EFRON'S COINS AND THE LINIAL ARRANGEMENT

GÁBOR HETYEI

To (Richard P. S.)^2

Abstract. We characterize the tournaments that are dominance graphs of sets of (unfair) coins in which each coin displays its larger side with greater probability. The class of these tournaments coincides with the class of tournaments whose vertices can be numbered in a way that makes them semiacyclic, as defined by Postnikov and Stanley. We provide an example of a tournament on nine vertices that can not be made semiacyclic, yet it may be represented as a dominance graph of coins, if we also allow coins that display their smaller side with greater probability. We conclude with an example of a tournament with 81 vertices that is not the dominance graph of any system of coins.

Introduction

A fascinating paradox in probability theory is due to B. Efron, who devised a four element set of nontransitive dice [5]: the first has four faces labeled 4 and two faces labeled 0, the second has all faces labeled 3, the third has four faces labeled 2 and two faces labeled 6, the fourth has three faces labeled 1 and three faces labeled 5. On this list, die number i defeats die number $i+1$ in the cyclic order in the following sense: when we roll the pair of dice simultaneously, die number i is more likely to display the larger number than die number $i+1$. The paradox arose the interest of Warren Buffet, it inspired several other similar constructions and papers in game theory and probability: sample references include [1, 3, 6, 7, 13].

This paper intends to investigate a hitherto unexplored aspect of Efron's original example: on each of its dice, only at most two numbers appear. They could be replaced with unfair coins, which display one of two numbers with a given probability. This restriction seems to be strong enough that we should be able to describe exactly which tournaments can be realized as *dominance graphs* of collections of unfair coins, where the direction of the arrows indicates which coin of a given pair is more likely to display the larger number.

Our paper gives a complete characterization in the case when we restrict ourselves to the use of *winner* coins: these are coins that are more likely to display their larger number. The answer, stated in Theorem 3.2, is that a tournament has such a representation exactly when its vertices may be numbered in a way that it becomes a *semiacylic tournament*. These tournaments were introduced by Postnikov and Stanley [12], the number of semiacyclic tournaments on n numbered vertices is the same as the number of regions of the $(n-1)$-dimensional Linial arrangement. Our result allows to construct

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an example of a tournament on 9-vertices that can not be represented using winner coins only. On the other hand, we will see that this example is representable if we also allow the use of *loser* coins, that is, coins that are more likely to display their smaller number. However, as we will see in Theorem 4.3, any tournament that is not representable with a set of winner coins only, gives rise via a direct product operation to a tournament that is not representable by any set of coins. In particular, we obtain an example of a tournament on 81 vertices that can not be represented by any set of coins as a dominance graph. Our results motivate several open questions, listed in the concluding Section 5.

1. Preliminaries

A hyperplane arrangement is a finite collection of codimension one hyperplanes in a finite dimensional vectorspace, together with the induced partition of the space into regions. The number of these regions may be expressed in terms of the Möbius function in the intersection poset of the hyperplanes, using Zaslavsky's formula [15].

The *Linial arrangement* $\mathcal{L}_{n-1}$ is the hyperplane arrangement

$$x_i - x_j = 1, \quad 1 \leq i < j \leq n \tag{1.1}$$

in the $(n-1)$-dimensional vector space $V_{n-1} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \ : \ x_1 + \cdots + x_n = 0\}$. We will use the combinatorial interpretation of the regions of $\mathcal{L}_{n-1}$ in terms of *semiacyclic tournaments*, due to Postnikov and Stanley [12]. A *tournament* on the vertex set $\{1, \ldots, n\}$ is a directed graph with no loops nor multiple edges, such that for each 2-element subset $\{i, j\}$ of $\{1, \ldots, n\}$, exactly one of the directed edges $i \to j$ and $j \to i$ belongs to the graph. We may think of a tournament as the visual representation of the outcomes of all games in a championship, such that each team plays against each other team exactly once, and there is no "draw".

Definition 1.1. *A directed edge $i \to j$ is called an* ascent *if $i < j$ and it is a* descent *if $i > j$. For any directed cycle $C = (c_1, \ldots, c_m)$ we denote the number of directed edges that are ascents in the cycle by* $\operatorname{asc}(C)$*, and the number of directed edges that are descents by* $\operatorname{desc}(C)$*. A cycle is* ascending *if it satisfies* $\operatorname{asc}(C) \geq \operatorname{desc}(C)$*. A tournament on $\{1, \ldots, n\}$ is* semiacyclic *if it contains no ascending cycle.*

To each region R in $\mathcal{L}_{n-1}$ we may associate a tournament on $\{1, \ldots, n\}$ as follows: for each $i < j$ we set $i \to j$ if $x_i > x_j + 1$ and we set $j \to i$ if $x_i < x_j + 1$. Postnikov and Stanley, and independently Shmulik Ravid, gave the following characterization of the tournaments arising this way [12, Proposition 8.5]

Proposition 1.2. *A tournament T on $\{1, \ldots, n\}$ corresponds to a region R in $\mathcal{L}_{n-1}$ if and only if T is semiacyclic. Hence the number $r(\mathcal{L}_{n-1})$ of regions of $\mathcal{L}_{n-1}$ is the number of semiacyclic tournaments on $\{1, \ldots, n\}$.*

2. The coin model and its elementary properties

In this paper we will study n element sets of (fair and unfair) coins. Each coin is described by a triplet of real parameters (a_i, b_i, x_i) where $a_i \leq b_i$ and $x_i > 0$ hold (here $i = 1, 2, \ldots, n$). The ith coin has the number a_i on one side and b_i on the other. After flipping it, it shows the number a_i with probability $1/(1 + x_i)$, equivalently it shows the number b_i with probability $x_i/(1+x_i)$. Note that, as x_i ranges over the set of all positive real numbers, the probability $1/(1+x_i)$ ranges over all numbers in the open interval $(0,1)$. We call the triplet (a_i, b_i, x_i) the *type* of the coin. We say that coin i *dominates* coin j if, after tossing both at the same time, the probability that coin i displays a strictly larger number than coin j is greater than the probability that coin j displays a strictly larger number. In other words, when we flip both coins, the one displaying the larger number "wins", the other one "loses", and we consider both coins displaying the same number a "draw". The coin that is more likely to win, dominates the other.

We represent the domination relation as the *dominance graph*, whose vertices are the coins and there is a directed edge $i \to j$ exactly when coin i dominates coin j. We will be interested in the question, which tournaments may be represented as the dominance graph of a set of n coins.

Up to this point we made one, inessential simplification: we assume that x_i can not be zero or infinity, that is, no coin can land on the same side with probability 1. If we have such a coin, we may replace it with a coin of type $(a, a, 1)$, that is, a fair coin that has the same number written on both sides. Since we are interested only in dominance graphs as tournaments we will also require that *for each pair of coins one dominates the other*. After fixing the parameters a_i and b_i for each coin, this restriction will exclude the points of $\binom{n}{2}$ hypersurfaces of codimension one from the set of possible values of $(x_1, x_2, \ldots, x_n)$, each hypersurface being defined by an equation involving a pair of variables $\{x_i, x_j\}$. The equations defining these surfaces may be obtained by replacing the inequality symbols with equal signs in the inequalities stated in Table 1 below. In particular, we assume that *different coins have different types*. In the rest of this section we will describe in terms of the types when the ith coin dominates the jth coin. To reduce the number of cases to be considered we first show that we may assume that no coin has the same number written on both sides. This is a direct consequence of the next lemma.

Lemma 2.1.

Suppose there is a coin of type (a_i, b_i, x_i), satisfying $a_i = b_i$. Replace this coin with a coin of type $(a_i', b_i, 1)$ where a_i' is any real number that is less than a_i but larger than any element in the intersection of the set $\{a_1, b_1, a_2, b_2, \ldots, a_n, b_n\}$ with the open interval $(-\infty, a_i)$. Then modified system of coins has the same dominance graph.

Proof. We only need to verify that another coin j, of type (a_j, b_j, x_j) dominates a coin of type (a_i, a_i, x_i) if and only if it dominates a coin of type $(a_i', a_i, 1)$. This is certainly the case when neither of a_j and b_j is equal to a_i, as these numbers compare to a_i the same way as to a_i'. We are left to consider the case when exactly one of a_j and b_j equals a_i (both can not equal because then there is no directed edge between i and j in the dominance graph).

Case 1: $a_j = a_i$ and so $a_i' < a_i = a_j < b_j$ hold. In the original system, as well as in the modified one, only coin j can win against coin i and it does so with a positive probability. We have $j \to i$ in both dominance graphs.

Case 2: $b_j = a_i$ and so $a_j < a_i' < a_i = b_j$. In the original system only coin j can lose (when it displays a_j) so we have $i \to j$ in the dominance graph. In the modified system,

i can also lose sometimes, exactly when it displays a_i' (with probability $1/2$) and coin j displays a_j. The probability of this event is $1/(1+x_j)/2$. A draw can also occur, exactly when both coins display $a_i = b_j$, the probability of this event is $x_j/(1+x_j)/2$. By subtracting these probabilities from 1 we obtain that the probability that j loses is exactly $1/2$. This is more than the probability of i losing, as $1/(1+x_j)/2 < 1/2$. Thus $i \rightarrow j$ still holds in the dominance graph of the modified system. □

From now on we will assume that the type (a_i, b_i, x_i) of each coin satisfies $a_i < b_i$. If the type also satisfies $x_i > 1$ then we call the coin a *winner*, if it satisfies $x_i < 1$, we call it a *loser*. Note that a loser coin can dominate a winner coin under the appropriate circumstances, these terms refer to the fact whether the coin is more likely to display its larger or smaller value. Note that a coin satisfies $x_i = 1$ exactly when it is a *fair coin*.

In several results we will list our coins in *increasing lexicographic order of their types*.

Definition 2.2. *We say that the type (a_i, b_i, x_i) is lexicographically smaller than the type (a_j, b_j, x_j), if one of the following holds:*

(1) $a_i < a_j$;
(2) $a_i = a_j$ *and* $b_i < b_j$;
(3) $a_i = a_j$, $b_i = b_j$ *and* $x_i < x_j$.

Theorem 2.3. *Assume we list the coins in increasing lexicographic order by their types, comparing the coordinates left to right. Then, for $i < j$, we have $i \rightarrow j$ if and only if exactly one of the following conditions is satisfied:*

(1) $a_i = a_j < b_i < b_j$ *and* $1/x_j > 1/x_i + 1$;
(2) $a_i < a_j < b_j < b_i$ *and* $x_i > 1$;
(3) $a_i < a_j < b_i = b_j$ *and* $x_i > x_j + 1$;
(4) $a_i < a_j < b_i < b_j$ *and* $(1/x_i + 1)(x_j + 1) < 2$.

Proof. Assuming that $(a_i, b_i) \leq (a_j, b_j)$ holds in the lexicographic order, the six cases corresponding to the six lines of Table 1 below are a complete and pairwise mutually exclusive list of possibilities. The statement on the first line of Table 1 is obvious, and

Relation between (a_i, b_i) and (a_j, b_j)	$i \rightarrow j$ exactly when
$(a_i, b_i) = (a_j, b_j)$	$x_i > x_j$
$a_i = a_j < b_i < b_j$	$1/x_j > 1/x_i + 1$
$a_i < a_j < b_j < b_i$	$x_i > 1$
$a_i < a_j < b_i = b_j$	$x_i > x_j + 1$
$a_i < a_j < b_i < b_j$	$(1/x_i + 1)(x_j + 1) < 2$
$a_i < b_i \leq a_j < b_j$	never

TABLE 1. Characterization of the dominance relation in terms of the types

$x_i > x_j$ can not happen as (a_i, b_i, x_i) comes before (a_i, b_i, x_j) in the lexicographic order exactly when $x_i < x_j$ holds. The description on the third line is also clear, coin i wins exactly when it displays b_i, otherwise it loses. The last line is also obvious: coin j can not lose in that case and it wins with a positive probability. The remaining lines require just a little more attention.

To prove the statement on the second line, observe first that coin i wins exactly when it displays b_i and coin j displays a_j, and it loses exactly when coin j displays b_j (regardless the outcome of tossing coin i). Thus $i \to j$ exactly when

$$\frac{x_i}{x_i+1} \cdot \frac{1}{x_j+1} > \frac{x_j}{x_j+1}$$

which is equivalent to $1/x_i + 1 < 1/x_j$. The statement on the fourth line is completely analogous, only easier.

We are left to prove the statement on the fifth line. In this case, coin i wins exactly when it displays b_i and coin j displays a_j, and it loses in all other cases (there is never a draw). Therefore coin i dominates coin j if and only if the probability of i winning is more than $1/2$, that is, we have

$$\frac{x_i}{x_i+1} \cdot \frac{1}{x_j+1} > \frac{1}{2}.$$

This is obviously equivalent to the statement on the fifth line.

□

3. Winner coins represent semiacyclic tournaments

In this section we give a complete description of all tournaments that may be realized as the domination graph of a system of coins that has only winner and fair coins. In proving our main result, the following lemma plays an important role.

Lemma 3.1. *Assume that the ith coin, of type (a_i, b_i, x_i) dominates the jth coin, of type (a_j, b_j, x_j) and that the jth coin is not a loser coin. Then we must have $b_i \geq b_j$.*

Proof. Assume by way of contradiction that $b_j > b_i$ holds. The jth coin wins whenever it displays b_j, and this happens with probability $x_j/(1 + x_j) \geq 1/2$. We obtain that $i \to j$ can not hold, a contradiction. □

Theorem 3.2. *Assume a set of n winner and fair coins is listed in increasing lexicographic order of their types. If the domination graph is a tournament, it must be semiacyclic. Conversely every semiacyclic tournament is the domination graph of a set of winner coins.*

Proof. Assume we are given an n-element set of winner and fair coins whose domination graph is a tournament and that the coins are listed in increasing lexicographic order of their types. Consider a cycle $(i_1, i_2, \ldots, i_k)$ in the domination graph. Repeated use of Lemma 3.1 yields $b_{i_1} \geq b_{i_2} \geq \cdots \geq b_{i_k} \geq b_{i_1}$, implying $b_{i_1} = \cdots = b_{i_k}$. As a consequence, using the fourth line in Table 1, we have $x_{i_{j+1}} < x_{i_j} - 1$ whenever $i_j < i_{j+1}$ and $x_{i_{j+1}} < x_{i_j} + 1$ whenever $i_{j+1} < i_j$, for $j = 1, \ldots, k-1$. Similarly, we have $x_{i_1} < x_{i_k} - 1$ whenever $i_k < i_1$ and $x_{i_1} < x_{i_k} + 1$ whenever $i_1 < i_k$. In other words, the last coordinate of the type decreases by more than 1 at each ascent, and it increases by less than 1 at each descent. Therefore, a cycle can not be an ascending cycle, and the tournament must be semiacyclic.

Conversely, consider a semiacyclic tournament T on $\{1, \ldots, n\}$. By Proposition 1.2 this tournament corresponds to a region of the Linial arrangement $\mathcal{L}_{n-1}$ in the following way. For each $i < j$ we have $i \to j$ if $x_i > x_j + 1$ and we have $j \to i$ if $x_i <$

x_j+1, where $(x_1,\ldots,x_n)$ is an arbitrary point in the region. Note that some of the coordinates must be negative or zero here, as we have $x_1+\cdots+x_n=0$. Introducing $r:=\max\{c-x_1,\ldots,c-x_n\}$ for some $c>1$, and setting $x_i'=x_i+r$, we obtain a vector $(x_1',\ldots,x_n')$ whose coordinates are all greater than 1 and satisfy $i\rightarrow j$ if $x_i'>x_j'+1$ and $j\rightarrow i$ if $x_i'<x_j'+1$, for each $i<j$. Consider now the set of n coins where the ith coin has type $(i,n+1,x_i')$. By the fourth line of Table 1, the dominance graph of this set of coins is precisely T. □

The proof of Theorem 3.2 also has the following consequence.

Corollary 3.3. *If a set of coins does not contain loser coins and its dominance graph is a tournament, then this tournament is also the dominance graph of a set of winner coins that all have the same number on one of their sides.*

Analogous results may also be stated for loser and fair coins. In analogy to Lemma 3.1 we can make the following observation.

Lemma 3.4. *Assume that the ith coin, of type (a_i,b_i,x_i) dominates the jth coin, of type (a_j,b_j,x_j) and that the ith coin is not a winner coin. Then we must have $a_i\geq a_j$.*

The proof is completely analogous and omitted. Lemma 3.4 may be used to prove the following result.

Theorem 3.5. *Assume a set of n loser and fair coins is listed in increasing lexicographic order of their types. If the domination graph is a tournament, it must be semiacyclic. Conversely every semiacyclic tournament is the domination graph of a set of loser coins.*

Proof. The proof is analogous to that of Theorem 3.2. Consider first the dominance graph of a set of n loser and fair coins and assume it is a tournament. We may use Lemma 3.4 to show that, in this dominance graph, all vertices contained in a cycle must have the same first coordinate. The role played by the fourth line of Table 1 is taken over by the second line, which may be used to show that the reciprocal of the third coordinate of the type increases by more than 1 after each ascent, and it decreases by less than 1 after each descent. Again we obtain that the dominance graph can not contain an ascending cycle.

Conversely, given a semiacyclic tournament T on the vertex set $\{1,\ldots,n\}$, consider a point $(x_1,\ldots,x_n)$ satisfying $x_1+\cdots+x_n=0$, such that for each $i<j$ we have $i\rightarrow j$ if $x_i>x_j+1$ and we have $j\rightarrow i$ if $x_i<x_j+1$. Let us set $r:=\max\{x_1+c,\ldots,x_n+c\}$ for some $c>1$, and let us define x_i' by $x_i'=1/(r-x_i)$ for $i=1,\ldots,n$. Consider the set of n coins where the ith coin has type $(0,i,x_i')$ for $i=1,\ldots,n$. Since $1/x_i'=r-x_i>1$ holds for all i, all coins are loser coins. Furthermore $1/x_j'>1/x_i'+1$ is equivalent to $r-x_j>r-x_i+1$, that is, $x_i>x_j+1$ for each $i<j$. Hence the dominance graph of this set of coins is T. □

In analogy to Corollary 3.3, the proof of Theorem 3.5 also has the following consequence.

Corollary 3.6. *If a set of coins does not contain winner coins and its dominance graph is a tournament, then this tournament is also the dominance graph of a set of loser coins that all have the same number on one of their sides.*

We conclude this section by an example of a tournament T whose vertices can not be labeled in an order that would make T semiacyclic. As a consequence, T can not be the dominance graph of a set of coins that does not contain winner, as well as loser coins. Our example will use a direct product construction, which we will reuse later, thus we make a separate definition.

Definition 3.7. *Given two tournaments T_1 and T_2 on the vertex sets V_1 and V_2 respectively, we define their* direct product $T_1 \times T_2$ *as follows. Its vertex set is $V_1 \times V_2$ and we set $(u_1, u_2) \to (v_1, v_2)$ if either $u_1 \to v_1$ belongs to T_1 or we have $u_1 = v_1$ and $u_2 \to v_2$ belongs to T_2.*

This direct product operation is not commutative, but it is associative in the following sense: given k tournaments $T_1, \ldots, T_k$ on their respective vertex sets $V_1, \ldots, V_k$, the vertex set of $T_1 \times \cdots \times T_k$ (where parentheses may be inserted in any order) is identifiable with the set of all k-tuples $(v_1, \ldots, v_k)$, where, for each i, v_i belongs to V_i. The edge $(u_1, \ldots, u_k) \to (v_1, \ldots, v_k)$ belongs to $T_1 \times \cdots \times T_k$ exactly when $u_i \to v_i$ belongs to T_i for the least i such that $u_i \neq v_i$.

Proposition 3.8. *Let C_3 denote the 3-cycle. Then for any numbering of the vertex set of $C_3 \times C_3$, the resulting labeled tournament is not semiacyclic.*

Proof. Let us identify the vertices of C_3 with 0, 1 and 2 in the order that $0 \to 1$, $1 \to 2$, $2 \to 0$ belong to C_3. The vertex set of $C_3 \times C_3$ is then the set of ternary strings of length 2. Assume, by way of contradiction, that there is a labeling of these ternary strings that makes $C_3 \times C_3$ a semiacyclic tournament.

For each $u \in \{0, 1, 2\}$ the cyclic permutation $(u0, u1, u2)$ is an automorphism of the tournament $(C_3 \times C_3)$. Hence, without loss of generality, we may assume that in the set $\{u0, u1, u2\}$ the vertex $u0$ has the least label. As a consequence, the edge $u0 \to u1$ is an ascent for each u. The cycle $(00, 01, 10, 11, 20, 21)$ has 6 edges, and contains at least 3 ascents, thus it is an ascending cycle, in contradiction of having a semiacyclic tournament. □

4. General sets of coins

We begin with showing that the tournament $C_3 \times C_3$ introduced in Proposition 3.8 can be represented with a set of coins, if we allow both winner and loser coins.

Example 4.1. We represent the vertices of $C_3 \times C_3$ with the set of coins given in Table 2.

vertex coin (a, b, x)	00 $(3, 6, 1 + \delta)$	01 $(2, 6, 3/2)$	02 $(1, 6, 2 + 2\delta)$
vertex coin (a, b, x)	20 $(5, 12, 1/(s + 1 + 2\varepsilon))$	21 $(5, 11, 1/(s + 1/2)))$	22 $(5, 10, 1/(s + \varepsilon))$
vertex coin (a, b, x)	10 $(4, 9, 1/(r + 1 + 2\varepsilon))$	11 $(4, 8, 1/(r + 1/2))$	12 $(4, 7, 1/(r + \varepsilon))$

Table 2. A coin-representation of the vertices of $C_3 \times C_3$

Here r, s, ε and δ are positive real numbers, whose values we determine below. Initially we only require $r>1$ and $s>1$ which makes all coins loser except for the ones representing $(0,0)$, $(0,1)$ and $(0,2)$.

To assure $00 \rightarrow 01 \rightarrow 02 \rightarrow 00$ we need

$$2+2\delta<3/2+1, \quad 3/2<1+\delta+1, \quad \text{and} \quad 2+2\delta>1+\delta+1.$$

All three are satisfied if and only if we have

$$0<\delta<1/4. \tag{4.1}$$

To assure $20 \rightarrow 21 \rightarrow 22 \rightarrow 20$ we need

$$r+1/2<r+\varepsilon+1, \quad r+1+2\varepsilon<r+1/2+1, \quad \text{and} \quad r+1+2\varepsilon>r+\varepsilon+1.$$

All three are satisfied if and only if we have

$$0<\varepsilon<1/4. \tag{4.2}$$

Note that condition (4.2) is independent of the value of r, and we may replace r with s in the above calculations. Therefore to assure $10 \rightarrow 11 \rightarrow 12 \rightarrow 10$ we also need to make sure (4.2), and this is a sufficient condition.

Note next, that any coin $(4,b_1,x_1)$ representing a vertex of the form $2v_1$ is dominated by any coin $(5,b_2,x_2)$ representing a vertex of the form $1v_2$. Indeed, with probability $1/(1+x_1)>1/2$ the first coin displays 4 and loses.

Next we make sure that any coin $(4,b_1,x_1)$ representing a vertex of the form $2v_1$ dominates any coin $(a_2,6,x_2)$ representing a vertex of the form $0v_2$. We have $a_2<4<6<b_1$ and, using line 5 of Table 1 we get that

$$(1+1/x_2)(1+x_1)>2$$

needs to be satisfied. Substituting the least value o x_1 and the largest value of x_2 we get

$$\left(1+\frac{1}{2+2\delta}\right)\left(1+\frac{1}{r+1+2\varepsilon}\right)>2$$

This inequality is equivalent to

$$r<\frac{2}{1+\delta}-2\varepsilon. \tag{4.3}$$

Finally, we want to make sure that any coin $(5,b_1,x_1)$ representing a vertex of the form $2v_1$ is dominated by any coin $(a_2,6,x_2)$ representing a vertex of the form $0v_2$. We have $a_2<5<6<b_1$ and, using line 5 of Table 1 we get that

$$(1+1/x_2)(1+x_1)<2$$

needs to be satisfied. Substituting the largest value o x_1 and the least value of x_2 we get

$$\left(1+\frac{1}{1+\delta}\right)\left(1+\frac{1}{s+\varepsilon}\right)<2.$$

This inequality is equivalent to

$$s>\frac{2+\delta}{\delta}-\varepsilon. \tag{4.4}$$

The conditions (4.1), (4.2), (4.3) and (4.4) may be simultaneously satisfied, by setting $\delta=0.1$, $\varepsilon=0.1$, $r=1.6$ and $s=22$, for example.

Example 4.1 proves that among the dominance graphs of systems of coins there are some that can not be labeled to become semiacyclic tournaments. On the other hand, Theorems 3.2 and 3.5 imply an important necessary condition for a tournament to be the dominance graph of a system of coins.

Corollary 4.2. *If a tournament T may be represented as the dominance graph of a system of coins, then its vertex set V may be written as a union $V = V_1 \cup V_2$, such that the full subgraphs induced by V_1 and V_2, respectively, may be labeled to become semiacyclic tournaments.*

Indeed, given a system of coins whose dominance graph is T, we may choose V_1 to be the set of winner and fair coins and V_2 to be the set of loser and fair coins (we do not have to include the fair coins on both sets) and then apply Theorems 3.2 and 3.5.

We conclude this section with an example of a tournament that can not be the dominance graph of any system of coins. Our example is $C_3 \times C_3 \times C_3 \times C_3$. The fact this tournament is not the dominance graph of any system of coins, is a direct consequence of Proposition 3.8 and the next theorem.

Theorem 4.3. *Suppose the tournaments T_1 and T_2 have the property that they are not semiacyclic for any ordering of their vertex sets. Then the tournament $T_1 \times T_2$ can not be the dominance graph of any system of coins.*

Proof. Let us denote the vertex set of T_1 and T_2, respectively, by V_1 and V_2, respectively. Assume, by way of contradiction, that $T_1 \times T_2$ is the dominance graph of a system of coins. As noted in Corollary 4.2, we may then write $V_1 \times V_2$ as a union $V_1 \times V_2 = W_1 \cup W_2$ such that the restriction of $T_1 \times T_2$ to either of W_1 or W_2 can be labeled to become a semiacyclic tournament. For a fixed $v_1 \in V_1$, the restriction of $T_1 \times T_2$ to the set $\{(v_1, v_2) : v_2 \in V_2\}$ is isomorphic to T_2. Indeed, the first coordinate is the same for all vertices in the set and identifying each (v_1, v_2) with v_2 yields an isomorphism. Since T_2 can not be ordered to be semiacyclic, we obtain that $\{(v_1, v_2) : v_2 \in V_2\}$ can not be entirely contained in W_2. As a consequence, for each $v_1 \in V_1$ we can pick an $f(v_1) \in V_2$ such that $(v_1, f(v_1))$ belongs to W_1. Consider now the restriction of $T_1 \times T_2$ to the set $\{(v_1, f(v_1)) : v_1 \in V_1\}$. This restriction is isomorphic to T_1, an isomorphism is given by $(v_1, f(v_1)) \mapsto v_1$. Since T_1 can not be labeled to become a semiacyclic tournament, neither can the restriction of $T_1 \times T_2$ to the set W_1 that properly contains $\{(v_1, f(v_1)) : v_1 \in V_1\}$. We obtained a contradiction. □

5. Concluding remarks

The problem of representing tournaments with sets of dice is closely related to representing tournaments by *voting preference patterns*. In the latter setup the n vertices of the tournament correspond to n candidates. There are m voters, and each has a linearly ordered preference list of all candidates. Candidate i defeats candidate j if the majority of voters prefers i over j. Equivalently we may instruct voter i to assign the "score" $(i-1) \cdot n + n + 1 - k$ to the the kth candidate on their preference list. We may then associate an m-sided fair die to each candidate in such a way that each face corresponds to a voter and is marked by the score the voter assigned to the candidate. The dominance graph of this set of dice is identical with the tournament of the voting preference patterns. It has been shown by McGarvey [8] that every tournament on n vertices can be represented as a preference pattern of n candidates and $n(n-1)$ voters. A lower bound of $0.55n/\log(n)$ on the minimally necessary number of voters to be able to represent all tournaments on n vertices was given by Stearns [14]. Erdős and Moser [4] have shown that any tournament on n vertices may be also realized as a preference pattern of $O(n/\log(n))$ voters. As a consequence, any tournament may be represented as the dominance graph of a set of dice with $O(n/\log(n))$ faces. However, not all sets of dice need to arise in connection with voting preference patterns, and Bednay and Bozóki [3] showed that every tournament on n vertices can be realized with a set of dice such that each die has $\lfloor 6n/5 \rfloor$ faces. Our paper shows that 2 faces do not suffice, even if we allow the "two-sided dice" (better known as coins) to be unfair, and even if we allow "ties" between two coins and only require the dominating coin to display the larger number with a greater probability. On the other hand, our research indicates that describing the classes of tournaments that may be represented by sets of dice with a fixed number of sides could be an interesting question.

Theorem 3.2 motivates the question, how to describe those tournaments that can not be made semiacyclic, no matter how we order their vertices. Postnikov and Stanley have given a list of five cycles [12, Theorem 8.6], one of which must appear if the tournament (with numbered vertices) is not semiacyclic. Our question is different here, as we are allowed to choose our own numbering on the vertices.

Conjecture 5.1. *There is a finite list of tournaments that are minimally not semiacyclic in the sense, that removing any of their vertices allows the numbering of the remaining vertices in a semiacyclic way.*

We wish to make the analogous conjecture for dominance graphs of systems of coins.

Conjecture 5.2. *There is a finite list of tournaments that are minimally not dominance graphs of systems of coins in the sense, that after removing any of their vertices we obtain the dominance graph of a system of coins.*

Proving Conjecture 5.2 may be helped by looking at generalizations of hyperplane arrangements to hypersurface arrangements, that allow adding equations of hypersurfaces of the form $1/x_j = 1/x_i + 1$ and of the form $(1/x_i + 1)(x_j + 1) = 2$, see Table 1. Such a study would only allow finding an analogue of semiacyclic tournaments (with numbered vertices), the question would still be open which subgraphs would obstruct numbering the vertices in a way that allows them to be represented in the desired way.

The number of regions in the Linial arrangement is listed as sequence A007889 in the OEIS [10]. These numbers count alternating trees and binary search trees as well. Several ways are known to find these numbers, see [2, 11, 12] neither of which seems to be related to counting semiacyclic tournaments directly. In view of the role played by these tournaments, it may be interesting to find a way to count them directly.

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Department of Mathematics and Statistics, UNC-Charlotte, Charlotte NC 28223-0001. WWW: http://www.math.uncc.edu/~ghetyei/.