GENERALIZED GONČAROV POLYNOMIALS

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Dedicated to Ronald Graham on the occasion of his 80th birthday

Abstract. We introduce the sequence of generalized Gončarov polynomials, which is a basis for the solutions to the Gončarov interpolation problem with respect to a delta operator. Explicitly, a generalized Gončarov basis is a sequence $(t_n(x))_{n\geq 0}$ of polynomials defined by the biorthogonality relation $\varepsilon_{z_i}(\mathfrak{d}^i(t_n(x))) = n!\,\delta_{i,n}$ for all $i, n \in \mathbf{N}$, where $\mathfrak{d}$ is a delta operator, $\mathcal{Z} = (z_i)_{i\geq 0}$ a sequence of scalars, and ε_{z_i} the evaluation at z_i. We present algebraic and analytic properties of generalized Gončarov polynomials and show that such polynomial sequences provide a natural algebraic tool for enumerating combinatorial structures with a linear constraint on their order statistics.

1. Introduction

This paper is a work combining three areas: interpolation theory, finite operator calculus, and combinatorial enumeration. Lying in the center is a sequence of polynomials, the generalized Gončarov polynomials, that arose from the Gončarov Interpolation problem in numerical analysis.

The classical Gončarov Interpolation problem is a special case of Hermite-like interpolation. It asks for a polynomial $f(x)$ of degree n such that the ith derivative of $f(x)$ at a given point a_i has value b_i, for $i = 0, 1, \ldots, n$. The problem was introduced by Gončarov [4, 5] and Whittaker [26], and the solution is obtained by taking linear combinations of the (classical) Gončarov polynomials, or the Abel-Gončarov polynomials, which have been studied extensively by analysts, due to their considerable significance in the interpolation theory of smooth and analytic functions, see for instance [4, 13, 3, 6] and references therein.

Surprisingly, Gončarov polynomials also play an important role in combinatorics due to their close relations to parking functions, A parking function is a sequence $(a_1, a_2, \ldots, a_n)$ of positive integers such that for every $i = 1, 2, \ldots, n$, there are at least i terms that are less than or equal to i. Parking functions are one of the most fundamental objects in combinatorics and are related to many different structures, for example, labeled trees and graphs, linear probing in computer algorithms, hyperplane arrangements, non-crossing partitions, and diagonal harmonics and representation theory, to name a few. See, for example, [28] for a comprehensive survey on parking functions. It is shown by Kung and Yan [12] that Gončarov polynomials are the natural basis of polynomials for working with parking functions, and the enumeration of parking functions and their generalizations can be obtained using Gončarov polynomials. Khare, Lorentz and Yan [9] investigated a multivariate Gončarov interpolation problem and defined sequences of multivariate Gončarov polynomials, which

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are solutions to the interpolation problem and enumerate k-tuples of integer sequences whose order statistics are bounded by certain weight function along lattice paths in $\mathbf{N}^k$.

In [21], Rota, Kahaner and Odlyzko introduced a unified theory of special polynomials by exploiting to the hilt the duality between x and d/dx. The main technique is a rigorous version of symbolic calculus, also called *finite operator calculus*, since it has an emphasis on operator methods. This algebraic theory is particularly useful in dealing with polynomial sequences of binomial type, which occur in a large variety of combinatorial problems when one wants to enumerate objects that can be pieced together out of small, disjoint objects. Each polynomial sequence of binomial type can be characterized by a linear operator called *delta operator*, which possesses many properties of the differential operator. A few basic principles of delta operators lead to a series of expansion and isomorphism theorems on families of special polynomials, which in turn lead to new identities and solutions to combinatorial problems.

Inspired by the rich theory on delta operators, we extend the Gončarov interpolation problem by replacing the differential operator with a delta operator and consider the following interpolation.

> **Generalized Gončarov Interpolation.** Given two sequences $z_0, z_1, \ldots, z_n$ and $b_0, b_1, \ldots, b_n$ of real or complex numbers and a delta operator $\mathfrak{d}$, find a (complex) polynomial $p(x)$ of degree n such that
>
> $$\varepsilon_{z_i}\mathfrak{d}^i(p(x)) = b_i \qquad \text{for each } i = 0, 1, \ldots, n. \tag{1}$$

The solution of this problem rises the generalized Gončarov polynomial. When $\mathfrak{d} = D$, we recover the classical Gončarov polynomials. Generalized Gončarov polynomials enjoy many nice algebraic properties and carry a combinatorial interpretation that combines the ideas of binomial enumeration and order statistics. Roughly speaking, if each combinatorial object is associated with a sequence of numbers and we rearrange those numbers in non-decreasing order, then the generalized Gončarov polynomial counts those objects of which the non-decreasing rearrangements are bounded by a predetermined sequence. Such structures give a new generalization of the classical parking functions.

The main objective of this paper is to present the algebraic and combinatorial properties of generalized Gončarov polynomials. The paper is organized as follows. In Section 2 we recall the basic theory of delta operators, polynomial sequences of binomial type, and polynomial sequences biorthogonal to a sequence of linear operators. Using this theory we introduce the sequence of generalized Gončarov polynomials $t_n(x; \mathfrak{d}, \mathcal{Z})$ associated with a delta operator $\mathfrak{d}$ and a grid $\mathcal{Z}$. In the subsequent Sections 3-5, we discuss the algebraic properties, present a combinatorial formula for $t_n(x; \mathfrak{d}, \mathcal{Z})$, and describe the combinatorial interpretation by counting reluctant functions, a kind of combinatorial structures arising from the study of binomial enumeration, with constraints on the order statistics. Many examples are given in Section 6. We finish the paper with some further remarks in the last section.

2. Delta Operators and Generalized Gončarov polynomials

2.1. Delta operator and basic polynomials.

We start by recalling the basic theory of delta operators and their associated sequence of basic polynomials, as developed by Mullin and Rota [21].

Consider the vector space $\mathbb{K}[x]$ of all polynomials in the variable x over a field $\mathbb{K}$ of characteristic zero. For each $a \in \mathbb{K}$, let E_a denote the shift operator $\mathbb{K}[x] \to \mathbb{K}[x] : f(x) \mapsto f(x+a)$. A linear operator $\mathfrak{s} : \mathbb{K}[x] \to \mathbb{K}[x]$ is called *shift-invariant* if $\mathfrak{s}E_a = E_a\mathfrak{s}$ for all $a \in \mathbb{K}$.

Definition 1. *A* delta operator $\mathfrak{d}$ *is a shift-invariant operator satisfying* $\mathfrak{d}(x) = a$ *for some nonzero constant* a.

Delta operators possess many of the properties of the differentiation operator D. For example, $\deg(\mathfrak{d}(f)) = \deg(f) - 1$ for any $f \in \mathbb{K}[x]$ and $\mathfrak{d}(a) = 0$ for every constant a.

We say that a shift-invariant operator $\mathfrak{s}$ is *invertible* if $\mathfrak{s}(1) \neq 0$. Note that delta operators are not invertible.

Definition 2. *Let* $\mathfrak{d}$ *be a delta operator. A polynomial sequence* $(p_n(x))_{n\geq 0}$ *is called the* sequence of basic polynomials*, or the* basic sequence*, of* $\mathfrak{d}$ *if*

(1) $p_0(x) = 1$*;*
(2) $p_n(0) = 0$ *whenever* $n \geq 1$*;*
(3) $\mathfrak{d}(p_n(x)) = np_{n-1}(x)$*.*

Every delta operator has a unique sequence of basic polynomials, which is a sequence of binomial type, i.e., satisfies

$$p_n(x+y) = \sum_{k\geq 0} \binom{n}{k} p_k(x)p_{n-k}(x) \qquad \text{for all } n. \tag{2}$$

Conversely, every sequence of polynomials of binomial type is the basic sequence for some delta operator.

Let $\mathfrak{s}$ be a shift-invariant operator, and $\mathfrak{d}$ a delta operator with basic sequence $p_n(x)$. Then $\mathfrak{s}$ can be expanded as a formal power series of $\mathfrak{d}$, as

$$\mathfrak{s} = \sum_{k\geq 0} \frac{a_k}{k!}\mathfrak{d}^k \tag{3}$$

with $a_k = \varepsilon_0(\mathfrak{s}(p_k(x)))$. We will say that the formal power series $f(t) = \sum_{k\geq 0} \frac{a_k}{k!}t^k$ is the $\mathfrak{d}$-indicator of $\mathfrak{s}$. In fact, there exists an isomorphism from the ring $\mathbb{K}[\![t]\!]$ of formal power series in the variable t over $\mathbb{K}$ onto the ring Σ of shift-invariant operators, which carries

$$f(t) = \sum_{k\geq 0} \frac{a_k}{k!}t^k \qquad \text{into} \qquad \sum_{k\geq 0} \frac{a_k}{k!}\mathfrak{d}^k. \tag{4}$$

Under this isomorphism, a shift-invariant operator $\mathfrak{s}$ is invertible if and only if its $\mathfrak{d}$-indicator $f(t)$ satisfies $f(0) \neq 0$, and $\mathfrak{s}$ is a delta operator if and only if $f(0) = 0$ and $f'(0) \neq 0$.

Another result we will need is the generating function for the sequence of basic polynomials $\{p_n(x)\}$ associated to a delta operator $\mathfrak{d}$. Let $q(t)$ be the D-indicator of $\mathfrak{d}$, i.e., $q(t)$ is a formal power series satisfying $\mathfrak{d} = q(D)$. Let $q^{-1}(t)$ be the compositional inverse of $q(t)$. Then

$$\sum_{n\geq 0} \frac{p_n(x)}{n!}t^n = e^{xq^{-1}(t)}. \tag{5}$$

2.2. **Biorthogonal sequences.**

Generalized Gončarov polynomials are defined by a biorthogonality condition posted in the Gončarov interpolation problem. Many properties of these polynomials follow from a general theory of *sequences of polynomials biorthogonal to a sequence of linear functionals*. The idea behind this theory is well-known (for examples, see [1, 2]), and an explicit description for the differential operator D is given in [12, section 2]. Here we briefly describe this theory with a general delta operator $\mathfrak{d}$. The proofs are analogous to the ones in [12] and hence omitted.

Let $\mathfrak{d}$ be a delta operator with the basic sequence $\mathcal{P} = (p_n(x))_{n\geq 0}$. Let Φ_i, $i = 0, 1, 2, \ldots$, be a sequence of shift-invariant operators of the form $\sum_{j\geq 0} a_j^{(i)} \mathfrak{d}^{i+j}$, where $(a_j^{(i)}) \in \mathbb{K}$ and $a_0^{(i)} \neq 0$. Then we have:

Theorem 2.1. (1) *There exists a unique sequence $\mathcal{F} = (f_n(x))_{n\geq 0}$ of polynomials such that $f_n(x)$ is of degree n and*

$$\varepsilon_0(\Phi_i(f_n(x))) = n!\delta_{i,n} \qquad \text{for all } i, n \in \mathbf{N}$$

In addition, for every n we have

$$f_n(x) = \frac{n!}{a_0^{(0)} \cdots a_0^{(n)}} \det_{\mathbb{K}[x]}(\Lambda^{(n)}),$$

with $\Lambda^{(n)}$ the $(n+1)$-by-$(n+1)$ matrix whose (i, j)-entry, for $0 \leq i, j \leq n$, is given by

$$\lambda_{i,j}^{(n)} := \begin{cases} a_{j-i}^{(i)}, & \text{if } 0 \leq i \leq \min(j, n-1) \\ \frac{1}{j!} p_j(x), & \text{if } i = n \\ 0, & \text{otherwise.} \end{cases}$$

(2) *The sequence $\mathcal{F}$ defined above forms a basis of $\mathbb{K}[x]$. For every $f(x) \in \mathbb{K}[x]$ it holds*

$$f(x) = \sum_{n\geq 0} \frac{\varepsilon_0(\Phi_i(f))}{n!} f_n(x) = \sum_{n=0}^{\deg(f)} \frac{\varepsilon_0(\Phi_i(f))}{n!} f_n(x). \tag{6}$$

2.3. **Generalized Gončarov polynomials.**

Let $\mathcal{Z} = (z_i)_{i\geq 0}$ be a fixed sequence with values in $\mathbb{K}$; in this context, we may refer to $\mathcal{Z}$ as an (interpolation) $\mathbb{K}$-grid (or simply a grid), and to the scalars $z_i \in \mathbb{K}$ as (interpolation) nodes. For every $a \in \mathbb{K}$, E_a is an invertible shift-invariant operator and hence can be expressed as $E_a = f_a(\mathfrak{d})$ where $f_a(t) \in \mathbb{K}[\![t]\!]$ with $f_a(0) \neq 0$. It follows from Theorem 2.1 that there is a unique sequence of polynomials $\mathcal{T} = (t_n(x))_{n\geq 0}$ biorthogonal to the sequence of operators $\{\Phi_i = E_{z_i}\mathfrak{d}^i : i \geq 0\}$. More precisely, $t_n(x)$ satisfies

$$\varepsilon_{z_i}\mathfrak{d}^i(t_n(x)) = n!\delta_{i,n}. \tag{7}$$

Definition 3. *We call the polynomial sequence $\mathcal{T} = (t_n(x))_{n\geq 0}$ determined by Eq. (7) the* sequence of generalized Gončarov polynomials*, or the* generalized Gončarov basis*, associated with the pair $(\mathfrak{d}, \mathcal{Z})$, and $t_n(x)$ the n-th generalized Gončarov polynomial relative to the same pair. Accordingly, (7) will be referred to as the biorthogonality property of generalized Gončarov bases.*

By Theorem 2.1(2), for any polynomial $f(x) \in \mathbb{K}[x]$, we have the expansion formula

$$f(x) = \sum_{i \geq 0} \frac{\varepsilon_{z_i} \mathfrak{d}^i(f)}{i!} t_i(x) = \sum_{i=0}^{\deg(f)} \frac{\varepsilon_{z_i} \mathfrak{d}^i(f)}{i!} t_i(x). \tag{8}$$

In particular, the solution of the generalized Gončarov interpolation (1) described in Section 1 is given by the polynomial

$$p(x) = \sum_{i=0}^{n} \frac{b_i}{i!} t_i(x).$$

In some cases, to emphasize the dependence of $\mathcal{T}$ on $\mathfrak{d}$ and $\mathcal{Z}$, we write $t_n(x)$ as $t_n(x; \mathfrak{d}, \mathcal{Z})$. When the delta operator is the differentiation D, we get the classical Gončarov polynomials, which were studied in [12]. We reserve the symbols $g_n(x)$ and $g_n(x; \mathcal{Z})$, respectively, for the classical Gončarov polynomials to avoid confusion when we compare the results of generalized Gončarov polynomials with those of the classical case. Another special case that has been considered is the difference Gončarov polynomials [11], for which $\mathfrak{d}$ is the backward difference operator $\Delta_{0,-1} = I - E_{-1}$. The present paper is the first one describing the theory of biorthogonal polynomials with an arbitrary delta operator, and hence connecting the theory of interpolation to finite operator calculus. In the next sections we will describe the algebraic properties of the generalized Gončarov polynomials, and reveal a deeper connection between binomial enumerations and structure of order statistics.

We remark that every generalized Gončarov basis is a sequence of biorthogonal polynomials, but the converse is not true in general.

3. Algebraic properties of generalized Gončarov polynomials

Let $\mathcal{Z} = (z_i)_{i \geq 0}$ be a fixed $\mathbb{K}$-grid. We denote by $\mathcal{Z}^{(j)}$ the grid whose i-th term is the $(i+j)$-th term of $\mathcal{Z}$, and call $\mathcal{Z}^{(j)}$ the j-th shift of the grid $\mathcal{Z}$. The zero grid, herein denoted by $\mathcal{O}$, is the one with $z_i = 0$ for all i.

Proposition 3.1. *If $\mathcal{T} = (t_n(x))_{n \geq 0}$ is the generalized Gončarov basis associated with the pair $(\mathfrak{d}, \mathcal{Z})$, then $t_0(x) = 1$ and $t_n(z_0) = 0$ for all $n \geq 1$.*

Proof. This follows from the biorthogonality property (7) with $i = 0$. □

The next proposition is a generalization of the differential relations satisfied by the classical Gončarov polynomials [12, p. 23].

Proposition 3.2. *Let $\mathcal{T} = (t_n(x))_{n \geq 0}$ be the generalized Gončarov basis associated with the pair $(\mathfrak{d}, \mathcal{Z})$. Fix $j \in \mathbf{N}$ and define for each $n \in \mathbf{N}$ the polynomial $t_n^{(j)}(x)$ by letting*

$$t_n^{(j)}(x) := \frac{1}{(n+j)_{(j)}} \mathfrak{d}^j t_{n+j}(x), \tag{9}$$

where $n_{(j)} = n(n-1)\cdots(n-j+1)$ is the j-th lower factorial function. Then, $(t_n^{(j)}(x))_{n \geq 0}$ is the generalized Gončarov basis associated with the pair $(\mathfrak{d}, \mathcal{Z}^{(j)})$. In particular, we have

$$\mathfrak{d}^j t_n(x) = n_{(j)} t_{n-j}^{(j)}(x). \tag{10}$$

Proof. First, notice that $t_n^{(j)}(x)$ is a polynomial of degree n, since $t_{n+j}(x)$ is a polynomial of degree $n+j$ and delta operators reduce degrees by one.

Next, pick $i, n \in \mathbf{N}$ with $i \leq n$, and let $z_i^{(j)}$ denote the i-th node of the grid $\mathcal{Z}^{(j)}$. We just need to verify that

$$\varepsilon_{z_i^{(j)}}(\mathfrak{d}^i(t_n^{(j)}(x))) = n!\delta_{i,n}.$$

Since $z_i^{(j)} = z_{i+j}$ and $\delta_{i,n} = \delta_{i+j,n+j}$, the above equation is equivalent to

$$\frac{1}{(n+j)_{(j)}}\varepsilon_{z_{i+j}}(\mathfrak{d}^{i+j}(t_{n+j}(x))) = n!\delta_{i+j,n+j},$$

which follows from the equations $(n+j)_{(j)}n! = (n+j)!$ and $\varepsilon_{z_l}(\mathfrak{d}^l t_k(x)) = k!\delta_{l,k}$ for all $l, k \in \mathbf{N}$. The last statement is obtained by replacing $n+j$ by n in (9). $\square$

Following Proposition 3.2 we see that sequences of polynomials of binomial type are a special case of generalized Gončarov polynomials.

Proposition 3.3. *The basic sequence of the delta operator $\mathfrak{d}$ is the generalized Gončarov basis associated with the pair $(\mathfrak{d}, \mathcal{O})$.*

Proof. Let $(p_n(x))_{n\geq 0}$ be the basic sequence of the delta operator $\mathfrak{d}$. Then iterating the equation $\mathfrak{d}(p_n) = np_{n-1}$ yields $\mathfrak{d}^i(p_n(x)) = n_{(i)}p_{n-i}(x)$, which, when evaluated at $x = 0$, is $n!p_0(x) = n!$ if $n = i$, and $n_{(i)}p_{n-i}(0) = 0$ if $i \neq n$. $\square$

Corollary 3.4. *Let $\mathcal{P} = (p_n(x))_{n\geq 0}$ be a sequence of polynomials with $\deg(p_i) = i$ for all i. Then, $\mathcal{P}$ is of binomial type if and only if $\mathcal{P}$ is the generalized Gončarov basis associated with the pair $(\mathfrak{d}, \mathcal{O})$ for a suitable choice of $\mathfrak{d}$,.*

Proof. The necessity follows from Proposition 3.3 and Theorem 1(b) of [21], which states that any sequence of polynomials of binomial type is a basic sequence for some delta operator.

Conversely, let $\mathcal{P} = (p_n(x))_{n\geq 0}$ be the generalized Gončarov basis associated with the pair $(\mathfrak{d}, \mathcal{O})$. If $(p_n^{(1)}(x))_{n\geq 0}$ denotes the generalized Gončarov basis associated with the pair $(\mathfrak{d}, \mathcal{O}^{(1)})$, then by Proposition 3.2, $\mathfrak{d}(p_n) = np_{n-1}^{(1)}$ for all $n \geq 1$, which in turn implies $\mathfrak{d}(p_n) = np_{n-1}$, since $\mathcal{O}^{(1)} = \mathcal{O}$. This, together with Proposition 3.1, concludes the proof. $\square$

Next we investigate the behavior of a generalized Gončarov basis with respect to a transformation of the interpolation grid. Proposition 3.5 extends the shift-invariance property for classical Gončarov polynomials and difference Gončarov polynomials.

Proposition 3.5. *Let $\mathcal{W} = (w_i)_{i\geq 0}$ be a translation of the grid $\mathcal{Z}$ by $\xi \in \mathbb{K}$, i.e., $w_i = z_i + \xi$ for all i. Assume that $\mathcal{T} = (t_n(x))_{n\geq 0}$ and $\mathcal{H} = (h_n(x))_{n\geq 0}$ are the generalized Gončarov bases associated with the pairs $(\mathfrak{d}, \mathcal{Z})$ and $(\mathfrak{d}, \mathcal{W})$, respectively. Then, $h_n(x+\xi) = t_n(x)$ for all n.*

Proof. Clearly $h_n(x+\xi) = E_\xi(h_n(x))$ is a polynomial of degree n, so by the uniqueness of the Gončarov basis associated with the pair $(\mathfrak{d}, \mathcal{Z})$, it suffices to prove that

$$\varepsilon_{z_i}(\mathfrak{d}^i E_\xi(h_n(x))) = n!\delta_{i,n}.$$

Note that $\mathfrak{d}^i E_\xi = E_\xi \mathfrak{d}^i$ because any two shift-invariant operators commute. Therefore

$$\varepsilon_{z_i}(\mathfrak{d}^i(E_\xi(h_n(x)))) = \varepsilon_{z_i} E_\xi(\mathfrak{d}^i(h_n(x))) = \varepsilon_{z_i+\xi}(\mathfrak{d}^i(h_n(x))) = \varepsilon_{w_i}(\mathfrak{d}^i(h_n(x))) = n!\delta_{i,n}.$$

□

Proposition 3.6. *Fix $\xi \in \mathbb{K}$ and let $\mathcal{H} = (h_n(x))_{n\geq 0}$ be the generalized Gončarov basis associated with the pair $(\mathfrak{d}, \mathcal{W})$, where $\mathcal{W} = (w_i)_{i\geq 0}$ is the grid given by $w_i = z_i + i\xi$ for all $i \geq 0$. Then, $\mathcal{H}$ is also the generalized Gončarov basis associated with the pair $(E_\xi \mathfrak{d}, \mathcal{Z})$.*

Proof. One checks that

$$\varepsilon_{z_i}((E_\xi \mathfrak{d})^i h_n(x)) = n!\delta_{i,n}.$$

To this end, first observe that $(E_\xi \mathfrak{d})^i = E_\xi^i \mathfrak{d}^i = E_{i\xi} \mathfrak{d}^i$, since any two shift-invariant operators commute and $E_a E_b = E_{a+b}$ for all $a, b \in \mathbb{K}$. It follows that

$$\varepsilon_{z_i}((E_\xi \mathfrak{d})^i h_n(x)) = \varepsilon_{z_i} E_{i\xi}(\mathfrak{d}^i h_n(x)) = \varepsilon_{z_i+i\xi}(\mathfrak{d}^i h_n(x)) = \varepsilon_{w_i}(\mathfrak{d}^i h_n(x)) = n!\delta_{i,n}.$$

The last equation is true because $\mathcal{H}$ is the generalized Gončarov basis associated with $(\mathfrak{d}, \mathcal{W})$. □

The next proposition gives a relation between the generalized Gončarov polynomials and the basic polynomials of the same delta operator. It provides a linear recurrence that can be used to compute efficiently the explicit formulas for the generalized Gončarov basis if the basic sequence is known, for example, as in the classical case where the basic sequence is $(x^n)_{n\geq 1}$, or in the case of difference Gončarov polynomials where the n-term of the basic sequence is the lower factorial $(x - n + 1)_{(n)}$, see [11].

Proposition 3.7. *Let $\mathcal{T} = (t_n(x))_{n\geq 0}$ be the generalized Gončarov basis associated with the pair $(\mathfrak{d}, \mathcal{Z})$, and let $(p_n(x))_{n\geq 0}$ be the sequence of basic polynomials of the delta operator $\mathfrak{d}$. Then, for all $n \in \mathbf{N}$ it holds*

$$p_n(x) = \sum_{i=0}^{n} \binom{n}{i} p_{n-i}(z_i) t_i(x),$$

and hence

$$t_n(x) = p_n(x) - \sum_{i=0}^{n-1} \binom{n}{i} p_{n-i}(z_i) t_i(x). \tag{11}$$

Proof. Let $n \in \mathbf{N}$. Substituting $f(x)$ with $p_n(x)$ in Eq. (8) we obtain

$$p_n(x) = \sum_{i=0}^{n} \frac{\varepsilon_{z_i}(\mathfrak{d}^i p_n)}{i!} t_i(x) = \sum_{i=0}^{n} \frac{(n)_{(i)} p_{n-i}(z_i)}{i!} t_i(x) = \sum_{i=0}^{n} \binom{n}{i} p_{n-i}(z_i) t_i(x),$$

where we use the fact that $\mathfrak{d}^i(p_n(x)) = (n)_{(i)} p_{n-i}(x)$ for all $i = 0, 1, \ldots, n$.

□

Proposition 3.8 generalizes the binomial expansion for classical Gončarov polynomials.

Proposition 3.8. *Let $(t_n^{(j)}(x))_{n\geq 0}$ be the generalized Gončarov basis associated with the pair $(\mathfrak{d}, \mathcal{Z}^{(j)})$, and let $(p_n(x))_{n\geq 0}$ be the sequence of basic polynomials of the delta operator $\mathfrak{d}$. Then, for all $\xi \in \mathbb{K}$ and $n \in \mathbf{N}$ we have the following "binomial identity":*

$$t_n(x+\xi) = t_n^{(0)}(x+\xi) = \sum_{i=0}^{n} \binom{n}{i} t_{n-i}^{(i)}(\xi) p_i(x). \tag{12}$$

In particular, letting $\xi = 0$ we have

$$t_n(x) = t_n^{(0)}(x) = \sum_{i=0}^{n} \binom{n}{i} t_{n-i}^{(i)}(0) p_i(x). \tag{13}$$

Proof. Fix $\xi \in \mathbb{K}$ and $n \in \mathbf{N}$. Since $(p_i(x))_{0\leq i\leq n}$ is a basis of the linear subspace of $\mathbb{K}[x]$ of polynomials of degree $\leq n$, there exist $c_0, \dots, c_n \in K$ such that $E_\xi t_n(x) = \sum_{i=0}^{n} c_i p_i(x)$, where $t_n := t_n^{(0)}$.

Pick an integer $j \in [0, n]$. One computes

$$\varepsilon_\xi(\mathfrak{d}^j t_n(x)) = \varepsilon_0(E_\xi \mathfrak{d}^j(t_n(x))) = \varepsilon_0(\mathfrak{d}^j(E_\xi t_n(x))) = \sum_{i=0}^{n} c_i \varepsilon_0(\mathfrak{d}^j p_i(x)). \tag{14}$$

From Proposition 3.2 we have $\mathfrak{d}^j t_n(x) = (n)_{(j)} t_{n-j}^{(j)}(x)$, so that $\varepsilon_\xi(\mathfrak{d}^j t_n(x)) = (n)_{(j)} t_{n-j}^{(j)}(\xi)$. On the other hand, the sequence $(p_n(x))_{n\geq 0}$, being the basic polynomials of $\mathfrak{d}$, satisfies $\varepsilon_0(\mathfrak{d}^j p_i) = i!\delta_{i,j}$ for all $i \in \mathbf{N}$. Combining the above we obtain from (14) that

$$c_j = \frac{(n)_{(j)}}{j!} t_{n-j}^{(j)}(\xi) = \binom{n}{j} t_{n-j}^{(j)}(\xi),$$

which proves (12) . □

Corollary 3.9. *Assume $\mathcal{Z}$ is a constant grid, namely $z_i = z_j$ for all i, j. Let $(t_n(x))_{n\geq 0}$ be the generalized Gončarov basis associated with the pair $(\mathfrak{d}, \mathcal{Z})$, and $(p_n(x))_{n\geq 0}$ the sequence of basic polynomials of the delta operator $\mathfrak{d}$. Then, for all $\xi \in K$ and $n \in \mathbf{N}$ we have*

$$t_n(x+\xi) = \sum_{i=0}^{n} \binom{n}{i} t_{n-i}(\xi) p_i(x) \quad \textit{and} \quad t_n(x) = \sum_{i=0}^{n} \binom{n}{i} t_{n-i}(0) p_i(x). \tag{15}$$

Proof. Immediate by Proposition 3.8 and the fact that $\mathcal{Z} = \mathcal{Z}^{(j)}$ for all j. □

The next proposition gives an extension of the integral formula for classical Gončarov polynomials, see [12, p. 23].

Proposition 3.10. *Let $\mathfrak{d}$ be a delta operator. Then $\mathfrak{d}$ has a right inverse, i.e., there exists a linear operator $\mathfrak{d}^{-1} : \mathbb{K}[x] \to \mathbb{K}[x]$ such that $\mathfrak{d}(\mathfrak{d}^{-1}(f(x))) = f$ for all $f(x) \in \mathbb{K}[x]$, $\deg(\mathfrak{d}^{-1}(f(x))) = 1 + \deg(f(x))$ for $f \neq 0$, and $\mathfrak{d}^{-1}(f)(0) = 0$.*

Proof. Let $\mathcal{P} = (p_n(x))_{n\geq 0}$ denote the sequence of basic polynomials of the delta operator $\mathfrak{d}$. Then we can define an operator $\mathfrak{d}^{-1} : \mathbb{K}[x] \to \mathbb{K}[x]$ as follows: Given a polynomial $f(x) \in \mathbb{K}[x]$ of degree n, let $a_0, \dots, a_n \in K$ be such that $f(x) = \sum_{i=0}^{n} a_i p_i(x)$, set

$$\mathfrak{d}^{-1}(f(x)) := \sum_{i=0}^{n} \frac{a_i}{i+1} p_{i+1}(x).$$

It is seen that $\mathfrak{d}^{-1}$ is a linear operator on $\mathbb{K}[x]$, and since $\mathfrak{d}(p_n(x)) = np_{n-1}(x)$ and $p_n(0) = 0$ for all $n \in \mathbf{N}^+$, we have as well that $\mathfrak{d}(\mathfrak{d}^{-1}(f(x))) = f(x)$ and $\mathfrak{d}^{-1}(f)(0) = 0$ for every $f(x) \in \mathbb{K}[x]$. The rest is trivial, when considering that $p_n(x)$ is, for each $n \in \mathbf{N}^+$, a polynomial of degree n. □

Thus, we have the following generalization of the integral formula for classical Gončarov polynomials, see [12, p. 23].

Proposition 3.11. *Let $\mathfrak{d}$ be a delta operator and $\mathfrak{d}^{-1}$ its right inverse (which exists by Proposition 3.10), and let $(t_n(x))_{n\geq 0}$ be the generalized Gončarov basis associated with the pair $(\mathfrak{d}, \mathcal{Z})$. Then, for every $n, k \in \mathbf{N}$ with $k \leq n$, it holds that $t_n(x) = (n)_{(k)} \cdot \mathcal{I}_k(t_{n-k}^{(k)}(x))$, where $(t_i^{(n)}(x))_{n\geq 0}$ is the generalized Gončarov basis associated with the pair $(\mathfrak{d}, \mathcal{Z}^{(k)})$ and $\mathcal{I}_k$ the linear operator $\prod_{i=0}^{k-1}(1 - \varepsilon_{z_i})\mathfrak{d}^{-1}$.*

Proof. Fix $n, k \in \mathbf{N}$ with $0 \leq k \leq n$. If $k = 0$, the claim is trivial, because $\mathcal{I}_0$ is the identity operator. Now, suppose we have already confirmed that the statement is true for $0 \leq k < n$; then, by induction, we are just left to show that it continues to be true for $k + 1$.

For this, it is enough to prove that $t_{n-k}^{(k)}(x) = (n - k) \cdot (1 - \varepsilon_{z_k})\mathfrak{d}^{-1}(t_{n-k-1}^{(k+1)}(x))$. It follows from the facts that (i) both $t_{n-k}^{(k)}(x)$ and $(1 - \varepsilon_{z_k})\mathfrak{d}^{-1}(t_{n-k-1}^{(k+1)}(x))$ are zero when evaluated at z_k, and (ii) we have $\mathfrak{d}(t_{n-k}^{(k)}(x)) = (n - k)t_{n-k-1}^{(k+1)}(x)$ by Proposition 3.2, and

$$\mathfrak{d}((1 - \varepsilon_{z_k})\mathfrak{d}^{-1}(t_{n-k-1}^{(k+1)}(x))) = \mathfrak{d}(\mathfrak{d}^{-1}t_{n-k-1}^{(k+1)}(x)) - \mathfrak{d}(\varepsilon_{z_k}\mathfrak{d}^{-1}(t_{n-k-1}^{(k+1)}(x))) = t_{n-k-1}^{(k+1)}(x),$$

where we used Proposition 3.10 and the fact that $\varepsilon_{z_k}\mathfrak{d}^{-1}(t_{n-k-1}^{(k+1)}(x))$ is a constant (and $\mathfrak{d}$ applied to a constant is zero). □

In addition, we have the following generalization of the "perturbation formulas" obtained in [12, p. 24] and [11, p. 5].

Proposition 3.12. *Let $\mathfrak{d}$ be a delta operator, and let $\mathcal{Z} = (z_i)_{i\geq 0}$ and $\mathcal{Z}' = (z_i')_{i\geq 0}$ be $\mathbb{K}$-grids such that $z_k \neq z_k'$ for a given $k \in \mathbf{N}$ and $z_i = z_i'$ for $i \neq k$. Then we have, for $n > k$, that*

$$t_n(x; \mathfrak{d}, \mathcal{Z}') = t_n(x; \mathfrak{d}, \mathcal{Z}) - \binom{n}{k} t_{n-k}(z_k'; \mathfrak{d}, \mathcal{Z}^{(k)}) t_k(x; \mathfrak{d}, \mathcal{Z}), \tag{16}$$

while $t_n(x; \mathfrak{d}, \mathcal{Z}') = t_n(x; \mathfrak{d}, \mathcal{Z})$ for $n \leq k$.

Proof. Let $n \in \mathbf{N}$ and denote by $f_n(x)$ the polynomial on the right-hand side of (16). The claim is straightforward if $n \leq k$, essentially because we get by Theorem 2.1(1) that $t_n(x; \mathfrak{d}, \mathcal{Z})$ and $t_n(x; \mathfrak{d}, \mathcal{Z}')$ depend only on the first n nodes of $\mathcal{Z}$ and $\mathcal{Z}'$, respectively. Accordingly, assume in what follows that $n > k$ and fix $i \in \mathbf{N}$. By the unicity of the generalized Gončarov basis associated to the pair $(\mathfrak{d}, \mathcal{Z}')$, we just have to prove that $\varepsilon_{z_i'}(\mathfrak{d}^i f_n(x)) = n!\delta_{i,n}$. This is immediate if $i > n$, since then $\mathfrak{d}^i f_n(x) = 0$, and for $i \leq n$ it is a consequence of Proposition 3.2 (we omit further details). □

We conclude the present section by proving that generalized Gončarov bases obey an Appell relation, which extends an analogous result from [12, Section 3].

Proposition 3.13. *Let $(t_n(x))_{n\geq 0}$ be the generalized Gončarov basis associated with the pair $(\mathfrak{d}, \mathcal{Z})$. In addition, denote by $d(t)$ the compositional inverse of the D-indicator of $\mathfrak{d}$ in $\mathbb{K}[\![t]\!]$. Then the following identity holds,*

$$e^{x\,d(t)} = \sum_{n\geq 0} \frac{1}{n!} t_n(x) e^{z_n d(t)} t^n. \tag{17}$$

In particular, if $\mathfrak{d} = D$ then $e^{xt} = \sum_{n\geq 0} \frac{1}{n!} t_n(x) e^{z_n t} t^n$.

Proof. Let $(p_n(x))_{n\geq 0}$ the sequence of basic polynomials of $\mathfrak{d}$. By Proposition 3.7

$$p_n(x) = \sum_{i=0}^{n} \binom{n}{i} p_{n-i}(z_i) t_i(x) \qquad \text{for all } n \in \mathbf{N},$$

whence we get that

$$\sum_{n\geq 0} \frac{p_n(x)}{n!} t^n = \sum_{n\geq 0} \sum_{i=0}^{n} \frac{1}{i!(n-i)!} p_{n-i}(z_i) t_i(x) t^n = \sum_{i\geq 0} \left(\frac{1}{i!} t_i(x) t^i \sum_{j\geq 0} \frac{p_j(z_i)}{j!} t^j \right). \tag{18}$$

On the other hand, Formula (5) gives

$$e^{x\,d(t)} = \sum_{n\geq 0} \frac{p_n(x)}{n!} t^n,$$

which, together with (18), implies (17). The rest is trivial, when considering that the D-indicator of D over $\mathbb{K}[\![t]\!]$ is just t. $\square$

Using Proposition 3.13, we obtain the following characterization of sequences of binomial type, which is complementary to Corollary 3.4.

Proposition 3.14. *Let $\mathcal{T} = (t_n(x))_{n\geq 0}$ be the generalized Gončarov basis associated with the pair $(\mathfrak{d}, \mathcal{Z})$. Then $\mathcal{T}$ is of binomial type if and only if $\mathcal{Z}$ is an arithmetic grid of initial term 0.*

Proof. Suppose first that $\mathcal{T}$ is a sequence of binomial type. We get by [21, Section 3, Corollary 3] that there is a formal power series $f \in \mathbb{K}[\![t]\!]$ with $f(0) = 0$ and $f'(0) \neq 0$ such that

$$e^{xf(t)} = \sum_{n=0}^{\infty} \frac{t_n(x)}{n!} t^n. \tag{19}$$

On the other hand, we have from Proposition 3.13 that

$$e^{x\,d(t)} = \sum_{n\geq 0} \frac{1}{n!} t_n(x) e^{z_n d(t)} t^n, \tag{20}$$

where $d(t)$ is the compositional inverse of the D-indicator of the delta operator $\mathfrak{d}$ in $\mathbb{K}[\![t]\!]$.

Let $h \in \mathbb{K}[\![t]\!]$ be the compositional inverse of f, which exists by the assumption that $f(0) = 0$ and $f'(0) \neq 0$ [23]. Then $h(f(t)) = f(h(t)) = t$. Using the change of variable $y \mapsto h(d(t))$ in (19) yields that

$$e^{xd(t)} = \sum_{n=0}^{\infty} \frac{1}{n!} t_n(x) (h(d(t)))^n.$$

Comparing this with (20) implies, for all $n \in \mathbf{N}$, that

$$(h(d(y)))^n = e^{z_n d(y)} y^n,$$

which holds as an identity between formal power series in $\mathbb{K}[\![t]\!]$, and is in turn possible, by the further change of variables $t \mapsto d^{-1}(t)$, if and only if

$$(h(t))^n = e^{z_n t}(d^{-1}(t))^n. \tag{21}$$

Combining with $(h(t))^{n+1} = e^{z_{n+1}t}(d^{-1}(t))^{n+1}$ for all $n \in \mathbf{N}$, we find that

$$h(t) = \frac{(h(t))^{n+1}}{(h(t))^n} = e^{(z_{n+1}-z_n)t} d^{-1}(t).$$

It follows that $z_{n+1}-z_n$ is a constant independent of n, viz. there exists $b \in K$ such that $z_{n+1}-z_n = b$ for all $n \in \mathbf{N}$. Then $z_n = z_0 + nb$ for all $n \in \mathbf{N}$. But evaluating (21) at $n = 0$ gives $1 = e^{z_0 t}$, which implies $z_0 = 0$. Thus $\mathcal{Z}$ is an arithmetic grid of initial term 0.

As for the converse, assume now that $\mathcal{Z}$ is an arithmetic grid of common difference $b \in \mathbb{K}$ and initial term 0. Then $\mathcal{T}$ is, by Proposition 3.6, the generalized Gončarov basis associated with the pair $(E_b\mathfrak{d}, \mathcal{O})$, and hence by Corollary 3.4 is a sequence of binomial type. □

4. A combinatorial formula for generalized Gončarov polynomials

Let $\mathfrak{d}$ be a delta operator and $\mathcal{Z}$ a $\mathbb{K}$-grid. Assume $\mathcal{T} = (t_n(x))_{n\geq 0}$ is the generalized Gončarov basis associated with the pair $(\mathfrak{d}, \mathcal{Z})$. The main purpose of this section is to provide a combinatorial interpretation of the coefficients of $t_n(x)$. By Eq. (13) it is sufficient to consider only the constant terms. We will give an explicit combinatorial formula of $t_n(0)$ as a summation of ordered partitions.

Given a finite set S with n elements, an *ordered partition*, or *preferential arrangement*, of S is an ordered list $(B_1, \ldots, B_k)$ of disjoint nonempty subsets of S such that $B_1 \cup \cdots \cup B_k = S$.

If $\rho = (B_1, \ldots, B_k)$ is an ordered partition of S, then we set $|\rho| = k$. For every $i = 0, 1, \ldots, k$ we let $b_i := b_i(\rho) := |B_i|$ and $s_i := s_i(\rho) := \sum_{j=1}^{i} b_i$. In particular, $s_0(\rho) = 0$.

Let $\mathcal{R}[n]$ be the set of all ordered partitions of the set $[n] := \{1, 2, \ldots, n\}$. It is shown in [12, Theorem 4.2] that the constant coefficient for $g_n(x; \mathcal{Z})$, the classical Gončarov polynomial associated to $(D, \mathcal{Z})$, can be expressed as

$$g_n(0; z_0, \ldots, z_{n-1}) = \sum_{\rho} (-1)^{|\rho|} \prod_{i=0}^{k-1} z_{s_i}^{b_{i+1}} = \sum_{\rho \in \mathcal{R}[n]} (-1)^{|\rho|} z_0^{b_1} \cdots z_{s_{k-1}}^{b_k}. \tag{22}$$

A similar formula holds for the generalized Gončarov polynomials associated to the pair $(\mathfrak{d}, \mathcal{Z})$.

Theorem 4.1. *Let $(t_n(x))_{n\geq 0}$ be the generalized Gončarov basis associated with the pair $(\mathfrak{d}, \mathcal{Z})$, and $(p_n(x))_{n\geq 0}$ be the sequence of basic polynomials of $\mathfrak{d}$. Then for $n \geq 1$,*

$$t_n(0) = \sum_{\rho \in \mathcal{R}[n]} (-1)^{|\rho|} \prod_{i=0}^{k-1} p_{b_{i+1}}(z_{s_i}) = \sum_{\rho \in \mathcal{R}[n]} (-1)^{|\rho|} p_{b_1}(z_0) \cdots p_{b_k}(z_{s_{k-1}}). \tag{23}$$

Proof. Using Proposition 3.7 and noting that $p_n(0) = 0$ for $n \geq 1$, we have

$$t_n(0) = -\sum_{i=0}^{n-1} \binom{n}{i} p_{n-i}(z_i) t_i(0). \tag{24}$$

Denote by $\mathcal{T}(n)$, for $n \geq 1$, the right-hand side of Eq. (23), and let $\mathcal{T}(0) = 1$, which agrees with $t_0(0)$. We show by induction that $(\mathcal{T}(n))_{n\geq 0}$ satisfies the same recurrence as (24), i.e.,

$$\mathcal{T}(n) = -\sum_{i=0}^{n-1} \binom{n}{i} p_{n-i}(z_i)\mathcal{T}(i) \tag{25}$$

To see this, we divide $\mathcal{R}[n]$ into disjoint subsets $\mathcal{R}[n,i]$, where

$$\mathcal{R}[n,i] := \{(B_1,\ldots,B_k) \in \mathcal{R}[n] : |B_k| = n-i\} \qquad \text{for } i = 0,1,\ldots,n-1.$$

Given $\rho \in \mathcal{R}[n,i]$ with a fixed last block B_k, we can write ρ as the concatenation of B_k and an ordered partition of a set with i elements. So we get from the inductive hypothesis that

$$\sum_{\rho' \in \mathcal{R}[i]} (-1)^{|\rho'|} \prod_{i=0}^{k-2} p_{b_{i+1}}(z_{s_i}) = \mathcal{T}(i).$$

Since there are $\binom{n}{i}$ ways to choose the elements of B_k, it follows that the total contribution of ordered partitions in $\mathcal{R}[n,i]$ to $\mathcal{T}(n)$ is

$$-\binom{n}{i} p_{n-i}(z_i)\mathcal{T}(i).$$

Then, summing over $i = 0,1,\ldots,n-1$ proves the desired recurrence (25). $\square$

One way of obtaining $t_n(x;\mathfrak{d},\mathcal{Z})$ is to use the shift-invariance property (Proposition 3.5) as to write $t_n(x;\mathfrak{d},\mathcal{Z}) = t_n(0;\mathfrak{d},\mathcal{Z}-x)$. Another way is to use Eq. (13):

$$t_n(x;\mathfrak{d},\mathcal{Z}) = \sum_{i=0}^{n} \binom{n}{i} t_{n-i}(0;\mathfrak{d},\mathcal{Z}^{(i)})p_i(x),$$

and apply Theorem 4.1 to each $t_{n-i}(0;\mathfrak{d},\mathcal{Z}^{(i)})$. Comparing this with the analogous equation for classical Gončarov polynomials:

$$g_n(x;\mathcal{Z}) = \sum_{i=0}^{n} \binom{n}{i} g_{n-i}(0;z_i,\ldots,z_{n-1})x^i,$$

we notice that $t_n(x;\mathfrak{d},\mathcal{Z})$ can be obtained from $g_n(x;\mathcal{Z})$ by replacing x^i with $p_i(x)$ and z_k^i by $p_i(z_k)$. For example, we have the following formulas, which the reader may want to compare with the ones for $g_n(x;\mathcal{Z})$ in [12, p. 23]:

$$\begin{aligned}
t_0(x;\mathfrak{d},\mathcal{Z}) &= 1,\\
t_1(x;\mathfrak{d},\mathcal{Z}) &= p_1(x) - p_1(z_0),\\
t_2(x;\mathfrak{d},\mathcal{Z}) &= p_2(x) - 2p_1(z_1)p_1(x) + 2p_1(z_0)p_1(z_1) - p_2(z_0),\\
t_3(x;\mathfrak{d},\mathcal{Z}) &= p_3(x) - 3p_1(z_2)p_2(x) + (6p_1(z_1)p_1(z_2) - 3p_2(z_1))p_1(x)\\
&\qquad - p_3(z_0) + 3p_2(z_0)p_1(z_2) - 6p_1(z_0)p_1(z_1)p_1(z_2) + 3p_1(z_0)p_2(z_1).
\end{aligned}$$

For a generic delta operator $\mathfrak{d}$ and an arbitrary grid $\mathcal{Z}$, the generalized Gončarov polynomials do not usually have a simple closed formula. However, an interesting exception to this "rule" occurs when $\mathcal{Z}$ is an arithmetic progression with $z_i = a + bi$, in which case we refer to the corresponding generalized

Gončarov polynomials as $\mathfrak{d}$-Abel polynomials. In fact, $\mathfrak{d}$-Abel polynomials can be expressed in terms of the basic polynomials of $\mathfrak{d}$, as implied by the following:

Theorem 4.2. *Let $\mathfrak{d}$ be a delta operator with basic sequence $(p_n(x))_{n\geq 0}$, and let $\mathcal{Z}$ be the arithmetic grid $(a+bi)_{i\geq 0}$, where $a, b \in \mathbb{K}$. Then the $\mathfrak{d}$-Abel polynomial $t_n(x; \mathfrak{d}, \mathcal{Z})$ can be obtained by*

$$t_n(x; \mathfrak{d}, \mathcal{Z}) = \frac{(x-a)p_n(x-a-nb)}{x-a-nb}. \tag{26}$$

Proof. By the shift-invariance formula we have $t_n(x; \mathfrak{d}, \mathcal{Z}) = t_n(x-a; \mathfrak{d}, (bi)_{i\geq 0})$. Using Proposition 3.6, the generalized Gončarov polynomials associated to $(\mathfrak{d}, \mathcal{W})$ with $w_i = bi$ is also the generalized Gončarov polynomials associated to $(E_b\mathfrak{d}, \mathcal{O})$. Let $q_n(x)$ be the basic sequence of $E_b\mathfrak{d}$. Hence we have $t_n(x-a; \mathfrak{d}, (bi)_{i\geq 0}) = q_n(x-a)$.

Now, we have from [17, Theorem 4(3)], along with the fact that shift-invariant operators commute with each other, that

$$q_n(x) = x(E_b\mathfrak{d})^{-n}(x^n) = xE_{-nb}\mathfrak{d}^{-n}(x^n). \tag{27}$$

On the other hand, a further application of [17, Theorem 4(3)] yields that $p_n(x) = x\mathfrak{d}^{-n}(x^n)$, and hence $\mathfrak{d}^{-1}(x^n) = p_n(x)/x$. Together with (27), this in turn implies that

$$q_n(x) = xE_{-nb}\left(\frac{p_n(x)}{x}\right) = \frac{xp_n(x-nb)}{x-nb}.$$

So putting it all together, Eq. (26) follows immediately. □

We note that Niederhausen has also obtained formula (26) in [18], but with other means; he calls the procedure $\mathfrak{d} \mapsto E_a\mathfrak{d}$, for a fixed $a \in \mathbb{K}$, the abelization of the delta operator $\mathfrak{d}$.

Bivariate extensions of $\mathfrak{d}$-Abel polynomials, which are solutions to a multivariate Gončarov Interpolation problem with respect to an affine grid, are further studied and characterized in [15] for $\mathfrak{d} = D$, and in [14] for general delta operators.

5. Reluctant functions and order statistics

In the classical paper *Finite Operator Calculus*, Rota, Kahaner, and Odlyzko presented a unified theory of special polynomials via operator methods. One open question arose from this algebraic theory is to find the "statistical, probabilistic and combinatorial interpretations of the identities" (of the polynomials), see Problem 5 of Section 14, [21]. For polynomial sequences of binomial type, Mullin and Rota [17] provided a combinatorial interpretation through counting binomial type structures such as *reluctant functions*. The ideas of binomial enumeration and reluctant functions also provide a combinatorial setting for generalized Gončarov polynomials: We show in this section that generalized Gončarov polynomials are the natural polynomial basis for counting the number of binomial type structures subject to a linear constraint on their order statistics.

To start, let S and X be finite disjoint sets, and $f : S \to S \cup X$ a function. We say that f is a *reluctant function* from S to X if, for every $s \in S$, there is a positive integer $k = k(s)$ such that $f^k(s) \in X$, in which case we refer to $f^k(s)$ as the *final image* of s (under f). It is easy to see that for any given s, the integer $k(s)$, if it exists, is unique.

Accordingly, we take the *final range* of f, here denoted by $\underline{\mathrm{Im}}(f)$, to be the set of all $\xi \in X$ such that ξ is the final image of some $s \in S$. Given $\xi \in \underline{\mathrm{Im}}(f)$, we let the *final inverse image* of ξ, which we write as $f^{(-1)}(\xi)$, be the set of all the elements in S whose final image is ξ.

From a combinatorial point of view, the final inverse image of an element $\xi \in X$ can be regarded in a canonical way as a rooted forest: The nodes are just the elements of S, the roots are the elements of the inverse image $f^{-1}(\xi) = \{s \in S : \ f(s) = \xi\}$. In a rooted tree we say that a vertex is of depth k if the unique path from u to the root contains k edges. The root itself is of depth 0. Then for each $s_0 \in f^{-1}(\xi)$ and $k \in \mathbf{N}$ the vertices of depth k in a tree of $f^{(-1)}(\xi)$ rooted at s_0 are those elements $s \in S$ such that $f^k(s) = s_0$ and hence $f^{k+1}(s) = \xi$.

The *final coimage* of f is the partition $\{f^{(-1)}(\xi) : \xi \in \underline{\mathrm{Im}}(f)\}$ of S. Based on the above discussion the final coimage carries over a natural structure, T_f, of a rooted forest defined on each block of the partition. Furthermore, each block of the final coimage can be further partitioned into connected components (relative to T_f); the resulting partition is a refinement of the final coimage and has the additional property that each block has the structure of a rooted tree. This finer partition together with the rooted tree structure is called the *final preimage* of the reluctant function f.

Remark 1. What we call "final range", "final coimage", and "final preimage" of a reluctant function are actually called "range", "coimage", and "preimage" by Mullin and Rota in [17]. However, these latter terms are already used in the everyday practice of mathematics with a different meaning. We add the word "final" to avoid potential misunderstanding.

A *binomial class* $\mathcal{B}$ of reluctant functions is defined as follows. To every pair of finite sets S and X we assign a set $F(S, X)$ of reluctant functions from S to X, where $F(S, X)$ is isomorphic to $F(S', X')$ whenever S is isomorphic to S and X is isomorphic to X'. Consequently, the size of $F(S, X)$ depends only on the sizes of S and X, but not the content of these sets. Let $X \oplus Y$ stand for the disjoint union of X and Y. For every reluctant function f from S to $X \oplus Y$, let $A = \{s \in S : \text{the final image of } s \text{ is in } X\}$ and f_A is the restriction of f to A. Similarly $f_{S\setminus A}$ is the restriction of f to the set $S \setminus A$. The class $\mathcal{B}$ is a *binomial class* if $f_A \in F(A, X)$, $f_{S\setminus A} \in F(S \setminus A, Y)$ and the above decomposition leads to a natural isomorphism

$$\mu : F(S, X \oplus Y) \to \bigcup_{A \subseteq S}^{\cdot} (F(A, X) \otimes F(S \setminus A, Y)) \tag{28}$$

by letting $\mu(f) := (f_A, f_{S\setminus A})$, where $\otimes$ denotes the operation of piecing functions f_A and $f_{S\setminus A}$ together, and $\dot\cup$ a disjoint union.

Let $p_n(x)$ denote the size of the set $F(S, X)$ when $|S| = n$ and $|X| = x$. Then for a binomial class, $p_n(x)$ is a well-defined polynomial in the variable x of degree n, and the sequence $(p_n(x))_{n\geq 0}$ is of binomial type:

$$p_n(x + y) = \sum_{k\geq 0} \binom{n}{k} p_k(x) p_{n-k}(y).$$

The above construction gives a family of polynomial sequences of binomial types with combinatorial significance–they count the number of reluctant functions in a binomial class. For example, let $\mathcal{B}$ contains all the reluctant functions for which each block of the final preimage is a singleton. Then $p_n(x) = x^n$. Another example is that $\mathcal{B}$ contains all possible reluctant functions. In this case

$p_n(x) = x(x+n)^{n-1}$, the Abel polynomial $x(x-an)^{n-1}$ with $a=-1$. This result can be proved by using Prüfer codes, see e.g. [23, Prop. 5.3.2]. More examples of binomial classes are discussed in the next section.

For a sequence of numbers $(a_0, \ldots, a_{n-1})$, let $a_{(0)} \leq \cdots \leq a_{(n)}$ be the non-decreasing rearrangements of the terms a_i. The value of $a_{(i)}$ is called the i-th order statistic of the sequence. We will combine the notations of reluctant functions and order statistics. From now on we assume $S = \{s_0, \ldots, s_{n-1}\}$, x is a positive integer, and $X = \{1, \ldots, x\}$. Associated with any reluctant function f from S to X a sequence $\vec{x} = (x_0, \ldots, x_{n-1}) \in X^n$ where x_i is the final image of s_i.

Assume that $z_0 \leq \cdots \leq z_{n-1}$ are integers in X. For a binomial class $\mathcal{B}$ of reluctant functions enumerated by $p_n(x)$, define $\mathcal{O}rd(z_0, \ldots, z_{n-1})$, *the set of reluctant functions of length n whose order statistics are bounded by $\mathcal{Z}$*, by letting

$$\mathcal{O}rd(z_0, \ldots, z_{n-1}) = \{f \in F(S,X) : x_{(0)} \leq z_0, \ldots, x_{(n-1)} \leq z_{n-1}\}.$$

Then we have the following equation about $ord(z_0, \ldots, z_{i-1}) = |\mathcal{O}rd(z_0, \ldots, z_{n-1})|$.

Theorem 5.1. *With the notation as above, it holds that, for every $n \in \mathbf{N}$,*

$$p_n(x) = \sum_{i=0}^{n} \binom{n}{i} p_{n-i}(x - z_i) \cdot ord(z_0, \ldots, z_{i-1}). \tag{29}$$

Proof. For any reluctant function $f \in F(S,X)$, let $\kappa(f)$ be the maximal index i such that

$$x_{(0)} \leq z_0, \ldots, x_{(i-1)} \leq z_{i-1}. \tag{30}$$

The maximality of i implies that

$$z_i < x_{(i)} \leq x_{(i+1)} \leq \cdots \leq x_{(n-1)}.$$

Let X_1 be the subset of X consisting of $\{1, \ldots, z_i\}$, and $X_2 = X \setminus X_1 = \{z_i + 1, \ldots, x\}$. Assume $(x_{r_0}, \ldots, x_{r_{i-1}})$ is the subsequence of $\vec{x}$ from which the sequence $(x_{(0)}, \ldots, x_{(i-1)})$ was obtained by rearrangement, and let $A = \{s_{r_0}, \ldots, s_{r_{i-1}}\} \subseteq S$. Then the reluctant function f is obtained by piecing two functions, f_A and $f_{S\setminus A}$, the restrictions of f on A and $S \setminus A$, together. Since $\mathcal{B}$ is a binomial class, we have that $f_A \in F(A, X_1), f_{S\setminus A} \in F(S \setminus A, X_2)$. Furthermore, the function f_A has the property that its order statistics are bounded by $\mathcal{Z}$, and hence belongs to $\mathcal{O}rd(z_0, \ldots, z_{i-1})$. The function $f_{S\setminus A}$ can be any reluctant function from $S \setminus A$ to X_2.

Conversely, any pair of subsequences as described above can be reassembled into a reluctant function from S to X. In other word, the decomposition $f \to (f_A, f_{S\setminus A})$ defines a bijection from the set $F(S,X)$ to

$$\dot{\bigcup_{i=0}^{n}} \dot{\bigcup_{A=\{r_0,\ldots,r_{i-1}\}}} F_{ord}(A, \{1, \ldots, z_i\}) \otimes F(S \setminus A, \{z_i + 1, \ldots, x\}) \tag{31}$$

where $F_{ord}(A, \{1, \ldots, z_i\})$ is the set of reluctant functions from the set $\{s_i : i \in A\}$ to $\{1, \ldots, z_i\}$ whose order statistics are bounded by $\mathcal{Z}$.

Now counting the number of elements in the disjoint union of (31), we get Eq. (29). $\square$

Comparing Eq. (29) with the linear recurrence of Proposition 3.7, we obtain a combinatorial interpretation of the generalized Ginčarov basis.

Theorem 5.2. *Let $p_n(x)$ count the number of reluctant functions in a binomial class $\mathcal{B}$. Assume $p_n(x)$ is the sequence of basic polynomials of the delta operator $\mathfrak{d}$, and $t_n(x;\mathfrak{d},\mathcal{Z})$ is the nth generalized Gončarov polynomial associated to the pair $(\mathfrak{d},\mathcal{Z})$ with $\mathcal{Z}=(z_0,z_1,\ldots)$. Then*

$$ord(z_0,\ldots,z_{n-1}) = t_n(x;\mathfrak{d},(x-z_i)_{i\geq 0}) = t_n(0;\mathfrak{d},-\mathcal{Z}). \tag{32}$$

That is, $t_n(0;\mathfrak{d},-\mathcal{Z})$ counts the number of reluctant functions of the binomial class $\mathcal{B}$ whose order statistics are bounded by $\mathcal{Z}$.

6. Examples

In this section we give some examples of sequences of polynomials of binomial type that enumerate binomial classes. In each case, we describe the delta operator, the associated generalized Gončarov polynomials, and their combinatorial significance. We also compute $\mathfrak{d}$-Abel polynomials.

In [17] Mullin and Rota introduced two important families of binomial classes of reluctant functions. The first one is class $\mathcal{B}(T)$ where T is a family of rooted trees. The class $\mathcal{B}(T)$ consists of all reluctant functions whose final preimages are labeled rooted forests on S each of whose components is isomorphic to a rooted tree in the family T. Examples 1–5 belong to this family. The second family of binomial classes is formed by taking a subclass of $\mathcal{B}(T)$: one only allows those reluctant functions in $\mathcal{B}(T)$ having the property that their final coimage coincides with their final preimage. In other words, each rooted tree in the final preimage is mapped to a distinct element in X. Such a subclass, denoted by $\mathcal{B}_m(T)$, is called the *monomorphic class associated to* $\mathcal{B}(T)$ and is ultimately a generalization of the notion of injective function. Example 6 and 7 belong to the monomorphic family.

The combinatorial interpretation of generalized Gončarov polynomials are closely related to vector-parking functions. Hence we recall the necessary notations on parking functions. More results and theories of parking functions can be found in the survey paper [28]. Let $\mathbf{u}=(u_i)_{i\geq 1}$ be a sequence of non-decreasing positive integers. A $\mathbf{u}$-parking function of length n is a sequence $(x_1,\ldots,x_n)$ of positive integers whose order statistics satisfy $x_{(i)}\leq u_i$. When $u_i=i$, we get the classical parking functions, which was originally introduced by Konheim and Weiss [10]. Classical parking functions have a "parking description" as follows.

> There are n cars $C_1,\ldots,C_n$ that want to park on a one-way street with ordered parking spaces $1,\ldots,n$. Each car C_i has a preferred space a_i. The cars enter the street one at a time in the order $C_1,\ldots,C_n$. A car tries to park in its preferred space. If that space is occupied, then it parks in the next available space. If there is no space then the car leaves the street (without parking). The sequence $(a_1,\ldots,a_n)$ is called a *parking function of length* n if all the cars can park, i.e., no car leaves the street.

An equivalent definition for classical parking functions is that at least i cars prefer the parking spaces of labels i or less. Similarly a $\mathbf{u}$-parking function of length n, where $\mathbf{u}=(u_1,\ldots,u_n)$ is a vector of positive integers, can be viewed as a parking preference sequence in which at least i cars prefer the parking spaces of labels $\leq u_i$ (out of a total of $x\geq u_n$ parking spaces).

As explained in the following examples, the classical Gončarov polynomial associated to the differential operator D enumerate **u**-parking functions, while generalized Gončarov polynomials associated to other delta operators for binomial class $\mathcal{B}(T)$ enumerate variant forms of the parking scheme, in which the cars arrive in groups with certain special structures, cars in the same group have the same preferred space, and there are at least i cars preferring spaces of label u_i or less. Similarly, the generalized Gončarov polynomials associated to other delta operators for the monomorphic class $\mathcal{B}_m(T)$ enumerate those with the additional property that different groups have different preferences.

6.1. **The standard power polynomials.** The sequence $(x^n)_{n\geq 0}$ is the basic polynomials of the differential operator D. It enumerates the binomial class $\mathcal{B}(T_0)$, where T_0 consists of a single tree with only one vertex (viz., an isolated vertex which is also the root of the tree).

A reluctant function in this class is just a usual function from S to X, which can be represented by the sequence $\vec{x} = (f(s_1), \ldots, f(s_n))$. Gončarov polynomials $g_n(x; \mathcal{Z})$ associated to $(D, \mathcal{Z})$ are the classical ones studied in [12], for which $(-1)^n g_n(0; \mathcal{Z})$ enumerates the number of **z**-parking functions of length n, [12, Theorem 5.4].

If $\mathcal{Z}$ is the arithmetic progression $z_i = a + bi$, then $g_n(x; \mathcal{Z})$ is a shift of the classical Abel polynomials, or more explicitly,

$$g_n(x; (a + bi)_{i\geq 0}) = (x - a)(x - a - nb)^{n-1}.$$

It follows that $P_n(a, a + b, \ldots, a + (n - 1)b) = a(a + nb)^{n-1}$, where $P_n(z_0, \ldots, z_{n-1})$ is the number of **z**-parking functions, i.e., positive integer sequences whose order statistics are weakly bounded by **z**. In particular, for $a = b = 1$ we recover the formula for ordinary parking functions $P_n(1, \ldots, n) = (n + 1)^{n-1}$.

6.2. **Abel polynomials.** The Abel polynomial with parameter a, namely $A_n(x; a) = x(x-na)^{n-1}$, is the n-th basic polynomial of the delta operator $\mathfrak{d} = E_a D = DE_a$. When $a = -1$, $A_n(x; -1) = x(x + n)^{n-1}$. This polynomial counts the reluctant functions in the binomial class $\mathcal{B}(T)$ where T contains all possible rooted trees. In fact, such reluctant functions from S to X are represented as rooted forests on vertex set $S \cup X$ with X being the root set, whose number can be computed by using Prüfer codes. We remark that the rooted forest corresponding to a reluctant function in $\mathcal{B}(T)$ is different from the rooted forest T_f in the final coimage of f. The latter is a rooted tree on S only. The rooted forest corresponding to f can be obtained from T_f by replacing every root s_0 of T_f with an edge from s_0 to $f(s_0) \in X$, and letting every vertex of X be a root.

Similarly, if $a = -k$ for a positive integer k, then $A_n(x; -k) = x(x + nk)^{n-1}$ enumerates the reluctant functions in the binomial class $\mathcal{B}(T_k)$, where T_k contains all the rooted k-trees, which are rooted trees each of whose edge is colored by one of the colors $0, 1, \ldots, k-1$. Such trees were studied in [22, 27]. In particular the reluctant functions in $\mathcal{B}(T_k)$ can be represented as sequences of rooted k-forests of length x, which are defined in [27] and proved to be in bijection with the (a, b)-parking functions with $a = x$ and $b = k$.

By Proposition 3.6, $t_n(x; E_{-k}D, \mathcal{Z})$, the generalized Gončarov polynomial associated to $(E_{-k}D, \mathcal{Z})$ is the same as $g_n(x; (z_i - ki)_{i\geq 0})$, where $g_n(x; \mathcal{Z})$ is the classical Gončarov polynomial.

The polynomial $t_n(x; E_{-k}D, -\mathcal{Z}) = g_n(x; (-z_i - ki)_{i\geq 0})$, when evaluated at 0, has two combinatorial interpretations. On one hand, it gives the number of $\mathbf{u}$-parking functions with $\mathbf{u} = (u_i = ki + z_i)_{i\geq 0}$. On the other hand, by Theorem 5.2 $t_n(0; E_{-k}D, -\mathcal{Z})$ also counts the number of ways that n cars form disjoint groups, each group is equipped with a structure of rooted k-tree, cars in the same group have the same preference, and the order statistics of the parking sequence are bounded by $\mathcal{Z}$.

The $\mathfrak{d}$-Abel polynomial is of the form $t_n(x; E_{-k}D, (a + bi)_{i\geq 0}) = (x - a)(x - a - nb + nk)^{n-1}$.

6.3. **Laguerre polynomials.** The nth Laguerre polynomial $L_n(x)$ is given by the formula

$$L_n(x) = \sum_{k\geq 0} \frac{n!}{k!}\binom{n-1}{k-1}(-x)^k. \tag{33}$$

This is the n-th basic polynomial of the Laguerre delta operator $K := D(D - I)^{-1} = -\sum_{i\geq 0} D^i$.

The coefficients $\frac{n!}{k!}\binom{n-1}{k-1}$ are called the (unsigned) Lah numbers, which are also the coefficients expressing rising factorials in terms of falling factorials. See Example 6.7.

The polynomial $L_n(-x)$ enumerates the binomial class $\mathcal{B}(T_P)$, where T_P is the set of all rooted trees which is a path rooted at one of its leaves. To see this, consider all such reluctant functions whose final preimage contains exactly k rooted paths. To get such k paths, we can linearly order all the elements of S in a row and then cut it into k nonempty segments, for each segment let the first element be the root. There are $n!\binom{n-1}{k-1}$ ways. Since the paths are unordered, we divide $k!$ to get the number of sets of k rooted paths. Then multiplying x^k to get all functions from the set of k paths to X.

Let $t_n(x; K, \mathcal{Z})$ be the generalized Gončarov polynomial associated to $(K, \mathcal{Z})$. By Theorem 5.1 and its proof, we get that the number of reluctant functions in $\mathcal{B}(T_P)$ whose order statistics are bounded by $\mathcal{Z}$ is given by $t_n(-x; K, -x + \mathcal{Z}) = t_n(0; K, \mathcal{Z})$. Equivalently, $t_n(0; K, \mathcal{Z})$ counts the number of parking schemes in which n cars want to park in a parking lot of x spaces such that (i) the cars arrive in disjoint groups,(ii) each group forms a queue, (iii) all cars in the same queue prefer the same space, and (iv) the order statistics of the preference sequence is bounded by $\mathcal{Z}$.

The n-th $\mathfrak{d}$-Abel polynomial associated with the operator $\mathfrak{d} = K$ and the arithmetic grid $\mathcal{Z} = (a + bi)_{i\geq 0}$ is given by

$$t_n(x; K, (a + bi)_{i\geq 0}) = (a - x)\sum_{k=1}^{n} \frac{n!}{k!}\binom{n-1}{k-1}(a + nb - x)^{k-1}.$$

In particular, $t_n(0; K, (a + bi)_{i\geq 0}) = a\sum_{k=1}^{n} \frac{n!}{k!}\binom{n-1}{k-1}(a + nb)^{k-1}$, which, for $a = b = 1$, supplies the sequence $1, 5, 46, 629, 11496, \ldots$, namely A052873 in the On-Line Encyclopedia of Integer Sequences (OEIS) [19], where one can find the exponential generating function and an asymptotic formula.

6.4. **Inverse of the Abel polynomial** $A_n(x; -1)$**.** Let $p_n(x) := \sum_{k\geq 0}\binom{n}{k}k^{n-k}x^k$. Then $(p_n(x))_{n\geq 0}$ is a sequence of binomial type: This is actually the basic sequence of the delta operator $\mathfrak{d}$ whose D-indicator is the compositional inverse of $f(t) = te^t$ (often referred to as the Lambert w-function), and it is also the inverse sequence of the Abel polynomials $(A_n(x; -1))_{n\geq 0}$ under the umbral composition.

In fact, the sequence $(p_n(x))_{n\geq 0}$ enumerates the binomial class $\mathcal{B}(T_1)$, where T_1 contains all the rooted trees with depth at most 1, i.e., stars $\{S_k : \ k \geq 0\}$ where S_k is the tree on vertices $\{v_0, \ldots, v_k\}$ with root v_0 and edges $\{v_0, v_i\}$ for $i = 1, \ldots, k$, [17, Sec. 7].

By Theorem 5.2, the generalized Gončarov polynomial associated to $(\mathfrak{d}, \mathcal{Z})$ gives a formula for the number of parking schemes such that the cars arrives in disjoint groups, each group has a leader, all cars in the same group prefer the same space, and the order statistics of the preference sequence is bounded by $\mathcal{Z}$.

In addition, it follows from the above and Theorem 4.2 that the n-th $\mathfrak{d}$-Abel polynomial associated wit h the arithmetic grid $\mathcal{Z} = (a + bi)_{i\geq 0}$ is given by

$$t_n(x; \mathfrak{d}, (a + bi)_{i\geq 0}) = (x - a) \sum_{k=1}^{n} \binom{n}{k} k^{n-k} (x - a - nb)^{k-1}.$$

In particular, $t_n(0; \mathfrak{d}, (-a - bi)_{i\geq 0}) = a \sum_{k=1}^{n} \binom{n}{k} k^{n-k} (a + nb)^{k-1}$; for $a = b = 1$, this yields the sequence $1, 5, 43, 549, 9341, \ldots$, which is A162695 in OEIS, where one can find the exponential generating function and an asymptotic formula.

6.5. **Exponential polynomials.** The n-th exponential polynomial, also called the *Touchard polynomial* or the *Bell polynomial*, is given by $b_n(x) = \sum_{k=1}^{n} S(n, k)x^k$, where the coefficients $S(n, k)$ are the familiar Stirling numbers of the second kind, see [21, pp. 747–750].

Introduced by J. F. Steffensen in his 1927 treatise on interpolation [24] and later reconsidered, in particular, by J. Touchard [25] for their combinatorial and arithmetic properties, the exponential polynomials are the basic polynomials of the delta operator

$$\mathfrak{b} := \log(I + D) := \sum_{i\geq 1} (-1)^{i+1} \frac{1}{i} D^i. \tag{34}$$

The exponential polynomials also enumerate a binomial class $\mathcal{B}(T)$ of reluctant functions from S to X, where we require that elements in S are totally ordered, i.e., there is an order such that $s_1 < \cdots < s_n$. Now let T be the family of rooted paths labeled by S such that the labels are monotone along the path, with the root having the largest label. Correspondingly, the generalized Gončarov polynomial $t_n(x; \mathfrak{b}, \mathcal{Z})$ gives the enumeration of parking schemes in which n cars arrive in disjoint groups, all cars in the same group prefer the same space, and the order statistics of the preference sequence are bounded by $\mathcal{Z}$.

Thus, we get from the above and Theorem 4.2 that the n-th $\mathfrak{d}$-Abel polynomial associated with the arithmetic grid $\mathcal{Z} = (a + bi)_{i\geq 0}$ is

$$t_n(x; \mathfrak{b}, (a + bi)_{i\geq 0}) = (x - a) \sum_{k=1}^{n} S(n, k)(x - a - nb)^{k-1}.$$

In particular, $t_n(0; \mathfrak{b}, (-a - bi)_{i\geq 0}) = a \sum_{k=1}^{n} S(n, k)(a + nb)^{k-1}$; for $a = b = 1$, this gives the sequence $1, 4, 29, 311, 4447 \ldots$, which, after a shift of index, is A030019 in OEIS. The sequence also has other combinatorial interpretations, for example, as hypertrees on n labeled vertices [19]. It would be interesting to find bijections between the parking sequences and the hypertrees.

The next two examples correspond to monomorphic classes associated to some binomial class $\mathcal{B}(T)$. As pointed out in [17], if the reluctant functions in $\mathcal{B}(T)$ are counted by $p_n(x)$ where

$$p_n(x) = \sum_{k=0}^{n} a_k x^k,$$

then the basic sequence counting $\mathcal{B}_m(T)$ is

$$q_n(x) = \sum_{k=0}^{n} a_k x_{(k)}.$$

6.6. **Lower factorial polynomials.** The lower factorial $x_{(n)} = \prod_{i=0}^{n-1}(x-i)$ is the basic polynomial for the monomorphic class $\mathcal{B}_m(T_0)$, where T_0 is as described in Example 6.1. The corresponding delta operator is the forward difference operator $\Delta_{1,0} = E_1 - E_0 = E_1 - I$. It counts the number of one-to-one functions from S to X.

By Theorem 5.1, the generalized Gončarov polynomials $t_n(x)$ associated to $(\Delta_{1,0}, \mathcal{Z})$ give the number of one-to-one functions from S to X whose order statistics are bounded by $\mathcal{Z}$ via Eq. (32). Assume $x \geq n$. Note that any sequence $1 \leq x_1 < \cdots < x_n \leq x$ can be represented geometrically as a strictly increasing lattice path in the plane from $(0,0)$ to $(x-1,n)$ using only steps $E = (1,0)$ and $N = (0,1)$: one simply takes the N-steps from $(x_i - 1, i-1)$ to $(x_i - 1, i)$, and connects the N-steps with E-steps, (see Figure 1).

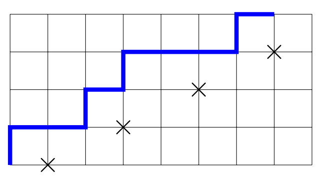

FIGURE 1. Lattice path corresponding to the sequence $(1,3,4,7)$ with $x = 8$. The stars indicate the right boundary $(1,3,5,7)$.

Clearly there are $n!$ one-to-one functions from S to X whose images are $\{x_1, \ldots, x_n\}$. Thus Theorem 5.2 implies that $\frac{1}{n!} t_n(0; \Delta_{1,0}, -\mathcal{Z})$ counts the number of strictly increasing lattice paths from $(0,0)$ to $(x-1,n)$ with strict right boundary $(z_0, \ldots, z_{n-1})$, viz. with right boundary $(z_0, \ldots, z_{n-1})$ that never touch the points $\{(z_i, i) : 0 \leq i < n\}$.

In particular, we have $t_n(x; \Delta_{1,0}, (a+bi)_{i\geq 0}) = (x-a)(x-a-nb-1)_{(n-1)}$, so the number of strictly increasing lattice paths from $(0,0)$ to $(x-1,n)$ with the affine right boundary $(a, a+b, a+2b, \ldots)$, where $a > 0$ and $b \geq 0$ are integers, is given by

$$\frac{1}{n!} a(a+nb-1)_{(n-1)} = \frac{a}{a+nb}\binom{a+nb}{n}.$$

When $a = b = 1$, the above number is $\frac{1}{n+1}\binom{n+1}{n} = 1$, since there is only one strictly increasing lattice path from $(0,0)$ to $(x-1,n)$ that is bounded by $(1, \ldots, n)$.

6.7. **Upper factorial polynomials.** The upper factorial $x^{(n)} = (x+n-1)_{(n)} = \prod_{i=0}^{n-1}(x+i)$ is related to the Laguerre polynomials by the equation

$$x^{(n)} = \sum_{k\geq 0} \frac{n!}{k!}\binom{n-1}{k-1} x_{(k)}.$$

Hence $x^{(n)}$ is the basic polynomial for the monomorphic class $\mathcal{B}_m(T_p)$ where T_p is as described in Example 6.3. The corresponding delta operator for $(x^{(n)})_{n\geq 0}$ is the backward difference operator $\Delta_{0,-1} = E_0 - E_{-1} = I - E_{-1}$.

Monomorphic reluctant functions in $\mathcal{B}_m(T_p)$ can also be described by lattice paths. Assume the final preimages of a reluctant function in $\mathcal{B}_m(T_p)$ consists of k paths of lengths $p_1, \ldots, p_k$ whose images are $x_1 < \cdots < x_k$. It corresponds to a lattice path from $(0,0)$ to $(x-1,n)$ whose consecutive vertical runs are given by p_1 N-steps at $y = x_1 - 1$, followed by p_2 N-steps at $y = x_2 - 1$, and so on. See Figure 2 for an example.

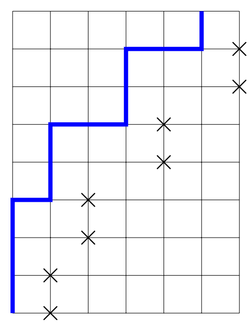

FIGURE 2. Lattice path corresponding to the images $(1,1,1,2,2,4,4,6)$ with $x = 6$. The stars indicate the right boundary $(1,1,2,2,4,4,6,6)$.

The labels on each path can be recorded by labeling the N-steps of the lattice path. Again there are $n!$ many labels possible for each lattice path. Hence by Theorem 5.1 we know that if $t_n(x;\Delta_{0,-1},\mathcal{Z})$ is the generalized Gončarov polynomial associated to the pair $(\Delta_{0,-1},\mathcal{Z})$, then $\frac{1}{n!}t_n(0;\Delta_{0,-1},-\mathcal{Z})$ is the number of lattice paths with the right boundary $\mathcal{Z}$, a result first established in [11].

In particular, $t_n(x;\Delta_{0,-1},(a+bi)_{i\geq 0}) = (x-a)(x-a-nb+1)^{(n-1)}$, hence the number of lattice paths from $(0,0)$ to $(x-1,n)$ with strict affine right boundary $(a, a+b, a+2b, \ldots)$ for some integers $a > 0$ and $b \geq 0$ is

$$\frac{1}{n!}a(a+nb+1)^{(n-1)} = \frac{a}{a+n(b+1)}\binom{a+n(b+1)}{n}, \tag{35}$$

a well-known result, see e.g. [16, p. 9]. In particular, for $a = 1$ and $b = k$ for some positive integer k, it counts the number of lattice paths from the origin to (kn, n) that never pass below the line $x = yk$. In this case, (35) gives $\frac{1}{1+(k+1)n}\binom{1+(k+1)n}{n} = \frac{1}{1+kn}\binom{(k+1)n}{n}$, which is the n-th k-Fuss-Catalan number.

7. Further remarks

In the theory of binomial enumeration, it is not really necessary to restrict oneself to reluctant functions, and we can in fact consider a more general setting, as outlined in [20, Sec. 2]. Following Joyal [8], a *species* B is a covariant endofunctor on the category of finite sets and bijections. Given a finite set E, an element $s \in B(E)$ is called a B-structure on E. A *k-assembly of B-structures on E* is then a partition π of the set E into k blocks such that each block of π is endowed with a B-structure. Let $B_k(E)$ denote the set of all such k-assemblies. For example, when B is a set of rooted trees, a k-assembly of B-structures on E is a k-forest of rooted trees with vertex set E. But we can also take B to be other structures, such as permutations, graphs, posets, etc.

To enumerate the number of assemblies of B-structures, we define sequences of nonnegative integers by

$$b_{n,k} = \begin{cases} |B_k([n])|, & k \leq n, \\ 0, & k > n. \end{cases}$$

Mullin and Rota's work [17] establishes that if $b_n(x) = \sum b_{n,k} x^k$ is the enumerator for assemblies of B-structures on $[n]$, then $(b_n(x))_{n \geq 0}$ is a polynomial sequence of $\mathbb{K}[x]$ of binomial type. Now we can interpret the factor x^k in $b_n(x)$ by considering all functions (or monomorphic functions if one replaces x^k with $x_{(k)}$) from the blocks of a k-assembly to a set X of size x. When X is totally ordered, i.e., X is isomorphic to the poset $[s]$ with numerical order, where $s = |X|$, we can consider all such functions whose order statistics are bounded by a given sequence. The enumeration for such k-assemblies with an order-statistics constraint is captured by the associated generalized Gončarov polynomials.

In principle, the above combinatorial description only applies to polynomial sequences $(p_n(x))_{n \geq 0}$ of binomial type with nonnegative integer coefficients. Mullin and Rota hinted at a generalization of their theory to include polynomials with negative coefficients, and Ray [20] developed a concept of weight functions on the partition category which allows one to realize any binomial sequence, over any commutative ring with identity, as the enumerator of weights. It would be interesting to investigate the role of generalized Gončarov polynomials in such weighted counting, as well as in other dissecting schemes as described in Henle [7], and to find connections to rook polynomials, order invariants, Tutte invariants of combinatorial geometries, and symmetric functions.

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