Sizes of the extremal girth five graphs of orders from 40 to 49.

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Abstract

The Turán type numbers for graphs without 3-cycles and 4-cycles are determined for vertex numbers from 40 to 49 inclusive. Hence, now, 43 of the first 50 numbers of OEIS **A006856** are known. Estimates for the remaining seven numbers are presented.

Keywords: Turán type number; girth; extremal graph; Hoffman-Singleton graph.

1 Introduction.

In this note, all considered graphs G are simple and undirected, with orders (numbers of vertices) $n(G)$ and sizes (numbers of edges) $e(G)$, respectively. The main object is the extremal graphs of girth at least five, and in particular their sizes. Thus, we consider the Turán type numbers

$$T(C_{\leqslant 4}; n) = \max(e \ : \ \exists \text{ graph } G \text{ of order } n, \text{ size } e, \text{ and girth } \geqslant 5).$$

They form item **A006856** in *The on-line encyclopedia of integer sequences* ([1]), and for $n \leqslant 32$ were determined by Garnick, Kwong, Lazebnik, Nieuwejaar, McKay, Codish, Miller, Prosser, and Stuckey ([5], [6], [3]).

Moreover, Dutton and Brigham ([4]), and independently the authors of [6], also found a "theoretical upper bound"

$$e(G) \leqslant 0.5n(G)\sqrt{n(G)-1}, \tag{1}$$

and the latter authors also noted that this bound is attained for the rather symmetric Hoffman-Singleton graph HS ([8]); whence they deduced that

$$T(C_{\leqslant 4}; 50) = e(HS) = 175.$$

Now, in my experience, the series of extremal graphs of increasing orders for similar kinds of conditions often contain unique and highly symmetric graphs for some orders; and, when they do, all the extremal graphs of the nearest preceeding orders usually are induced subgraphs of these symmetric graphs. Thus, I suspected the same to be true in this case.

In this note, I prove this, but probably not to the fullest possible extent. The two first theorems do imply that for orders $40 \leqslant n(G) \leqslant 49$ there are extremal graphs which are subgraphs of HS; but I have succeeded to prove that there also are no other extremal graphs only for $n \in \{40, 45, 47, 48, 49\}$. For these n, the extremal graphs also are unique (up to isomorphisms). Unicity does not hold for all the remaining 5 values, but I find it likely that also in these cases all the extremal graphs are subgraphs of Hoffman-Singleton graphs. Moreover, for several lower values of n, the corresponding questions are open.

Recall that the vertex set $V(HS)$ of HS may be partitioned into two parts V' and V'', such that the induced HS subgraph $HS[V^{(\nu)}]$ on either part is isomorphic to the disjoint union $5C_5$ of five 5-cycles, and that the induced subgraphs on the unions of one 5-cycle from each part are Petersen graphs. If we choose one 5-cycle from V'', and successively remove its vertices from HS, the resulting induced HS subgraphs realise the lower bounds

$$T(C_{\leqslant 4}; 45) \geqslant 145, \text{ and } T(C_{\leqslant 4}; n) \geqslant 6n - 126 \text{ for } n = 46, \dots, 49. \tag{2}$$

(All but the first one of these lower bounds also were found by Garnick, Kwong, and Lazebnik in [6].)

Note, that when the entire 5-cycle is removed, the resulting graph G has minimal degree $\delta(G) = 6$, maximal degree $\Delta(G) = 7$, induced graph $G_6 = G[V_6] \simeq 5C_5$, and induced graph $G_7 = G[V_7] \simeq 4C_5$, where $V_i = V_i(G) := \{v \in V(G) : \deg v = i\}$. If we repeat the procedure, by choosing one of the 5-cycles in G_6, and successively removing its vertices, we get further HS subgraphs, yielding

$$T(C_{\leqslant 4}; 40) \geqslant 120, \text{ and } T(C_{\leqslant 4}; n) \geqslant 5n - 81 \text{ for } n = 41, \dots, 44. \tag{3}$$

The main object of this article is to prove equalities in (2) and (3), i.e., to prove

Theorem 1. *$T(C_{\leqslant 4}; 45) = 145$, and $T(C_{\leqslant 4}; n) = 6n - 126$ for $n = 46, \dots, 49$.*

and

Theorem 2. *$T(C_{\leqslant 4}; 40) = 120$, and $T(C_{\leqslant 4}; n) = 5n - 81$ for $n = 41, \dots, 44$.*

The main technical tools for proving the theorems are the following main lemmata:

Lemma 1.1. *If G is a graph with $\text{girth}(G) \geqslant 5$, $n(G) = 45$, and $e(G) = 145$, then $\delta(G) = 6$, $\Delta(G) = 7$, and each vertex has exactly two neighbours with the same degree as itself.*

and

Lemma 1.2. *If G is a graph with $\text{girth}(G) \geqslant 5$, $n(G) = 40$, and $e(G) = 120$, then G is 6-regular (and hence a cage).*

A more precise structural result is

Theorem 3. *For any fixed order* $n \in \{40, 45, 47, 48, 49\}$*, the girth* $\geqslant 5$ *graphs of maximal size are isomorphic, and subgraps of Hoffman-Singleton graphs.*

The case $n = 40$ follows directly from lemma 1.2, and from P.-K. Wong's precise characterisation of (6,5) cages as 'Hoffman-Singleton minus Petersen' graphs in [10]. For the higher n values, the claims are consequences of lemma 1.1 and theorem 1, as seen in section 5.

Remark 4. The unicity part of theorem 3 may be reformulated as

$$a(40) = a(45) = a(47) = a(48) = a(49) = 1,$$

where a is the function listed in the OEIS item **A159847** ([2]).

Remark 5. The order in which the results are listed probably appears to be counter-intuitive. However, I found the analysis of $T(C_{\leqslant 4}; n)$ easier for values closer to 50, than for those a bit lower. Moreover, both the auxiliary lemma 1.1 and theorem 1 are proved by "intrinsic" means, which do not depend on lower $T(C_{\leqslant 4}; n)$ values or bounds, and nor on structure results for small extremal girth 5 graphs.

In fact, instead of investigating the rather large number of potentially possible degree sequences one by one, I have as far as I was able 'linearised' the influence of differences in degree sequences. In an intermediate step, we shall work with some 'virtual degree sequences', where actually some entries may be negative. This enables a reduction of the main part of the proof to a kind of 'elementary calculus', rather than a division into a cumbersome number of cases. In particular, the proof presented here does not in any manner depend on computer calculations.

On the other hand, I have found no independent way to prove the corresponding auxiliary result lemma 1.2, but instead partly had to resort to the traditional 'recursive' use of established values of and upper bounds for $T(C_{\leqslant 4}; n)$ for $n < 40$, and to case divisions. The best bounds (to my knowledge) for 'intermediate' n (i.e., for $n = 33, \ldots, 39$) are given in table 3; they contain a few improvements. However, it should be noted that such bounds to some extent are based on exact values which seem to be announced but not proven in the literature.

Notwithstanding, theorem 2 does not depend on theorem 1; but the former might be used to abbreviate the proof of a minor part lemma 1.1, by proving (8) faster.

2 Auxiliary results.

For any graph G and vertex $v \in V(G)$, let $S(v) = \{w \in V(G) : v \text{-} w \in E(G)\}$ (the 'unit sphere' centred at v, or the open neighbourhood of v), $B(v) = B(v; 1) = S(v) \cup \{v\}$ (the 'unit ball' centred at v, or the closed neighbourhood of v), $B(v; 2) = \bigcup_{w \in S(v)} B(w)$ (the 'radius two ball' centred at v), and

$$\deg^2(v) = \deg^2_G(v) = \sum_{w \in S(v)} \deg(w)$$

(the ‘second degree’ of v).

For any $W_1, W_2, W_3 \subseteq V$, let $p_3(W_1, W_2, W_3)$ be the number of ordered paths of order 3 (and thus size 2), which have the i'th vertex in W_i for $i = 1, 2, 3$. Our main application of this counter will be $p(U, W) := 0.5p_3(W, U, W)$, the number of (unordered) 3-vertex paths with middle vertex in U and end vertices in W (or, equivalently, with middle vertex in U and both edges in $E(U, W)$).

If G is a graph of girth $\geqslant 5$, and $v \in V(G)$, then any two different paths of size two and starting from v will have different end points. This simple observation is sufficient for deducing the following fairly well-known lemmata.

Lemma 2.1. *If G is a graph with* $\operatorname{girth}(G) \geqslant 5$ *and* $v \in V(G)$*, then* $|B(v; 2)| = \deg^2(v) + 1$*. In particular, thus,* $\deg^2(v) \leqslant n(G) - 1$. □

Lemma 2.2. *If G is a graph with* $\operatorname{girth}(G) \geqslant 5$*,* $v \in V(G)$*,* $l \in \mathbf{N}$*, and* $w_1, \ldots, w_l$ *are distinct* $S(v)$ *elements, then* $e(G - \{v, w_1, \ldots, w_l\}) = e(G) - \deg(v) - \sum_{i=1}^{l} \deg(w_i) + l$*. In particular, then* $e(G) \leqslant T(C_{\leqslant 4}; n - l - 1) + \deg(v) + \sum_{i=1}^{l} \deg(w_i) - l$. □

Lemma 2.3. *If G is a graph with* $\operatorname{girth}(G) \geqslant 5$ *and* $v \in V(G)$*, then* $e(G - B(v)) = e(G) - \deg^2(v)$*. In particular,* $e(G) \leqslant T(C_{\leqslant 4}; n - \Delta(G) - 1) + n(G) - 1$. □

Lemma 2.4. *If G is a graph with* $\operatorname{girth}(G) \geqslant 5$ *and P is a path of order r (and thus size $r - 1$) in G, then* $\sum_{v \in V(P)} \deg(v) \geqslant e(G) - T(C_{\leqslant 4}; n - r) + r - 1$. □

By counting 3-vertex walks either by their first or by their middle vertex, we get

Lemma 2.5. $\sum_{v \in W} \deg^2_{G[W]}(v) = p_3(W, W, W) - 2e(W) = \sum_{v \in W} \deg_{G[W]}(v)^2$*, for any* $W \subseteq V$. □

If P_3 is a 3-vertex path with vertex set $\{p_1, p_2, p_3\}$, of which $p_1, p_3 \in W$, then $\{p_1, p_3\}$ is one of the $\binom{|W|}{2}$ 2-sets of vertices in W; however, not one of the $e(W)$ edges, and nor two common neighbours of a third element in V. Thus, we have

Lemma 2.6. *If G is a graph with* $\operatorname{girth}(G) \geqslant 5$ *and* $U, W \subseteq V$*, then*

$$p(U, W) \leqslant p(V, W) \leqslant \binom{|W|}{2} - e(G[W]).$$ □

Lemma 2.7. *If G is a graph with* $\operatorname{girth}(G) \geqslant 5$*,* $U, W \subseteq V$*, and U is the disjoint union of U' and U'', then* $p(U, W) = p(U', W) + p(U'', W)$. □

We mainly shall apply $p(U, W)$ analysis in situations where moreover U and W are disjoint. In these cases, $p(U, W) \leqslant \binom{|W|}{2} - e(W) - p(W, W)$ by lemma 2.7; whence $p(W, W)$ estimates provide upper bounds of $p(U, W)$. For a lower bound, given $|U| = m$ and $|E(U, W)| = |\{\text{edges between } U \text{ and } W\}| = z$ (say), note, that $p(U, W)$ is minimal, if the $|W \cap S(u)|$ are distributed as evenly as possible, when u runs through U. Thus, if in addition $rm \leqslant z \leqslant (r+1)m$ for an integer q, then

$$p(U, W) \geqslant ((q+1)m - z)\binom{q}{2} + (z - qm)\binom{q+1}{2},$$

since the RHS (right hand side) equals $p(U, W)$, if $|W \cap S(u)| \in \{r, r+1\}$ for each u in U. Actually, a little reflection should convince the reader that this inequality also holds, if $z < qm$ or $z > (q+1)m$, although then one of the RHS terms is negative. Hence, without further restrictions, the following lemma holds:

Lemma 2.8. *If G is a graph with* $\mathrm{girth}(G) \geqslant 5$*, V' and V'' are disjoint subsets of $V(G)$, and r and s are natural numbers, then*

$$\begin{array}{ll} & ((r+1)|U| - |E(U,W)|)\binom{r}{2} + (|E(U,W)| - r|U|)\binom{r+1}{2} \\ \leqslant & p(U,W) \\ \leqslant & \binom{|W|}{2} - e(G[W]) - ((s+1)|W| - 2e(G[W]))\binom{s}{2} - (2e(G[W]) - s|W|)\binom{s+1}{2} \end{array} \qquad \square$$

3 Proof of the first main lemma.

For each potentially possible vertex degree i, let $V_i = \{v \in V \mid \deg v = i\}$, and let $n_i = |V_i|$ (the cardinality of V_i). The degree sequence is $\mathbf{s} = \mathbf{s}(G) = (n_0, n_1, \ldots, n_{44})$, which may be considered as an element in $\mathbf{R}^{45}$. (In practice, of course, most entries in $\mathbf{s}$ are zero.)

As a vector, $\mathbf{s}$ satisfies the linear restrictions $\sum_i n_i = 45$ and $\sum_i in_i = 2e = 290$, or, in other words,

$$\mathbf{s} \cdot \mathbf{u} = 45 \text{ and } \mathbf{s} \cdot \mathbf{w} = 290, \tag{4}$$

where $\mathbf{u} = (1, 1, \ldots, 1)$ and $\mathbf{w} = (0, 1, 2, \ldots, 44)$. Let $\mathcal{V} = \{\mathbf{t} \in \mathbf{R}^{45} | \mathbf{t} \cdot \mathbf{u} = 45$ and $\mathbf{t} \cdot \mathbf{w} = 290\}$, the set of *virtual degree sequences*, and let $\mathcal{D} = \{\mathbf{t} \in \mathbf{R}^{45} | \mathbf{t} \cdot \mathbf{u} = \mathbf{t} \cdot \mathbf{w} = 0\}$, the linear $\mathbf{R}^{45}$ subspace of *(degree sequence) deviations*. Clearly, $\mathbf{t}, \mathbf{t}' \in \mathcal{V} \implies \mathbf{t} - \mathbf{t}' \in \mathcal{D}$, and more precisely

$$\mathcal{V} = \overline{\mathbf{s}} + \mathcal{D}, \tag{5}$$

where $\overline{\mathbf{s}} = (\cdots 0, 25, 20, 0 \cdots)$ is the 'globally most even degree distribution', satisfying $\mathbf{s}(G) = \overline{\mathbf{s}} \iff \delta(G) = 6 \wedge \Delta(G) = 7$.

Put $V' := V_{\leqslant 6} := \{v' \in V : \deg v \leqslant 6\} = \bigcup_{i \leqslant 6} V_i$, $V'' := V_{\geqslant 7} := \{v' \in V : \deg v \geqslant 7\} = \bigcup_{i \geqslant 7} V_i$, $G'' = G[V'']$, $n'' = |V''| = n(G'')$, $n' = |V'| = 45 - n''$, $z = |E(V', V'')|$, $e'' = e(G'')$, $e' = e(G')$, $a = \sum_{v \in V'} 6 - \deg v = \sum_{i \leqslant 5} (6-i) n_i$, and $b = \sum_{v \in V''} \deg(v) - 7 = \sum_{i \geqslant 8} (i-7) n_i$, and for each $v \in V$, let $\deg''(v)$ be the number of v neighbours in V''.

The idea is to give a lower estimate p_l and an upper estimate p_u for the quantity $p := p(V', V'')$ (whence necessarily $p_l \leqslant p \leqslant p_u$), and to show that, on the other hand, always $p_l \geqslant p_u$, with equality only for the prescribed form of G. For our first estimates, we let p_l and p_u be the lower and the upper p bounds in lemma 2.8, with $r = 4$ and $s = 2$, respectively. This yields

$$p_l = 4z - 10n' = 6n' + 4(z - 4n'),$$

and

$$p_u = \binom{n''}{2} - 14.5n'' - 2.5b + 2.5z,$$

where for the second equality we also use the fact that

$$e'' = \left(\sum_{v \in V''} \deg(v) - |E(V', V'')| \right) /2 = 0.5n'' + 0.5b - 0.5z\,.$$

We thus indeed have $p_l \leqslant p \leqslant p_u$, where the second inequality is an equality if and only if $2 \leqslant \deg''(v) \leqslant 3$ for each $v \in V''$.

For $i \neq 6, 7$, we define the elementary deviation $\mathbf{d}_i = (d_{i,0}, d_{i,1}, \ldots, d_{i,44}) \in \mathcal{D}$ in the following manner:
$\mathbf{d}_5 = (0, 0, 0, 0, 0, 1, -2, 1, 0 \cdots)$ and $\mathbf{d}_8 = (0, 0, 0, 0, 0, 0, 1, -2, 1, 0 \cdots)$.
For $0 \leqslant i \leqslant 4$, let $d_{i,i} = 1$, $d_{i,5} = -(6 - i)$, $d_{i,6} = 5 - i$, and all other $d_{i,j} = 0$. Similarly, for $9 \leqslant i \leqslant 44$, let the non-zero entries of $\mathbf{d}_i$ be $d_{i,7} = i - 8$, $d_{i,8} = -(i - 7)$, and $d_{i,i} = 1$. Then, elementary vector calculation shows that

$$\mathbf{s}(G) - \bar{\mathbf{s}} = \sum_{i=0}^{4} n_i \mathbf{d}_i + a\mathbf{d}_5 + b\mathbf{d}_8 + \sum_{i=9}^{44} n_i \mathbf{d}_i \in \mathcal{D}. \tag{6}$$

Note, that all $\mathbf{d}_*$ coefficients in the right hand side are non-negative.

Let $\tilde{\mathbf{s}}(G) = (0, 0, 0, 0, 0, a, n' - 2a + b, n'' + a - 2b, b, 0 \cdots) = \bar{\mathbf{s}} + a\mathbf{d}_5 + b\mathbf{d}_8 \in \mathcal{V}$, the *trunkated virtual degree sequence of* G, and note that $\mathbf{s}(G) = \tilde{\mathbf{s}}(G)$ if and only if $5 \leqslant \delta(G)$ and $\Delta(G) \leqslant 8$.

For any $\mathbf{t} = (t_0, \ldots, t_{44}) \in \mathbf{R}^{45}$, we may put $n'(\mathbf{t}) = \sum_{j=0}^{6} t_j$, $n''(\mathbf{t}) = \sum_{j=7}^{44} t_j$, $a(\mathbf{t}) = \sum_{j=0}^{5} (6-j) t_j$, and $b(\mathbf{t}) = \sum_{j=8}^{44} (j-7) t_j$, making $n'(G) = n'(\mathbf{s}(G))$, *et cetera*. Note, that these four functions are linear, and have $n'(\bar{\mathbf{s}}) = 25$, $n''(\bar{\mathbf{s}}) = 20$, and $a(\bar{\mathbf{s}}) = b(\bar{\mathbf{s}}) = 0$. Moreover, $n'(\mathbf{d}_5) = -1$, $n''(\mathbf{d}_5) = 1$, $a(\mathbf{d}_5) = 1$, and $b(\mathbf{d}_5) = 0$; $n'(\mathbf{d}_8) = 1$, $n''(\mathbf{d}_8) = -1$, $a(\mathbf{d}_8) = 0$, and $b(\mathbf{d}_8) = 1$; and $n'(\mathbf{d}_i) = n''(\mathbf{d}_i) = a(\mathbf{d}_i) = b(\mathbf{d}_i) = 0$ for all other i.

In particular, $n''(\mathbf{s} - \bar{\mathbf{s}}) + b(\mathbf{s} - \bar{\mathbf{s}}) - a(\mathbf{s} - \bar{\mathbf{s}}) = 0$ for each $\mathbf{s} - \bar{\mathbf{s}} \in \mathcal{D}$, whence

$$n''(\mathbf{s}) + b(\mathbf{s}) - a(\mathbf{s}) = n''(\bar{\mathbf{s}}) + b(\bar{\mathbf{s}}) - a(\bar{\mathbf{s}}) = 20 + 0 + 0 = 20 \text{ for each } \mathbf{s} \in \mathcal{V}. \tag{7}$$

Finally, also put $f = f(G) := 20 - n'' = b - a$.

We now collect a few preliminary results, under these assumptions and with this notation.

Lemma 3.1. *$z \geqslant 100 + 2b$.*

Proof. For each $v \in V''$, since $\deg^2(v) \leqslant 44$ by lemma 2.1, there are at least $7 \deg v - 44$ edges $v \text{-} w$ between v and V', "counted with multiplicity", where the multiplicity of such a $v \text{-} w$ is $7 - \deg w$. Thus and by (7), indeed

$$z \geqslant 5n'' + 7b - \sum_{i=0}^{6} (6 - i) i n_i \geqslant 5n'' + 7b - 5a = 5(n'' + b - a) + 2b = 100 + 2b.$$

□

Since on the other hand $z \leqslant 6n' - a \leqslant 6n'$, we directly get

Corollary 6. $n' \geqslant \lceil \frac{100}{6} \rceil = 17 \implies n'' \leqslant 28$. $\square$

Note, that

$$p_u - p_l = \binom{n''}{2} - 14.5n'' - 2.5b + 2.5z - (4z - 10n') = 450 + \binom{n''}{2} - 24.5n'' - 2.5b - 1.5z\,.$$

For any fixed $\tilde{\mathbf{s}}$, this is a decreasing function of z, whence and by lemma 3.1 we get

$$p_u - p_l \leqslant 450 + \binom{n''}{2} - 24.5n'' - 2.5b - 150 - 3b = 0.5 \cdot ((n'' - 20)(n'' - 30) - 11b);$$

and thus have deduced

Lemma 3.2. $11b \leqslant (n'' - 20)(n'' - 30) = f^2 + 10f$. $\square$

By lemma 2.1 $\sum_{w \in V' \cap S(v)} (6 - \deg(w)) \geqslant 6\deg(v) - 44$ for any $v \in V(G)$; whence we have

Lemma 3.3. $a \geqslant 6\Delta - 44$. $\square$

Thus, and by direct counting and (7),

Lemma 3.4. $\Delta \geqslant 7 + \lceil \frac{b}{n''} \rceil = 7 + \left\lceil \frac{(20-n'')+a}{n''} \right\rceil \geqslant \frac{140-6f+a}{20-f} \geqslant 6 + \frac{6\Delta-24}{20-f}$. $\square$

We now may prove lemma 1.1. First, by corollary 6, $f \geqslant -8 > -10$, but $0 \leqslant f(f+10)$ by lemma 3.2. Hence

$$f \geqslant 0, \tag{8}$$

and if $f = 0$, then we indeed must have "equalities everywhere", and may deduce the conclusions of the lemma.

Thus, when we, for a while and for a contradiction, assume that G satisfies the prerequisites but not all the conclusions in lemma 1.1, then $1 \leqslant f = 20 - n'' \leqslant 19$, and

$$b = f + a \geqslant 1 + 0 = 1\,. \tag{9}$$

Thus $\Delta \geqslant 8$ by lemma 3.4, and moreover we may eliminate a and b from the inequalities in lemmata 3.2 and L:a, and get

$$\frac{1 + \sqrt{264\Delta - 1935}}{2} \leqslant f,$$

which together with lemma 3.4 (and the bounds we already have deduced for f and Δ) yields

$$11\Delta^3 - 243\Delta^2 + 1728\Delta - 4344 \leqslant 0,$$

and thus that $\Delta = 8$. However, then $f \geqslant 0.5\lceil 1 + \sqrt{177} \rceil = 8$ $a \geqslant 6 \cdot 8 - 44 = 4$, $f \geqslant 8 + 4 = 12$, and $\frac{b}{20-f} \leqslant 1$, whence we must have equalities 'everywhere'. Thus, in particular, $V' = V_8$ and $n_8 = f = n'' = 12$, $V_{\leqslant 5} \neq \emptyset$, and $\deg''(v) = 12 > 5 \geqslant \deg(v)$ for each $v \in V_{\leqslant 5}$, the sought contradiction. $\square$

4 Proof of theorem 1.

The lower bounds are proven in the introduction.

Now, assume for a contradiction that n were minimal in $\{45, 46, 47, 48, 49\}$ with the property that $T(C_{\leqslant 4}; n)$ were strictly larger than the lower bound given in (2). Choose a G with $\mathrm{girth}(G) \geqslant 5$, $n(G) = n$, and $e(G)$ exactly one more than that bound. Since then $e(G) < 3.5n$, there is a vertex $v \in V(G)$ with $\deg v = \delta(G) \leqslant 6$. Now, if $\delta(G) > 0$ then let w be a neighbour of v and put $u := v$; else, choose any edge $u \text{-} w$ in $E(G)$. Put $G' := G - u \text{-} v$, and note, that therein v has degree $\delta(G') \leqslant 5$.

However, if $n = 45$, then $\delta(G') = 6$ by lemma 1.1. Thus, instead, $n \geqslant 46$. By the minimality of n, the conclusions of the theorem would hold for $G'' := G - v$. Since on the other hand $e(G'') = e(G) - \deg v \geqslant e(G) - 6$, we must have $n = 46$, $e(G) = 151$, and G'' must satisfy the assumptions and thus the conclusions of lemma 1.1.

In particular, G'' would contain 25 vertices of degree 6, and each one of these would have two neighbours of the same degree. Pick any such vertex v' and neighbour w'', such that neither v' nor w'' were adjacent to v. Then $v' \text{-} w'$ were an edge between two vertices of degree 6 in G; whence $G - v'$ were a graph with $\mathrm{girth}(G - v') \geqslant 5$, $n(G - v') = 45$, and $e(G - v') = 145$, but with $\delta(G - v') = 5$, in contradiction to lemma 1.1. □

5 Unicity.

We now study the more precise structures of extremal graphs realising the Turán type numbers we just have established. First, again consider a G with $\mathrm{girth}(G) \geqslant 5$, $n(G) = 45$, and $e(G) = 145$.

By lemma 1.1, both G_6 and G_7 are 2-regular, and thus consist of disjoint unions of cycles. We start by proving that each one of these cycles has length 5; i.e., that $G_7 \simeq 4C_5$ and $G_6 \simeq 5C_5$; and continue by determining $E(V_6, V_7)$, and the sets of vertices with mutual distances 3.

Let C_ℓ be a G_7 component, and, for a contradiction, assume $\ell \neq 5$, whence $\ell \geqslant 6$. Put $\{c_1, \ldots, c_\ell\} := V(C_\ell)$, $X_i := V_6 \cap S(c_i)$, and $Y(v) := V(C_\ell) \cap S(v)$, for $1 \leqslant i \leqslant \ell$ and $v \in V_6$. For $\ell = 6$ (or $\ell = 7$), each $|Y(v)| \leqslant 2$, with equality in at most 3 (7) cases, causing

$$25 = n_6 \geqslant 5\ell - 3 \ (5\ell - 7) = 27 \ (28, \quad \text{respectively}),$$

in either case a contradiction; whence instead $\ell \geqslant 8$. Now consider $(X_{i+1}, \ldots, X_{i+8})$, for any fixed i (counting indices modulo ℓ). Then, for each higher index, we get at least 5, 5, 5, 4, 3, 2, 1, or 0 ‘new’ elements (i. e., elements in the respective $X_{i+j} \setminus \bigcup_{k=1}^{j-1} X_k$), respectively; since the sum of these amounts is $25 = n_6$, we must have equalities. In particular, X_{i+8} has one distinct member in each one of $X_{i+1}, \ldots, X_{i+5}$. Analogously, X_i has one distinct member in each of $X_{i+3}, \ldots, X_{i+8}$. In particular, $X_1 \cap X_8 \neq \emptyset \implies \ell \geqslant 10$. Moreover, hence, each $Y(v) \neq \emptyset$, the ‘index gap lengths modulo ℓ’ in $Y(v)$ all belong to $\{3, \ldots, 7\}$, and

$$\forall\, i \in \{1,\dots,\ell\}\,\forall\, j \in \{3,\dots,7\}\exists!\, v \in V_6 : c_i, c_{i+j} \in Y(v).$$

Applying this for $i, j = 3$, there were a $v \in V_6$ such that $c_3, c_6 \in Y(v)$. If $\{v', v''\} := V_6 \cap A(v)$, then $Y(v) \cap Y(v^{(\nu)}) = \emptyset$, and, in fact, $(c_i \in Y(v) \wedge c_k \in Y(v^{(\nu)}) \implies |i-k| \geqslant 2)$, for $\nu = 1, 2$; forcing $c_1, c_8 \in Y(v') \cap Y(v'')$, and contradiction.

Thus, instead, indeed, $G_7 \simeq 4C_5$. Name the G_7 components C_5, C_5', C_5'', and C_5''', with $V(C_5^{(\nu)}) = \{c_1^{(\nu)}, \dots, c_5^{(\nu)}\}$ for $\nu = 0, \dots, 3$. Moreover, let $X_i^{(\nu)} := V_6 \cap S(c_i^{(\nu)})$. For each ν and i, $\{X_i^{(\nu)}\}_{i=1}^5$ is a 5-partition of V_6, such that each vertex in an $X_i^{(\nu)}$ has its two V_6 neighbours in $X_{i-2}^{(\nu)}$ and $X_{i+2}^{(\nu)}$. Hence, for any G_6 component C_ℓ, exactly every fifth vertex belongs to $X_1^{(\nu)}$; whence $5|\ell$. Moreover, $1 \geqslant |C_\ell \cap (X_1 \cup X_1')| = 0.2\ell$, whence indeed $\ell = 5$.

Thus, indeed, $G_6 \simeq 5C_5$, and each $X_i^{(\nu)}$ intersects each one of these five C_5 in exactly one vertex. In fact, for each choice of one of the G_7 components and one of the G_6 components, the induced subgraph on these ten vertices is a Petersen graph. This makes it natural to present the G_6 components as pentagrams rather than pentagons. Thus, we may let these components be $D_l = \{d_{l,1}, \dots, d_{l,5}\}$ for $l = 0, \dots, 4$, with the unusual prescriptions that their edges be $d_{l,i} \text{-} d_{l,i+2}$, with the second index interpreted modulo 5.

In this manner and without loss of generality, we get

$$X_i = \{d_{0,i},, \dots, d_{4,i}\}$$

for $i = 1, \dots, 5$. We also may reindex the vertices in each remaining complonent $C_5^{(\nu)}$ $(\nu = 1, 2, 3)$ of G_7, in such a way, that

$$\bigcap_{\nu=0}^{3} X_i^{(\nu)} = \{d_{0,i}\},\ i = 1, \dots, 5.$$

As a consequence (and all the time employing the girth condition), there must be a 'shift function' $\phi : \{1,2,3\} \times \{1,2,3,4\} \longrightarrow \{1,2,3,4\}$, such that, for each $\nu \in \{1,2,3\}$, $l \in \{1,2,3,4\}$, and $i \in \{1,2,3,4,5\}$, we have

$$X_i^{(\nu)} \cap D_l = \{d_{l,i+\phi(\nu,l)}\}.$$

If necessary, by means of some further reindexing, we now may determine G uniquely. Indeed, for some l', both $d_{l',1}$ and $d_{l',2}$ must share V_7 neighbours with $d_{0,1}$, i. e., they must belong to $\bigcup_\nu X_1^{(\nu)}$; and we may rearrange the l and the ν to have $l' = 1$, $\phi(1,1) = 1$, and $\phi(2,l) \equiv 2l \pmod 5$ for $l = 1, \dots, 4$. Now, since $\{c_1'', d_{0,1}, c_1', d_{2,1+\phi(1,2)}\}$ is not the vertex set of a C_4 in G, we must have $\phi(1,2) \neq 4$; and similarly considering

$$\{c_2, d_{1,2}, c_1', d_{2,1+\phi(1,2)}\} \text{ and } \{c_5'', d_{1,2}, c_1', d_{2,1+\phi(1,2)}\}$$

yields $\phi(1,2) \neq 1, 3$; whence instead $\phi(1,2) = 2$. In the same manner, we get $\phi(1,l) = l$, for all l. Furthermore, $D_1 \cap X_1'''$ is either $\{d_{1,4}\}$ or $\{d_{1,5}\}$, i.e., $\phi(2,1) \in \{3,4\}$; and similar girth analysis as before shows that then either $\phi(3,l) \equiv 3l \pmod 5$ or $\phi(3,l) \equiv 4l$

(mod 5), for all l. Finally, the second case by some reindexing can be seen to be isomorphic to the first one.

Thus, we get $\phi(\nu, l) \equiv \nu l$; or, in other words, up to isomorphisms, we have

$$E(V_7, V_6) = \{c_i^{(\nu)} \text{-} d_{l,i+\nu l} : i = 1, \ldots, 5,\ \nu = 0, \ldots, 3,\ l = 0, \ldots, 4\};$$

which completely determines G. Moreover, the only pairs of elements in G of distance greater than 2 are the pairs belonging to the same part in a 5-partition of V_6, consisting of "the missing $X_*^{(*)}$", namely $X_1^{(4)}, \ldots, X_5^{(4)}$, where

$$d_{l,k} \in X_i^{(4)} \iff k \equiv i + 4l \pmod 5.$$

Now, it is obvious that G can be extended to a Hoffman-Singleton graph, by adding a new C_5 with vertices $c_1^{(4)}, \ldots, c_5^{(4)}$, and with further edges from $c_i^{(4)}$ to the $X_i^{(4)}$ elements. However, we may do better. If G' is a girth 5 supergraph of G with $45 < n(G') < 50$ and $e(G') = T(C_{\leqslant 4}; n(G'))$, then each $v \in V(G') \setminus V(G)$ must have several neighbours in $V(G)$; and these neighbours must have distance 3 in G and thus belong to the same $X_i^{(4)}$. This yields

Lemma 5.1. *If G is a graph with* $\operatorname{girth}(G) \geqslant 5$, $45 \leqslant n(G) \leqslant 50$, $e(G) = T(C_{\leqslant 4}; n(G))$, *and containing a subgraph of order 45 and size 145, then G is a subgraph of a Hoffman-Singleton graph.* □

This includes the $n = 45$ case of theorem 3. We also get the cases $n = 47, 48, 49$, as soon as we prove that extremal graphs of these orders indeed contain extremal subgraphs of order 45. (Note, however, that the corresponding statement for $n = 46$ is not true; removing a bipartite graph $K_{1,3}$ from HS yields an extremal order 46 graph, all of whose order 45 subgraphs have sizes < 145.)

Let G be a graph of girth $\geqslant 5$, order $n = 47$, and size $6n - 126 = 156$. By lemma 2.2 (for $l = 0$), $\delta(G) \geqslant 6$; whence $\Delta(G) \leqslant 7$ by lemma 2.1, whence $\mathbf{s}(G) = (\cdots 0, 17, 30, 0 \cdots)$. Now, if there were no edges in G_6, then we would have $|E(V_6, V_7)| = 102$ and $e(G_7) = 54$, and lemma 2.8 (for $r = 6$ and $s = 3$) would yield

$$255 = 17 \cdot 15 \leqslant p(V_6, V_7) \leqslant 435 - 54 - 12 \cdot 3 - 18 \cdot 6 = 237,$$

a contradiction. Thus, instead, we may choose a $v \in V_6$, with a neighbour w_1, which also has degree 6. By lemma 2.2 (this time with $l = 1$), thus, indeed, $G - \{v, w_1\}$ is a subgraph of order 45 and size 145, and lemma 5.1 applies.

Now, if instead $48 \leqslant n = n(G) \leqslant 49$, but still $e(G) = 6n - 126$, then G contains a vertex v of degree 6, and $G - \{v\}$ is an extremal graph of order $n - 1$, which 'recursively' is a supergraph of an extremal order 45 graph. Thus, and by inspecting the few induced subgraphs of HS of orders $\geqslant 47$, theorem 3 is proven for all $n \neq 40$.

Table 1: *The lower known $T(C_{\leqslant 4}; n)$ (OEIS* **A006856** *in October, 2015).*

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$T(C_{\leqslant 4}; n)$	0	1	2	3	5	6	8	10	12	15	16	18	21	23	26	28

n	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
$T(C_{\leqslant 4}; n)$	31	34	38	41	44	47	50	54	57	61	65	68	72	76	80	85

Table 2: *The higher known $T(C_{\leqslant 4}; n)$.*

n	40	41	42	43	44	45	46	47	48	49	50
$T(C_{\leqslant 4}; n)$	120	124	129	134	139	145	150	156	162	168	175

6 Bounds for $T(C_{\leqslant 4}; n)$, for $33 \leqslant n \leqslant 39$.

For proving lemma 1.2, we shall make recursive use of tables 1 and 3. The former is well-known; the latter proven in this section.

Actually, also six of the seven lower bounds are known, since they were tabled in [5]; the exception is that only $T(C_{\leqslant 4}; 35) \geqslant 94$ was established there. On the other hand, for all $n \geqslant 34$, the lower bounds given in table 3 are realised by HS subgraphs; but for $n = 33$ I only found such subgraphs of sizes $\leqslant 86$.

In the rest of this section, we consider the upper bounds in table 3, and we let G be a girth $\geqslant 5$ graph with $n(G) = n$, $e(G) = e$, $\delta(G) = \delta$, and $\Delta(G) = \Delta$. The upper bounds are determined by increasing n; whence we may employ the bounds on the $T(C_{\leqslant 4}; m)$ for $m < n$ when treating n. Some bounds follow by just the standard argument that a higher e or a lower δ would violate the inequalities

$$\left\lfloor \frac{2e}{n} \right\rfloor \geqslant \delta \geqslant e - T(C_{\leqslant 4}; n-1).$$

If $(n, e) = (33, 90)$, then $5 \geqslant \delta \geqslant e - T(C_{\leqslant 4}; 32) = 5 \implies \delta = 5$. Thus, any $v \in V_{\geqslant 7}$ would have $\deg^2(v) \geqslant 7\delta = 35 > 32$ contradicting lemma 2.1, whence instead $\Delta = 6$. Hence, repressing all trailing zeros in degree sequences, $\mathbf{s}(G) = (18, 15)$; and all neighbours

Table 3: *To my knowledge the narrowest known intermediate $T(C_{\leqslant 4}; n)$ potential value ranges.*

n	33	34	35	36	37	38	39
$T(C_{\leqslant 4}; n)$	$87-89$	$90-93$	$95-98$	$99-103$	$104-107$	$109-112$	$114-116$

w of a $v \in V_5$ must be of degree 6, since $\deg(v) + \deg(w) \geqslant e(G) - T(C_{\leqslant 4}; 31) + 1 = 11$ by lemma 2.2. Thus, G were bipartite and (5,6) biregular, with parts V_5 and V_6. For a $v \in V_5$, this would force $p_3(\{v\}, V_6, V_5) = 25 > 17 = |V_5 \setminus \{v\}|$, a contradiction.

Eliminating $(n, e) = (34, 94)$ takes more effort. Step by step, we provide more and more structural conditions on the induced subgraphs

$$G_i = G_i(G) := G[V_i(G)] := G[\{v \in V(G) : \deg(v) = i\}],$$

on *their* induced subgraphs $G_j(G_i)$, on unions of corresponding vertex sets, *et cetera*, at last deducing a contradiction.

Indeed, arguing as in the $n = 33$ case, $\delta = 5$ and $\Delta = 6$, and by lemma 2.2 also $\Delta(G_5) \leqslant 1$. Thus, with $e_5 := e(G_5)$ and $z := |E(V_5, V_6)|$, we have

$$\mathbf{s}(G) = (16, 18) \wedge (z, e(G_6)) = (80 - 2e_5, 14 + e_5) \wedge 0 \leqslant e_5 \leqslant 8\,.$$

Thus, and by lemma 2.8 applied for $(V', V'') = (V_5, V_6)$, and since, for $e_5 = 8$, the first and last expressions in lemma 2.8 differ just 1, forcing an as even as possible induced degree sequence in G_6,

$$e_5 = 8 \wedge \mathbf{s}(G_6) = (10, 8) \wedge G_5 = G_1(G_5) = 8P_2\,, \tag{10}$$

the disjoint union of eight order two paths, which we may name P_2', P_2'', …, $P_2^{(8)}$.

Now, ($v \in V_3(G_6) \implies \deg^2(v) = 33 = n - 1 \implies V_5 \subset V = B(v; 2)$) by lemma 2.1; which, for such a v, since $p_3(\{v\}, V_5, V_5) = |V_5 \cap S(v)| = 3$, forces $p_3(\{v\}, V_6, V_5) = 16 - 3 - 3 = 10$, and thus the induced second degree $\deg^2_{G_6}(v) = 6\,|V_6 \cap S(v)| - 10 = 8$; i. e., that

$$G_3(G_6) = G_2(G_3(G_6)) = C_8$$

(the 8-cycle graph, with $V(C_8) = \{c_1, \ldots, c_8\}$, say). Similarly, for any $v \in V_2(G_6)$ there is a single $z \in V$ at distance 3 from v, of degree either 5 or 6, and with v having $\deg^2_{G_6}(v) = 5$ or 4, respectively; whence

$$V_2(G_6) = V_1(G_2(G_6)) \cup V_2(G_2(G_6)) \wedge |V_1(G_2(G_6))| = 8 \wedge |V_2(G_2(G_6))| = 2,$$

whence in particular

$$G_2(G_6) \text{ has three or two } P_2 \text{ components;} \tag{11}$$

the amount depending on whether or not $G_2(G_2(G_6))$ is connected.

The girth property and (10) yield that

$$\text{the } V_5 \text{ neighbours of any edge in } E(V_6) \text{ are in different } G_5 \text{ components.} \tag{12}$$

Next, we explicitly match the eight $c_i \in V(C_8) = V_3(G_6)$ and the eight components $P_2^{(i)}$ of G_5 (in both cases counting indices modulo 8). Let $\{d_i\} := V_2(G_6) \cap S(c_i)$. c_i is adjacent to three G_5 components. Since each degree 5 vertex is the end vertex of exactly one P_3 or P_2 from c_i, the V_5 neighbours of d_i, c_{i-1}, and c_{i+1} together form the remaining five G_5 components. More precisely, d_i is adjacent to four of these, while the fifth component, and only that component, is adjacent to both c_{i-1} and c_{i+1}.

Thus, any three successive elements in C_8 must be adjacent to $3 + 3 + (3 - 1) = 8$ of the G_5 components, i. e., to all of them. Hence, if we consider four successuve elements c_{i-1}, c_i, c_{i+1}, and c_{i+2}, then both c_{i-1} and c_{i+2} must be adjacent to both the G_5 components adjacent to neither c_i nor c_{i+1}; whence c_{i-1} and c_{c+2} must have exactly these two components as common component neighbours; the third P_2 adjacent to c_{i+2} also is adjacent to c_i, and thus by (12) not to c_{i-1} .

Thus, c_i shares two of its three adjacent G_5 components with c_{i+3}, and two with c_{i-3}, whence, by the principle of inclusion-exclusion, all three of c_i and the $c_{i\pm 3}$ share one G_5 component neighbour, say $P_2^{(\nu(i))}$. On the other hand, by (12), $P_2^{(\nu(i))}$ has no other neighbour in C_8. Thus, ν is injective, and hence bijective, whence without loss of generality we may let it be the identity. Summing up, and also employing that $S(c_{i-3}) \cap S(c_{i+3})$ consists of *only* c_{i+4} , we find that

$$\text{the } P_2^{(i)} \text{ neighbours in } V_3(G_6) \text{ are } \{c_{i-3}, c_i, c_{i+3}\}, \text{ and } P_2^{(i)} \subset S(c_{i-3}) \cup S(c_{i+3}), \qquad (13)$$

for all i.

However, by (11), $G_{6,2}$ contains a P_2 component, which without loss of generality be $\{d_1, d_i\}$, where $i \in \{3, \ldots, 7\}$ by the girth condition. Each G_5 component is adjacent to exactly one P_2 vertex; and by (12) applied for c_1-d_1 and for c_i-d_i, respectively, P_2', $P_2^{(4)}$, and $P_2^{(6)}$ are adjacent to d_i, but $P_2^{(i)}$ and the $P_2^{(i\pm 3)}$ to d_1. Thus $\{1, 4, 6\} \cap \{i-3, i, i+3\} = \emptyset$, forcing $i = 5$. Thus and without loss of generality, $p_1^{(5)} \in P_2^{(5)} \cap S(d_1)$; but by (13) $p_1^{(5)}$ also is adjacent to a $c_j \in \{c_2, c_8\}$, causing the existence of a 4-cycle $\{c_1, d_1, p_1^{(5)}, c_j\}$, and thus the sought contradiction.

Thus and by the standard argument, $T(C_{\leqslant 4}; 35) \leqslant 98$ and $T(C_{\leqslant 4}; 36) \leqslant 103$.

For the next few bounds, we use the fact (from [9]) that (6,5) cages have order 40, whence G cannot be 6-regular for $n < 40$.

If $(n, e) = (37, 108)$ and $\Delta = 6$ (whence $\mathbf{s}(G) = (6, 31)$), then $(v, v' \in V_5 \implies \mathrm{dist}(v, v') \geqslant 3)$ by lemma 2.4, whence we could make a 6-regular girth $\geqslant 5$ realiser G' out of G, by adding one vertex and edges from that to all V_5 vertices; yielding the contradiction $n(G') = 38 < 40$.

Thus, instead, for such graphs we should have $\Delta = 7$, but $\deg(v) = 7 \implies |V_5 \cap S(v)| \geqslant 5$; contradicting $T(C_{\leqslant 4}; 31) < 81$, by lemma 2.2 with $l = 5$.

Thus, instead, $T(C_{\leqslant 4}; 37) \leqslant 107$, whence (and by the standard argument) $T(C_{\leqslant 4}; 38) \leqslant 112$.

For a while and for a contradiction, assume that $(n, e) = (39, 117)$. As before, G is not 6-regular, whence, instead and by the standard arguments,

$$\delta = 5 \text{ and } \Delta = 7\,.$$

Next, consider a $v \in V_7$, with neighbours $w_1, \ldots, w_7$, say, where we may assume $\deg(w_1) \leqslant \ldots \leqslant \deg(w_7)$. Since then, on the one hand, $38 \geqslant \deg^2(v)$ by lemma 2.1, while on the other hand

$$\sum_{i=1}^{5} \deg(w_i) \geqslant 26$$

by lemma 2.2, then $\deg(w_1) = \deg(w_2) = \deg(w_3) = \deg(w_4) = 5$, but $\deg(w_5) = \deg(w_6) = \deg(w_7) = 6$. In particular, each $v \in V_7$ has four neighbours in V_5, and all its neighbours in $V_{\leqslant 6}$. Hence, $n_7 > 0 \implies n_7 \geqslant n_5 \geqslant 4$.

Thus, in applying lemma 2.8 for $(V', V'') := (V_7, V_5)$,

$$\binom{n_5}{2} \geqslant \binom{4}{2} n_7 \geqslant 6n_5 \implies 13 \leqslant n_5 \leqslant n_7 \,.$$

On the other hand, since there are $7n_7$ edges between V_7 and $V_{\leqslant 6}$, $n_7 \leqslant 16$; and for each of the potential four values of n_7, lemma 2.8 applied for $(V', V'') := (V_{\leqslant 6}, V_7)$ yields a contradiction.

7 Proof outline for the second main lemma.

This proof of lemma 1.2 is fairly eclectic. It uses both the more 'independent' methods of the proof of lemma 1.1, and more 'conventional' methods, including references to some upper bounds for lower orders, and case division. Where the proof mainly re-uses already presented ideas, it just is outlined.

We retain the notation from section 3, *mutatis mutandis*. Thus, this time, the globally most even degree distribution is $\bar{\mathbf{s}} = (\cdots, 0, 40, 0 \cdots) \in \mathbf{R}^{40}$, and we find that

$$\mathbf{s} = \mathbf{s}(G) = \bar{\mathbf{s}} \iff \delta = \delta(G) = 6 \iff \Delta = \Delta(G) = 6.$$

For the rest of the proof (and for a contradiction), we assume the converse, i. e., that $\delta \leqslant 5$ and $\Delta \geqslant 7$. Actually, since $T(C_{\leqslant 4}; 31) \leqslant 80 < e - 39$, and by lemma 2.3, we must have $\Delta = 7$, i. e., that

$$a = n'' = n_7 \text{ and } b = 0\,. \tag{14}$$

Likewise, employing $T(C_{\leqslant 4}; 39) \leqslant 116$, $T(C_{\leqslant 4}; 37) \leqslant 107$, and $T(C_{\leqslant 4}; 38) \leqslant 112$, and putting $\tilde{n} = |V_{\leqslant 5}|$, we get

$$\delta \geqslant 4,\ c := n_4 \leqslant 2,\ e(G_4) = 0,\ \text{and } a = \tilde{n} + c\,. \tag{15}$$

In other words,

$$\mathbf{s} = \mathbf{s}(G) = (\cdots 0, c, a - 2c, 40 - 2a + c, a, 0 \cdots),\ \text{whence } a \leqslant 0.5(40 + c) \leqslant 21\,. \tag{16}$$

Moreover, if both v and three of its neighbours would have degree 7, then the sum of the degrees of the other four neighbours of v by lemma 2.1 would not exceed $39 - 3 \cdot 21 = 18$, which would contradict lemma 2.2, since $T(C_{\leqslant 4}; 35) \leqslant 98 < e - 7 - 18 + 4$. Thus, instead, and by counting,

$$\Delta(G_7) \leqslant 2 \implies e'' \leqslant a \wedge z = 7a - 2e'' \geqslant 5a\,. \tag{17}$$

We shall make no further 'recursive' reference to bounds for $T(C_{\leqslant 4}; m)$ for any $m < 40$.

This time, we shall have no use for virtually trunkated degree sequences.
For $p := p(V', V'') = p(V_{\leqslant 6}, V_7)$, consider the upper estimate

$$p_u := \binom{a}{2} - e'' - \max(0, 2(e'' - 0.5a)) = 0.5 \min(a^2 - a - 2e'', a^2 + a - 6e'') \geqslant p,$$

the minimum of the two upper bounds in lemma 2.8 got by choosing $s = 0$ and $s = 1$, respectively.

As our first crude lower p estimate, we similarly may combine the lower estimates of that lemma for $r = 1, 2, 3$, putting

$$\begin{aligned} p_l &:= \max(z - n', (3n' - z) \cdot 1 + (z - 2n') \cdot 3, (4n' - z) \cdot 3 + (z - 3n') \cdot 6) \\ &= \max(8a - 2e'' - 40, 17a - 4e'' - 120, 27a - 6e'' - 240). \end{aligned}$$

In particular, and by (17), employing the $s = 1$ and the $r = 3$ clauses,

$$0 \leqslant p_u - p_l \leqslant 0.5(a^2 + a - 6e'') - (27a - 6e'' - 240) = 0.5(a^2 - 53a + 6e'' + 480) \leqslant 0.5(a - 15)(a - 32).$$

Thus, and by (16),

$$a \leqslant 15; \tag{18}$$

and for $a = 15$, only the maximal $e'' = a$ would be possible. Similarly, by means of the $r = 2$ and both the s clauses, we get restrictions on e'' for $a \geqslant 10$:

$$\begin{aligned} a \geqslant 10 \implies e'' &\geqslant \max\left(\lceil \tfrac{1}{6}(35a - a^2 - 240)\rceil, \lceil \tfrac{1}{2}(33a - a^2 - 240)\rceil\right) \\ &= \max\left(2a - 18, 16 - \tbinom{17-a}{2}\right). \end{aligned}$$

Both (16) and (18) depend only on the crude p_l we got, assuming an even distribution of $\deg''$ on $V_{\leqslant 6}$. However, in average, $\deg''$ is higher on $V_{\leqslant 5}$ than on V_6, which leads to sharper bounds, by considering $p(V_{\leqslant 5}, V_7)$ and $p(V_6, V_7)$ separately, and noting that

$$p = p(V_{\leqslant 5}, V_7) + p(V_6, V_7)$$

by lemma 2.7. In fact, with $\tilde{z} := |E(V_{\leqslant 5}, V_7)|$,

$$\tilde{z} \geqslant 3a - 4c + 2e''. \tag{19}$$

This enables us to work with improved higher p_l, and (by worst case analyses for the various a, combined with analysis of the impact of rising e'' above the (19) bound), we may sharpen (18) to

$$a \leqslant 8.$$

Moreover, the same analysis yields

$$a = 8 \implies c = 2 \wedge e'' = 0 \wedge \max_{v \in V_{\leqslant 5}} \deg''(v) = 3 \wedge \tilde{z} = 16.$$

However, this $\tilde{z}$ value is the minimal one allowed in (19), and a closer analysis of that inequality reveals that the term $-4c$ therein may be replaced by

$$-\sum_{v\in V_4} \deg''(v);$$

a quantity that thus on the one hand were at least -6, and on the other hand were equal to -8; a contradiction. Thus, instead, and summing up,

$$a \leqslant 7. \tag{20}$$

For the lowest a values, also work with $\tilde{p} := p(V_7, V_{\leqslant 5})$. In order to force a larger $\tilde{z}$ than given by (19), we sometimes may employ a *discharging technique*: We start by giving each element in $\binom{V_5}{2}$ (i. e., 2-subset in V_5) a charge $\frac{1}{3}$, and every other element in $\binom{V_{\leqslant 5}}{2}$ a charge 1. Next, we move the charge of any 2-subset of $V_{\leqslant 5}$ with a common neighbour $v \in V_7$ to that v. Then, after discharging, each vertex of degree 7 has received a total charge $\geqslant 1$, since v either has precisely two neighbours in $V_{\leqslant 5}$, of which at least one has degree 4, or has at least three neighbours in V_5. Moreover, the number of 3-subsets of n_5 belonging to the neighbourhoods of different v_7 elements is at most

$$\left\lfloor \tfrac{n_5}{3} \cdot \lfloor 0.5(n_5-1)\rfloor \right\rfloor - \varepsilon, \text{ where } \varepsilon = \begin{cases} 1 & \text{if } n_5 \equiv 5 \pmod 6 \\ 0 & \text{else} \end{cases},$$

since different such 3-sets share at most one vertex, and by [7, theorem 1].

This eliminates almost all the remaining possibilities. The few exceptions are treated by structure determination and case division, eventually leading to dismissal. The most complex of these treatments occur for the cases where $a = 7$, $c = 2$, and $e'' = 0$; let us briefly consider them. In these cases, we may put $V_7 = \{v_1, \ldots, v_7\}$, $V_4 = \{u_1, u_2\}$, $V_5 = \{w_1, w_2, w_3\}$, and $Z_i = V_{\leqslant 5} \cap S(v_i)$ for $i = 1, \ldots, 7$. Note, that then $\tilde{z} \geqslant 14$, since each $|S(v_i)| \geqslant 2$. On the other hand, $\tilde{z} \leqslant 15$, since $p(V_7, V_{\leqslant 5}) \leqslant \binom{5}{2} < 5 \cdot 1 + 2 \cdot 3$. This yields two cases: $\tilde{z} = 14$ and $\tilde{z} = 15$. In either case,

$$p(V_6, V_7) \geqslant z - \tilde{z} - n_6 = 21 - \tilde{z} \implies p(V_{\leqslant 5}, V_7) \leqslant \tilde{z}.$$

However, if here $\tilde{z} = 14$, then each one of the seven elements in V_7 must be adjacent to V_4, whence and without loss of generality $S(u_1) \subset V_7$, whence each one of the four 2-subsets of $V_{\leqslant 5}$ containing u were employed as some Z_i, whence in particular some $Z_i = \{u_1, u_2\}$, whence $S(u_2) \subset V_7$, too; yielding

$$p(V_{\leqslant 5}, V_7) = 3 \cdot 1 + 2 \cdot 6 = 15 > 14 = \tilde{z},$$

a contradiction.

In the remaining case, $\tilde{z} = 15$; any $S(u_i) \subset V_7$ again would yield a too high $p(V_{\leqslant 5}, V_7)$, and contradiction; whence instead (and without loss of generality) $Z_7 = V_5$, and

$$Z_i = \{u_j, w_k\} \text{ with } i \equiv j \pmod 2 \text{ and } i \equiv k \pmod 3$$

for $j = 1, \ldots, 6$. By the girth condition and (15),

$$E(V_{\leqslant 5}) = E(V_4) = \emptyset.$$

Thus, we have determined the graph structure of $G(V_{\leqslant 5} \cup V_7)$ and may continue to the degree 6 vertices. Put

$$A := V_6 \cap \bigcup_{i=1}^{3} S(w_i),\ B = \{b_1, \ldots, b_4\} := V_6 \cap S(v_7),\ C := V_6 \setminus (A \cup B),$$

and put $C_i := V_6 \cap S(b_i)$ for $i = 1, \ldots, 4$. By the girth conditions, $A \cap B = E(A, B) = \emptyset$ and $|A| = 3 \cdot 2$; whence (A, B, C) is a tripartition of V_6, with

$$(|A|, |B|, |C|) = (4, 6, 18).$$

On the other hand, the b_i together have 20 neighbours apart from v_7, and all these neighbours are different. Hence,

$$\bigcup_{i=1}^{4} S(b_i) = C \cup V_4 \cup \{v_7\}.$$

Thus, $\{C_1, C_2, C_3, C_4\}$ is a partition of C into four parts, of sizes 5+5+4+4 or 5+5+5+3.

We now continue with the degree 6 neighbours of the v_i, for $1 \leqslant i \leqslant 6$. For these i, $|V_6 \cap S(v_i)| = 5$. However, $B \cap S(v_i) = \emptyset$ (girth reasons), and v_i has at most one neighbour in each C_j. Moreover, some $u_l \in S(v_i)$, and some $b_{j'} \in S(u_k)$, and v_i has no neighbour in $C_{j'}$ (girth reasons). Thus, $|C \cap S(v_i)| \leqslant 3$; whence $|A \cap S(i)| \geqslant 2$. Summing up, and by counting,

$$|B \cap C| = 18 \wedge E(V_{\leqslant 5}, C) = \emptyset \wedge |E(V_7, C)| \leqslant 18 \wedge |E(A, C)| \leqslant 18 \implies e(C) \geqslant 27.$$

On the other hand, no C vertex could have a neighbour in its own part; and for $1 \leqslant i < j \leqslant 4$, $|E(C_i, C_j)| \leqslant \min(|C_i|, |C_j|)$, whence

$$e(C) \leqslant \max(1 \cdot 5 + 5 \cdot 4, 3 \cdot 5 + 3 \cdot 3) = 25 < 27,$$

the sought contradiction.

After similarly having eliminated all remaining cases with $\Delta = 7$, thus, indeed, only the possibility that G is 6-regular remains. □

In particular and by [10] we have

Corollary 7. *The graphs treated in lemma 1.2 are unique (up to isomorphisms), and are subgraphs of Hoffman-Singleton graphs.* □

In particular, this concludes the proof of teorem 3.

8 Proof of theorem 2.

From lemma 1.2, we indeed may deduce theorem 2 in a few steps, which we briefly indicate. In each case, we assume that G has girth $\geqslant 5$, order n, and size one more that the proposed value of $T(C_{\leqslant 4}; n)$. We sometimes refer to the lemma 2.8 upper and lower bounds of $p := p(V_{\leqslant 6}, V_7)$ as p_u and p_l, respectively.

Start by noting that indeed $T(C_{\leqslant 4}; 40) < 121$; proven as for $T(C_{\leqslant 4}; 45)$. Thus:

$n = 41$ (and $e(G) = 124 + 1$) $\implies \Delta \geqslant 7 \implies \delta = 5 \wedge n_5 \geqslant 2$; but then $v \in V_5 \implies G - \{v\}$ not 6-regular, contradicting lemma 1.2. Thus:

$n = 42 \implies \Delta \geqslant 7$, contradicting $\delta > 5$. Thus:

$n = 43 \implies \Delta = 7 \wedge G_7 \simeq \overline{K_{12}}$ (the edge-free graph of order 12), yielding $p_u - p_l \leqslant 66 - (9 \cdot 1 + 22 \cdot 3) = -9 < 0$, which contradicts $p_l \leqslant p \leqslant p_u$. Thus, finally:

$n = 44 \implies \Delta = 7 \wedge n_7 = 16 \wedge e(G_7) \leqslant 8 \implies p_u - p_l \leqslant (120 - 8) - (16 \cdot 3 + 12 \cdot 6) = -8 < 0$, a contradiction. □

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