dots \right),$$

seems to be related to Catalan numbers. Due to the special form of our polynomial Pol (given by (40)), it is in fact simple to prove that

$$X_0 = \frac{1 - \sqrt{1 - 16t^2}}{4t}. \quad (44)$$

Indeed, we observe that

$$\begin{aligned} (x^2 + 1)\text{Pol} - x(x^2 - 1) \frac{\partial \text{Pol}}{\partial x} - (2x + x_0 + x^2x_0) \frac{\partial \text{Pol}}{\partial x_0} & = -2x(1 - 2t(x + \bar{x})) \\ & \times (x + x_0) \left(x + \bar{x} + t(x - \bar{x})^2x_1 + x_0(x_0 + x + \bar{x}) \left(x + \bar{x} - t(x - \bar{x})^2 \right) \right). \end{aligned}$$

Since the three series $X \equiv X_i$ cancel Pol and its partial derivatives, each of them must satisfy

$$1 - 2t(X + 1/X) = 0, \quad (45)$$

or

$$X + S(X) = 0,$$

or

$$X + 1/X + t(X - 1/X)^2S_1 + S(X)(S(X) + X + 1/X) \left(X + 1/X - t(X - 1/X)^2 \right) = 0. \quad (46)$$

Using the initial values (42-43), we conclude that X_0 satisfies (45) (from which the expression (44) follows), and that X_1 and X_2 satisfy (46).

Elimination. Eliminating $S(X)$ and X between $\text{Pol}(S(X), S_1, P_0, t, X)$, $\text{Pol}_{x_0}(S(X), S_1, P_0, t, X)$ and (45) gives a first polynomial equation between S_1 and P_0 . A second equation is obtained by eliminating $S(X)$ and X between $\text{Pol}(S(X), S_1, P_0, t, X)$, $\text{Pol}_{x_0}(S(X), S_1, P_0, t, X)$ and (46). (When several factors occur, one determines the correct one using the first coefficients of $S(x)$, S_1 , P_0 and the X_i 's.) A further elimination (first of P_0 , then of S_1) between these two equations gives polynomial equations for each of these two series. Both are found to be of degree 4 over $\mathbb{Q}(t)$:

$$19683t^6S_1^4 + 2187t^4(20t^2 - 1)S_1^3 + 81t^2(11t^2 - 1)(38t^2 - 1)S_1^2 + (92t^2 - 1)(11t^2 - 1)^2S_1 + t^2(1331t^4 - 107t^2 + 1) = 0, \quad (47)$$

and

$$387420489t^6P_0^4 + 3188646t^4(284t^4 - 113t^2 - 1)P_0^3 + 8748t^2(31570t^8 - 96755t^6 + 7251t^4 + t^2 + 1)P_0^2 + (29962144t^{12} - 441273288t^{10} + 87261432t^8 - 4754122t^6 + 64860t^4 - 687t^2 - 8)P_0 + t^4(1102736t^{10} - 53770928t^8 + 4286896t^6 - 58740t^4 + 751t^2 + 8) = 0.$$

From this and (36), we derive that $S(x) = txM(0, x)$ is algebraic over $\mathbb{Q}(t, x)$. Then, (30) implies that the same holds for $M(x, 0)$. Finally, the algebraicity of $M(x, y)$ follows from (24), and that of $P(x, y)$ from (22). We have thus proved the algebraicity of the series $A(x, y)$ given by (21), which, by definition, is

$$C(x, y) - \frac{1}{3}Q(x, y) + \frac{1}{3}\bar{x}^2Q(\bar{x}, y) + \frac{1}{3}\bar{y}^2Q(x, \bar{y}).$$

2.8. RATIONAL PARAMETRIZATIONS AND DEGREES

The equations obtained above for S_1 and P_0 can be parametrized by introducing the unique series $T \in \mathbb{Q}[[t]]$, with constant term 1, satisfying (7). Indeed, both equations factor when replacing t^2 by $(T - 1)(T + 3)^3/(256T^3)$, and extracting the correct factor gives:

$$S_1 = \frac{(T - 1)(11 + 6T - T^2)}{(T + 3)^3} \quad (48)$$

and

$$P_0 = \frac{(T - 1)^2(41 + 331T + 106T^2 + 38T^3 - 3T^4 - T^5)}{128T^3(T + 3)^3}.$$

We recall that parametrizations of algebraic curves (of genus 0) can be computed using the Maple command `parametrization`.

We now plug these expressions in the equation (36) defining $S(x)$. This gives a cubic equation for $S(x)$ over $\mathbb{Q}(t, x, T)$. Eliminating T gives an irreducible polynomial of degree 12 in $S(x)$ over $\mathbb{Q}(t, x)$: since $S(x) = txM(0, x)$, we conclude that $M(0, x)$ has degree 12 as well.

We now return to the cubic equation satisfied by $S(x)$ over $\mathbb{Q}(t, x, T)$, and replace x by xt . Due to the structure of the square lattice, $S(xt)/t = xtM(0, xt)$ is an even function of t : this allows us to replace t^2 by its rational expression in terms of T , and gives a cubic equation for $S(xt)/t$ over $\mathbb{Q}(x, T)$. Then, introducing the parametrization (8) of x factors this cubic equation into a linear factor and a quadratic one. The one that vanishes is found to be the linear one. This gives a rational expression of $S(xt)/t$ in terms of T and U , which is equivalent to the expression (76) of $tM(0, xt)$.

We now want to express $M(x, 0)$, or equivalently the series $R(x) = tM(x, 0)$, using (32). Using the cubic equation (over $\mathbb{Q}(t, x, T)$) found for $S(x)$, we first find that the term $D(x) := (S(x) - 2S(\bar{x}) - \bar{x})$ has degree 6 over $\mathbb{Q}(t, x, T)$, but is in fact bicubic. Thus (32) gives automatically an equation of degree 6 for $R(x)$ over $\mathbb{Q}(t, x, T)$. Eliminating T shows that $R(x)$ has degree 24 over $\mathbb{Q}(t, x)$. Since $R(x) = tM(x, 0)$, the same holds for $M(x, 0)$.

Finally, we replace x by xt in the equation of degree 6 satisfied by $R(x)$ over $\mathbb{Q}(t, x, T)$ (the series $R(xt)$ is an even function of t). Then we parametrize t^2 and T by $Z = \sqrt{T}$, and find that $R(xt)$ is cubic over $\mathbb{Q}(x, Z)$. We finally parametrize x by U (as in (8)): the equation factors into a linear term and a quadratic one. The one that cancels turns out to be linear, and this gives for $R(xt)$ a rational expression in Z and U which is equivalent to our expression (75) of $tM(xt, 0)$.

It remains to show that $M(x, y)$ and $P(x, y)$ have degree 72 over $\mathbb{Q}(x, y, t)$. It suffices to prove this for $tM(xt, yt)$ and $P(xt, yt)$, which are even series in t .

First, it follows from (24) and the rational expressions of $tM(0, xt)$ and $tM(xt, 0)$ that $tM(xt, yt)$ belongs to $\mathbb{Q}(Z, U, \tilde{U})$, where \tilde{U} is the counterpart of U for the variable y instead of x . Since Z has degree 8 over $\mathbb{Q}(t)$, and U has degree 3 over $\mathbb{Q}(x, Z)$, it follows that $tM(xt, yt)$ has degree at most 72 over $\mathbb{Q}(t, x, y)$. Computing its minimal equation over $\mathbb{Q}(x, y, t)$ (in practice, for $x = 2$ and $y = -2$ for instance, to avoid extremely heavy computations) shows that this bound is tight.

We proceed similarly with $P(x, y)$, expressed in terms of $M(x, y)$ and its specializations thanks to (22).

We have now completed the proof of Theorem 1.

2.9. WALKS ENDING AT A PRESCRIBED POSITION

Let us now prove Corollary 2. We want to show that for $i, j \geq 0$, the coefficients of $x^i y^j$ in $M(x, y)$ and $P(x, y)$, which we denote by $M_{i,j}$ and $P_{i,j}$ respectively, belong to $t^{i+j+1}\mathbb{Q}(Z)$ (resp. to $t^{i+j}\mathbb{Q}(Z)$). First, the connection (5) between P and M gives

$$P_{i,j} = M_{i+1,j} + M_{j+1,i},$$

and shows that it suffices to prove the property for the series $M_{i,j}$. Then, extracting from the functional equation (6) satisfied by M the coefficient of $x^i y^j$ shows that for $i, j \geq 0$, the series $M_{i,j+1}$ can be expressed as a linear combination of series $M_{k,\ell}$ such that $\ell \leq j$ and/or $k = 0$, with coefficients in $\mathbb{Q}[t, 1/t]$. Hence it suffices to prove our results for the series $M_{i,0}$ and $M_{0,j}$. Equivalently, by definition (31) of the series R and S , it suffices to prove that the coefficient of x^i in $\tilde{R}(x) := R(xt) = tM(xt, 0)$ and $\tilde{S}(x) := S(xt)/(tx) = tM(0, xt)$ (which are both even functions of t) belong to $\mathbb{Q}(Z)$.

Recall from the previous subsection that \tilde{R} satisfies a cubic equation over $\mathbb{Q}(Z, x)$. This equation reads

$$Z^{24}(Z^2 + 3)^3 \tilde{R} = Z^{24}(Z^2 - 1)(11 + 6Z^2 - Z^4) + x \text{Pol}(x, Z, \tilde{R}) \quad (49)$$

for some polynomial Pol . This gives

$$[x^0] \tilde{R} = [x^0] R(x) = tM_{0,0} = \frac{(Z^2 - 1)(11 + 6Z^2 - Z^4)}{(Z^2 + 3)^3}$$

(which we already obtained in (48)), and shows, by induction on $i \geq 0$, that the coefficient of x^i in $\tilde{R}(x)$ is a rational function of Z .

For the series $\tilde{S}(x) = tM(0, xt)$, we have to go one step further in the application of Newton's polygon method. We start from the cubic equation satisfied by this series over $\mathbb{Q}(T, x)$, which we write as an equation over $\mathbb{Q}(Z, x)$ with $T = Z^2$. Writing $\tilde{S}(x) = \tilde{S}_0 + x\tilde{S}_1 + x^2\hat{S}(x)$, we first determine (by setting $x = 0$ in the equation)

the value of \tilde{S}_0 (once again equivalent to (48)). Then there are two possible values for \tilde{S}_1 ; after checking the first coefficients, we conclude that the correct one is

$$\tilde{S}_1 = t^2 M_{0,1} = \frac{4(Z-1)^2(Z^2+1)^2(1+2Z-Z^2)}{Z^3(3+Z^2)^3}.$$

For the remaining series $\hat{S}(x)$, we find an equation over $\mathbb{Q}(Z, x)$ which, as (49), has degree 1 in \hat{S} when $x = 0$. One can then compute recursively the coefficient of x^i in $\hat{S}(x)$, which belongs to $\mathbb{Q}(Z)$.

Finally, the nature of the series $C_{i,j}$ follows from the fact that $Q_{i,j}$ is D-finite but transcendental for $i, j \geq 0$. Indeed, the asymptotic behaviour its n th coefficient, in $4^n n^{-3}$, is not compatible with algebraicity [12].

3. THE DIAGONAL SQUARE LATTICE

We now adapt the calculations of Section 2 to walks on the *diagonal* square lattice (Figure 2, right). That is, walks now take steps $(\pm 1, \pm 1)$. One difference from the square lattice case is an extra term in the basic functional equation, corresponding to the forbidden move from $(0, 0)$ to $(-1, -1)$. Otherwise the argument is very similar, and we give much fewer details.

We adopt the same notation as before. In particular, $C(x, y)$ denotes the generating function of walks on the diagonal square lattice, starting from $(0, 0)$ and confined to \mathcal{C} . As in the square lattice case, the expression of $C(x, y)$ involves the generating function $Q(x, y)$ of walks confined to the first quadrant, which is now [10]:

$$Q(x, y) = \sum_{i, j, n \geq 0} \frac{(i+1)(j+1)}{\left(1 + \frac{n+i}{2}\right)\left(1 + \frac{n+j}{2}\right)} \binom{n}{\frac{n+i}{2}} \binom{n}{\frac{n+j}{2}} x^i y^j t^n, \quad (50)$$

where the sum is restricted to values of i, j and n having the same parity.

Theorem 4. *The generating function of walks with steps $(\pm 1, \pm 1)$, starting at $(0, 0)$, confined to \mathcal{C} and ending in the first quadrant (resp. at a negative abscissa) is*

$$\frac{1}{3} Q(x, y) + P(x, y), \quad \left(\text{resp.} \quad -\frac{1}{3} \bar{x}^2 Q(\bar{x}, y) + \bar{x} M(\bar{x}, y) \right),$$

where $M(x, y)$ and $P(x, y)$ are algebraic series of degree 72 over $\mathbb{Q}(x, y, t)$.

More precisely, P can be expressed in terms of M by:

$$P(x, y) = \bar{x}(M(x, y) - M(0, y)) + \bar{y}(M(y, x) - M(0, x)), \quad (51)$$

M satisfies the functional equation

$$(1 - t(x + \bar{x})(y + \bar{y}))(2M(x, y) - M(0, y)) = 2x/3 - 2t\bar{y}(x + \bar{x})M(x, 0) \\ + t(x - \bar{x})(y + \bar{y})M(0, y) + t(1 + \bar{y}^2)M(y, 0) - t\bar{y}M_{1,0} \quad (52)$$

where $M_{1,0}$ is the coefficient of $x^1 y^0$ in $M(x, y)$, and the specializations $M(x, 0)$ and $M(0, y)$ have respective degrees 24 and 12 over $\mathbb{Q}(t, x)$ and $\mathbb{Q}(t, y)$.

Moreover, these algebraic series admit rational parametrizations. Let us define the series T and Z as in Theorem 1, and let V be the only series in t , with constant term 0, satisfying

$$1 - T + 3V + VT = xV^2(3 + V + T - VT). \quad (53)$$

Then the series $\sqrt{x}M(\sqrt{x}, 0)$ and $t\sqrt{x}M(0, \sqrt{x})$ (both even series in t with polynomial coefficients in x) admit rational expressions in terms of Z and V , given in Appendix A.2.

By combining the above results and *singularity analysis* of algebraic (and D-finite) series [13], one can derive from the above theorem that the number of n -step walks on the diagonal square lattice confined to \mathcal{C} is

$$[t^n]C(1,1) \sim \frac{2^3 \sqrt{3}}{3^2 \Gamma(2/3)} \frac{4^n}{n^{1/3}}.$$

Corollary 5 (Walks ending at a prescribed position). *Let T be the unique series in t with constant term 1 satisfying (7), and let $Z = \sqrt{T}$.*

For $j \geq 0$, the series $C_{-1,j}$ belongs to $t\mathbb{Q}(T)$, and is thus algebraic. More generally, for $i \geq 1$ and $j \geq 0$ having the same parity, the series $C_{-i,j}$ is D-finite, of the form

$$-\frac{1}{3}Q_{i-2,j} + t^{\min(i,j)} \text{Rat}(Z)$$

for some rational function Rat. It is transcendental as soon as $i \geq 2$.

Finally, for $i \geq 0$ and $j \geq 0$ having the same parity, the series $C_{i,j}$ is of the form

$$\frac{1}{3}Q_{i,j} + t^{\min(i,j)} \text{Rat}(Z).$$

It is D-finite and transcendental.

Here are some examples:

$$tC_{-1,1} = \frac{(T-1)(11+6T-T^2)}{(T+3)^3}, \quad (54)$$

$$C_{-2,0} = -\frac{1}{3}Q_{0,0} + \frac{32}{3} \frac{Z^3(1+Z+3Z^2-Z^3)}{(Z+1)(Z^2+3)^3},$$

$$C_{0,0} = \frac{1}{3}Q_{0,0} + \frac{64}{3} \frac{Z^3(1+Z+3Z^2-Z^3)}{(Z+1)(Z^2+3)^3}.$$

The similarity between the last two expressions comes from (51). Finally, we have found closed form expressions for walks ending on the boundary of the cone \mathcal{C} . For instance:

$$\begin{aligned} c_{0,0}(2n) &= \frac{16^n}{9} \left(3 \frac{(1/2)_n^2}{(2)_n^2} + 8 \frac{(1/2)_n(7/6)_n}{(2)_n(4/3)_n} - 2 \frac{(1/2)_n(5/6)_n}{(2)_n(5/3)_n} \right), \\ c_{-2,0}(2n) &= \frac{16^n}{9} \left(-3 \frac{(1/2)_n^2}{(2)_n^2} + 4 \frac{(1/2)_n(7/6)_n}{(2)_n(4/3)_n} - \frac{(1/2)_n(5/6)_n}{(2)_n(5/3)_n} \right), \\ c_{-4,0}(2n) &= \frac{16^n}{3^5} \left(-3^5 n \frac{(1/2)_n^2}{(2)_n(2)_{n+1}} + 4(21n^2 + 30n - 14) \frac{(1/2)_n(7/6)_n}{(2)_{n+1}(4/3)_{n+1}} \right. \\ &\quad \left. - 7(3n^2 + 3n - 10) \frac{(1/2)_n(5/6)_n}{(2)_{n+1}(5/3)_{n+1}} \right). \end{aligned} \quad (55)$$

It seems that this pattern persists, that is, that $c_{-2i,0}$ is a sum of three hypergeometric terms (we have checked this for $0 \leq i \leq 4$.) There is no such expression for walks ending at $(-3,1)$, $(-1,1)$, $(1,1)$, $(-2,2)$, nor $(0,2)$.

Our starting point is of course the functional equation obtained by constructing walks recursively. It reads

$$K(x,y)C(x,y) = 1 - t\bar{y}(x+\bar{x})C_-(\bar{x}) - t\bar{x}(y+\bar{y})C_-(\bar{y}) - t\bar{x}\bar{y}C_{0,0},$$

where $C_-(\bar{x})$ is still given by (11), the kernel is $K(x,y) = 1 - t(x+\bar{x})(y+\bar{y})$, and $C_{0,0}$ is the coefficient of x^0y^0 in $C(x,y)$. As in the square lattice case, the kernel is invariant by the transformations $x \mapsto \bar{x}$ and $y \mapsto \bar{y}$.

3.1. REDUCTION TO AN EQUATION WITH ORBIT SUM ZERO

Let us compare this equation to the one that describes walks confined to \mathcal{Q} :

$$K(x, y)Q(x, y) = 1 - t\bar{y}(x + \bar{x})Q_+(x) - t\bar{x}(y + \bar{y})Q_+(y) + t\bar{x}\bar{y}Q_{0,0},$$

with $Q_{0,0} := Q(0, 0)$ and $Q_+(x) = Q(x, 0)$. The orbit equations of Q and C are still given by (15) and (17), and in particular they have the same right-hand side $(x - \bar{x})(y - \bar{y})$. This leads us to introduce a series $A(x, y)$ defined by (18), again with $a = 1/3$. The equation satisfied by A (the counterpart of (20)) is

$$K(x, y)A(x, y) = (2 + \bar{x}^2 + \bar{y}^2)/3 - t\bar{y}(x + \bar{x})A_-(\bar{x}) - t\bar{x}(y + \bar{y})A_-(\bar{y}) - t\bar{x}\bar{y}A_{0,0}.$$

The corresponding orbit sum is of course zero.

3.2. REDUCTION TO A QUADRANT-LIKE PROBLEM

In the series A , we separate the contributions of the three quadrants by introducing the series P and M given by (21). Given that the orbit sum of A is zero, these two series are still related by (22). We now follow the lines of Section 2.3 to obtain the quadrant-like equation (52) for $M(x, y)$. It is the diagonal counterpart of (24).

3.3. CANCELLING THE KERNEL: AN EQUATION BETWEEN $M(\mathbf{0}, x)$, $M(\mathbf{0}, \bar{x})$ AND $M(x, \mathbf{0})$

As a polynomial in y , the kernel K admits only one root in the ring of formal series in t :

$$Y = \frac{1 - \sqrt{1 - 4t^2(x + \bar{x})^2}}{2t(x + \bar{x})} = (x + \bar{x})t + (x + \bar{x})^3t^3 + O(t^5). \quad (56)$$

We follow the steps of Section 2.4 to obtain the counterpart of (30):

$$(x + \bar{x})(Y - 1/Y)(xM(0, x) - 2\bar{x}M(0, \bar{x})) - 2\bar{x}Y/t + 3(x + \bar{x})M(x, 0) + 3M_{1,0} = 0. \quad (57)$$

3.4. AN EQUATION BETWEEN $M(\mathbf{0}, x)$ AND $M(\mathbf{0}, \bar{x})$

The discriminant occurring in the expression (56) of Y is

$$1 - 4t^2(x + \bar{x})^2 = 1 - 4t^2(x^2 + 1)(\bar{x}^2 + 1).$$

It is an even function of x . The same holds for the series $xM(x, 0)$ and $xM(0, x)$. This suggests to define:

$$\Delta = 1 - 4t^2(x + 1)(\bar{x} + 1),$$

$$R(x) = t^2/\sqrt{x}M(\sqrt{x}, 0), \quad S(x) = t\sqrt{x}M(0, \sqrt{x}). \quad (58)$$

Then $R(x)$ and $S(x)$ both belong to $\mathbb{Q}[x][[t^2]]$, and (57) gives:

$$\sqrt{\Delta(x)}\left(S(x) - 2S(\bar{x}) - \frac{1}{1+x}\right) = 3(x+1)R(x) + 3R_0 - \frac{1}{1+x}, \quad (59)$$

with $R_0 = R(0) = t^2M_{1,0}$. Expanding around $x = -1$ gives

$$3R_0 + S(-1) = 0, \quad (60)$$

which will be useful later.

As in Section 2.5, we square (59) and then extract the negative part. This gives

$$\Delta(x)\left(S(\bar{x})^2 + \frac{S(\bar{x})}{1+x}\right) - [x^{<}] (\Delta(x)S(x)S(\bar{x})) = \bar{x}t^2 + \frac{S(-1)}{x+1}.$$

As before, we denote by P_0 the coefficient of x^0 in $\Delta(x)S(x)S(\bar{x})$. Reconstructing this series finally gives the counterpart of (35):

$$\Delta(x) \left(S(x)^2 + S(\bar{x})^2 - S(x)S(\bar{x}) + \frac{S(x)}{\bar{x}+1} + \frac{S(\bar{x})}{x+1} \right) = S(-1) - P_0 + t^2(x + \bar{x}).$$

3.5. AN EQUATION FOR $M(\mathbf{0}, \mathbf{x})$ ONLY

We multiply the previous equation by $S(x) + S(\bar{x}) + 1$. The non-negative part of the resulting equation reads:

$$\Delta(x) \left(S(x)^3 + \frac{2x+1}{x+1} S(x)^2 + \frac{S(x)}{\bar{x}+1} \right) = (S(-1) - P_0 + t^2(x + \bar{x}))(S(x) + 1) + t^2 S_1 - \frac{S(-1)(S(-1) + 1)}{x+1} - t^2 \bar{x}$$

with $S_1 := S'(0)$. Extracting the constant term in x gives $P_0 + S(-1)^2 = 2t^2 S_1$, which allows us to rewrite the above equation as

$$\Delta(x) \left(S(x)^3 + \frac{2x+1}{x+1} S(x)^2 + \frac{S(x)}{\bar{x}+1} \right) = (t^2(x + \bar{x}) - F_0)(S(x) + 1) + t^2 S_1 - \frac{2t^2 S_1 - F_0}{x+1} - t^2 \bar{x}, \quad (61)$$

with $F_0 = P_0 - S(-1)$.

3.6. THE GENERALIZED QUADRATIC METHOD: ALGEBRAICITY OF $A(\mathbf{x}, \mathbf{y})$

We now apply the generalized quadratic method of Section 2.7. We denote

$$\text{Pol}(x_0, x_1, x_2, t, x) = \Delta(x) \left(x_0^3 + \frac{2x+1}{x+1} x_0^2 + \frac{x_0}{\bar{x}+1} \right) - (t^2(x + \bar{x}) - x_2)(x_0 + 1) - t^2 x_1 + \frac{2t^2 x_1 - x_2}{x+1} + t^2 \bar{x}.$$

Then (37) holds with $A_1 = S_1$ and $A_2 = F_0$. We find that (38) now admits two solutions X_0 and X_1 . Computing their first few coefficients leads us to conjecture that

$$X_0 = \frac{1 - 2t - \sqrt{1 - 4t}}{2t}, \quad X_1 = -\frac{1 + 2t - \sqrt{1 + 4t}}{2t}.$$

This is proved by eliminating x_1 and x_2 between Pol , Pol_{x_0} and Pol_x , since

$$2\text{Pol} - (1 + 2x_0)\text{Pol}_{x_0} - (x^2 - 1)\text{Pol}_x = -(1 - 2t - t(x + \bar{x}))(1 + 2t + t(x + \bar{x})) \frac{(1 + 2x_0)^2(x + x_0(1 + x))}{x+1}.$$

The series X_0 and X_1 cancel the first and second factor, respectively.

It remains to say that the discriminant of $\text{Pol}(x_0, A_1, A_2, t, x)$ with respect to x_0 admits roots at $x = X_0$ and $x = X_1$. This gives two polynomial relations between A_1 and A_2 , that is, between S_1 and F_0 , from which we derive:

$$19683t^6 S_1^4 + 2187t^4 (20t^2 - 1) S_1^3 + 81t^2 (11t^2 - 1) (38t^2 - 1) S_1^2 + (92t^2 - 1) (11t^2 - 1)^2 S_1 + t^2 (1331t^4 - 107t^2 + 1) = 0$$

and

$$27F_0^4 + 27(8t^2 - 1)F_0^3 + 9(2t + 1)(2t - 1)(10t^2 - 1)F_0^2 + (224t^6 - 68t^4 + 16t^2 - 1)F_0 + t^2(48t^6 + 88t^4 - 20t^2 + 1) = 0.$$

Note that the equation satisfied by S_1 is the same as in the square lattice case (see (47)).

From this point on, we conclude that the series $S(x)$ (or $M(0, x)$), $R(x)$ (or $M(x, 0)$), $M(x, y)$ and finally $P(x, y)$ and $A(x, y)$ are algebraic, using, in this order (61), (59), (52), (51) and (21).

3.7. RATIONAL PARAMETRIZATION AND DEGREES

The end of the proof of Theorem 4 is very similar to Section 2.8. The above equations for S_1 and F_0 factor when t^2 is parametrized by T , and one obtains

$$S_1 = \frac{(T - 1)(11 + 6T - T^2)}{(3 + T)^3},$$

while

$$F_0 = \frac{(1 - T)(3T^3 - 29T^2 - 15T + 9)}{128T^3}.$$

We plug these expressions in the equation (61) defining $S(x)$, parametrize x by the series V defined by (53), and obtain a rational expression of $S(x)$ in terms of T and V , which is equivalent to the expression (78) of $t\sqrt{x}M(0, \sqrt{x})$.

We then consider (59). Our first task is to determine R_0 , or equivalently $S(-1)$ (see (60)). In order to do so, we replace x by -1 in the cubic equation defining $S(x)$ over $\mathbb{Q}(x, T)$. The resulting equation factors once we write $T = Z^2$, giving

$$R_0 = -\frac{1}{3}S(-1) = \frac{(Z - 1)(-Z^3 + 3Z^2 + Z + 1)}{24Z^3}.$$

Then, the term $D(x) = S(x) - 2S(\bar{x}) - 1/(1 + x)$ is found to be bicubic over $\mathbb{Q}(T, x)$. Combined with the above value of R_0 , this gives for $R(x)$ an equation of degree 6 over $\mathbb{Q}(Z, x)$, which factors into two cubic terms, and factors even further when parametrizing x by T (or Z) and V . This gives a rational expression for $R(x)$ in terms of Z and V , which is equivalent to the expression (77) of $t/\sqrt{x}M(\sqrt{x}, 0)$.

It remains to prove that $M(x, y)$ and $P(x, y)$ have degree 72 over $\mathbb{Q}(x, y, t)$. We have proved that $M(x, 0)$ and $M(0, x)$ belong to $\mathbb{Q}(t, x, Z, V_2)$, where V_2 denotes the series V with x replaced by x^2 . The functional equations (52) and (51) defining $M(x, y)$ and $P(x, y)$ prove that both series belong to $\mathbb{Q}(t, x, y, Z, V_2, \tilde{V}_2)$, where \tilde{V}_2 is V_2 with x replaced by y . Since V_2 is cubic over $\mathbb{Q}(x, Z)$, it follows that $M(x, y)$ and $P(x, y)$ have degree at most 72 over $\mathbb{Q}(x, y, t)$. Computing, by successive eliminations, their minimal equation when $x = 2$ and $y = 3$ shows that this bound is tight.

3.8. WALKS ENDING AT A PRESCRIBED POSITION

The argument is similar to that of Section 2.9. The result boils down to proving that $R(x)$, seen as a series in x , has coefficients in $\mathbb{Q}(Z)$, while $S(x)$ has coefficients in $\mathbb{Q}(T)$.

The (cubic) equation over $\mathbb{Q}(x, Z)$ satisfied by $R(x)$ has degree 1 in R when $x = 0$, and this permits a recursive computation of the coefficients of $R(x)$ in $\mathbb{Q}(Z)$. A similar statement holds for the equation over $\mathbb{Q}(x, T)$ satisfied by $S(x)/x$.

4. STARTING AT $(-1, 0)$ ON THE SQUARE LATTICE

We now return to the ordinary square lattice, but change the starting point to $(-1, 0)$. The x/y symmetry is lost, which complicates the derivation a bit. On the other hand, the series $C(x, y)$ counting walks confined to \mathcal{C} now satisfies a functional equation with orbit sum zero, and turns out to be algebraic. That algebraicity is sensitive to the starting point is not a new phenomenon: for instance, quadrant walks with steps $\nearrow, \downarrow, \leftarrow$, known as Kreweras' walks, are algebraic when starting at $(0, 0)$ [14], but transcendental when starting at $(1, 0)$ [28].

Theorem 6. *The generating function of square lattice walks starting at $(-1, 0)$ and confined to the cone \mathcal{C} is algebraic. Let $P(x, y)$ (resp. $\bar{x}L(\bar{x}, y)$, $\bar{y}B(x, \bar{y})$) denote the generating function of such walks ending in the first quadrant (resp. at a negative abscissa, at a negative ordinate). These three series are algebraic of degree 72 over $\mathbb{Q}(x, y, t)$.*

More precisely, P can be expressed in terms of L and B by:

$$P(x, y) = \bar{x}(L(x, y) - L(0, y)) + \bar{y}(B(x, y) - B(x, 0)), \quad (62)$$

the series L and B satisfy

$$(1 - t(x + \bar{x} + y + \bar{y}))(2L(x, y) - L(0, y)) = 1 - 2t\bar{y}L(x, 0) + t(x - \bar{x})L(0, y) + t\bar{y}B(0, y) + t\bar{y}L_{0,0} - t\bar{y}B_{0,0} \quad (63)$$

and

$$(1 - t(x + \bar{x} + y + \bar{y}))(2B(x, y) - B(x, 0)) = t(y - \bar{y})B(x, 0) - 2t\bar{x}B(0, y) + t\bar{x}L(x, 0) - t\bar{x}L_{0,0} + t\bar{x}B_{0,0}, \quad (64)$$

where $L_{0,0} = L(0, 0)$ and $B_{0,0} = B(0, 0)$, and each specialization $L(x, 0)$, $L(0, x)$, $B(x, 0)$ and $B(0, x)$ has degree 24 over $\mathbb{Q}(t, x)$.

Moreover, these algebraic series have rational parametrizations. Defining the series T , Z and U as in Theorem 1, the series $L(xt, 0)$, $L(0, xt)$, $B(xt, 0)$, $B(0, xt)$ admit rational expressions in terms of Z and U , given in Appendix A.3.

Corollary 7 (Walks ending at a prescribed position). *Let T be the unique series in t with constant term 1 satisfying (7), and let $Z = \sqrt{T}$.*

For any (i, j) in \mathcal{C} , the series $C_{i,j}$ belongs to $t^{i+j-1}\mathbb{Q}(Z)$, and is thus algebraic.

Sometimes $C_{i,j}$ even belongs to $t^{i+j-1}\mathbb{Q}(T)$. Here are some examples:

$$\begin{aligned} tC_{0,0} &= -\frac{(T-1)(T^2-6T-11)}{(T+3)^3}, & tC_{-2,0} &= 16\frac{(T-1)}{(T+3)^3}, \\ tC_{0,-2} &= \frac{(T-1)^2(5-T)}{(T+3)^3}, & C_{-1,0} &= 64\frac{Z^3}{(Z^2+3)^3}, \\ tC_{-1,1} &= -16\frac{Z^2(Z-1)(Z-3)}{(Z^2+3)^3}, & C_{0,-1} &= -32\frac{Z^3(Z-1)(Z^2-2Z-1)}{(Z+1)(Z^2+3)^3}. \end{aligned}$$

From these expressions, we can look for hypergeometric forms of the coefficients. In this way, we find

$$\begin{aligned} c_{-1,0}(2n) &= \frac{16^n}{3} \left(2 \frac{(1/2)_n(7/6)_n}{(2)_n(4/3)_n} + \frac{(1/2)_n(5/6)_n}{(2)_n(5/3)_n} \right), \\ c_{0,-1}(2n) &= \frac{2 \cdot 16^n}{3} \left(\frac{(1/2)_n(7/6)_n}{(2)_n(4/3)_n} - \frac{(1/2)_n(5/6)_n}{(2)_n(5/3)_n} \right), \end{aligned}$$

but no other simple expression in the vicinity of the origin. The reflection principle directly relates the number of Gessel walks ending at $(0, 0)$ to the above numbers (see Section 6).

4.1. REDUCTION TO TWO QUADRANT-LIKE PROBLEMS

Let $C(x, y)$ be the generating function of square lattice walks starting at $(-1, 0)$ and confined to \mathcal{C} . It satisfies the functional equation

$$K(x, y)C(x, y) = \bar{x} - t\bar{y}C_{-0}(\bar{x}) - t\bar{x}C_{0-}(\bar{y}), \quad (65)$$

where $K(x, y) = 1 - t(x + \bar{x} + y + \bar{y})$ is the kernel,

$$C_{-0}(\bar{x}) = \sum_{i < 0, n \geq 0} c_{i,0}(n)x^i t^n \quad (66)$$

and

$$C_{0-}(\bar{y}) = \sum_{j < 0, n \geq 0} c_{0,j}(n)y^j t^n. \quad (67)$$

Due to the constant term \bar{x} in (65) (instead of 1 in Section 2), the orbit sum vanishes:

$$xyC(x, y) - \bar{x}yC(\bar{x}, y) + \bar{x}\bar{y}C(\bar{x}, \bar{y}) - x\bar{y}C(x, \bar{y}) = 0. \quad (68)$$

We write

$$C(x, y) = P(x, y) + \bar{x}L(\bar{x}, y) + \bar{y}B(x, \bar{y}),$$

where $P(x, y)$, $L(x, y)$ and $B(x, y)$ belong to $\mathbb{Q}[x, y][[t]]$ (the letter P stands for *positive*, and the letters L and B for *left* and *below*, respectively). We plug this expression of C in the orbit equation (68), and extract the positive part in x and y . This gives the expression (62) of P in terms of L and B . We can thus express C in terms of L and B as well.

We now return to the equation (65) defining C , and replace C by its expression in terms of L and B . Extracting from the resulting equation the negative part in x gives

$$\bar{x}K(x, y)L(\bar{x}, y) = -t\bar{x}\bar{y}L(\bar{x}, 0) - tL(0, y) + t\bar{x}\bar{y}B(0, y) + t\bar{x}L_x(0, y) - t\bar{x}\bar{y}B_{0,0} + \bar{x},$$

with $L_x = \partial L / \partial x$. Extracting from this the coefficient of \bar{x} gives an expression of $L_x(0, y)$ in terms of $L(0, y)$ and $B(0, y)$, which, plugged in the previous equation, leads to:

$$\begin{aligned} \bar{x}K(x, y)(L(\bar{x}, y) - L(0, y)/2) &= \bar{x}/2 - t\bar{x}\bar{y}L(\bar{x}, 0) \\ &\quad + t(\bar{x}^2 - 1)L(0, y)/2 + t\bar{x}\bar{y}B(0, y)/2 + t\bar{x}\bar{y}L_{0,0}/2 - t\bar{x}\bar{y}B_{0,0}/2, \end{aligned}$$

which is equivalent to (63), upon replacing x by \bar{x} .

Repeating the procedure with y instead of x leads to the equation (64) satisfied by B .

Equations (63) and (64) both involve the series L and B . To obtain two decoupled equations, we define

$$M(x, y) = L(x, y) + B(y, x) \quad \text{and} \quad N(x, y) = L(x, y) - B(y, x). \quad (69)$$

Then

$$K(x, y)(2M(x, y) - M(0, y)) = 1 - 2t\bar{y}M(x, 0) + t(x - \bar{x})M(0, y) + t\bar{y}M(y, 0)$$

and

$$K(x, y)(2N(x, y) - N(0, y)) = 1 - 2t\bar{y}N(x, 0) + t(x - \bar{x})N(0, y) - t\bar{y}N(y, 0) + 2t\bar{y}N_{0,0}.$$

These equations are extremely close to (24), and we solve them in exactly the same way. Below we give a few details on some steps of the procedure, but we otherwise refer to the Maple session available on the author's [webpage](#). Having found $M(x, y)$

and $N(x, y)$, we reconstruct L and B thanks to (69). The remaining results (degrees, nature of the coefficients) are established as in Section 2.

4.2. SOLVING THE EQUATION FOR $M(x, y)$

We introduce the series R and S related to M by (31). Following the steps of Sections 2.4 to 2.6 leads to a cubic equation for $S(x)$ which, as (36), involves two additional unknown series, namely $R_0 = S_1$ and S_2 .

We apply to this equation the generalized quadratic method of Section 2.7. Two series cancel Pol_{x_0} . One of them is

$$X_0 = \frac{1 - \sqrt{1 - 16t^2}}{4t},$$

and the other satisfies $X_1 = X$ with

$$(2t + X + 1/X - t(X^2 + 1/X^2))S(X)(2 + S(X)) + t + (X + 1/X) - t(X^2 + 1/X^2)/2 = 0.$$

Proceeding exactly as in Section 2.7, we derive from this that S_1 and S_2 have degree 8 over $\mathbb{Q}(t)$. After introducing the series Z , we obtain for S_1/t and S_2 rational expressions in terms of Z .

Then one finds that $S(xt) = xt^2M(0, xt)$ (which is an even function of t) has degree 24 over $\mathbb{Q}(x, t)$, and is cubic over $\mathbb{Q}(Z, x)$. It can be expressed rationally in terms of the series Z and U .

For the series $R(xt)/t$, the degree is only 12 over $\mathbb{Q}(x, t)$, and 3 over $\mathbb{Q}(T, x)$. Again, introducing U factors the equation and gives a rational expression of $R(xt)/t = M(xt, 0)$ in terms of T and U .

4.3. SOLVING THE EQUATION FOR $N(x, y)$

We introduce series R and S related to N in the same way they were related to M before (see (31)). Following the steps of Sections 2.4 to 2.6 leads to a cubic equation for $S(x)$, which now involves only one additional unknown series, namely $R_0 = S_1$.

We apply to this equation the generalized quadratic method of Section 2.7. One series X cancels Pol_{x_0} , and we derive from its existence an equation of degree 8 for S_1 over $\mathbb{Q}(t)$. Again, S_1/t has a rational expression in terms of the series Z .

Then one finds that $S(xt) = xt^2N(0, xt)$ (which is an even function of t) has degree 12 over $\mathbb{Q}(x, t)$, and is cubic over $\mathbb{Q}(T, x)$. It can be expressed rationally in terms of the series T and U .

For the series $R(xt)/t$, the degree is 24 over $\mathbb{Q}(x, t)$, and 3 over $\mathbb{Q}(Z, x)$. Introducing U factors the equation and gives a rational expression of $R(xt)/t = N(xt, 0)$ in terms of Z and U .

5. STARTING AT $(-2, 0)$ ON THE DIAGONAL SQUARE LATTICE

We finally return to the diagonal lattice and change the starting point to $(-2, 0)$. The orbit sum associated with the generating function $C(x, y)$ is non-zero, and the generating function $Q(x, y)$ counting quadrant walks with diagonal steps, given by (50), enters the picture again. The generating function for walks starting from $(-2, 0)$ and confined to the cone \mathcal{C} now differs from

$$-\frac{1}{3} \left(Q(x, y) - \bar{x}^2 Q(\bar{x}, y) - \bar{y}^2 Q(x, \bar{y}) \right)$$

by an algebraic series. Comparing with Theorem 4 shows that the above D-finite part is the *opposite* of what it was when starting from $(0, 0)$.

Theorem 8. *The generating function of walks on the diagonal square lattice that start from $(-2, 0)$, remain in \mathcal{C} , and end in the first quadrant (resp. at a negative abscissa, at a negative ordinate) is*

$$-\frac{1}{3}Q(x, y) + P(x, y) \quad \left(\text{resp. } \frac{1}{3}\bar{x}^2Q(\bar{x}, y) + \bar{x}L(\bar{x}, y), \quad \frac{1}{3}\bar{y}^2Q(x, \bar{y}) + \bar{y}B(x, \bar{y}) \right),$$

where $P(x, y)$, $L(x, y)$ and $B(x, y)$ are algebraic of degree 72 over $\mathbb{Q}(x, y, t)$.

More precisely, P can be expressed in terms of L and B by:

$$P(x, y) = \bar{x}(L(x, y) - L(0, y)) + \bar{y}(B(x, y) - B(x, 0)), \quad (70)$$

the series L and B satisfy

$$(1 - t(x + \bar{x})(y + \bar{y}))(2L(x, y) - L(0, y)) = 4x/3 - 2t\bar{y}(x + \bar{x})L(x, 0) + t(x - \bar{x})(y + \bar{y})L(0, y) + t(1 + \bar{y}^2)B(0, y) - t\bar{y}B_{0,1} \quad (71)$$

and

$$(1 - t(x + \bar{x})(y + \bar{y}))(2B(x, y) - B(x, 0)) = -2y/3 + t(x + \bar{x})(y - \bar{y})B(x, 0) - 2t\bar{x}(y + \bar{y})B(0, y) + t(1 + \bar{x}^2)L(x, 0) - t\bar{x}L_{1,0}, \quad (72)$$

where $L_{1,0}$ (resp. $B_{0,1}$) denotes the coefficient of x^1y^0 (resp. x^0y^1) in $L(x, y)$ (resp. $B(x, y)$). The specializations $L(0, x)$ and $B(x, 0)$ have degree 12 over $\mathbb{Q}(t, x)$, while $L(x, 0)$ and $B(0, x)$ have degree 24.

Moreover, these algebraic series have rational parametrizations. Defining the series T , Z and V as in Theorem 4, the series $\sqrt{x}L(\sqrt{x}, 0)$, $t\sqrt{x}L(0, \sqrt{x})$, $t\sqrt{x}B(\sqrt{x}, 0)$, and $\sqrt{x}B(0, \sqrt{x})$ (which belong to $\mathbb{Q}[x][[t^2]]$) admit rational expressions in terms of Z and V , given in Appendix A.4.

Corollary 9 (Walks ending at a prescribed position). *Let T be the unique series in t with constant term 1 satisfying (7), and let $Z = \sqrt{T}$.*

For $j \geq 0$, the series $C_{-1,j}$ and $C_{j,-1}$ belong to $t\mathbb{Q}(T)$, and are thus algebraic. More generally, for $i \geq 1$ and $j \geq 0$ having the same parity, the series $C_{-i,j}$ and $C_{j,-i}$ are D -finite, of the form

$$\frac{1}{3}Q_{i-2,j} + t^{\min(i,j)} \text{Rat}(Z)$$

for some rational function Rat . They are transcendental as soon as $i \geq 2$.

Finally, for $i \geq 0$ and $j \geq 0$, having the same parity, the series $C_{i,j}$ is of the form

$$-\frac{1}{3}Q_{i,j} + t^{\min(i,j)} \text{Rat}(Z).$$

It is D -finite and transcendental.

Here are some examples:

$$\begin{aligned} tC_{-1,1} &= \frac{16(T-1)}{(T+3)^3}, & tC_{-1,3} &= \frac{64(T-1)^2(T+1)(7-T)}{(T+3)^6}, \\ C_{-2,0} &= \frac{1}{3}Q_{0,0} + \frac{32Z^3(5+5Z-3Z^2+Z^3)}{3(Z+1)(Z^2+3)^3}, \\ C_{0,0} &= -\frac{1}{3}Q_{0,0} + \frac{32Z^3(1+Z+3Z^2-Z^3)}{3(Z+1)(Z^2+3)^3}, \\ C_{0,-2} &= \frac{1}{3}Q_{0,0} - \frac{64Z^3(2+2Z-3Z^2+Z^3)}{3(Z+1)(Z^2+3)^3}, & tC_{1,-1} &= \frac{(T-1)^2(5-T)}{(T+3)^3}. \end{aligned}$$

We found hypergeometric expressions for the number of n -step walks starting from $(-2, 0)$ and ending at a prescribed point of the boundary of \mathcal{C} :

$$\begin{aligned} c_{0,0}(2n) &= \frac{16^n}{9} \left(-3 \frac{(1/2)_n^2}{(2)_n^2} + 4 \frac{(1/2)_n(7/6)_n}{(2)_n(4/3)_n} - \frac{(1/2)_n(5/6)_n}{(2)_n(5/3)_n} \right), \\ c_{-2,0}(2n) &= \frac{16^n}{9} \left(3 \frac{(1/2)_n^2}{(2)_n^2} + 2 \frac{(1/2)_n(7/6)_n}{(2)_n(4/3)_n} + 4 \frac{(1/2)_n(5/6)_n}{(2)_n(5/3)_n} \right), \\ c_{0,-2}(2n) &= \frac{16^n}{9} \left(2 \frac{(1/2)_n^2}{(2)_n^2} - 5 \frac{(1/2)_n(7/6)_n}{(2)_n(4/3)_n} + 2 \frac{(1/2)_n(5/6)_n}{(2)_n(5/3)_n} \right), \\ c_{-4,0}(2n) &= \frac{16^n}{3^5} \left(3^5 n \frac{(1/2)_n^2}{(2)_n(2)_{n+1}} + 2(21n^2 + 30n + 16) \frac{(1/2)_n(7/6)_n}{(2)_{n+1}(4/3)_{n+1}} \right. \\ &\quad \left. + 4(39n^2 + 66n - 10) \frac{(1/2)_n(5/6)_n}{(2)_{n+1}(5/3)_{n+1}} \right). \end{aligned}$$

This pattern persists at least up to $c_{-8,0}(2n)$ and $c_{0,-8}(2n)$.

The proof of these results combines those of the last two sections: Section 3, which dealt with walks starting from $(0, 0)$ on the diagonal lattice, and Section 4, which dealt with walks starting at $(-1, 0)$ on the square lattice. There is one new difficulty in the application of the generalized quadratic method, because none of the auxiliary series X_i is easy to guess. We explain how to overcome this problem, but otherwise simply write down some intermediate results of the derivation.

5.1. REDUCTION TO TWO QUADRANT-LIKE PROBLEMS

Let $C(x, y)$ be the generating function of walks starting at $(-2, 0)$ and confined to the cone \mathcal{C} in the diagonal square lattice. It satisfies the functional equation

$$K(x, y)C(x, y) = \bar{x}^2 - t\bar{y}(x + \bar{x})C_{-0}(\bar{x}) - t\bar{x}(y + \bar{y})C_{0-}(\bar{y}) - t\bar{x}\bar{y}C_{0,0},$$

where $K(x, y) = 1 - t(x + \bar{x})(y + \bar{y})$ is the kernel, and the series $C_{-0}(\bar{x})$ and $C_{0-}(\bar{y})$ are defined by (66-67). The orbit sum is now $-(x - \bar{x})(y - \bar{y})$. This is the opposite of the orbit sum for quadrant walks (see Section 3.1), and this leads us to introduce the series

$$A(x, y) := C(x, y) + \frac{1}{3} \left(Q(x, y) - \bar{x}^2 Q(\bar{x}, y) - \bar{y}^2 Q(x, \bar{y}) \right).$$

The equation satisfied by A reads

$$K(x, y)A(x, y) = \frac{1}{3} \left(1 + 2\bar{x}^2 - \bar{y}^2 \right) - t\bar{y}(x + \bar{x})A_{-0}(\bar{x}) - t\bar{x}(y + \bar{y})A_{0-}(\bar{y}) - t\bar{x}\bar{y}A_{0,0}, \quad (73)$$

and now the orbit sum vanishes. We write

$$A(x, y) = P(x, y) + \bar{x}L(\bar{x}, y) + \bar{y}B(x, \bar{y}),$$

where $P(x, y), L(x, y)$ and $B(x, y)$ belong to $\mathbb{Q}[x, y][[t]]$. We plug this expression of A in the above equation, and extract the positive part in x and y . This gives the expression (70) of P in terms of L and B . We can thus express A in terms of L and B as well.

We plug this expression of A in the equation (73). Extracting the negative part in x gives an equation which is equivalent to (71), upon replacing x by \bar{x} .

Symmetrically, extracting the negative part in y leads to the equation (72) satisfied by B .

As in the square lattice case, we can decouple the series L and B by defining

$$M(x, y) = L(x, y) + B(y, x) \quad \text{and} \quad N(x, y) = L(x, y) - B(y, x). \quad (74)$$

Then

$$K(x, y)(2M(x, y) - M(0, y)) = 2x/3 - 2t\bar{y}(x + \bar{x})M(x, 0) \\ + t(x - \bar{x})(y + \bar{y})M(0, y) + t(1 + \bar{y}^2)M(y, 0) - t\bar{y}M_{1,0}$$

and

$$K(x, y)(2N(x, y) - N(0, y)) = 2x - 2t\bar{y}(x + \bar{x})N(x, 0) \\ + t(x - \bar{x})(y + \bar{y})N(0, y) - t(1 + \bar{y}^2)N(y, 0) + t\bar{y}N_{1,0}.$$

The first equation is exactly the one we met when counting walks starting at $(0, 0)$ on the diagonal lattice (see (52)), and we only have to solve the other one. Below we give a few details on some steps of the procedure, but we again refer to the Maple session available on the author's [webpage](#). Having found $M(x, y)$ and $N(x, y)$, we reconstruct L and B thanks to (74). The remaining results (degrees, nature of the coefficients) are established as in Section 3.

5.2. SOLVING THE EQUATION FOR $N(x, y)$

We introduce series R and S related to N in the same way they were related to M in (58). Following the steps of Sections 3.3 to 3.5 leads to a cubic equation for $S(x)$ which involves two additional unknown series, namely S_1 and

$$F_0 := [x^0](\Delta(x)S(x)S(\bar{x})) - 3S(-1),$$

with $\Delta(x) = 1 - 4t(1+x)(1+\bar{x})$. This equation (which is the counterpart of (61)) reads $\text{Pol}(S(x), S_1, F_0, t, x) = 0$ with

$$\text{Pol}(x_0, x_1, x_2, t, x) = \Delta(x) \left(x_0^3 - \frac{3}{x+1}x_0^2 + \frac{2-x}{x+1}x_0 \right) \\ - \left(16t^2x_1 - x_2 + t^2\bar{x} + t^2x \right) x_0 + 7t^2x_1 - x_2 + t^2x + \frac{x_2 + 2t^2x_1}{x+1}.$$

We apply to this equation the generalized quadratic method of Section 2.7. Two series, denoted X_0 and X_1 , cancel Pol_{x_0} , and their first coefficients are:

$$X_0 = 2 - \frac{21}{2}t^2 - \frac{117}{8}t^4 + O(t^6), \quad X_1 = \frac{9}{2}t^2 + \frac{261}{8}t^4 + \frac{5067}{16} + O(t^8).$$

These coefficients do not suggest any obvious values for these series. To obtain equations satisfied by S_1 and F_0 , one has to actually work with the system of 6 equations

$$\text{Pol}(S(X_i), S_1, F_0, t, X_i) = 0, \\ \text{Pol}_{x_0}(S(X_i), S_1, F_0, t, X_i) = 0, \\ \text{Pol}_x(S(X_i), S_1, F_0, t, X_i) = 0,$$

as explained in [9, Sec. 9]. The most effective way seems to use Theorem 14 from [9], which says that the discriminant of $\text{Pol}(x_0, S_1, F_0, t, x)$ with respect to x_0 admits X_0 and X_1 as *double* roots. Up to a denominator and a factor $\Delta(x)$ (which does not vanish at X_0 nor X_1), this discriminant is a polynomial of degree 4 in $s := x + \bar{x}$, with coefficients in $\mathbb{Q}(t, S_1, F_0)$. This polynomial in s has two double roots, namely $X_0 + 1/X_0$ and $X_1 + 1/X_1$, and thus it must be the square of a polynomial in s of degree 2. This gives us two conditions on S_1 and F_0 , from which we obtain equations of degree 4 for each of these two series. As before, they can be expressed rationally in terms of the series T :

$$S_1 = \frac{(T-1)(21-6T+T^2)}{(T+3)^3},$$

$$F_0 = \frac{(T-1)(5T^3 - 11T^2 + 135T - 33)}{128T^3}.$$

From there one finds that $S(x) = t\sqrt{x}N(0, \sqrt{x})$ (a series of $\mathbb{Q}[x][[t^2]]$) has degree 12 over $\mathbb{Q}(x, t)$, and is cubic over $\mathbb{Q}(T, x)$. It can be expressed rationally in terms of T and V .

For the series $R(x) = t^2/\sqrt{x}N(\sqrt{x}, 0)$, the degree is 24 over $\mathbb{Q}(x, t)$, and 3 over $\mathbb{Q}(Z, x)$. Introducing V factors the equation and gives a rational expression of $R(x)$ in terms of Z and V .

6. SQUARE LATTICE WALKS IN A 135° WEDGE

We now return to Ira Gessel's ex-conjecture (2) about square lattice walks starting and ending at $(0, 0)$ and remaining in the (convex) cone $\{(i, j) : i + j \geq 0 \text{ and } j \geq 0\}$ (Figure 1, left). More generally, let us denote by $g_{i,j}(n)$ the number of n -step walks in this cone, starting at $(0, 0)$ and ending at (i, j) . A step by step construction of Gessel's walks gives

$$(1 - t(x + \bar{x} + y + \bar{y}))G(x, y) = 1 - t\bar{y}G(x, 0) - t(\bar{x} + \bar{y})G^\Delta(\bar{x}y) + t\bar{y}G_{0,0},$$

where

$$G^\Delta(x) := \sum_{j, n \geq 0} g_{-j, j}(n) x^j t^n$$

counts walks ending on the diagonal $i + j = 0$. Hence it suffices to determine the series $G(x, 0)$ and $G^\Delta(x)$.

One of our motivations for studying walks in a three-quadrant cone was to attack the enumeration of Gessel's walks by the reflection principle. Indeed, let us denote by $c_{i,j}(n)$ the number of n -step walks going from $(-1, 0)$ to (i, j) on the square lattice, and avoiding the negative quadrant. The generating function of these numbers is given in Theorem 6. Then the reflection principle (Figure 3) implies that, for $j \geq 0$ and $i < j$,

$$c_{i,j}(n) - c_{j,i}(n) = g_{-i-1, j}(n).$$

In particular, the case $j = 0$ allows us to compute the specialization $G(x, 0)$ in terms of the series L and B of Theorem 6:

$$G(x, 0) = L(x, 0) - B(0, x).$$

Using the rational expressions (79) and (80) of $L(xt, 0)$ and $B(0, xt)$ in terms of Z and U , we recover the parametrized expression of $G(xt, 0)$ given in [2, 6].

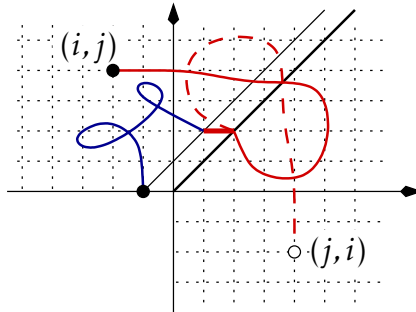


FIGURE 3. The reflection principle: a walk from $(-1, 0)$ to (i, j) that crosses the line $y = x + 1$ can be transformed bijectively into a walk ending at (j, i) .

In order to determine the series G^Δ , we have to extract from $C(x, y)$ the quasi-diagonal terms $c_{i,i+1}(n)$ and $c_{i+1,i}(n)$. To avoid this extraction, we can use instead our results on the *diagonal* square lattice obtained in Section 5. Indeed, let us denote by $\tilde{c}_{i,j}(n)$ the number of n -step walks going from $(-2, 0)$ to (i, j) on the diagonal square lattice and avoiding the negative quadrant. The generating function of these numbers is given by Theorem 8 (we use the tilde because we are mixing results for the square lattice and the diagonal square lattice). The reflection principle (Figure 4) now gives, for $j \geq 0$ and $i < j$:

$$\tilde{c}_{i,j}(n) - \tilde{c}_{j,i}(n) = g_{k,\ell}(n)$$

with $k = \frac{i+j}{2} + 1$ and $\ell = \frac{i-i}{2} - 1$. In particular, the case $j = 0$ gives us the value

$$G^\Delta(x) = \frac{1}{\sqrt{x}} \left(\tilde{L}(\sqrt{x}, 0) - \tilde{B}(0, \sqrt{x}) \right),$$

where \tilde{L} and \tilde{B} are the series denoted L and B in Theorem 8. The series $\sqrt{x}\tilde{L}(\sqrt{x}, 0)$ and $\sqrt{x}\tilde{B}(0, \sqrt{x})$ have rational expressions in terms of Z and V (see (81) and (82)), and we thus recover the parametrized expression of $G^\Delta(x)$ given in [2, 6].

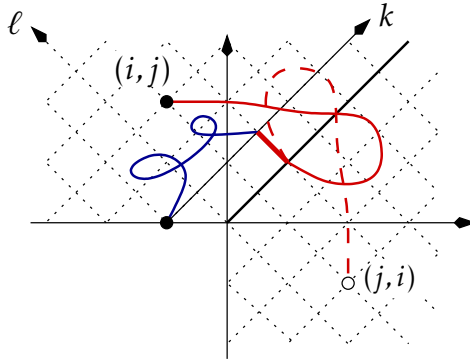


FIGURE 4. Second application of the reflection principle.

This solution of Gessel’s model is only short because we have spent much effort solving three-quadrant problems. The self-contained proofs of [6, 1] remain more direct.

7. QUESTIONS, PERSPECTIVES

7.1. ABOUT THE PRESENT PAPER

The first obvious problem raised by this paper is finding more combinatorial proofs of our results. Since these results include a solution to Gessel’s famously difficult problem, this is not likely to be easy. However, at least one question that arises from the first step of our approach should be easier.

Consider square lattice walks starting from $(0, 0)$ and confined to \mathcal{C} (Theorem 1). The first two equations in this theorem, namely (4) and (5), come at once by forming the orbit equation of $C(x, y)$, and they imply that, for $i, j \geq 0$,

$$C_{i,j} = Q_{i,j} + C_{-i-2,j} + C_{i,-j-2}.$$

Given that forming the orbit equation is essentially taking reflections in the coordinate axes, is there a simple explanation for this identity? Note that it holds

verbatim for walks starting at $(0,0)$ on the diagonal square lattice. For square lattice walks starting at $(-1,0)$, the term in Q disappears, leaving

$$C_{i,j} = C_{-i-2,j} + C_{i,-j-2}.$$

For walks on the diagonal square lattice starting from $(-2,0)$, the term in Q remains, but its sign changes:

$$C_{i,j} = -Q_{i,j} + C_{-i-2,j} + C_{i,-j-2}.$$

Another result that may be studied *per se* is the fact that square lattice walks confined to \mathcal{C} , starting at $(0,0)$ and ending at $(-1,0)$ are equinumerous with walks on the *diagonal* square lattice, confined to \mathcal{C} and joining $(0,0)$ to $(-1,1)$ (see (9) and (54)).

7.2. PERSPECTIVES

As mentioned in the introduction, we hope that this paper will be the starting point of a systematic study of walks with small steps confined to \mathcal{C} , analogous to what has been achieved in the past decade for walks confined to the first quadrant \mathcal{Q} . By small steps, we mean steps taken from $\{-1,0,1\}^2$.

Let us recall some of the quadrant results: given a set \mathcal{S} of small steps, the generating function $Q(x,y)$ that counts walks starting from $(0,0)$, confined to \mathcal{Q} and taking their steps in \mathcal{S} is D-finite if and only if a certain group of rational transformations is finite. This happens for 23 inherently different step sets, among which exactly 4 even lead to an algebraic generating function (Figure 5). There remain 56 inherently different non-D-finite step sets, among which 5 are called *singular*: this means that all their elements (i,j) satisfy $i+j \geq 0$. Two generic approaches prove the non-D-finiteness of the 51 non-singular models [20, 5], and the remaining 5 are proved non-D-finite in a more *ad hoc* way [24, 22].

Now what happens for walks with small steps confined to the three-quadrant cone \mathcal{C} ?

- One can check that all models that are trivial or simple when counting walks confined to the first quadrant (and have a rational or algebraic generating function for elementary reasons [10]) are still trivial or simple when counting walks avoiding the negative quadrant. This leaves us, as in the quadrant problem, with 79 inherently different models.
- The case of singular step sets is particularly simple: all walks formed of such steps remain in the half-plane $i+j \geq 0$, and *a fortiori* in \mathcal{C} . Hence the associated generating function is rational, equal to $(1 - t \sum_{(i,j) \in \mathcal{S}} x^i y^j)^{-1}$. A simple start!
- Could it be that for any step set associated with a finite group, the generating function $C(x,y)$ is D-finite? and, maybe, differs from a simple D-finite series related to $Q(x,y)$ by an algebraic series?
- In particular, could it be that for the four step sets of Figure 5, for which $Q(x,y)$ is known to be algebraic, $C(x,y)$ is also algebraic? We could not resist trying a bit of guessing on these models, and algebraicity seems very plausible. At least, we have guessed in each case an algebraic equation for the series $C_{0,0}$ that counts walks starting and ending at $(0,0)$. The degree is, from left to right, 6, 6, 16, 24, which should be compared to the values 3, 3, 4, 8 obtained for quadrant walks.
- To what extent can the approach of this paper be adapted to other step sets associated with a finite group? A first candidate would be the set of all eight small steps, which has the same symmetries as the models studied here, and is quite likely to be solvable by the same approach.

- Could it be that for non-singular step sets associated with an infinite group, the series $C(x, y)$ is non-D-finite? Can this be proved using asymptotic enumeration, as has been done for quadrant walks [5] using the results of [11]? (This question has been answered positively by Mustapha after this paper appeared on arXiv [25].)
- Can the powerful analytic approach of [29] be adapted to walks avoiding a quadrant? This approach was the first to yield non-D-finiteness results for the 51 models non-singular with an infinite group [20].

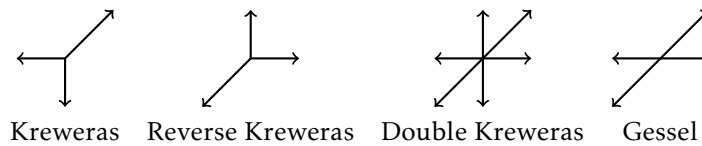


FIGURE 5. The four algebraic quadrant models.

To finish, let us mention that the reflection principle relates several three-quadrant models to quadrant models, as exemplified in Section 6. More precisely, counting walks with Kreweras steps in \mathcal{C} gives a solution of walks with *reverse* Kreweras steps in \mathcal{Q} , and vice-versa. Similarly, counting walks with double Kreweras steps in \mathcal{C} also solves walks with double Kreweras steps in \mathcal{Q} .

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REFERENCES

- [1] O. Bernardi, M. Bousquet-Mélou, and K. Raschel. Counting quadrant walks with Tutte’s invariant method (extended abstract). In *Formal power series and algebraic combinatorics (FPSAC 2016)*, DMTCS Proceedings, pages 203–214, 2016. [ArXiv:1511.04298](#).
- [2] A. Bostan and M. Kauers. Automatic classification of restricted lattice walks. In *21st International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2009)*, Discrete Math. Theor. Comput. Sci. Proc., AK, pages 201–215. Assoc. Discrete Math. Theor. Comput. Sci., Nancy, 2009. [ArXiv:0811.2899](#).
- [3] A. Bostan and M. Kauers. The complete generating function for Gessel walks is algebraic. *Proc. Amer. Math. Soc.*, 138(9):3063–3078, 2010. With an appendix by Mark van Hoeij. [ArXiv:0909.1965](#).
- [4] A. Bostan, I. Kurkova, and K. Raschel. A human proof of Gessel’s lattice path conjecture. *Transactions of the AMS*, to appear. [ArXiv:1309.1023](#).
- [5] A. Bostan, K. Raschel, and B. Salvy. Non-D-finite excursions in the quarter plane. *J. Combin. Theory Ser. A*, 121:45–63, 2014. [ArXiv:1205.3300](#).
- [6] M. Bousquet-Mélou. An elementary solution of Gessel’s walks in the quadrant. [ArXiv:1503.08573](#).
- [7] M. Bousquet-Mélou. Counting walks in the quarter plane. In *Mathematics and computer science 2, (Versailles, 2002)*, Trends Math., pages 49–67. Birkhäuser, Basel, 2002.
- [8] M. Bousquet-Mélou. Walks in the quarter plane: Kreweras’ algebraic model. *Ann. Appl. Probab.*, 15(2):1451–1491, 2005. [Arxiv:math/0401067](#).
- [9] M. Bousquet-Mélou and A. Jehanne. Polynomial equations with one catalytic variable, algebraic series and map enumeration. *J. Combin. Theory Ser. B*, 96:623–672, 2006. [Arxiv:math/0504018](#).
- [10] M. Bousquet-Mélou and M. Mishna. Walks with small steps in the quarter plane. In *Algorithmic probability and combinatorics*, volume 520 of *Contemp. Math.*, pages 1–39. Amer. Math. Soc., Providence, RI, 2010. [Arxiv:0810.4387](#).
- [11] D. Denisov and V. Wachtel. Random walks in cones. *Ann. Probab.*, 43(3):992–1044, 2015. [Arxiv:1110.1254](#).
- [12] P. Flajolet. Analytic models and ambiguity of context-free languages. *Theoret. Comput. Sci.*, 49(2-3):283–309, 1987.

- [13] P. Flajolet and R. Sedgewick. *Analytic combinatorics*. Cambridge University Press, Cambridge, 2009.
- [14] I. M. Gessel. A probabilistic method for lattice path enumeration. *J. Statist. Plann. Inference*, 14(1):49–58, 1986.
- [15] I. M. Gessel and D. Zeilberger. Random walk in a Weyl chamber. *Proc. Amer. Math. Soc.*, 115(1):27–31, 1992.
- [16] R. K. Guy, C. Krattenthaler, and B. E. Sagan. Lattice paths, reflections, & dimension-changing bijections. *Ars Combin.*, 34:3–15, 1992.
- [17] OEIS Foundation Inc. The on-line encyclopedia of integer sequences. <http://oeis.org>.
- [18] M. Kauers, C. Koutschan, and D. Zeilberger. Proof of Ira Gessel’s lattice path conjecture. *Proc. Nat. Acad. Sci. USA*, 106(28):11502–11505, 2009.
- [19] M. Kauers and D. Zeilberger. The quasi-holonomic Ansatz and restricted lattice walks. *J. Difference Equ. Appl.*, 14:1119–1126, 2008. [ArXiv:0806.4318](https://arxiv.org/abs/0806.4318).
- [20] I. Kurkova and K. Raschel. On the functions counting walks with small steps in the quarter plane. *Publ. Math. Inst. Hautes Études Sci.*, 116:69–114, 2012. [ArXiv:1107.2340](https://arxiv.org/abs/1107.2340).
- [21] L. Lipshitz. D-finite power series. *J. Algebra*, 122:353–373, 1989.
- [22] S. Melczer and M. Mishna. Singularity analysis via the iterated kernel method. *Combin. Probab. Comput.*, 23(5):861–888, 2014. [ArXiv:1303.3236](https://arxiv.org/abs/1303.3236).
- [23] M. Mishna. Classifying lattice walks restricted to the quarter plane. *J. Combin. Theory Ser. A*, 116(2):460–477, 2009. [ArXiv:math/0611651](https://arxiv.org/abs/math/0611651).
- [24] M. Mishna and A. Rechnitzer. Two non-holonomic lattice walks in the quarter plane. *Theoret. Comput. Sci.*, 410(38-40):3616–3630, 2009. [ArXiv:math/0701800](https://arxiv.org/abs/math/0701800).
- [25] S. Mustapha. Non-D-finite excursions walks in a three-quadrant cone. Preprint 2016.
- [26] H. Niederhausen. The ballot problem with three candidates. *European J. Combin.*, 4(2):161–167, 1983.
- [27] M. Petkovšek, H. S. Wilf, and D. Zeilberger. *A = B*. A K Peters Ltd., Wellesley, MA, 1996.
- [28] K. Raschel. Private communication. 2015.
- [29] K. Raschel. Counting walks in a quadrant: a unified approach via boundary value problems. *J. Eur. Math. Soc. (JEMS)*, 14(3):749–777, 2012. [ArXiv:1003.1362](https://arxiv.org/abs/1003.1362).

APPENDIX A. PARAMETRIZED EXPRESSIONS

Our parametrizing series T , Z , U and V are defined as follows. First, T is the unique series in t with constant term 1 satisfying:

$$T = 1 + 256t^2 \frac{T^3}{(T+3)^3},$$

and $Z = \sqrt{T}$. We have:

$$T = 1 + 4t^2 + 36t^4 + 396t^6 + 4788t^8 + O(t^{10}),$$

$$Z = 1 + 2t^2 + 16t^4 + 166t^6 + 1934t^8 + O(t^{10}).$$

In fact, Z is the sum of two hypergeometric series:

$$Z = \sum_{n \geq 0} 16^n \left(2 \frac{(-1/2)_n (1/6)_n}{(1)_n (1/3)_n} - \frac{(-1/2)_n (5/6)_n}{(1)_n (2/3)_n} \right) t^{2n}.$$

Then, U is the only power series in t with constant term 1 (and coefficients in $\mathbb{Q}[x]$) satisfying

$$16T^2(U^2 - T) = x(U + UT - 2T)(U^2 - 9T + 8TU + T^2 - TU^2).$$

Finally, V is the only series in t with constant term 0 (and coefficients in $\mathbb{Q}[y]$) satisfying

$$1 - T + 3V + VT = yV^2(3 + V + T - VT).$$

We have

$$U = 1 + 2t^2 + 16t^4 + (166 + 2x)t^6 + (2x^2 + 40x + 1934)t^8 + O(t^{10}),$$

$$V = t^2 + (8 + x)t^4 + (2x^2 + 16x + 82)t^6 + (5x^3 + 48x^2 + 227x + 944)t^8 + O(t^{10}).$$

A.1. WALKS STARTING AT $(0, 0)$ ON THE SQUARE LATTICE

The generating function of walks ending on the negative x -axis (resp. at abscissa -1) is $\bar{x}M(\bar{x}, 0) - \bar{x}^2Q(\bar{x}, 0)/3$ (resp. $\bar{x}M(0, y)$) where $Q(x, y)$ is given by (1) and

$${}^tM(xt, 0) = \frac{N_1(Z, U)}{3T(Z-1)(T+3)^3(U+Z)^2(U^2-9T+8TU+T^2-TU^2)} \quad (75)$$

and

$${}^tM(0, xt) = \frac{(TU-2T+U)^2N_2(T, U)}{T(T+3)^3D_2(T, U)}, \quad (76)$$

with

$$\begin{aligned} N_1(z, u) = & -(z^2+1)^2(z+1)^3(z-1)^4u^8 \\ & - 2z(z^2+1)(z^4-10z^3-14z-1)(z+1)^2(z-1)^3u^7 \\ & + 4z^2(z+1)(10z^7-35z^6+14z^5-115z^4-10z^3-57z^2-14z+15)(z-1)^2u^6 \\ & + 2z^3(z-1)(z^{10}+14z^9-77z^8+252z^7-66z^6+1224z^5+106z^4+68z^3 \\ & + 33z^2-534z+3)u^5+2z^4(z-1)(z^{10}+8z^9-115z^8-400z^7-1154z^6 \\ & -1728z^5-5890z^4-1520z^3-2607z^2-456z+549)u^4+2z^5(z^{11}-11z^{10} \\ & -207z^9+149z^8+2946z^7+2202z^6+8506z^5+9266z^4-5571z^3+4017z^2 \\ & -3627z-1287)u^3-4z^6(14z^{10}-z^9-465z^8-684z^7+3704z^6+2034z^5 \\ & +11274z^4+6756z^3-702z^2+3159z-513)u^2-2z^7(z^{11}+13z^{10}-115z^9 \\ & -95z^8+1346z^7+3722z^6-8334z^5-4470z^4-24291z^3-12663z^2 \\ & -351z-3915)u-z^8(z^{11}-z^{10}-37z^9+293z^8+382z^7-894z^6-4614z^5 \\ & +7686z^4+5409z^3+17631z^2+9099z-2187), \end{aligned}$$

$$\begin{aligned} N_2(t, u) = & t^5 + (-2u^2+32u-73)t^4 + (u^4-16u^3+90u^2-96u-177)t^3 \\ & + (-u^4+82u^2-192u-135)t^2 - u^2(u^2-16u+42)t + u^4, \end{aligned}$$

and

$$D_2(t, u) = -t^4 + 2(u-2)(u-6)t^3 - (u-3)(u^3+3u^2-15u+3)t^2 + 6tu^2 + u^4.$$

A.2. WALKS STARTING AT $(0, 0)$ ON THE DIAGONAL SQUARE LATTICE

The generating function for walks ending on the negative x -axis (resp. at abscissa -1) is $\bar{x}M(\bar{x}, 0) - \bar{x}^2Q(\bar{x}, 0)/3$ (resp. $\bar{x}M(0, y)$) where $Q(x, y)$ is given by (50) and

$$\frac{t^2}{\sqrt{x}}M(\sqrt{x}, 0) = \frac{V(VT-T-V-3)N(Z, V)}{48Z^3(V+1)^2(VZ-V-Z-1)^2}, \quad (77)$$

$${}^t\sqrt{x}M(0, \sqrt{x}) = \frac{(TV-T+3V+1)(-TV^2-2TV+V^2+T+2V+3)}{2(V+1)(T^2V^2-2T^2V+2TV^2+T^2-3V^2+2T-6V-3)} \quad (78)$$

with

$$\begin{aligned} N(z, v) = & -(z^2+3)(z-1)^2v^3 + (z-1)(3z^3+9z^2+z+11)v^2 \\ & + (z+1)(3z^3-9z^2+z-11)v - (z^2+3)(z+1)^2. \end{aligned}$$

A.3. WALKS STARTING AT $(-1, 0)$ ON THE SQUARE LATTICE

The generating function for walks ending on the negative x -axis (resp. at abscissa -1 , on the negative y -axis, at ordinate -1) is $\bar{x}L(\bar{x}, 0)$ (resp. $\bar{x}L(0, \bar{y})$, $\bar{y}B(0, \bar{y})$, $\bar{y}B(x, 0)$) where

$$\begin{aligned} L(xt, 0) &= \frac{256Z^4(UZ^2 - 2Z^2 + U)}{(UZ + Z^2 + U - 3Z)(U + Z)(Z - 1)(Z^2 + 3)^3}, \quad (79) \\ xL(0, xt) &= \frac{256Z^6(U - Z)(UZ - Z^2 - U - 3Z)(U^2Z^2 - Z^4 - 4UZ^2 + U^2 + 3Z^2)}{D(Z, U)(Z - 1)(Z^2 + 3)^3}, \\ xB(xt, 0) &= \frac{512Z^6(UZ^2 - 2Z^2 + U)(U - Z)N_1(Z, U)}{(Z^2 - 1)(Z^2 + 3)^3D(Z, U)}, \end{aligned}$$

and

$$B(0, xt) = \frac{16Z^2(UZ^2 - 2Z^2 + U)N_2(Z, U)}{(U + Z)(1 - Z^2)(Z^2 + 3)^3(UZ - Z^2 - U - 3Z)} \quad (80)$$

with

$$\begin{aligned} D(z, u) &= z^8 - 2(u - 2)(u - 6)z^6 + (u - 3)(u^3 + 3u^2 - 15u + 3)z^4 - 6u^2z^2 - u^4, \\ N_1 &= z^4 + 2(u - 3)z^3 + (u + 1)(u - 3)z^2 - u^2, \\ N_2(z, u) &= (z + 1)(z - 1)^2u^2 - 4z(3z + 1)(z - 1)u - z^2(z^3 + 3z^2 - 25z - 11). \end{aligned}$$

A.4. WALKS STARTING AT $(-2, 0)$ ON THE DIAGONAL SQUARE LATTICE

The generating function for walks ending on the negative x -axis (resp. at abscissa -1 , on the negative y -axis, at ordinate -1) is $\bar{x}L(\bar{x}, 0) + \bar{x}^2Q(\bar{x}, 0)/3$ (resp. $\bar{x}L(0, \bar{y})$, $\bar{y}B(0, \bar{y}) + \bar{y}^2Q(\bar{y}, 0)/3$, $\bar{y}B(x, 0)$) where $Q(x, y)$ is given by (50) and

$$\begin{aligned} \frac{1}{\sqrt{x}}L(\sqrt{x}, 0) &= \frac{32VZ^3(VT - T - V - 3)N_1(Z, V)}{3(1 + V)^2(T - 1)(T + 3)^3(VZ - V - Z - 1)^2}, \quad (81) \\ t\sqrt{x}L(0, \sqrt{x}) &= \frac{(TV - T + 3V + 1)(V - 1)(TV - T - V - 3)}{2(V + 1)(T^2V^2 - 2T^2V + 2TV^2 + T^2 - 3V^2 + 2T - 6V - 3)}, \\ t\sqrt{x}B(\sqrt{x}, 0) &= \frac{V(TV - T + 3V + 1)(2 - TV + V)}{(V + 1)(T^2V^2 - 2T^2V + 2TV^2 + T^2 - 3V^2 + 2T - 6V - 3)}, \\ \frac{1}{\sqrt{x}}B(0, \sqrt{x}) &= \frac{16VZ^3(VT - T - V - 3)N_2(Z, V)}{3(V + 1)^2(T - 1)(T + 3)^3(VZ - V - Z - 1)^2} \quad (82) \end{aligned}$$

with

$$N_1(z, v) = -(z^2 + 3)(z - 1)^2v^3 + 4(z + 2)(z - 1)v^2 + 4(z + 1)(z - 2)v - (z^2 + 3)(z + 1)^2$$

and

$$\begin{aligned} N_2(z, v) &= (z^2 + 3)(z - 1)^2v^3 + (3z^2 + 12z + 5)(z - 1)^2v^2 \\ &\quad + (3z^2 - 12z + 5)(z + 1)^2v + (z^2 + 3)(z + 1)^2. \end{aligned}$$

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