THE PASCAL RHOMBUS AND THE GENERALIZED GRAND MOTZKIN PATHS

JOSÉ L. RAMÍREZ

Abstract. In the present article, we find a closed expression for the entries of the Pascal rhombus. Moreover, we show a relation between the entries of the Pascal rhombus and a family of generalized grand Motzkin paths.

1. Introduction

The Pascal rhombus was introduced by Klostermeyer et al. [6] as a variation of the well-known Pascal triangle. It is an infinite array $\mathcal{R} = [r_{i,j}]_{i=0,j=-\infty}^{\infty,\infty}$ defined by

$$r_{i,j} = r_{i-1,j-1} + r_{i-1,j} + r_{i-1,j+1} + r_{i-2,j}, \quad i \geq 2, \quad j \in \mathbb{Z}, \tag{1.1}$$

with the initial conditions

$$r_{0,0} = r_{1,-1} = r_{1,0} = r_{1,1} = 1, \quad r_{0,j} = 0 \ (j \neq 0), \quad r_{1,j} = 0, \ (j \neq -1, 0, 1).$$

The first few rows of $\mathcal{R}$ are

					1					
				1	1	1				
			1	2	4	2	1			
		1	3	8	9	8	3	1		
	1	4	13	22	29	22	13	4	1	
1	5	19	42	72	82	72	42	19	5	1
⋮		⋮		⋮		⋮		⋮		⋮

Table 1. Pascal Rhombus.

Klostermeyer et al. [6] studied several identities of the Pascal rhombus. Goldwasser et al. [4] proved that the limiting ratio of the number of ones to the number of zeros in $\mathcal{R}$, taken modulo 2, approaches zero. This result was generalized by Mosche [7]. Recently, Stockmeyer [9] proved four conjectures about the Pascal rhombus modulo 2 given in [6].

The Pascal rhombus corresponds with the entry A059317 in the On-Line Encyclopedia of Integer Sequences (OEIS) [8], where it is possible to read: *There does not seem to be a simple expression for* $r_{i,j}$.

In the present article, we find an explicit expression for $r_{i,j}$. In particular, we prove that

$$r_{i,j} = \sum_{m=0}^{i} \sum_{l=0}^{i-j-2m} \binom{2m+j}{m} \binom{l+j+2m}{l} \binom{l}{i-j-2m-l}.$$

For this we show that $r_{i,j}$ is equal to the number of 2-generalized grand Motzkin paths.

2. The Main Result

A *Motzkin path* of length n is a lattice path of $\mathbb{Z} \times \mathbb{Z}$ running from $(0,0)$ to $(n,0)$ that never passes below the x-axis and whose permitted steps are the up diagonal step $U=(1,1)$, the down diagonal step $D=(1,-1)$ and the horizontal step $H=(1,0)$, called rise, fall and level step, respectively. The number of Motzkin paths of length n is the n-th *Motzkin number* m_n, (sequence A001006). Many other examples of bijections between Motzkin numbers and others combinatorial objects can be found in [1]. A *grand Motzkin path* of length n is a Motzkin path without the condition that never passes below the x-axis. The number of grand Motzkin paths of length n is the n-th *grand Motzkin number* g_n, sequence A002426. A 2-*generalized Motzkin path* is a Motzkin path with an additional step $H_2=(2,0)$. The number of 2-generalized Motzkin paths of length n is denoted by $m_n^{(2)}$. Analogously, we have 2-*grand generalized Motzkin paths*, and the number of these paths of length n is denoted by $g_n^{(2)}$.

Lemma 2.1. *The generating function of the 2-generalized Motzkin numbers is given by*

$$B(x) := \sum_{i=0}^{\infty} m_i^{(2)} x^i = \frac{1-x-x^2-\sqrt{1-2x-5x^2+2x^3+x^4}}{2x^2} \tag{2.1}$$

$$= \frac{F(x)}{x} C(F(x)^2), \tag{2.2}$$

where $F(x)$ and $C(x)$ are the generating functions of the Fibonacci numbers and Catalan numbers, i.e.,

$$F(x) = \frac{x}{1-x-x^2}, \quad C(x) = \frac{1-\sqrt{1-4x}}{2x}.$$

Proof. From the first return decomposition any nonempty 2-generalized Motzkin path T may be decomposed as either $UT'DT''$, HT', or H_2T', where T', T'' are 2-generalized Motzkin paths (possible empty). Making use of the Flajolet's symbolic method (cf. [3]) we obtain

$$B(x) = 1 + (x+x^2)B(x) + x^2 B(x)^2.$$

Therefore Equation (2.1) follows. Moreover,

$$B(x) = \frac{1-x-x^2-\sqrt{(1-x-x^2)^2-4x^2}}{2x^2} = \frac{1-\sqrt{1-4\left(\frac{x}{1-x-x^2}\right)^2}}{\frac{2x^2}{1-x-x^2}}$$

$$= \frac{1}{1-x-x^2} \frac{1-\sqrt{1-4F(x)^2}}{2F(x)^2} = \frac{F(x)}{x} C(F(x)^2).$$

□

The height of a 2-generalized grand Motzkin path is defined as the final height of the path, i.e., the stopping y-coordinate. The number of 2-generalized grand Motzkin paths of length n and height j is denoted by $g_{n,j}^{(2)}$.

Theorem 2.2. *The generating function of the 2-generalized grand Motzkin paths of height j is*

$$M^{(j)}(x) := \sum_{i=0}^{\infty} g_{i,j}^{(2)} x^i = \frac{F(x)^{j+1} C(F(x)^2)^j}{x(1-2F(x)^2 C(F(x)^2))},$$

where $F(x)$ and $C(x)$ are the generating function of the Fibonacci numbers and Catalan numbers. Moreover,

$$g_{i,j}^{(2)} = \sum_{m=0}^{i} \sum_{l=0}^{i-j-2m} \binom{2m+j}{m}\binom{l+j+2m}{l}\binom{l}{i-j-2m-l}, \quad (0 \le j \le i).$$

Proof. Consider any 2-generalized grand Motzkin path P. Then any nonempty path P may be decomposed as either

$$UMDP', \quad DMUP', \quad HP', \quad H_2P', \quad \text{or} \quad UM_1UM_2\cdots UM_j,$$

where $M, M_1, \ldots, M_j$ are 2-generalized Motzkin paths (possible empty), P' is a 2-generalized grand Motzkin path (possible empty).

Schematically,

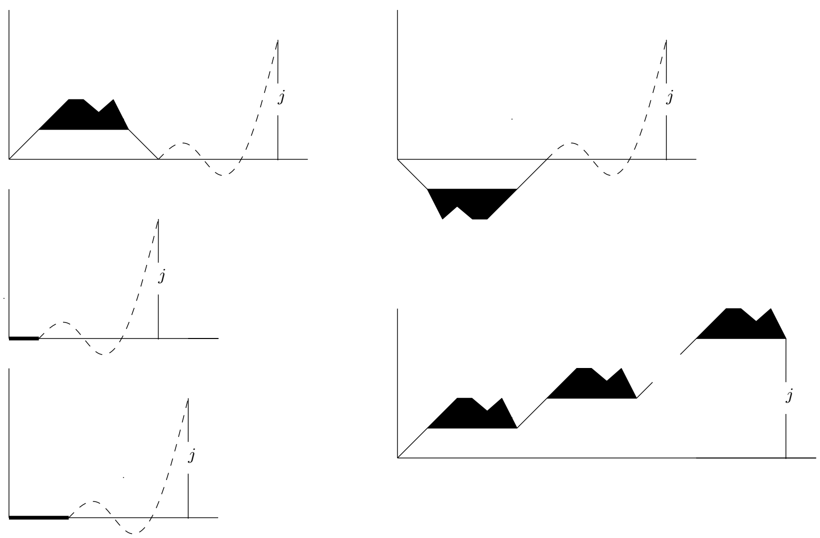

Figure 1. Factorizations of any 2-generalized grand Motzkin path.

From the Flajolet's symbolic method we obtain

$$M^{(j)}(x) = 2x^2B(x)M^{(j)}(x) + (x + x^2)M^{(j)}(x) + x^j(B(x))^j, \quad j \ge 0.$$

Therefore

$$M^{(j)}(x) = \frac{x^jB(x)^j}{1 - x - x^2 - 2x^2B(x)}.$$

From Lemma 2.1 we get

$$M^{(j)}(x) = \frac{x^j\left(\frac{F(x)}{x}C(F(x)^2)\right)^j}{1 - x - x^2 - 2x^2\frac{F(x)}{x}C(F(x)^2)} = \frac{F(x)^{j+1}C(F(x)^2)^j}{x(1 - 2F(x)^2C(F(x)^2))}.$$

On the other hand, from the following identity (see Ec. 2.5.15 of [10])

$$\frac{1}{\sqrt{1-4x}}\left(\frac{1-\sqrt{1-4x}}{x}\right)^k = \sum_{m=0}^{\infty}\binom{2m+k}{m}x^m$$

we obtain

$$\frac{C(x^2)^j}{1-2x^2C(x^2)} = \sum_{m=0}^{\infty}\binom{2m+j}{m}x^{2m}$$

Therefore

$$\begin{aligned}
M^{(j)}(x) &= \frac{F(x)^{j+1}(x)}{x}\sum_{m=0}^{\infty}\binom{2m+j}{m}F(x)^{2m} = \frac{1}{1-x-x^2}\sum_{m=0}^{\infty}\binom{2m+j}{m}F(x)^{2m+j}\\
&= \sum_{m=0}^{\infty}\binom{2m+j}{m}\frac{x^{2m+j}}{(1-x-x^2)^{2m+j+1}} = \sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\binom{2m+j}{m}\binom{l+j+2m}{l}(1+x)^l x^{2m+j+l}\\
&= \sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\sum_{s=0}^{l}\binom{2m+j}{m}\binom{l+j+2m}{l}\binom{l}{s}x^{2m+j+l+s},
\end{aligned}$$

Put $t = 2m+j+l+s$

$$M^{(j)}(x) = \sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\sum_{t=2m+j+l}^{2m+j+2l}\binom{2m+j}{m}\binom{l+j+2m}{l}\binom{l}{t-2m-j-l}x^t.$$

The result follows by comparing the coefficients. □

Theorem 2.3. *The number of 2-generalized grand Motzkin paths of length n and height j is equal to the entry (n,j) in the Pascal rhombus, i.e.,*

$$r_{n,j} = g_{n,j}^{(2)}.$$

Proof. The sequence $g_{n,j}^{(2)}$ satisfies the recurrence (1.1) and the same initial values. It is clear, by considering the positions preceding to the last step of any 2-generalized grand Motzkin path. □

Corollary 2.4. *The generating function of the jth column of the Pascal rhombus is*

$$L_j(x) = \frac{F(x)^{j+1}C(F(x)^2)^j}{x(1-2F(x)^2C(F(x)^2))},$$

where $F(x)$ and $C(x)$ are the generating function of the Fibonacci numbers and Catalan numbers. Moreover,

$$r_{i,j} = \sum_{m=0}^{i}\sum_{l=0}^{i-j-2m}\binom{2m+j}{m}\binom{l+j+2m}{l}\binom{l}{i-j-2m-l} \quad (0 \le j \le i).$$

The convolved Fibonacci numbers $F_j^{(r)}$ are defined by

$$(1-x-x^2)^{-r} = \sum_{j=0}^{\infty}F_{j+1}^{(r)}x^j, \quad r \in \mathbb{Z}^+.$$

If $r = 1$ we have the classical Fibonacci sequence.

Note that

$$F_{m+1}^{(r)} = \sum_{j_1+j_2+\cdots+j_r=m}F_{j_1+1}F_{j_2+1}\cdots F_{j_r+1}.$$

Moreover, using a result of Gould [5, p. 699] on Humbert polynomials (with $n = j, m = 2, x = 1/2, y = -1, p = -r$ and $C = 1$), we have

$$F_{j+1}^{(r)} = \sum_{l=0}^{\lfloor j/2 \rfloor} \binom{j+r-l-1}{j-l}\binom{j-l}{l}.$$

Corollary 2.5. *The following equality holds*

$$r_{i,j} = \sum_{m=0}^{\lfloor \frac{i-j}{2} \rfloor} \binom{2m+j}{m} F_{i-j-2m+1}^{(j+2m+1)}, \tag{2.3}$$

where $F_l^{(r)}$ are the convolved Fibonacci numbers.

Proof.

$$L_n(x) = \sum_{m=0}^{\infty} \binom{2m+n}{m} \frac{x^{2m+n}}{(1-x-x^2)^{n+2m+1}} = \sum_{m=0}^{\infty}\sum_{j=0}^{\infty} \binom{2m+n}{m} F_{j+1}^{(n+2m+1)} x^{2m+n+j},$$

Put $t = 2m + n + j$

$$L_n(x) = \sum_{m=0}^{\infty}\sum_{t=2m+n}^{\infty} \binom{2m+n}{m} F_{t-2m-n+1}^{(n+2m+1)} x^t.$$

The result follows by comparing the coefficients. □

Example 2.6. *The generating function of the central column of the Pascal rhombus (sequence A059345) is*

$$L_0(x) = \frac{1}{\sqrt{1-2x-5x^2+2x^3+x^4}} = 1 + x + 4x^2 + 9x^3 + 29x^4 + 82x^5 + 255x^6 + \cdots .$$

The generating function of the first few columns ($j = 1, 2, 3$) of the Pascal rhombus are:

$$L_1(x) = x + 2x^2 + 8x^3 + 22x^4 + 72x^5 + 218x^6 + 691x^7 + 2158x^8 + \cdots, \quad \textit{(A106053)}$$

$$L_2(x) = x^2 + 3x^3 + 13x^4 + 42x^5 + 146x^6 + 476x^7 + 1574x^8 + \cdots, \quad \textit{(A106050)}$$

$$L_3(x) = x^3 + 4x^4 + 19x^5 + 70x^6 + 261x^7 + 914x^8 + 3177x^9 + \cdots$$

Remark: The results of this article were discovered by using the Counting Automata Methodology, [2].

3. Acknowledgements

The author thanks the anonymous referee for his/her comments and remarks, which helped to improve the article.

References

[1] F. Bernhart, *Catalan, Motzkin, and Riordan numbers*, Discrete Math. **204**(1999), 73–112.
[2] R. De Castro, A. Ramrez and J. Ramrez, *Applications in enumerative combinatorics of infinite weighted automata and graphs*, Sci. Ann. Comput. Sci., **24.1** (2014), 137–171.
[3] P. Flajolet and R. Sedgewick, *Analytic Combinatorics*, Camdridge, 2009.
[4] J. Goldwasser, W. F. Klostermeyer, M. E. Mays and G. Trapp, *The density of ones in Pascal's rhombus*, Discrete Math. **204** (1999), 231–236.
[5] H. W. Gould, *Inverse series relations and other expansions involving Humbert polynomials*, Duke Math. J. **32.4** (1965), 697-711.

[6] W. F. Klostermeyer, M. E. Mays, L. Soltes and G. Trapp, *A Pascal rhombus*. The Fibonacci Quarterly, **35** (1997), 318–328.
[7] Y. Moshe. *The density of* 0*'s in recurrence double sequences*. J. Number Theory, **103** (2003), 109–121.
[8] OEIS Foundation Inc. (2011), The On-Line Encyclopedia of Integer Sequences, `http://oeis.org`.
[9] P. K. Stockmeyer, *The Pascal rhombus and the stealth configuration*, `http://arxiv.org/abs/1504.04404`, (2015).
[10] H. S. Wilf, *generatingfunctionology*, Academic Press, Second Edition, 1994.

MSC2010: 05A19, 11B39, 11B37.

Departamento de Matemáticas, Universidad Sergio Arboleda, Bogotá, Colombia
E-mail address: `josel.ramirez@ima.usergioarboleda.edu.co`