ELLIPTIC ROOK AND FILE NUMBERS

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ABSTRACT. Utilizing elliptic weights, we construct an elliptic analogue of rook numbers for Ferrers boards. Our elliptic rook numbers generalize Garsia and Remmel's q-rook numbers by two additional independent parameters a and b, and a nome p. These are shown to satisfy an elliptic extension of a factorization theorem which in the classical case was established by Goldman, Joichi and White and later was extended to the q-case by Garsia and Remmel. We obtain similar results for our elliptic analogues of Garsia and Remmel's q-file numbers for skyline boards. We also provide an elliptic extension of the j-attacking model introduced by Remmel and Wachs. Various applications of our results include elliptic analogues of (generalized) Stirling numbers of the first and second kind, Lah numbers, Abel numbers, and r-restricted versions thereof.

1. INTRODUCTION

The theory of rook numbers was introduced by Kaplansky and Riordan [26] in 1946, and since then it has been further studied and developed by many people. In 1975, Goldman, Joichi and White [19] proved the following result for rook numbers on a Ferrers board $B = B(b_1, \dots, b_n) \subset [n] \times \mathbb{N}$ (see Section 2 for the precise definition of rook numbers and of a Ferrers board):

$$\prod_{i=1}^{n}(z + b_i - i + 1) = \sum_{k=0}^{n} r_{n-k}(B)\, z(z-1)\dots(z-k+1). \tag{1.1}$$

We refer to an identity of the form (1.1) as a *product formula* or *factorization theorem.*

In 1986 Garsia and Remmel [17] established a q-analogue of rook numbers on Ferrers boards by considering a statistic involving *rook cancellation.* Among other results, they were in particular able to extend the product formula in (1.1) to the q-case. In 1991 Wachs and White [44] introduced a (p, q)-analogue of rook numbers which was later studied in more detail by Briggs and Remmel [3] and by Remmel and Wachs [32]. In 2001, Haglund and Remmel [23] considered a suitably modified statistic involving rook cancellation on *shifted* Ferrers boards. In this way they were able to develop a rook

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theory for partial matchings of the complete graph K_{2n} of $2n$ vertices and in particular proved a product formula analogous to (1.1).

In the present work, we construct elliptic analogues of the rook numbers on Ferrers boards by utilizing *elliptic weights*. Our elliptic rook numbers generalize Garsia and Remmel's q-rook numbers by two additional independent parameters a and b, and a nome p. They also contain the aforementioned (p, q)-rook numbers of [3, 44] as a special case. We show that the elliptic rook numbers satisfy an elliptic extension of (1.1). Our elliptic rook numbers can be used to define elliptic analogues of the Stirling numbers of the second kind, of the Lah numbers, and of certain "r-restricted" refinements of these numbers, by specializing Ferrers boards.

We also provide an elliptic extension of the $\mathfrak{j}$-attacking model which was considered by Remmel and Wachs [32]. As a consequence, we present elliptic extensions of the generalized Stirling numbers of the first and the second kinds.

Similarly, we construct elliptic analogues of the file numbers on skyline boards which in the classical case and in the q-case were first studied by Garsia and Remmel. We show that the elliptic file numbers satisfy a factorization theorem, extending an analogous result of Garsia and Remmel. Special cases of the elliptic file numbers include an elliptic extension of the unsigned Stirling numbers of the first kind, an elliptic enumeration of labeled forests of rooted trees which extends the work of Goldman and Haglund [20] to the elliptic setting, and again, elliptic extensions of r-restricted refinements of these numbers.

At this point, we would like to explain our motivation for this work. What is the reason for "going elliptic"? People working in enumerative combinatorics often encounter identities involving hypergeometric series (or more generally, special functions). On one hand, the theory of hypergeometric series serves as a tool for solving combinatorial problems, and on the other hand, combinatorial models can be used to prove or explain hypergeometric series identities. This phenomenon similarly also applies to other areas, such as algebra and geometry, or mathematics and physics. Problems which lay in the interface of two or more areas are often challenging and particularly interesting. The development of tools which combine two different areas is promising and may ultimately lead to a better understanding of both respective theories. Now, just as in many classical instances where hypergeometric series emerge from problems in enumerative combinatorics, q-hypergeometric or basic hypergeometric series frequently emerge from problems in enumerative combinatorics involving some q-statistics (which is a refined counting). On the contrary, given specific basic hypergeometric series identities, one can ask for suitable models where a combinatorial explanation can be provided. Now this is not the end of the story. There is in fact a natural hierarchy of hypergeometric series: *rational* (i.e. "ordinary"), *trigonometric* (or "q", i.e., "basic"), and *elliptic* (or "q, p", balanced and well-poised) hypergeometric series. Not such a long time ago, people working in hypergeometric series have realized that the following three term relation of theta functions,

$$\theta(xy, x/y, uv, u/v; p) - \theta(xv, x/v, uy, u/y; p) = \frac{u}{y}\,\theta(yv, y/v, xu, x/u; p) \tag{1.2}$$

(see Section 3 for the notation), can be used as a key relation to build up a theory of identities for series involving theta functions (see [15, 40, 41, 43] and the discussion in [17, Chapter 11]), analogous to the classical theories of hypergeometric and of basic hypergeometric series. This can be compared to the hierarchy of meromorphic solutions of the Yang–Baxter equation, being rational, trigonometric, or elliptic, described in [25]. (Whereas trigonometric functions are periodic, elliptic functions are doubly periodic. This cannot be pushed further, since by Liouville's theorem, meromorphic functions on $\mathbb{C}$ with three independent periods are constant.) In the last three decades the theory of theta and elliptic hypergeometric functions has been developed to a great extent from various points of view (including integrable systems, special functions, and biorthogonal functions). But despite of their original appearance in lattice models in statistical mechanics [9], elliptic hypergeometric series have not been studied much yet from a combinatorial point of view. In [34], one of us enumerated lattice paths with respect to suitable elliptic weight functions. This led to a combinatorial proof of the Frenkel–Turaev ${}_{10}V_9$ summation formula, a fundamental identity in the theory of elliptic hypergeometric series. Further results from [34] included the closed form elliptic enumeration of nonintersecting lattice paths. Similar elliptic weights have also subsequently been used in [8] and in [2] to enumerate dimers and lozenge tilings. In the quest of trying to better understand the connection between combinatorics and elliptic hypergeometric series it is just natural to look for suitable general combinatorial models where elliptic weights can be utilized. The goal is to obtain explicit results that generalize the existing ones to the elliptic level, but which are still "attractive" (such as results involving closed form products).

Although we were successful in our aim to extend the classical rook theory to the elliptic setting, we were somehow disappointed to find that elliptic hypergeometric series summations did not come out in this study. (See the discussion in Subsection 6.4.) This is probably inherent to the model. Already in the well-studied q-case the only basic hypergeometric series identities that arise in rook theory are those of Karlsson–Minton type (see [21], and again Subsection 6.4). Nevertheless, we are able for the first time to present a bunch of elliptic extensions of special numbers (Stirling, Lah, etc.). As these are new (and this territory opens up a new theory, namely of *elliptic special numbers*), we have put quite some attention to them in our exposition. A reader who is not so much interested in all of these new special numbers, is advised to mainly focus on the material leading to the main result of this paper, the product formula in Theorem 3.8 and to look at one or two specific examples of applications. Besides that, a reading of Section 6 in the end also serves to give an idea about the "big story".

The paper is outlined as follows. In Section 2 we give a gentle introduction to rook theory, state the product formula and a recursion for q-rook numbers. In Section 3, after introducing elliptic analogues of numbers and their properties, we generalize the results in Section 2 to the elliptic case and highlight several special cases of interest. In Section 4, we work out an elliptic extension of the $\mathfrak{j}$-attacking rook model. We consider elliptic file numbers in Section 5 and highlight several special cases of interest there as well. Lastly, we list some topics for future investigation in Section 6.

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2. Introduction to rook theory

Let $\mathbb{N}$ denote the set of positive integers and $\mathbb{N}_0$ the set of nonnegative integers. We consider a *board* to be a finite subset of the $\mathbb{N}\times\mathbb{N}$ grid, and label the columns from left to right with $1,2,3,\ldots$, and the rows from bottom to top with $1,2,3,\ldots$. We let (i,j) denote the cell in the i-th column from the left and the j-th row from the bottom. If a board has at most n columns and m rows, we consider it as a subset of the $[n]\times[m]$ grid, where $[n]=\{1,2,\ldots,n\}$ and $[m]=\{1,2,\ldots,m\}$. For technical reasons, in our proofs, we sometimes find it convenient to extend the $\mathbb{N}\times\mathbb{N}$ grid to $\mathbb{N}\times\mathbb{Z}$ where cells may have a zero or negative integer row index.

Let $B(b_1,\ldots,b_n)$ denote the set of cells

$$B=B(b_1,\ldots,b_n)=\{(i,j)\mid 1\le i\le n,\ 1\le j\le b_i\}.$$

If a board B can be represented by the set $B(b_1,\ldots,b_n)$ for some nonnegative integer b_i's, then the board B is called a *skyline board*. If in addition those b_i's are nondecreasing, then the board $B=B(b_1,\ldots,b_n)$ is called a *Ferrers board*.

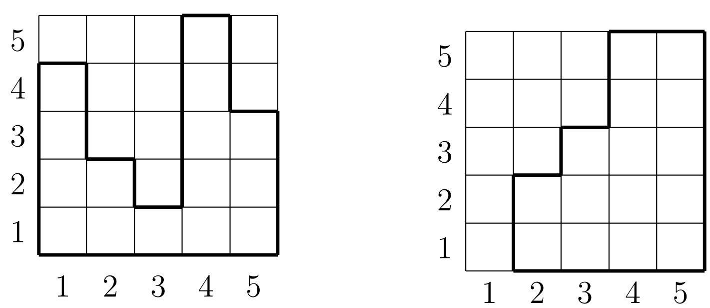


Figure 1. A skyline board $B(4,2,1,5,3)$ and a Ferrers board $B(0,2,3,5,5)$.

We say that we *place k nonattacking rooks* in B by choosing a k-subset of cells in B such that no two elements have a common coordinate, that is, no two rooks lie in the same row or in the same column. Let $\mathcal{N}_k(B)$ denote the set of all nonattacking placements of k rooks. The *k-th rook number* of B is defined by $r_k(B)=|\mathcal{N}_k(B)|$. To define the q-analogue of the rook numbers, we need the concept of *rook cancellation*. Given a rook placement $P\in\mathcal{N}_k(B)$, a rook in P cancels all the cells to the right in the same row and all the cells below it in the same column. Then Garsia and Remmel [17] defined the q-analogue of the rook numbers for Ferrers boards by

$$r_k(q;B)=\sum_{P\in\mathcal{N}_k(B)} q^{u_B(P)},$$

where q is an indeterminate and $u_B(P)$ counts the number of cells in B which are neither cancelled by rooks nor contain any rooks in a k-rook placement P. See Figure 2 for the set of cancelled cells (marked by thick dots) of a particular placement of four rooks (marked by X's) on the Ferrers board $B(0,2,3,5,5)$.

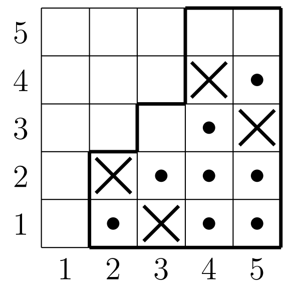


FIGURE 2. A rook cancellation in $B(0,2,3,5,5)$.

Note that for $B=[n]\times[n]$, a nonattacking rook placement of n rooks in B corresponds to a permutation of $1,2,\ldots,n$. By considering all the rook placements corresponding to the permutations of n numbers, it is not hard to see that

$$r_n(q;[n]\times[n])=[n]_q!. \tag{2.1}$$

Here the q-falling factorial and q-factorial are defined by

$$[n]_q\!\downarrow_k=[n]_q[n-1]_q\ldots[n-k+1]_q, \qquad \text{and} \qquad [n]_q!=[n]_q\!\downarrow_n$$

with $[n]_q\!\downarrow_0=1$, respectively, where

$$[n]_q=\frac{1-q^n}{1-q}$$

is the q-number of n. From (2.1) it can be concluded that the rook numbers satisfy the *Mahonian* property (first defined in [14]), i.e. the two statistics, the number of uncancelled cells of n nonattacking rooks on an $[n]\times[n]$ board and the number of inversions of permutations of n elements, have the same distribution.

For a given Ferrers board $B\subset[n]\times\mathbb{N}$, let us denote by $B^\infty\subset[n]\times\mathbb{Z}$ the Ferrers board obtained by appending below B the infinite board of width n. For convenience, we denote by $\mathfrak{g}$ the line separating B from the rest of B^∞ and refer to it as the *ground*. For a rook placement P in B^∞, we let $\max(P)$ denote the number of rows below the ground in which the lowest rook of P is located. Let $\max(P)=0$ if there are no rooks below $\mathfrak{g}$ in P. Garsia and Remmel [17, Equation (I.11)] showed the following identity.

Proposition 2.1. [17, Equation (I.11)] *For any Ferrers board $B=B(b_1,\ldots,b_n)$,*

$$\frac{1}{1-z}\sum_{P\in\mathcal{N}_n(B^\infty)}z^{max(P)}q^{u_B(P)}=\sum_{k\geq 0}z^k[k+b_1]_q[k+b_2-1]_q\cdots[k+b_n-n+1]_q. \tag{2.2}$$

Using this result, Garsia and Remmel [17, Equation (1.3)] proved the following factorization theorem for q-rook numbers on Ferrers boards which extends the result of Goldman, Joichi and White in (1.1).

Proposition 2.2. [17, Equation (1.3)] *Let $B=B(b_1,\ldots,b_n)\subset[n]\times\mathbb{N}$ be a Ferrers board. Then*

$$\prod_{i=1}^{n}[z+b_i-i+1]_q=\sum_{k=0}^{n}r_{n-k}(q;B)\,[z]_q\!\downarrow_k\,. \tag{2.3}$$

We recover (1.1) when $q \to 1$. By distinguishing whether there is a rook in the last column or not, Garsia and Remmel also showed the following recursion [17, Theorem 1.1].

Proposition 2.3. [17, Theorem 1.1] *Let B be a Ferrers board of height at most m and let $B \cup m$ denote the board obtained by adding a column of length m to B. Then for any nonnegative integer k, we have*

$$r_k(q; B \cup m) = q^{m-k} r_k(q; B) + [m-k+1]_q \, r_{k-1}(q; B). \tag{2.4}$$

Now we are ready to turn to the elliptic setting.

3. Elliptic analogues

A function is defined to be *elliptic* if it is meromorphic and doubly periodic. It is well known (cf. e.g. [45]) that elliptic functions can be built from quotients of theta functions.

Define a *modified Jacobi theta function* with argument x and nome p by

$$\theta(x;p) = \prod_{j \geq 0} ((1-p^j x)(1-p^{j+1}/x)), \qquad \theta(x_1, \ldots, x_m; p) = \prod_{k=1}^{m} \theta(x_k; p),$$

where $x, x_1, \ldots, x_m \neq 0$, $|p| < 1$. Further, we define the *theta shifted factorial* (or *q,p-shifted factorial*) by

$$(a;q,p)_n = \begin{cases} \prod_{k=0}^{n-1} \theta(aq^k; p), & n = 1, 2, \ldots, \\ 1, & n = 0, \\ 1/\prod_{k=0}^{-n-1} \theta(aq^{n+k}; p), & n = -1, -2, \ldots, \end{cases}$$

together with

$$(a_1, a_2, \ldots, a_m; q, p)_n = \prod_{k=1}^{m} (a_k; q, p)_n,$$

for compact notation. For $p = 0$ we have $\theta(x; 0) = 1 - x$ and, hence, $(a; q, 0)_n = (a; q)_n$ is a *q-shifted factorial* in base q (see [18] for classical q-series notation, [18, Chapter 11] treats the elliptic case). The parameters q and p in $(a; q, p)_n$ are called the *base* and *nome*, respectively.

The modified Jacobi theta functions satisfy the following basic properties which are essential in the theory of elliptic hypergeometric series:

$$\theta(x;p) = -x\,\theta(1/x;p), \tag{3.1a}$$

$$\theta(px;p) = -\frac{1}{x}\,\theta(x;p), \tag{3.1b}$$

and the *addition formula*

$$\theta(xy, x/y, uv, u/v; p) - \theta(xv, x/v, uy, u/y; p) = \frac{u}{y}\,\theta(yv, y/v, xu, x/u; p) \tag{3.1c}$$

(cf. [46, p. 451, Example 5]).

As a matter of fact, the three-term relation in (3.1c), containing four variables and four factors of theta functions in each term, is the "smallest" addition formula connecting products of theta functions with general arguments. Note that in the theta function $\theta(x;p)$ we cannot let $x \to 0$ (unless we first let $p \to 0$) for x is a pole of infinite order. This is the reason why elliptic analogues of q-series identities usually contain many parameters.

The elliptic identities we shall consider all involve terms which are elliptic (with the same periods) in all of its parameters (see e.g. Remark 3.1). Spiridonov [40] refers to such multivariate functions as *totally elliptic*, and they are by nature *well-poised* and *balanced* (see also [18, Chapter 11]).

Inspired by earlier work of the first author regarding weighted lattice paths and elliptic binomial coefficients [34, 35], we now define the *elliptic weights* $w_{a,b;q,p}(k)$ and $W_{a,b;q,p}(k)$, depending on two independent parameters a and b, base q, nome p, and integer parameter k by

$$w_{a,b;q,p}(k) = \frac{\theta(aq^{2k+1}, bq^k, aq^{k-2}/b;p)}{\theta(aq^{2k-1}, bq^{k+2}, aq^k/b;p)}q, \tag{3.2a}$$

and

$$W_{a,b;q,p}(k) = \frac{\theta(aq^{1+2k}, bq, bq^2, aq^{-1}/b, a/b;p)}{\theta(aq, bq^{k+1}, bq^{k+2}, aq^{k-1}/b, aq^k/b;p)}q^k, \tag{3.2b}$$

respectively. Observe that if k is a positive integer, Equations (3.2a) and (3.2b) imply that

$$W_{a,b;q,p}(k) = \prod_{j=1}^{k} w_{a,b;q,p}(j). \tag{3.2c}$$

We refer to the $w_{a,b;q,p}(k)$ as *small* weights and to the $W_{a,b;q,p}(k)$ as *big* weights. Note that the weights $w_{a,b;q,p}(k)$ and $W_{a,b;q,p}(k)$ also can be defined for arbitrary (complex) k which is clear from the definition.

Observe that

$$w_{a,b;q,p}(k+n) = w_{aq^{2k},bq^k;q,p}(n), \tag{3.3a}$$

and

$$W_{a,b;q,p}(k+n) = W_{a,b;q,p}(k)\, W_{aq^{2k},bq^k;q,p}(n), \tag{3.3b}$$

for all k and n, which are elementary identities we frequently make use of.

Remark 3.1. The small weight $w_{a,b;q,p}(k)$ (and so the big one) is indeed elliptic in its parameters (i.e., totally elliptic). If we write $q = e^{2\pi i\sigma}$, $p = e^{2\pi i\tau}$, $a = q^\alpha$ and $b = q^\beta$ with complex σ, τ, α, β and k, then the small weight $w_{a,b;q,p}(k)$ is clearly periodic in α with period σ^{-1}. A simple computation involving (3.1b) further shows that $w_{a,b;q,p}(k)$ is also periodic in α with period $\tau\sigma^{-1}$. The same applies to $w_{a,b;q,p}(k)$ as a function in β (or k) with the same two periods σ^{-1} and $\tau\sigma^{-1}$.

Remark 3.2. For $p \to 0$, the small and big weights reduce to

$$w_{a,b;q}(k) = \frac{(1-aq^{2k+1})(1-bq^k)(1-aq^{k-2}/b)}{(1-aq^{2k-1})(1-bq^{k+2})(1-aq^k/b)}q, \tag{3.4a}$$

and

$$W_{a,b;q}(k)=\frac{(1-aq^{1+2k})(1-bq)(1-bq^2)(1-aq^{-1}/b)(1-a/b)}{(1-aq)(1-bq^{k+1})(1-bq^{k+2})(1-aq^{k-1}/b)(1-aq^k/b)}q^k, \tag{3.4b}$$

respectively. In the $a,b;q$-weights in (3.4), we may let $b\to 0$ (or $b\to\infty$) to obtain "$a,0;q$-weights", or in short, "$a;q$-weights":

$$w_{a;q}(k)=\frac{(1-aq^{2k+1})}{(1-aq^{2k-1})}q^{-1}, \qquad \text{and} \qquad W_{a;q}(k)=\frac{(1-aq^{1+2k})}{(1-aq)}q^{-k}. \tag{3.5}$$

Note that by writing $q=e^{ix}$ and $a=e^{i(2c+1)x}$, $c\in\mathbb{N}$, the $a;q$-weights can be written as quotients of Chebyshev polynomials of the second kind.

Also, in (3.4), we may let $a\to 0$ (or $a\to\infty$) to obtain "$0,b;q$-weights". Importantly, if in (3.4) we first let $b\to 0$ and then $a\to\infty$ (or, equivalently, first let $a\to 0$ and then $b\to 0$), we obtain the familiar q-weights

$$w_q(k)=q \qquad \text{and} \qquad W_q(k)=q^k, \tag{3.6}$$

respectively.

Next, for a variable z, we define an *elliptic number* of z by

$$[z]_{a,b;q,p}=\frac{\theta(q^z,aq^z,bq^2,a/b;p)}{\theta(q,aq,bq^{z+1},aq^{z-1}/b;p)}. \tag{3.7}$$

Using the addition formula for theta functions (3.1c), it is not difficult to verify that the thus defined elliptic numbers satisfy

$$[z]_{a,b;q,p}=[z-1]_{a,b;q,p}+W_{a,b;q,p}(z-1). \tag{3.8a}$$

In case $z=n$ is a nonnegative integer, (3.8a) constitutes a recursion which, together with $W_{a,b;q,p}(0)=1$, uniquely defines any elliptic number $[n]_{a,b;q,p}$. More generally, by (3.1c) we have the following useful identity

$$[z]_{a,b;q,p}=[y]_{a,b;q,p}+W_{a,b;q,p}(y)[z-y]_{aq^{2y},bq^y;q,p} \tag{3.8b}$$

which reduces to (3.8a) for $y=z-1$.

Remark 3.3. In [35], the first author, in analogy to the q-binomial coefficients

$$\begin{bmatrix} n\\ k\end{bmatrix}_q:=\frac{(q^{1+k};q)_{n-k}}{(q;q)_{n-k}}=\frac{[n]_q!}{[k]_q![n-k]_q!},$$

defined the elliptic binomial coefficients

$$\begin{bmatrix} n\\ k\end{bmatrix}_{a,b;q,p}:=\frac{(q^{1+k},aq^{1+k},bq^{1+k},aq^{1-k}/b;q,p)_{n-k}}{(q,aq,bq^{1+2k},aq/b;q,p)_{n-k}}. \tag{3.9}$$

In [35] the elliptic binomial coefficients in (3.9) were shown to satisfy an elliptic binomial theorem involving "elliptic commuting" variables. They were also shown to satisfy a nice recursion, namely

$$\begin{bmatrix} 0\\ 0\end{bmatrix}_{a,b;q,p}=1, \qquad \begin{bmatrix} n\\ k\end{bmatrix}_{a,b;q,p}=0 \qquad \text{for } n\in\mathbb{N}_0, \text{ and } k\in -\mathbb{N} \text{ or } k>n, \tag{3.10a}$$

and

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_{a,b;q,p} = \begin{bmatrix} n \\ k \end{bmatrix}_{a,b;q,p} + \begin{bmatrix} n \\ k-1 \end{bmatrix}_{a,b;q,p} W_{aq^{k-1},bq^{2k-2};q,p}(n+1-k) \qquad \text{for } n,k \in \mathbb{N}_0. \tag{3.10b}$$

The recurrence in (3.10b) is a consequence of the addition formula (3.1c).

On the combinatorial side, the elliptic binomial coefficient in (3.9) can be interpreted in terms of weighted lattice paths in $\mathbb{Z}^2$ (see [34]). In fact, (3.9) is the area generating function for paths starting in $(0,0)$ and ending in $(k, n-k)$ composed of unit steps going north or east only, when the weight of each cell (with north-east corner (s,t)) "covered" by the path is defined to be $w_{aq^{s-1},bq^{2s-2};q,p}(t)$. Then it can be shown that the sum of weighted areas below the paths satisfies the same recursion (3.10b) by distinguishing the last step of the path which is either vertical or horizontal. The elliptic number $[n]_{a,b;q,p}$ is nothing but a short-hand notation for

$$[n]_{a,b;q,p} = \begin{bmatrix} n \\ 1 \end{bmatrix}_{a,b;q,p},$$

the weighted enumeration of all paths starting in $(0,0)$ and ending in $(1, n-1)$. Note that the elliptic binomial coefficients are in general *not* symmetric with respect to replacing k by $n-k$. However, the $a;q$-binomial coefficients

$$\begin{bmatrix} n \\ k \end{bmatrix}_{a;q} := \frac{(q^{1+k}, aq^{1+k}; q)_{n-k}}{(q, aq; q)_{n-k}} q^{k(k-n)} \tag{3.11}$$

obtained from (3.9) by formally letting $p \to 0$ followed by $b \to 0$, are symmetric, i.e., they satisfy

$$\begin{bmatrix} n \\ k \end{bmatrix}_{a;q} = \begin{bmatrix} n \\ n-k \end{bmatrix}_{a;q},$$

and this is a reason for the $a;q$-case being special (see e.g. Prop. 3.11). The q-binomial coefficient is recovered by letting $a \to \infty$.

Now we are ready to develop an elliptic analogue of the q-rook theory. We employ the same rook cancellation as Garsia and Remmel considered in the q-case, i.e., a rook cancels all the cells to the right and below of it. However, as already indicated, our definition of an elliptic-weighted rook number requires a refined statistic depending on the specific locations of the uncancelled cells.

Definition 3.4. *Given a Ferrers board $B = B(b_1, \dots, b_n)$, we define the elliptic analogue of the k-th rook number by*

$$r_k(a,b;q,p;B) = \sum_{P \in \mathcal{N}_k(B)} wt(P), \tag{3.12a}$$

with

$$wt(P) = \prod_{(i,j) \in U_B(P)} w_{a,b;q,p}\big(i - j - r_{(i,j)}(P)\big), \tag{3.12b}$$

where the elliptic weight $w_{a,b;q,p}(l)$ of an integer l is defined in (3.2a)*, $U_B(P)$ is the set of cells in B which are neither cancelled by rooks nor contain any rooks of P, and $r_{(i,j)}(P)$ is the number of rooks in P which are in the north-west region of (i,j).*

Example 3.5. Consider a Ferrers board $B = B(3,3,3)$ and let P be the placement of two rooks in $(1,3)$ and $(3,1)$ in B. Then the set of uncancelled cells $U_B(P)$ is

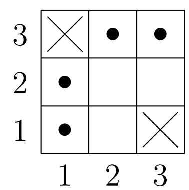


FIGURE 3. A rook placement in $(1,3)$ and $(3,1)$.

$\{(2,1),(2,2),(3,2)\}$. Note that for all the uncancelled cells $(i,j) \in U_B(P)$, $r_{(i,j)}(P) = 1$ due to the rook in $(1,3)$. Then $wt(P)$ is

$$\begin{aligned} wt(P) &= w_{a,b;q,p}(2-1-1) \cdot w_{a,b;q,p}(2-2-1) \cdot w_{a,b;q,p}(3-2-1) \\ &= \frac{\theta(aq^{-1}, bq^{-1}, aq^{-3}/b;p)}{\theta(aq^{-3}, bq, aq^{-1}/b;p)} \cdot \frac{\theta(aq, b, aq^{-2}/b;p)^2}{\theta(aq^{-1}, bq^2, a/b;p)^2} q^3. \end{aligned}$$

If we place three rooks in all possible ways in B and compute $r_3(a,b;q,p;B)$, then

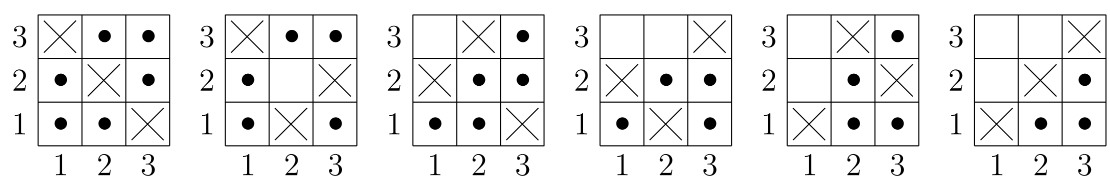


FIGURE 4. Rook placements of three rooks in $[3] \times [3]$.

$$\begin{aligned} &r_3(a,b;q,p;B) \\ &= 1 + w_{a,b;q,p}(-1) + w_{a,b;q,p}(-2) + 2 \cdot w_{a,b;q,p}(-2) w_{a,b;q,p}(-1) + w_{a,b;q,p}(-2) w_{a,b;q,p}(-1)^2 \\ &= (1 + w_{aq^{-6},bq^{-3};q,p}(1) + w_{aq^{-6},bq^{-3};q,p}(1) w_{aq^{-6},bq^{-3};q,p}(2))(1 + w_{aq^{-6},bq^{-3};q,p}(2)) \\ &= (1 + W_{aq^{-6},bq^{-3};q,p}(1) + W_{aq^{-6},q^{-3}b;q,p}(2))(1 + W_{aq^{-4},bq^{-2};q,p}(1)) \\ &= [3]_{aq^{-6},bq^{-3};q,p}[2]_{aq^{-4},bq^{-2};q,p}, \end{aligned}$$

where we used the property (3.3a). In general, for $B = B(n,n,\dots,n) = [n] \times [n]$, we have

$$r_n(a,b;q,p;B) = [n]_{aq^{-2n},bq^{-n};q,p}[n-1]_{aq^{2-2n},bq^{1-n};q,p} \cdots [1]_{aq^{-2},bq^{-1};q,p}. \tag{3.13}$$

See Corollary 3.9 for a proof.

The following lemma plays an essential role in the subsequent developments leading to the product formula in Theorem 3.8.

Lemma 3.6. *Let $B = B(b_1, \dots, b_n) \subset [n] \times \mathbb{N}$ be a board of n columns and let B_k denote the extended board by attaching an $[n] \times [k]$ board below B (the additional rows being indexed by $0, -1, \dots, -k+1$). Suppose that $Q \in \mathcal{N}_t(B_k)$ is a rook placement of t rooks in the first $i-1$ columns of B_k. Let $D_i(Q)$ denote the set of all rook placements which extend Q by adding a rook in column i. Then we have*

$$\sum_{P \in D_i(Q)} wt(P) = [b_i + k - t]_{aq^{2(i-1-b_i)}, bq^{i-1-b_i}; q,p} \, wt(Q). \tag{3.14}$$

Proof. Let $i = 1$. We want to show that

$$\sum_{P \in D_1(Q)} wt(P) = [b_1 + k]_{aq^{-2b_1}, bq^{-b_1}; q,p}.$$

If we consider all possible rook placements P in the first column and sum up all the weights of P, then we obtain

$$\begin{aligned} \sum_{P \in D_1(Q)} wt(P) &= 1 + w_{aq^{-2b_1}, bq^{-b_1}; q,p}(1) + \cdots + \prod_{j=1}^{b_1+k-1} w_{aq^{-2b_1}, bq^{-b_1}; q,p}(j) \\ &= 1 + \sum_{j=1}^{b_1+k-1} W_{aq^{-2b_1}, bq^{-b_1}; q,p}(j) \\ &= [b_1 + k]_{aq^{-2b_1}, bq^{-b_1}; q,p}, \end{aligned}$$

where the sum telescopes according to (3.8a).

Now given Q, a rook placement of t rooks in the first $i-1$ columns, we consider all possible rook placements of one additional rook in the i-th column. If we place the i-th rook in the topmost possible place, then it cancels all the empty cells below and so the weight coming from that rook placement is 1. Say we placed the i-th rook in the second topmost possible place. Then there is one empty cell which was the topmost possible cell to place a rook. If the coordinate of that cell is $(i, b_i - l_1)$, then that means there are l_1 many rooks in the north-west region of that cell. So the weight of this cell would be

$$\begin{aligned} w_{a,b;q,p}(i - b_i + l_1 - l_1) &= w_{a,b;q,p}(i - 1 - b_i + 1) \\ &= w_{aq^{2(i-1-b_i)}, bq^{i-1-b_i}; q,p}(1), \end{aligned}$$

by (3.3a). If we place the i-th rook in the third topmost place, then the weight of the second empty cell would be

$$\begin{aligned} w_{a,b;q,p}(i - b_i + l_1 + l_2 + 1 - l_1 - l_2) &= w_{a,b;q,p}(i - 1 - b_i + 2) \\ &= w_{aq^{2(i-1-b_i)}, bq^{i-1-b_i}; q,p}(2), \end{aligned}$$

where l_2 is the number of rows between the topmost empty cell and the second topmost empty cell. If we place the i-th rook in the bottom-most possible cell, the weight coming from that placement would be

$$\prod_{j=1}^{b_i+k-t-1} w_{aq^{2(i+n-b_i-1)}, bq^{i+n-b_i-1}; q,p}(j).$$

Hence by summing up all the weights coming from the all possible rook placements of the $(t+1)$-st rook in the i-th column, we get

$$1+\sum_{s=1}^{b_i+k-t-1}\prod_{j=1}^{s} w_{aq^{2(i-1-b_i)},bq^{i-1-b_i};q,p}(j) = 1+\sum_{s=1}^{b_i+k-t-1} W_{aq^{2(i-1-b_i)},bq^{i-1-b_i};q,p}(s)$$
$$= [b_i+k-t]_{aq^{2(i-1-b_i)},bq^{i-1-b_i};q,p}.$$

Combining this with the weights coming from the placement Q, we obtain (3.14). $\square$

The following proposition constitutes an elliptic extension of Proposition 2.1. As before, B^∞ denotes the Ferrers board obtained by appending below B the infinite board of width n, and for a rook placement P in B^∞, $\max(P)$ denotes the number of rows below the ground in which the lowest rook is located.

Proposition 3.7. *For a Ferrers board $B = B(b_1, b_2, \ldots, b_n)$, we have*

$$\frac{1}{1-z}\sum_{P\in\mathcal{N}_n(B^\infty)} z^{max(P)}\cdot wt(P) = \sum_{k\geq 0} z^k \prod_{i=1}^{n} [k+b_i-i+1]_{aq^{2(i-1-b_i)},bq^{i-1-b_i};q,p}. \tag{3.15}$$

Proof. We first show the following identity

$$\sum_{P\in\mathcal{N}_n(B^\infty)} wt(P)\cdot\chi(\max(P)\leq k) = \prod_{i=1}^{n}[k+b_i-i+1]_{aq^{2(i-1-b_i)},bq^{i-1-b_i};q,p}, \tag{3.16}$$

where χ is the truth function, i.e. $\chi(A) = 1$ if the statement A is true, otherwise $\chi(A) = 0$. This easily follows from Lemma 3.6 by iteration, using the fact that each column of B^∞ contains a rook. Then (3.15) is obtained by multiplying both sides of (3.16) by z^k and summing over all $k \geq 0$. $\square$

The following product formula is the main result of this section.

Theorem 3.8. *Let $B = B(b_1, \ldots, b_n)$ be a Ferrers board. Then we have*

$$\sum_{k=0}^{n} r_{n-k}(a,b;q,p;B)\prod_{j=1}^{k}[z-j+1]_{aq^{2(j-1)},bq^{j-1};q,p}$$
$$= \prod_{i=1}^{n}[z+b_i-i+1]_{aq^{2(i-1-b_i)},bq^{i-1-b_i};q,p}. \tag{3.17}$$

Proof. It suffices to prove the theorem for nonnegative integer values of z. The result follows then by analytic continuation.

We consider the extended board B_z by attaching an $[n]\times[z]$ board below B and compute

$$\sum_{P\in\mathcal{N}_n(B_z)} wt(P) \tag{3.18}$$

in two different ways. On one hand, (3.18) can be evaluated using the $k = z$ case of (3.16) which thus explains the right-hand side of (3.17). On the other hand, in (3.18) we can consider, for each $0 \leq k \leq n$, the contributions from the k-rook configurations below the ground, yielding $\prod_{j=1}^{k}[z-j+1]_{aq^{2(j-1)},bq^{j-1};q,p}$, and those from the $(n-k)$ rooks in B, yielding $r_{n-k}(a,b;q,p;B)$, separately. This explains the left-hand side of (3.17). □

The following corollary is an easy consequence of Theorem 3.8.

Corollary 3.9. *Let $B = B(b_1,\ldots,b_n)$ be a Ferrers board. Then we have*

$$r_n(a,b;q,p;B) = \prod_{i=1}^{n}[b_i-i+1]_{aq^{2(i-1-b_i)},bq^{i-1-b_i};q,p}. \tag{3.19}$$

In particular, for the square shape Ferrers board $B = B(n,n,\ldots,n) = [n]\times[n]$, we have

$$r_n(a,b;q,p;B) = [n]_{aq^{-2n},bq^{-n};q,p}[n-1]_{aq^{2-2n},bq^{1-n};q,p}\ldots[1]_{aq^{-2},bq^{-1};q,p}. \tag{3.20}$$

Proof. In Theorem 3.8 we let $z\to 0$. Since

$$\prod_{j=1}^{k}[1-j]_{aq^{2(j-1)},bq^{j-1};q,p} = \delta_{k,0},$$

the left-hand side of (3.17) reduces to one term only, corresponding to $k=0$. □

We also establish an elliptic analogue of Proposition 2.3, a recursion for elliptic rook numbers.

Theorem 3.10. *Let B be a Ferrers board with l columns of height at most m, and $B\cup m$ denote the board obtained by adding the $(l+1)$-st column of height m to B. Then, for any integer k, we have*

$$r_k(a,b;q,p;B) = 0 \qquad \text{for } k<0 \text{ or } k>l, \tag{3.21a}$$

$$r_0(a,b;q,p;B) = 1 \qquad \text{for } l=0, \text{ i.e. for } B \text{ being the empty board,} \tag{3.21b}$$

and

$$\begin{aligned} r_k(a,b;q,p;B\cup m) &= W_{aq^{2(l-m)},bq^{l-m};q,p}(m-k)\, r_k(a,b;q,p;B) \\ &\quad + [m-k+1]_{aq^{2(l-m)},bq^{l-m};q,p}\, r_{k-1}(a,b;q,p;B). \end{aligned} \tag{3.21c}$$

Proof. This recursion stems from a weighted enumeration of placements of k nonattacking rooks on $B\cup m$. We distinguish the cases whether there is a rook in the last column or not. The first term on the right-hand side of (3.21c) is obtained when there is no rook in the last column. The weight multiplied in front of $r_k(a,b;q,p;B)$ comes from the uncancelled $(m-k)$ cells in the last column. The second term on the right-hand side of (3.21c) is obtained when there is a rook in the last column. The coefficient in front of $r_{k-1}(a,b;q,p;B)$ is a consequence of Lemma 3.6. □

For $p\to 0$, followed by $b\to 0$ the above recurrence relation (3.21c) reads

$$r_k(a;q;B\cup m) = W_{aq^{2(l-m)};q}(m-k)\, r_k(a;q;B) + [m-k+1]_{aq^{2(l-m)};q}\, r_{k-1}(a;q;B), \tag{3.22}$$

where according to (3.5) and the $p \to 0$, then $b \to 0$ case of (3.7),

$$W_{a;q}(k) = \frac{(1-aq^{1+2k})}{(1-aq)} q^{-k}, \qquad \text{and} \qquad [z]_{a;q} = \frac{(1-q^z)(1-aq^z)}{(1-q)(1-aq)} q^{1-z}, \tag{3.23}$$

and the $a;q$-rook numbers are given by

$$r_k(a;q;B) = \lim_{b\to 0}\left(\lim_{p\to 0} r_k(a,b;q,p;B)\right),$$

or

$$r_k(a;q;B) = \sum_{P\in\mathcal{N}_k(B)}\left(\prod_{(i,j)\in U_B(P)} w_{a;q}(i-j-r_{(i,j)}(P))\right).$$

As an immediate consequence of this recursion, we have the following product formula for the $a;q$-rook numbers of a rectangular shape board $B=[l]\times[m]$ with l columns and m rows.

Proposition 3.11.

$$r_k(a;q;[l]\times[m]) = q^{\binom{k+1}{2}-lm}\begin{bmatrix} l\\ k\end{bmatrix}_q \frac{[m]_q!}{[m-k]_q!}\frac{(aq^{l-m-k};q)_k(aq^{1+2l-2m};q^2)_{m-k}}{(aq^{1-2m};q^2)_m}. \tag{3.24}$$

Proof. This follows by induction on l, the $l=0$ case being trivial. In the computation of $r_k(a;q;[l+1]\times[m])$ as a sum of two explicit terms according to the recurrence relation (3.22), after pulling out common factors, the sum of the two terms nicely factorizes due to the simple identity

$$(1-q^{l-k+1})(1-aq^{l-m-k})q^k + (1-q^k)(1-aq^{2l-m-k+1}) = (1-q^{l+1})(1-aq^{l-m}). \qquad \square$$

Note that the elliptic rook numbers and even the $a,b;q$-rook numbers (obtained from the elliptic rook numbers by letting $p\to 0$), nor the $0,b;q$-rook numbers, of rectangular shape boards in general do not factorize (unless $k=0$ or $k=l$). They in fact already don't factorize in the case $l=2$ and $k=1$ (and $m>1$).

We now take a close look at several special cases of elliptic rook numbers of particular interest.

3.1. **Elliptic Stirling numbers of the second kind.** The Stirling numbers of the second kind admit a nice rook theoretic interpretation when B is a staircase board $\mathsf{St}_n = B(0,1,\dots,n-1)$ (see [42, Corollary 2.4.2]). Namely, for each configuration of $n-k$ nonattacking rooks on St_n, we can associate a set partition of $[n]$ in k blocks. Whenever a cell (i,j) is occupied by a rook, i and j are put in the same block, and the numbers which are not contained in any block in this way correspond to single blocks. This describes a one-to-one correspondence between configurations of $n-k$ nonattacking rooks on St_n and set partitions of $[n]$ into k blocks. Garsia and Remmel [17] extended this to the q-case, thus providing a rook theoretic realization of Carlitz' [6, 7] q-Stirling numbers.

We consider the staircase board St_n to define an elliptic analogue of the Stirling numbers of the second kind. For $b_i = i-1$, $i=1,\dots,n$, Equation (3.17) becomes

$$([z]_{a,b;q,p})^n = \sum_{k=0}^{n} r_{n-k}(a,b;q,p;\mathsf{St}_n) \prod_{j=1}^{k} [z-j+1]_{aq^{2(j-1)},bq^{j-1};q,p}. \tag{3.25}$$

The $r_{n-k}(a,b;q,p;\mathsf{St}_n)$ are actually the *elliptic Stirling numbers of the second kind* $\mathcal{S}_{a,b;q,p}(n,k)$ which have recently been defined and studied (in a different setting) by Zsófia Kereskényiné Balogh and the first author [27].

By using the $y=k$ case of the elementary identity (3.8b), we obtain from (3.25) the following recursion

$$\mathcal{S}_{a,b;q,p}(n,k) = 0 \qquad \text{for } k<0 \text{ or } k>n,$$
$$\mathcal{S}_{a,b;q,p}(0,0) = 1,$$

and, for $k \geq 0$,

$$\mathcal{S}_{a,b;q,p}(n+1,k) = W_{a,b;q,p}(k-1)\mathcal{S}_{a,b;q,p}(n,k-1) + [k]_{a,b;q,p}\mathcal{S}_{a,b;q,p}(n,k), \tag{3.26}$$

which also can be derived from Theorem 3.10.

An explicit formula for the elliptic Stirling numbers $\mathcal{S}_{a,b;q,p}(n,k)$ has not yet been established. However, in [27] the following formulae for small k have been worked out.

$$\mathcal{S}_{a,b;q,p}(n,0) = \delta_{n,0}, \tag{3.27a}$$
$$\mathcal{S}_{a,b;q,p}(n,1) = 1-\delta_{n,0}, \tag{3.27b}$$
$$\mathcal{S}_{a,b;q,p}(n,2) = [2]_{a,b;q,p}^{n-1} - 1, \tag{3.27c}$$
$$\mathcal{S}_{a,b;q,p}(n,3) = \frac{1}{[2]_{aq^2,bq;q,p}} \left([3]_{a,b;q,p}^{n-1} - [2]_{aq^2,bq;q,p}[2]_{a,b;q,p}^{n-1} + w_{a,b;q,p}(2)\right). \tag{3.27d}$$

For $p \to 0$, followed by $a \to 0$ and $b \to 0$, these explicit evaluations can be easily seen to match the special instances $k=0,1,2,3$ of Carlitz' [7, Equation (3.3)] well-known formula

$$\mathcal{S}_q(n,k) = \frac{1}{[k]_q!} \sum_{j=0}^{k} (-1)^j q^{\binom{j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_q [k-j]_q^n. \tag{3.28}$$

As a matter of fact, the right-hand side of (3.28) can also be rewritten in terms of basic hypergeometric series (see [18] for definitions and notation). As such, the q-Stirling number of the second kind can be expressed as the following multiple of a basic hypergeometric series of Karlsson–Minton type:

$$\mathcal{S}_q(n,k) = \frac{[k]_q^n}{[k]_q!}\, {}_n\phi_{n-1}\left[\begin{matrix} q^{1-k}, q^{1-k}, \dots, q^{1-k} \\ q^{-k}, \dots, q^{-k} \end{matrix}; q, q^{k-n}\right]. \tag{3.29}$$

The existence of the latter series representation is not so surprising, if one recalls that a big class of (q-)rook numbers generally admit a representation in terms of (basic) hypergeometric series of Karlsson–Minton type, as revealed by Haglund [21].

Coming back to our quest for finding an explicit formula in the elliptic case, it is at this moment still not entirely clear how the pattern in (3.27) for the elliptic Stirling numbers can be extended to a formula for $\mathcal{S}_{a,b;q,p}(n,k)$ valid for general k.

3.2. **Elliptic r-restricted Stirling numbers of the second kind.** The r-restricted Stirling numbers of the second kind count the number of partitions of $[n]$ into k blocks such that each of the first r numbers $1,2\dots,r$ is in a different block (cf. [4] or [28]). The case $r=1$ (or $r=0$) gives the usual Stirling numbers of the second kind. In the literature, the r-restricted Stirling numbers of the second kind are usually just called r-Stirling numbers of the second kind. Nevertheless, in [38, see sequences A143494, A143495 and A143496] they are referred to as "r-restricted", a terminology which we adopt here, mainly to avoid confusion with the q-Stirling numbers of the second kind. These numbers admit a rook theoretic interpretation when B is a cut-off staircase board $\mathsf{St}_n^{(r)}=B(0,\dots,0,r,r+1,\dots,n-1)$ of n columns, the first r columns being empty. The correspondence between the $n-k$ nonattacking rook placements in $\mathsf{St}_n^{(r)}$ and the set partitions of $[n]$ in k blocks works exactly in the same way as for the board St_n. Then the shape of the board $\mathsf{St}_n^{(r)}$ puts $1,2,\dots,r$ automatically in different blocks.

We use $\mathsf{St}_n^{(r)}$ in (3.17) to define an elliptic extension of the r-restricted Stirling numbers of the second kind. For $b_i=0$ for $i=1,\dots,r$, and $b_i=i-1$ for $i=r+1,\dots,n$, Equation (3.17) becomes

$$\begin{aligned}&([z]_{a,b;q,p})^{n-r}\prod_{i=1}^{r}[z-i+1]_{aq^{2(i-1)},bq^{i-1};q,p}\\&=\sum_{k=0}^{n}r_{n-k}(a,b;q,p;\mathsf{St}_n^{(r)})\prod_{j=1}^{k}[z-j+1]_{aq^{2(j-1)},bq^{j-1};q,p}.\end{aligned}\tag{3.30}$$

Defining $\mathcal{S}^{(r)}_{a,b;q,p}(n,k):=r_{n-k}(a,b;q,p;\mathsf{St}_n^{(r)})$ to be the *elliptic r-restricted Stirling numbers of the second kind*, we obtain from Theorem 3.10 the following recursion

$$\begin{aligned}\mathcal{S}^{(r)}_{a,b;q,p}(n,k)&=0 &&\text{for } k<r-1 \text{ or } k>n,\\ \mathcal{S}^{(r)}_{a,b;q,p}(r-1,r-1)&=1 &&\text{(an artificial but felicitous initial condition),}\end{aligned}$$

and, for $k\geq r-1$,

$$\mathcal{S}^{(r)}_{a,b;q,p}(n+1,k)=W_{a,b;q,p}(k-1)\mathcal{S}^{(r)}_{a,b;q,p}(n,k-1)+[k]_{a,b;q,p}\mathcal{S}^{(r)}_{a,b;q,p}(n,k).\tag{3.31}$$

3.3. **Elliptic Lah numbers.** The q-Lah numbers $\mathcal{L}_{n,k}(q)$ have first been studied by Garsia and Remmel in [16] by carrying out a q-counting of placements of n distinguishable balls in k nonempty indistinguishable tubes which have a linear order on its elements. The same authors, in [17], subsequently gave a rook theoretic interpretation by considering the board $\mathsf{L}_n=[n]\times[n-1]$ of n columns, each of height $n-1$. In this case, Proposition 2.2 gives

$$[z]_q\!\uparrow_n=\sum_{k=0}^{n}r_{n-k}(q;B)[z]_q\!\downarrow_k,$$

where $[z]_q\!\uparrow_n=[z]_q[z+1]_q\cdots[z+n-1]_q$ and $[z]_q\!\downarrow_k=[z]_q[z-1]_q\cdots[z-k+1]_q$. If we let $\mathcal{L}_{n,k}(q)=r_{n-k}(q;\mathsf{L}_n)$, then the q-Lah numbers $\mathcal{L}_{n,k}(q)$ satisfy the recursion

$$\mathcal{L}_{n+1,k}(q)=q^{n+k-1}\mathcal{L}_{n,k-1}(q)+[n+k]_q\mathcal{L}_{n,k}(q).\tag{3.32}$$

This can be established by placing $n+1-k$ nonattacking rooks on L_{n+1} and distinguishing the cases whether there is a rook or not in the union of the top row and the last column. If there is no such rook, we remove the top row and last column (which contributes weight q^{n+k-1}) and consider $n+1-k$ nonattacking rooks on the smaller board L_n. If, for $1 \le j \le n+1$, there is a rook in the j-th position of the top row, the weight of the uncancelled cells coming from this rook (which are located to the left of the rook) will be q^{j-1}. We then remove the top row and j-th column and are left with a smaller board on which $n-k$ nonattacking rooks are placed. If there is no rook in the top row, there must be one in the last column (but not on the most top of that column). The weight of the uncancelled cells (the top row included) coming from this rook will be q^{n+l-1}, for some $2 \le l \le k$, depending on the position of the other $n-k$ rooks. The precise analysis is similar to that of the proof of Lemma 3.6. After removing the top row and last column (the possible weights adding up to $[n+k]_q$), we are again left with a smaller board on which $n-k$ nonattacking rooks are placed.

Using the recursion in (3.32), one can verify that $\mathcal{L}_{n,k}(q)$ has the following closed form

$$\mathcal{L}_{n,k}(q) = q^{k(k-1)} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{[n-1]_q!}{[k-1]_q!}. \tag{3.33}$$

Now we turn to the elliptic setting. For the board L_n, Theorem 3.8 gives

$$\begin{aligned}
&[z+n-1]_{aq^{2-2n},bq^{1-n};q,p}[z+n-2]_{aq^{4-2n},bq^{2-n};q,p}\cdots[z]_{a,b;q,p} \\
&= \sum_{k=1}^{n} r_{n-k}(a,b;q,p;\mathsf{L}_n)\ [z]_{a,b;q,p}[z-1]_{aq^2,bq^1;q,p}\cdots[z-k+1]_{aq^{2k-2},bq^{k-1};q,p}.
\end{aligned} \tag{3.34}$$

Let $\mathcal{L}_{n,k}(a,b;q,p)$ denote $r_{n-k}(a,b;q,p;\mathsf{L}_n)$. This defines an elliptic analogue of Lah numbers and matches those which have been defined and studied (in a different setting) by Kereskényiné Balogh and the first author [27]. Then by distinguishing whether there is a rook in the union of the top row and the last column of the board L_{n+1} or not, we obtain the following recursion for $\mathcal{L}_{n,k}(a,b;q,p)$:

$$\begin{aligned}
\mathcal{L}_{n+1,k}(a,b;q,p) &= W_{aq^{-2n},bq^{-n};q,p}(n+k-1)\,\mathcal{L}_{n,k-1}(a,b;q,p) \\
&\quad +[n+k]_{aq^{-2n},bq^{-n};q,p}\,\mathcal{L}_{n,k}(a,b;q,p).
\end{aligned} \tag{3.35}$$

Unfortunately, this elliptic analogue of Lah number does not have a nice closed form, but if we let $p \to 0$, followed by $b \to 0$, then it has the following closed form

$$\mathcal{L}_{n,k}(a;q) = q^{\binom{k}{2}-\binom{n}{2}-n(k-1)} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{[n-1]_q!}{[k-1]_q!} \frac{(aq^{k-n+1};q)_{n+k}}{(aq^{3-2n};q^2)_n(aq^2;q^2)_k}, \tag{3.36}$$

the formula being a consequence of the $(l,m,k) \mapsto (n,n-1,n-k)$ case of Proposition 3.11. It is not difficult to verify that the $a;q$-Lah numbers $\mathcal{L}_{n,k}(a;q)$ converge to the q-Lah numbers $\mathcal{L}_{n,k}(q)$ when $a \to \infty$.

Remark 3.12. Goldman, Joichi and White [19] observed that if the left-hand sides of the product formula (1.1) are equal for two different Ferrers boards B_1 and B_2, then also the rook numbers for B_1 and B_2 must be the same. In this case the two Ferrers boards B_1 and B_2 are called *rook equivalent*. By appealing to the q-analogue of the factorization theorem stated in Proposition 2.2, Garsia and Remmel [17] observed that Goldman, Joichi and White's observation readily extends to the q-case, i.e., two Ferrers boards that have the same rook numbers must also have the same q-rook numbers. For instance, the q-Lah number $\mathcal{L}_{n,k}(q)$ can also be obtained as the q-rook number of the Ferrers board $B(0,2,4,\dots,2n-2)$. Theorem 3.8 guarantees that this further extends to the elliptic case, namely, two rook equivalent Ferrers boards have the same elliptic rook numbers. In particular,

$$\mathcal{L}_{n,k}(a,b;q,p) = r_{n-k}(a,b;q,p;B_1) = r_{n-k}(a,b;q,p;B_2),$$

for $B_1 = \mathsf{L}_n = (n-1,\dots,n-1)$ (n occurrences of $n-1$) and $B_2 = B(0,2,4,\dots,2n-2)$. This appears to be not at all obvious from the combinatorial interpretation.

3.4. **Elliptic r-restricted Lah numbers.** The r-restricted Lah numbers count the number of placements of the elements $1,2,\dots,n$ into k nonempty tubes of linearly ordered elements such that $1,2,\dots,r$ are in distinct tubes, cf. [31] or [28]. The case $r=1$ (or $r=0$) gives the usual unsigned Lah numbers. In the literature, the r-restricted Lah numbers are usually just called r-Lah numbers. We use "restricted", in accordance with the terminology used in [38, see sequences A143497, A143498 and A143499], to avoid confusion with the q-Lah numbers. These numbers admit a rook theoretic interpretation when B is the board $\mathsf{L}_n^{(r)} = [n+r-1]\times[n-r]$ of $n+r-1$ columns, each of height $n-r$. In the following, we describe a simple correspondence between the rook configurations P of $n-k$ nonattacking rooks on the board $B=[n+r-1]\times[n-r]$ and the set of placements T of the elements $1,2,\dots,n$ into k nonempty tubes of linearly ordered elements such that the first r numbers $1,2,\dots,r$ are in distinct tubes: given a rook configuration of $n-k$ nonattacking rooks on $\mathsf{L}_n^{(r)}$, we have $n-k$ rows containing rooks and $k-r$ rows containing no rooks. We start with the trivial tube placement $T_0=\{(1),(2),\dots,(r)\}$ of singletons and want to successively build up the final tube placement by adding an element for each of the $n-r$ rows, depending on the existence and the positions of the rooks. Now, as mentioned, there are exactly $k-r$ rows without rooks, say in rows $l_1,\dots,l_{k-r}$ (without loss of generality, we may assume $n-r\ge l_1>l_2>\dots>l_{k-r}\ge 1$). These indices will determine the minimal elements of the new tubes which we append to T_0. These $k-r$ additional elements shall remain minimal elements, and we shall refer to them as *designated tube leaders*. (On the contrary, the elements $1,2,\dots,r$ do not necessarily remain as tube leaders in the final tube placement T.) We thus replace T_0 by $T_1=\{(1),(2),\dots,(r),(n+1-l_1),\dots,(n+1-l_{k-r})\}$, and that is a new placement of exactly k tubes of singletons. The $n-k$ remaining rows in P contain rooks and are indexed by $[n-r]\setminus\{l_1,\dots,l_{k-r}\}$. We add r to each of these indices, thus obtain the index set $I=([n]\setminus[r])\setminus\{r+l_1,\dots,r+l_{k-r}\}$ which contain exactly the numbers of $[n]$ which have not been already used in the tube T_1. We remove the top-most rook in P, say $\mathbf{r}_1$, and identity it with the smallest element in I, say ι_1. Since there are $n+r-1$ columns in B of which $n-k-1$ columns contain rooks below $\mathbf{r}_1$, there are exactly $k+r$ possibilities for $\mathbf{r}_1$ to be placed in its row. On the other side, there are exactly $(n+r-1)-(n-k-1)=k+r$ possible positions for the smallest element ι_1 of I to be added to T_1. That is, ι_1 can be placed on top of any element (which gives k possibilities), or below any element except the $k-r$ designated tube leaders (which gives r additional possibilities). In total we have $k+r$ possible positions to insert ι_1 in T_1, after which we obtain T_2. We now remove $\mathbf{r}_1$ from P and also delete ι_1 from I. In P, we turn to the next row from the top containing a rook, say $\mathbf{r}_2$, remove it and identify it with the next smallest element in $I \setminus \{\iota_1\}$ which we label ι_2. Now there are $k+r+1$ possibilities to place $\mathbf{r}_2$ in its row, and there are also exactly $k+r+1$ possibilities for ι_2 to be inserted in T_2. We iterate this, and in the end have $n+r-1$ possibilities for the $(n-k)$-th rook, say $\mathbf{r}_{n-k}$ to be placed in its row, and accordingly, $n+r-1$ possible positions to insert the maximal element of I, say ι_{n-k}, in the placement T_{n-k} of tubes after which we finally obtain the final tube placement T.

In total we have

$$\binom{n-r}{k-r}\frac{(n+r-1)!}{(k+r-1)!} = \binom{n+r-1}{k+r-1}\frac{(n-r)!}{(k-r)!}$$

such placements. This number matches the r-restricted Lah number.

For a concrete example of a rook configuration mapped to a placement of elements in tubes, see Figure 5, where we have chosen $n=8$, $r=2$, $k=4$. We consider a placement

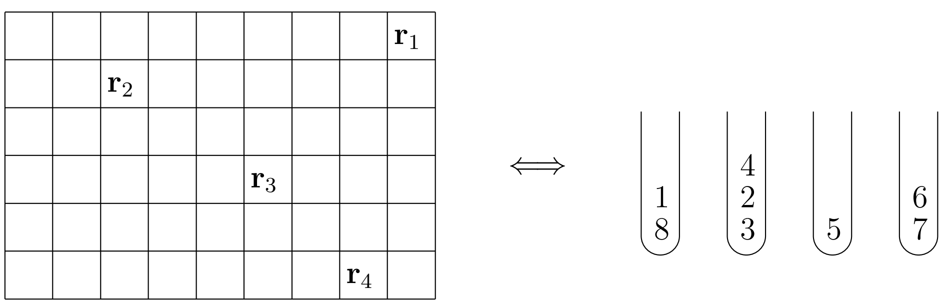


FIGURE 5. $n=8$, $r=2$, $k=4$; $B=[9]\times[6]$

of 4 nonattacking rooks on $B=[9]\times[6]$, the rooks being in the cells $(9,6)$, $(3,5)$, $(6,3)$, and $(8,1)$ (from top to bottom). We start with putting the numbers 1 and 2 into the (the first) two distinct tubes. Since the third and fifth row from the top of B contain no rooks, we put the third and fifth smallest numbers not already used, i.e. 5 and 7, into the third and fourth tubes, respectively. The numbers 5 and 7 are designated tube leaders. They will remain to be the minimal elements of their tubes. Now consider the top-most rook, $\mathbf{r}_1$, which is in $(9,6)$. If we remove it and decide to put it back into the top row, we have 6 possibilities. The possible positions are $(1,6)$, $(2,6)$, $(4,6)$, $(5,6)$, $(7,6)$ and $(9,6)$. From these $(9,6)$ is the sixth position. Accordingly, there are 6 different positions for the smallest number not already used in one of the tubes, i.e. 3, to be placed in one of the tubes. The six different choices are as follows: the element 3 can be put on top of the element 1, on top of the element 2, on top of the element 5, on top of the element 7, below the element 1, or below the element 2. (Nothing can be put below 5 or below 7 since they are designated tube leaders.) We take the sixth option (as $(9,6)$ was the sixth possible position of its row), which means that we put 3 below 2 (which is in the second tube). We chop off the top row of the board and consider the next rook from the top, $\mathbf{r}_2$, which is in position $(3,5)$. This rook occupies the third possible position in its row, of 7 possible positions in total. Accordingly, we put the element 4 (which is the smallest number not already used in the tube placement) on top of the element 2 which is the third possible position of 7 possibilities (on top of 1, on top of 3, on top of 2, on top of 5, on top of 7, below 1, or below 2.) We again chop off the top row and also the empty row below it and turn to the next rook from the top, $\mathbf{r}_3$, which is in position $(6, 3)$. This rook occupies the sixth possible position in its row, of 8 possible positions in total. Accordingly, we put the element 6 (which is the smallest number not already used in the tube placement) on top of the element 7 which is the sixth possible position of 8 possibilities (on top of 1, on top of 3, on top of 2, on top of 4, on top of 5, on top of 7, below 1, or below 3.) We again chop off the top row and also the empty row below it and turn to the last remaining rook, $\mathbf{r}_4$, which is in position $(8, 1)$. This is the eighth possible position of 9 possible positions in total. In the tube placement, the eight possibility is the position below the element 1. Thus we put the last element not yet appearing in the placements of tubes, i.e., the element 8, below 1. Finally we have arrived at the placement $\{(8, 1), (3, 2, 4), (5), (7, 6)\}$, written as a set of ordered lists.

Remark 3.13. Note that in the case when $r = 1$, this algorithm reduces to the case of the original Lah numbers $\mathcal{L}_{n,k}$ which counts the number of ways placing n distinguishable balls in k nonempty tubes. Garsia and Remmel [16] also provided a correspondence between $r_{n-k}(\mathsf{L}_n)$ and a placement of n balls in k tubes, but their correspondence is different from ours explained above.

We use the board $\mathsf{L}_n^{(r)}$ in Theorem 3.8 to define an elliptic analogue of the r-restricted Lah numbers. For $B = \mathsf{L}_n^{(r)} = [n+r-1] \times [n-r]$, i.e. $b_i = n-r$ for $i = 1, \dots, n+r-1$, Theorem 3.8 becomes

$$\begin{aligned}\sum_{k=2r-1}^{n+r-1} & r_{n+r-1-k}(a, b; q, p; \mathsf{L}_n^{(r)}) \prod_{j=1}^{k} [z-j+1]_{aq^{2(j-1)}, bq^{j-1}; q,p} \\ &= \prod_{i=1}^{n+r-1} [z+n-r-i+1]_{aq^{2(i-1-n+r)}, bq^{i-1-n+r}; q,p} \end{aligned} \tag{3.37}$$

where we can readily start the index of summation with $k = 2r-1$ since the lower terms vanish, and after shifting the index as $k \mapsto k+r-1$ and cancelling common factors on both sides of the sum we obtain

$$\begin{aligned} &\sum_{k=r}^{n} r_{n-k}(a, b; q, p; \mathsf{L}_n^{(r)}) \prod_{j=1}^{k} [z-r-j+2]_{aq^{2(j+r-2)}, bq^{j+r-2}; q,p} \\ &= \prod_{i=1}^{n-r} [z+n-r-i+1]_{aq^{2(i-1-n+r)}, bq^{i-1-n+r}; q,p} \prod_{i=1}^{r} [z-r-i+2]_{aq^{2(i+r-2)}, bq^{i+r-2}; q,p}. \end{aligned}$$

For a more compact result, we replace (a,b,z) by $(aq^{2(1-r)}, bq^{1-r}, z+r-1)$, after which we obtain

$$\sum_{k=r}^{n} r_{n-k}(aq^{2(1-r)}, bq^{1-r}; q, p; \mathsf{L}_n^{(r)}) \prod_{j=1}^{k} [z-j+1]_{aq^{2(j-1)}, bq^{j-1}; q,p}$$
$$= \prod_{i=1}^{n-r} [z+n-i]_{aq^{2(i-n)}, bq^{i-n}; q,p} \prod_{i=1}^{r} [z-i+1]_{aq^{2(i-1)}, bq^{i-1}; q,p}. \tag{3.38}$$

The *elliptic r-restricted Lah numbers* $\mathcal{L}_{n,k}^{(r)}(a,b;q,p) := r_{n-k}(aq^{2(1-r)}, bq^{1-r}; q, p; \mathsf{L}_n^{(r)})$ satisfy the recursion

$$\mathcal{L}_{n+1,k}^{(r)}(a,b;q,p) = W_{aq^{-2n}, bq^{-n}; q,p}(n+k-1)\, \mathcal{L}_{n,k-1}^{(r)}(a,b;q,p)$$
$$+ [n+k]_{aq^{-2n}, bq^{-n}; q,p}\, \mathcal{L}_{n,k}^{(r)}(a,b;q,p), \tag{3.39}$$

with initial conditions

$$\mathcal{L}_{n,k}^{(r)}(a,b;q,p) = 0 \qquad \text{for } k < r-1 \text{ or } k > n,$$
$$\mathcal{L}_{r-1,r-1}^{(r)}(a,b;q,p) = 1 \qquad \text{(an artificial but felicitous initial condition).} \tag{3.40}$$

As in the elliptic Lah-number case, this elliptic analogue of r-restricted Lah number does not have a nice closed form, but if we let $p \to 0$, followed by $b \to 0$, then it has the following closed form

$$\mathcal{L}_{n,k}^{(r)}(a;q) = q^{\binom{k}{2} - \binom{n}{2} - n(k-1) + 2\binom{r}{2}} \begin{bmatrix} n+r-1 \\ k+r-1 \end{bmatrix}_q \frac{[n-r]_q!}{[k-r]_q!} \frac{(aq^{1-n+k}; q)_{n-k} (aq^{1+2r}; q^2)_{k-r}}{(aq^{3-2n}; q^2)_{n-r}}, \tag{3.41}$$

the formula being a consequence of the $(a,l,m,k) \mapsto (aq^{2(1-r)}, n+r-1, n-r, n-k)$ case of Proposition 3.11. For $a \to \infty$ this $a;q$-analogue of r-restricted Lah numbers $\mathcal{L}_{n,k}^{(r)}(a;q)$ converges to the following q-analogue of r-restricted Lah numbers $\mathcal{L}_{n,k}^{(r)}(q)$

$$\mathcal{L}_{n,k}^{(r)}(q) = q^{k(k-1) - r(r-1)} \begin{bmatrix} n+r-1 \\ k+r-1 \end{bmatrix}_q \frac{[n-r]_q!}{[k-r]_q!}. \tag{3.42}$$

3.5. **$\mathfrak{p}, q$-Analogues.** Briggs and Remmel [3] defined the $\mathfrak{p}, q$-analogue[1] of rook numbers by using the (homogeneous) $\mathfrak{p}, q$-analogue of n and $n!$ defined by

$$[n]_{\mathfrak{p},q} := \mathfrak{p}^{n-1} + \mathfrak{p}^{n-2} q + \cdots + \mathfrak{p} q^{n-2} + q^{n-1} = \frac{\mathfrak{p}^n - q^n}{\mathfrak{p} - q}$$

and $[n]_{\mathfrak{p},q}! = [n]_{\mathfrak{p},q} [n-1]_{\mathfrak{p},q} \cdots [1]_{\mathfrak{p},q}$. They in particular proved that for a Ferrers board $B = B(b_1, \dots, b_n) \subseteq [n] \times \mathbb{N}$, one has

$$\prod_{i=1}^{n} [z + b_i - (i-1)]_{\mathfrak{p},q} = \sum_{k=0}^{n} r_{k,n}(B, \mathfrak{p}, q)\, \mathfrak{p}^{zk + \binom{k+1}{2}} \prod_{i=0}^{n-k} [z-i+1]_{\mathfrak{p},q}, \tag{3.43a}$$

[1] In this subsection, we use the Fraktur letter $\mathfrak{p}$ to denote the second base variable instead of the common Latin-script letter p since in our elliptic setting, we have reserved p to denote the *nome*.

where

$$r_{k,n}(B,\mathfrak{p},q) := \sum_{P\in\mathcal{N}_k(B)} q^{\alpha_B(P)+\epsilon_B(P)}\mathfrak{p}^{\beta_B(P)-(c_1+\cdots+c_k)}, \tag{3.43b}$$

for specifically defined $\alpha_B(P)$, $\beta_B(P)$ and $\epsilon_B(P)$, and where the $c_1, c_2, \ldots, c_k$ are the column labels of the k columns containing rooks of P. See [3] for the full details.

If in (3.2a) we let $p = 0$ and replace q by $q/\mathfrak{p}$, then the small weight function becomes

$$w_{a,b;q/\mathfrak{p},0}(k) = \frac{(\mathfrak{p}^{2k+1} - aq^{2k+1})(\mathfrak{p}^k - bq^k)(b\mathfrak{p}^{k-2} - aq^{k-2})}{(\mathfrak{p}^{2k-1} - aq^{2k-1})(\mathfrak{p}^{k+2} - bq^{k+2})(b\mathfrak{p}^k - aq^k)}\mathfrak{p}q, \tag{3.44}$$

while the $\mathfrak{p}, q$-numbers become

$$[z]_{a,b;q/\mathfrak{p},0} = \frac{(\mathfrak{p}^z - q^z)(\mathfrak{p}^z - aq^z)(\mathfrak{p}^2 - bq^2)(b-a)}{(\mathfrak{p}-q)(\mathfrak{p}-aq)(\mathfrak{p}^{z+1} - bq^{z+1})(b\mathfrak{p}^{z-1} - aq^{z-1})}. \tag{3.45}$$

The use of this weight in Theorem 3.8 yields an a, b-extension of the above result of Briggs and Remmel. By utilizing the weight function in (3.44) with the staircase board St_n, an a, b-extension of the $\mathfrak{p}, q$-Stirling numbers defined by Wachs and White [44] can be obtained.

4. $\hat{\mathfrak{j}}$-attacking rook model

We develop an elliptic analogue of the $\hat{\mathfrak{j}}$-*attacking rook model* of Remmel and Wachs [32]. We recall their setting first. For a fixed integer $\hat{\mathfrak{j}} \geq 1$, we say that a Ferrers board $B(b_1, \ldots, b_n)$ is a $\hat{\mathfrak{j}}$-*attacking board* if for all $1 \leq i < n$, $b_i \neq 0$ implies $b_{i+1} \geq b_i + \hat{\mathfrak{j}} - 1$. Suppose that $B(b_1, \ldots, b_n)$ is a $\hat{\mathfrak{j}}$-attacking board and P is a placement of rooks in $B(b_1, \ldots, b_n)$ which has at most one rook in each column of $B(b_1, \ldots, b_n)$. Then for any individual rook $\mathbf{r} \in P$, we say that $\mathbf{r}$ $\hat{\mathfrak{j}}$-*attacks* a cell $c \in B(b_1, \ldots, b_n)$ if c lies in a column which is strictly to the right of the column of $\mathbf{r}$ and c lies in the first $\hat{\mathfrak{j}}$ rows which are weakly above the row of $\mathbf{r}$ and which are not $\hat{\mathfrak{j}}$-attacked by any rook which lies in a column that is strictly to the left of $\mathbf{r}$. Figure 6 shows an example of $\hat{\mathfrak{j}}$-attack when $\hat{\mathfrak{j}} = 2$. In Figure 6, the cells which are attacked by the rook $\mathbf{r}_i$ are denoted by i in the cell. Let a rook $\mathbf{r}$ in $B(b_1, \ldots, b_n)$ cancel the cells below it and the cells which

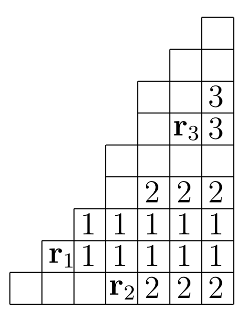


FIGURE 6. $\hat{\mathfrak{j}} = 2$, $B = B(1, 2, 3, 5, 7, 8, 9)$.

are $\hat{\mathfrak{j}}$-attacked by $\mathbf{r}$. A placement P of k rooks in B is called $\hat{\mathfrak{j}}$-*nonattacking* if each column contains at most one rook and each rook does not $\hat{\mathfrak{j}}$-attack other rooks. Given a $\hat{\mathfrak{j}}$-attacking board B, we let $\mathcal{N}_k^{\hat{\mathfrak{j}}}(B)$ be the set of all placements P of k $\hat{\mathfrak{j}}$-nonattacking rooks in B.

Let $B = B(b_1, \dots, b_n)$ be a $\hat{\mathfrak{j}}$-attacking board. For any placement $P \in \mathcal{N}_k^{\hat{\mathfrak{j}}}(B)$, denote the number of uncancelled cells in $B - P$ as $u_B^{\hat{\mathfrak{j}}}(P)$. We define the q-rook number of B by

$$r_k^{\hat{\mathfrak{j}}}(q; B) = \sum_{P \in \mathcal{N}_k^{\hat{\mathfrak{j}}}(B)} q^{u_B^{\hat{\mathfrak{j}}}(P)}.$$

Then Remmel and Wachs [32] proved the following product formula.

Theorem 4.1. [32] *Let $B = B(b_1, \dots, b_n)$ be a $\hat{\mathfrak{j}}$-attacking board. Then*

$$\prod_{i=1}^n [z + b_i - \hat{\mathfrak{j}}(i-1)]_q = \sum_{k=0}^n r_{n-k}^{\hat{\mathfrak{j}}}(q; B)[z]_q \downarrow_{k,\hat{\mathfrak{j}}},$$

where $[z]_q \downarrow_{0,\hat{\mathfrak{j}}} = 1$ and for $k > 0$, $[z]_q \downarrow_{k,\hat{\mathfrak{j}}} = [z]_q [z - \hat{\mathfrak{j}}]_q \cdots [z - (k-1)\hat{\mathfrak{j}}]_q$.

Remark 4.2. Remmel and Wachs defined $(\mathfrak{p}, q)$-rook numbers including one more parameter $\mathfrak{p}$. Here we have set $\mathfrak{p} = 1$. As we remarked in Section 3.5, we can modify the weight function to recover or extend the $\mathfrak{p}, q$-rook numbers.

Now we establish an elliptic analogue of the $\hat{\mathfrak{j}}$-attacking rook model. Given a $\hat{\mathfrak{j}}$-attacking board $B = B(b_1, \dots, b_n)$ and a placement $P \in \mathcal{N}_k^{\hat{\mathfrak{j}}}(B)$, let $U_B^{\hat{\mathfrak{j}}}(P)$ be the set of uncancelled cells in $B - P$. Then define

$$wt^{\hat{\mathfrak{j}}}(P) = \prod_{(i,j) \in U_B^{\hat{\mathfrak{j}}}(P)} w_{a,b;q,p}(\hat{\mathfrak{j}}(i-1) + 1 - j - \hat{\mathfrak{j}} r_{(i,j)}(P)),$$

where $r_{(i,j)}(P)$ is the number of rooks in P which are in the north-west region of (i, j), and define the k-th rook number of B by

$$r_k^{\hat{\mathfrak{j}}}(a, b; q, p; B) = \sum_{P \in \mathcal{N}_k^{\hat{\mathfrak{j}}}(B)} wt^{\hat{\mathfrak{j}}}(P).$$

Then we have the following elliptic analogue of Theorem 4.1.

Theorem 4.3. *Let $B = B(b_1, \dots, b_n)$ be a $\hat{\mathfrak{j}}$-attacking board. Then we have*

$$\begin{aligned} &\prod_{i=1}^n [z + b_i - \hat{\mathfrak{j}}(i-1)]_{aq^{2(\hat{\mathfrak{j}}(i-1)-b_i)}, bq^{\hat{\mathfrak{j}}(i-1)-b_i}; q,p} \\ &= \sum_{k=0}^n r_{n-k}^{\hat{\mathfrak{j}}}(a, b; q, p; B) \prod_{j=1}^k [z - \hat{\mathfrak{j}}(j-1)]_{aq^{2\hat{\mathfrak{j}}(j-1)}, bq^{\hat{\mathfrak{j}}(j-1)}; q,p}. \end{aligned} \tag{4.1}$$

Proof. The idea of proof is basically the same as in the proof of the product formula in the case when $\hat{\mathfrak{j}} = 1$. It is enough to prove (4.1) for all positive integers $z \geq \hat{\mathfrak{j}} n$. So fix a positive integer $z \geq \hat{\mathfrak{j}} n$ and consider B_z, the extended board obtained from B by attaching a $[n] \times [z]$ board below B. We shall consider nonattacking placements of n rooks in B_z. Recall that we denoted the line separating the board B and the extended part below by $\mathfrak{g}$ and called it *ground*. A rook $\mathbf{r}$ placed in B will $\hat{\mathfrak{j}}$-attack as described above, and thus it only $\hat{\jmath}$-attacks cells which are above the ground. If a rook $\mathbf{r}$ is placed below the ground, then it shall $\hat{\jmath}$-attack only the cells below the ground. More precisely, if a rook $\mathbf{r}$ is placed below the ground $\mathfrak{g}$, then it $\hat{\jmath}$-attacks cells strictly to the right of the column containing $\mathbf{r}$ and in the first $\hat{\jmath}$ rows which are weakly above the row containing $\mathbf{r}$ and below the ground which contain no cells that are $\hat{\jmath}$-attacked by any other rook $\mathbf{r}'$ to the left of $\mathbf{r}$ if there are such $\hat{\jmath}$ rows, and if there are $t < \hat{\jmath}$ such rows, then the rook $\mathbf{r}$ $\hat{\jmath}$-attacks those t rows and the first $\hat{\jmath} - t$ rows below the row of $\mathbf{r}$ which contain no cells that are $\hat{\jmath}$-attacked by any other rooks to the left of $\mathbf{r}$. Then we define that a rook placed below the ground cancels the cells below it and the cells which are $\hat{\jmath}$-attacked by the rook.

Now let $\mathcal{N}_n^{\hat{\jmath}}(B_z)$ denote the set of all placements P of n rooks in B_z such that there is at most one rook in each row and column and no rooks $\hat{\jmath}$-attack another rook. For a placement $P \in \mathcal{N}_n^{\hat{\jmath}}(B_z)$, let $U_{B_z}^{\hat{\jmath}}(P)$ be the set of uncancelled cells in $B_z - P$. Then we show (4.1) by computing the sum

$$\sum_{P \in \mathcal{N}_n^{\hat{\jmath}}(B_z)} wt^{\hat{\jmath}}(P), \tag{4.2}$$

where

$$wt^{\hat{\jmath}}(P) = \prod_{(i,j) \in U_{B_z}^{\hat{\jmath}}(P)} w_{a,b;q,p}(\hat{\jmath}(i-1) + 1 - j - \hat{\jmath} r_{(i,j)}(P)), \tag{4.3}$$

in two different ways. First, we place rooks column by column from the left to right and compute the contribution to the sum. If we place a rook in the first column in all possible ways, it gives the weights

$$\begin{aligned} &1 + w_{a,b;q,p}(1-b_1) + \cdots + w_{a,b;q,p}(1-b_1) \cdots w_{a,b;q,p}(z-1) \\ &= 1 + w_{aq^{-2b_1},bq^{-b_1}}(1) + w_{aq^{-2b_1},bq^{-b_1}}(1) \cdots w_{aq^{-2b_1},bq^{-b_1}}(b_1 + z - 1) \\ &= [b_1 + z]_{aq^{-2b_1},bq^{-b_1};q,p}. \end{aligned}$$

This first rook cancels $\hat{\jmath}$ rows to the right of it and so, placing the second rook in the second column gives $[b_2 + z - \hat{\jmath}]_{aq^{2(\hat{\jmath}-b_2)},bq^{\hat{\jmath}-b_2}}$. Note that the weights sum up to the elliptic integer since the factor $\hat{\jmath} r_{(i,j)}(P)$ compensates the possible gap in the row coordinates due to the cancellation from the rook to the left. Placing n rooks in this way gives the left-hand side of (4.1).

For the right hand side of (4.1), we place $n-k$ rooks in B and k rooks in the extended part below the ground and compute (4.2). Fix a placement $\mathcal{Q}$ of $n-k$ rooks in B. We want to compute

$$W(\mathcal{Q}) = \sum_{\substack{P \in \mathcal{N}_n^{\hat{\jmath}}(B_z), \\ P \cap B = \mathcal{Q}}} wt^{\hat{\jmath}}(P).$$

We put $wt^{\hat{\jmath}}(\mathcal{Q})$ for the weight contribution coming from the uncancelled cells in B and compute the weight coming from the uncancelled cells below the ground. Let s denote the first available column coordinate for the first rook below the ground. This means that there are $s-1$ rooks in the north-west region of this column. Then the possible placements of the first rook in this column give

$$1+w_{a,b;q,p}(\hat{\jmath}(s-1)+1-\hat{\jmath}(s-1))+\cdots+w_{a,b;q,p}(1)\cdots w_{a,b;q,p}(z-1)=[z]_{a,b;q,p}.$$

This rook cancels $\hat{\jmath}$ rows to the right and so the second rook contributes $[z-\hat{\jmath}]_{aq^{2\hat{\jmath}},bq^{\hat{\jmath}};q,p}$. Finally, placing k rooks below the ground gives

$$\prod_{j=1}^{k}[z-\hat{\jmath}(j-1)]_{aq^{2\hat{\jmath}(j-1)},bq^{\hat{\jmath}(j-1)};q,p}$$

to $wt^{\hat{\jmath}}(P)$. Hence,

$$\begin{aligned}\sum_{\mathcal{Q}\in\mathcal{N}^{\hat{\jmath}}_{n-k}(B)}W(\mathcal{Q})&=\sum_{\mathcal{Q}\in\mathcal{N}^{\hat{\jmath}}_{n-k}(B)}\ \sum_{\substack{P\in\mathcal{N}^{\hat{\jmath}}_{n}(B_z),\\ P\cap B=\mathcal{Q}}}wt^{\hat{\jmath}}(P)\\&=\sum_{\mathcal{Q}\in\mathcal{N}^{\hat{\jmath}}_{n-k}(B)}wt^{\hat{\jmath}}(\mathcal{Q})\prod_{j=1}^{k}[z-\hat{\jmath}(j-1)]_{aq^{2\hat{\jmath}(j-1)},bq^{\hat{\jmath}(j-1)};q,p}\\&=r^{\hat{\jmath}}_{n-k}(a,b;q,p;B)\prod_{j=1}^{k}[z-\hat{\jmath}(j-1)]_{aq^{2\hat{\jmath}(j-1)},bq^{\hat{\jmath}(j-1)};q,p}.\end{aligned}$$

We get the right hand side of (4.1) by summing this over $k=0,\dots,n$. □

It is clear that by taking $z=0$ in (4.1) the following product formula is obtained.

Corollary 4.4. *Let $B=B(b_1,\dots,b_n)$ be a $\hat{\jmath}$-attacking board. Then we have*

$$r^{\hat{\jmath}}_{n}(a,b;q,p;B)=\prod_{i=1}^{n}[b_i-\hat{\jmath}(i-1)]_{aq^{2(\hat{\jmath}(i-1)-b_i)},bq^{\hat{\jmath}(i-1)-b_i};q,p}.$$

4.1. **Elliptic analogue of generalized Stirling numbers of the second kind.** Here we consider the generalized $(\mathfrak{p},q)$-Stirling numbers of the second kind $\tilde{S}^{\hat{\imath},\hat{\jmath}}_{n,k}(\mathfrak{p},q)$ (here $\hat{\imath}$ is an additional nonnegative integer parameter) which were thoroughly investigated by Remmel and Wachs [32]. They are defined by

$$\tilde{S}^{\hat{\imath},\hat{\jmath}}_{n+1,k}(\mathfrak{p},q)=q^{\hat{\imath}+(k-1)\hat{\jmath}}\tilde{S}^{\hat{\imath},\hat{\jmath}}_{n,k-1}(\mathfrak{p},q)+\mathfrak{p}^{-(n+1)\hat{\jmath}}[k\hat{\jmath}+\hat{\imath}]_{\mathfrak{p},q}\tilde{S}^{\hat{\imath},\hat{\jmath}}_{n,k}(\mathfrak{p},q),\tag{4.4}$$

with $\tilde{S}^{\hat{\imath},\hat{\jmath}}_{0,0}(\mathfrak{p},q)=1$ and $\tilde{S}^{\hat{\imath},\hat{\jmath}}_{n,k}(\mathfrak{p},q)=0$ if $k<0$ or $k>n$. They also satisfy

$$[z+\hat{\imath}]^{n}_{\mathfrak{p},q}=\sum_{k=0}^{n}\tilde{S}^{\hat{\imath},\hat{\jmath}}_{n,k}(\mathfrak{p},q)\mathfrak{p}^{z(n-k)+\binom{n-k+1}{2}}[z]_{\mathfrak{p},q}\downarrow_{k,\hat{\jmath}}.$$

While for $\hat{\imath}=0$ and $\hat{\jmath}=1$ they were introduced by Wachs and White [44], the rescaled variant $S^{\hat{\imath},\hat{\jmath}}_{n,k}(\mathfrak{p},q)=\mathfrak{p}^{\binom{n-k+1}{2}\hat{\jmath}-(n-k)(\hat{\imath}-1)}q^{-k\hat{\imath}-\binom{k}{2}\hat{\jmath}}\tilde{S}^{\hat{\imath},\hat{\jmath}}_{n,k}(\mathfrak{p},q)$ was defined by de Médicis and Leroux [10] by using 0-1 tableaux.

From now on, we set $\mathfrak{p}=1$ and use the notation $\tilde{S}^{\hat{\imath},\hat{\jmath}}_{n,k}(q)=\tilde{S}^{\hat{\imath},\hat{\jmath}}_{n,k}(1,q)$ and $S^{\hat{\imath},\hat{\jmath}}_{n,k}(q)=S^{\hat{\imath},\hat{\jmath}}_{n,k}(1,q)$ for $\tilde{S}^{\hat{\imath},\hat{\jmath}}_{n,k}(q)=q^{k\hat{\imath}+\binom{k}{2}\hat{\jmath}}S^{\hat{\imath},\hat{\jmath}}_{n,k}(q)$.

Let $B_{\hat{\imath},\hat{\jmath},n} = B(\hat{\imath}, \hat{\imath}+\hat{\jmath}, \hat{\imath}+2\hat{\jmath}, \dots, \hat{\imath}+(n-1)\hat{\jmath})$. In [32], Remmel and Wachs showed that

$$\tilde{S}^{\hat{\imath},\hat{\jmath}}_{n,k}(q) = r^{\hat{\jmath}}_{n-k}(q; B_{\hat{\imath},\hat{\jmath},n}).$$

If we let $\alpha_B(P)$ denote the number of uncancelled cells of B which lie above a rook in P, then Remmel and Wachs also showed that

$$S^{\hat{\imath},\hat{\jmath}}_{n,k}(q) = \sum_{P \in \mathcal{N}^{\hat{\jmath}}_{n-k}(B_{\hat{\imath},\hat{\jmath},n})} q^{\alpha_B(P)}.$$

We use the board $B = B_{\hat{\imath},\hat{\jmath},n}$ in (4.1) to define an elliptic analogue of $\tilde{S}^{\hat{\imath},\hat{\jmath}}_{n,k}(q)$. For this board, the product formula becomes

$$([z+\hat{\imath}]_{aq^{-2\hat{\imath}},bq^{-\hat{\imath}};q,p})^n = \sum_{k=0}^{n} r^{\hat{\jmath}}_{n-k}(a,b;q,p;B_{\hat{\imath},\hat{\jmath},n}) \prod_{j=1}^{k} [z-\hat{\jmath}(j-1)]_{aq^{2\hat{\jmath}(j-1)},bq^{\hat{\jmath}(j-1)};q,p}. \tag{4.5}$$

If we define $\tilde{S}^{\hat{\imath},\hat{\jmath}}_{n,k}(a,b;q,p) := r^{\hat{\jmath}}_{n-k}(a,b;q,p;B_{\hat{\imath},\hat{\jmath},n})$, then up to whether there is a rook or not in the last column of $B_{\hat{\imath},\hat{\jmath},n}$, we get the following recursion

$$\begin{aligned} \tilde{S}^{\hat{\imath},\hat{\jmath}}_{n+1,k}(a,b;q,p) &= W_{aq^{-2\hat{\imath}},bq^{-\hat{\imath}};q,p}(\hat{\imath}+(k-1)\hat{\jmath}) \tilde{S}^{\hat{\imath},\hat{\jmath}}_{n,k-1}(a,b;q,p) \\ &\quad + [\hat{\imath}+k\hat{\jmath}]_{aq^{-2\hat{\imath}},bq^{-\hat{\imath}};q,p} \tilde{S}^{\hat{\imath},\hat{\jmath}}_{n,k}(a,b;q,p). \end{aligned} \tag{4.6}$$

With the initial conditions $\tilde{S}^{\hat{\imath},\hat{\jmath}}_{0,0}(a,b;q,p) = 1$ and $\tilde{S}^{\hat{\imath},\hat{\jmath}}_{n,k}(a,b;q,p) = 0$ for $k < 0$ or $k > n$, (4.6) can be used to characterize $\tilde{S}^{\hat{\imath},\hat{\jmath}}_{n,k}(a,b;q,p)$. In [32], Remmel and Wachs developed a combinatorial interpretation for $S^{\hat{\imath},\hat{\jmath}}_{n,k}(q)$ in terms of permutation statistics, colored partitions and restricted growth functions. We can modify their q-weight function to give a combinatorial interpretation for $S^{\hat{\imath},\hat{\jmath}}_{n,k}(a,b;q,p)$ where

$$\tilde{S}^{\hat{\imath},\hat{\jmath}}_{n,k}(a,b;q,p) = \left(\prod_{j=1}^{k} W_{aq^{-2\hat{\imath}},bq^{-\hat{\imath}};q,p}(\hat{\imath}+(j-1)\hat{\jmath}) \right) S^{\hat{\imath},\hat{\jmath}}_{n,k}(a,b;q,p).$$

We shall assume that $0 \le \hat{\imath} \le \hat{\jmath}$. Let $\mathcal{CP}$ be the collection of all set partitions of $\{0,1,\dots,n\}$ whose nonzero elements are colored with colors in the set $\{0,1,\dots,j-1\}$. We refer to the block of a colored partition that contains 0 as the *zero-block*. Define $\mathcal{CP}^{\hat{\imath},\hat{\jmath}}_{n,k}$ to be the subset of $\mathcal{CP}$ consisting of partitions with $k+1$ blocks where the elements are colored so that

(a) the nonzero elements of the zero-block have colors in $\{0,\dots,i-1\}$, and
(b) the smallest element of each block other than the zero-block has color 0.

Note that there is a natural way to encode the set partitions of $[n]$ as restricted growth functions. A *restricted growth function* is a word $w_1 \cdots w_n$ over the alphabet $[n]$ such that $w_1 = 1$ and for $s = 2, \dots, n$, we have $w_s \le 1 + max\{w_1, \dots, w_{s-1}\}$. To a partition $\pi = \langle \pi_1, \dots, \pi_k \rangle$, where $min(\pi_1) < \cdots < min(\pi_k)$, we associate the restricted growth function $w_1 w_2 \cdots w_n$, where $w_s = t$ if $s \in \pi_t$.

Now we generalize this encoding to colored partitions. Let $\pi = \langle \pi_0, \dots, \pi_k \rangle \in \mathcal{CP}^{\hat{\imath},\hat{\jmath}}_{n,k}$ where $min(\pi_0) < \cdots < min(\pi_k)$ and let $w(\pi) = w_0 w_1 \cdots w_n$ where for all $0 \le s \le n$, $w_s = t$ if $s \in \pi_t$. Then we color w with the same color that s was colored with in π. For example, if $\pi = \langle\{0, 1^1, 4^0\}, \{2^0, 5^1\}, \{3^0, 6^2\}\rangle \in \mathcal{CP}^{2,3}_{6,2}$ (here the exponents of the elements are the respective colors), then $w(\pi) = 00^1 1^0 2^0 0^0 1^1 2^2$. We let $\mathcal{RG}^{\hat{\imath},\hat{\jmath}}_{n,k} = \{w(\pi) = w_0 w_1^{e_1} \cdots w_n^{e_n} \mid \pi \in \mathcal{CP}^{\hat{\imath},\hat{\jmath}}_{n,k}\}$. We also express the colored word $w = w_0 w_1^{e_1} \cdots w_n^{e_n} \in \mathcal{RG}^{\hat{\imath},\hat{\jmath}}_{n,k}$ as a pair of words $(w_0 w_1 \cdots w_n : e_1 \cdots e_n)$. Remmel and Wachs [32, Theorem 18] showed that $S^{\hat{\imath},\hat{\jmath}}_{n,k}(1) = |\mathcal{RG}^{\hat{\imath},\hat{\jmath}}_{n,k}| = |\mathcal{CP}^{\hat{\imath},\hat{\jmath}}_{n,k}|$ by constructing a bijection $\phi : \mathcal{RG}^{\hat{\imath},\hat{\jmath}}_{n,k} \to \mathcal{N}^{\hat{\jmath}}_{n-k}(B_{\hat{\imath},\hat{\jmath},n})$ as follows. Let $(w : e) \in \mathcal{RG}^{\hat{\imath},\hat{\jmath}}_{n,k}$. We place rooks from left to right column by column so that in column s, a rook is placed in the $(\hat{\imath}+w_s\hat{\jmath}-e_s)$-th available cell (that is not $\hat{\jmath}$-attacked) from the bottom. If no such cell is available then we leave the column s empty. For example, $(w : e) = (0012012 : 100012)$ corresponds to the rook placement in Figure 7. In Figure 7, the $\hat{\jmath}$-attacked cells are denoted by $\bullet$'s and the uncancelled cells above rooks are denoted by $\circ$'s. We then define a q-statistic

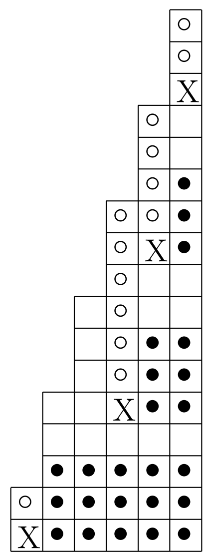


FIGURE 7. The rook placement corresponding to $(0012012 : 100012) \in \mathcal{RG}^{2,3}_{6,2}$.

for $S^{\hat{\imath},\hat{\jmath}}_q(n,k)$. For a given $\gamma = (w : e) \in \mathcal{RG}^{\hat{\imath},\hat{\jmath}}_{n,k}$, let $m_s(\gamma) = max\{w_1, \ldots, w_{s-1}\}$, for each $s = 1, \ldots, n$, and define

$$\mathcal{MAX} = \{s \in [n] : w_s > max\{w_0, \ldots, w_{s-1}\}\},$$

$$inv(w : e) = \hat{\jmath} \sum_{1 \le s < t \le n} \chi(w_s > w_t \text{ and } s \in \mathcal{MAX}(w : e)) + \sum_{s=1}^{n} e_s.$$

Then for

$$D^{\hat{\imath},\hat{\jmath}}_{n,k}(q) = \sum_{\gamma \in \mathcal{RG}^{\hat{\imath},\hat{\jmath}}_{n,k}} q^{inv(\gamma)},$$

Remmel and Wachs showed in [32, Theorem 19] that $D^{\hat{\imath},\hat{\jmath}}_{n,k}(q) = S^{\hat{\imath},\hat{\jmath}}_{n,k}(q)$.

Now for $\gamma = (w : e) = (w_0 w_1 \cdots w_n : e_1 \cdots e_n) \in \mathcal{RG}_{n,k}^{\hat{\imath},\hat{\jmath}}$, we define

$$D_{n,k}^{\hat{\imath},\hat{\jmath}}(a,b;q,p) = \sum_{\gamma \in \mathcal{RG}_{n,k}^{\hat{\imath},\hat{\jmath}}} \prod_{s=1}^{n} W_{aq^{-2\hat{\imath}},bq^{-\hat{\imath}};q,p}(\hat{\jmath}|\{t<s : w_t > w_s \text{ and } t \in \mathcal{MAX}(\gamma)\}| + e_s).$$

Proposition 4.5. *For each $\gamma \in \mathcal{RG}_{n,k}^{\hat{\imath},\hat{\jmath}}$, we have*

$$wt^{\hat{\jmath}}(\phi(\gamma)) = \prod_{s=1}^{n} W_{aq^{-2\hat{\imath}},bq^{-\hat{\imath}};q,p}(\hat{\jmath}|\{t<s : w_t > w_s \text{ and } t \in \mathcal{MAX}(\gamma)\}| + e_s),$$

and hence,

$$D_{n,k}^{\hat{\imath},\hat{\jmath}}(a,b;q,p) = S_{n,k}^{\hat{\imath},\hat{\jmath}}(a,b;q,p).$$

Proof. Let $\gamma = (w : e) \in \mathcal{RG}_{n,k}^{\hat{\imath},\hat{\jmath}}$. Observe that for each $s = 1,\ldots,n$, column s of $\phi(\gamma)$ has $\hat{\imath} + \hat{\jmath} m_s(\gamma)$ cells that are not $\hat{\jmath}$-attacked by any rooks on the left, since there would be $(s-1-m_s(\gamma))$ many rooks in the first $(s-1)$-columns, where $m_s(\gamma) = max\{w_0, w_1, \ldots, w_{s-1}\}$. This implies that the number of uncancelled cells above a rook in column s is

$$\begin{aligned}\hat{\imath} + \hat{\jmath} m_s(\gamma) - (\hat{\imath} + \hat{\jmath} w_s - e_s) &= \hat{\jmath}(m_s(\gamma) - w_s) + e_s \\ &= \hat{\jmath}|\{t<s : w_t > w_s \text{ and } t \in \mathcal{MAX}(\gamma)\}| + e_s.\end{aligned}$$

The weight of the top-most uncancelled cell is $w_{a,b;q,p}(1-\hat{\imath})$, and so the product of the weights of uncancelled cells in the column s is

$$\begin{aligned}&w_{a,b;q,p}(1-\hat{\imath}) w_{a,b;q,p}(2-\hat{\imath}) \cdots w_{a,b;q,p}(\hat{\jmath}|\{t<s : w_t > w_s \text{ and } t \in \mathcal{MAX}(\gamma)\}| + e_s - \hat{\imath}) \\ &= w_{aq^{-2\hat{\imath}},bq^{-\hat{\imath}};q,p}(1) \cdots w_{aq^{-2\hat{\imath}},bq^{-\hat{\imath}};q,p}(\hat{\jmath}|\{t<s : w_t > w_s \text{ and } t \in \mathcal{MAX}(\gamma)\}| + e_s) \\ &= W_{aq^{-2\hat{\imath}},bq^{-\hat{\imath}};q,p}(\hat{\jmath}|\{t<s : w_t > w_s \text{ and } t \in \mathcal{MAX}(\gamma)\}| + e_s).\end{aligned}$$

Thus $wt^{\hat{\jmath}}(\phi(\gamma))$ is obtained by multiplying the above weights over all $s = 1,\ldots,n$. □

5. Elliptic file numbers

In this section, we consider an elliptic analogue of the file numbers. The file numbers and their q-analogue were first considered by Garsia and Remmel in 1984 upon the introduction of q-rook numbers in [17] (but did not get into the final version of their paper). The first time they actually appeared in literature under the name "file numbers" was in [32] where already $\mathfrak{p}, q$-extensions were investigated. Other instances where they appear include [5] and [29].

Given a board $B \subset [n] \times \mathbb{N}$, let $\mathcal{F}_k(B)$ be the set of placements Q of k rooks in B such that no two rooks in Q lie in the same column. We refer to such a Q as a *file placement* of k rooks in B. Thus in a file placement Q, we do allow the possibility that two rooks lie in the same row. Given a placement $Q \in \mathcal{F}_k(B)$, we let each rook in Q cancel all the cells below it in B. Let $u_B(Q)$ be the number of cells in $B - Q$ which are not cancelled by any rook in Q. Then the q-file numbers are defined by

$$f_k(q;B) = \sum_{Q \in \mathcal{F}_k(B)} q^{u_B(Q)}. \tag{5.1}$$

Garsia and Remmel proved the following product formula involving the q-file numbers.

Theorem 5.1. *For any skyline board $B = B(c_1, \dots, c_n)$,*

$$\prod_{i=1}^{n} [z + c_i]_q = \sum_{k=0}^{n} f_{n-k}(q; B)([z]_q)^k. \tag{5.2}$$

This product formula can be proved by computing the sum

$$\sum_{Q \in \mathcal{F}_n(B_z)} q^{u_{B_z}(Q)},$$

where B_z again denotes the extended board obtained by attaching the board $[n] \times [z]$ below B, in two different ways. We omit the details since we will prove the elliptic extension of this result in Theorem 5.4.

By distinguishing the cases whether there is a rook or not in the last column, one can also obtain the following recursion.

Proposition 5.2. *Let B be a skyline board and let $B \cup m$ denote the board obtained by adding a column of length m to B. Then for any nonnegative integer k we have*

$$f_k(q; B \cup m) = q^m f_k(q; B) + [m]_q \, f_{k-1}(q; B). \tag{5.3}$$

We can define an elliptic analogue of the q-file numbers by assuming the same rook cancellation as in the q-case where elliptic weights are assigned to the uncancelled cells.

Definition 5.3. *Given a skyline board $B = B(c_1, \dots, c_n)$, we define the elliptic analogue of the k-th file number by*

$$f_k(a, b; q, p; B) = \sum_{Q \in \mathcal{F}_k(B)} wt_f(Q), \tag{5.4a}$$

with

$$wt_f(Q) = \prod_{(i,j) \in U_B(Q)} w_{a,b;q,p}(1 - j), \tag{5.4b}$$

where the elliptic weight $w_{a,b;q,p}(l)$ of an integer l is defined in (3.2a).

Note that in this case, the elliptic weight of a cell only depends on its row coordinate.

Theorem 5.4. *For any skyline board $B = B(c_1, \dots, c_n)$, we have*

$$\prod_{i=1}^{n} [z + c_i]_{aq^{-2c_i}, bq^{-c_i}; q, p} = \sum_{k=0}^{n} f_{n-k}(a, b; q, p; B)([z]_{a,b;q,p})^k. \tag{5.5}$$

Proof. As before, it suffices to prove the theorem for nonnegative integer values of z. We fix an integer z and consider the extended board B_z by attaching an $[n] \times [z]$ board below the board B and consider the n-file placements $\mathcal{F}_n(B_z)$ in B_z. Then (5.5) can be proved by computing the sum

$$\sum_{Q \in \mathcal{F}_n(B_z)} wt_f(Q) \tag{5.6}$$

in two ways. The left-hand side of (5.5) computes the above sum by placing rooks column by column. Since the elliptic weight used to define $wt_f(Q)$ does not depend on the column coordinate of the uncancelled cells, the weight sum in (5.6) is the product of the weight sums coming from the possible placements in each column, which is exactly the left-hand side of (5.5). The right-hand side computes (5.6) by considering the file placements in B and in the extended part separately. □

We have the following recursion for the elliptic file numbers, which is an elliptic extension of Proposition 5.2.

Theorem 5.5. *Let B be a skyline board, and $B \cup m$ denote the board obtained by adding a column of height m to B. Then, for any integer k, we have*

$$f_k(a,b;q,p;B) = 0 \qquad \text{for } k < 0, \tag{5.7a}$$

$$f_0(a,b;q,p;B) = 1 \qquad \text{for } B \text{ being the empty board}, \tag{5.7b}$$

and

$$\begin{aligned} f_k(a,b;q,p;B \cup m) &= W_{aq^{-2m},bq^{-m};q,p}(m)\, f_k(a,b;q,p;B) \\ &\quad + [m]_{aq^{-2m},bq^{-m};q,p}\, f_{k-1}(a,b;q,p;B). \end{aligned} \tag{5.7c}$$

Proof. This recursion stems from a weighted enumeration of a file placement of k rooks on $B \cup m$. We distinguish the cases whether there is a rook in the last column or not. The first term on the right-hand side of (5.7c) is obtained when there is no rook in the last column. The weight multiplied in front of $f_k(a,b;q,p;B)$ comes from the uncancelled m cells in the last column. The second term on the right-hand side of (5.7c) is obtained when there is a rook in the last column. The coefficient in front of f_{k-1} is a consequence of Lemma 3.6. □

Remark 5.6. It is analytically and combinatorially obvious that two skyline boards B_1 and B_2 for which the left-hand sides of (5.5) are equal (B_2 then must consist of the columns of B_1 which may be permuted) have the same elliptic file numbers. In this case we may refer to such boards B_1 and B_2 as *file equivalent*.

5.1. **Elliptic Stirling numbers of the first kind.** For the staircase board $\mathsf{St}_n = B(0,1,\dots,n-1)$, the product formula in Theorem 5.4 becomes

$$\prod_{i=1}^{n} [z+i-1]_{aq^{2(1-i)},bq^{1-i};q,p} = \sum_{k=0}^{n} f_{n-k}(a,b;q,p;B)([z]_{a,b;q,p})^k. \tag{5.8}$$

The file numbers $f_{n-k}(a,b;q,p;\mathsf{St}_n)$ are in fact the unsigned *elliptic Stirling numbers of the first kind* $\mathfrak{c}_{a,b;q,p}(n,k)$ which have recently been defined and studied (in a different setting) by Zsófia Keresként́yiné Balogh and the first author [27]. For a bijection of file placements of $n-k$ rooks in St_n and permutations of $[n]$ with k cycles, see the subsequent subsection, where we consider a refinement of the Stirling numbers of the first kind.

By using the $(a,b,y,z) \mapsto (aq^{-2n}, bq^{-n}, n, z+n)$ case of the elementary identity (3.8b), or by distinguishing whether there is a rook or not in the last column, we obtain from (5.8) the following recurrence relation

$$\mathfrak{c}_{a,b;q,p}(n+1,k) = [n]_{aq^{-2n},bq^{-n};q,p}\mathfrak{c}_{a,b;q,p}(n,k) + W_{aq^{-2n},bq^{-n};q,p}(n)\mathfrak{c}_{a,b;q,p}(n,k-1). \tag{5.9}$$

With the conditions $\mathfrak{c}_{a,b;q,p}(0,0) = 1$ and $\mathfrak{c}_{a,b;q,p}(n,k) = 0$ for $k < 0$ or $k > n$, the recurrence relation (5.9) uniquely determines $\mathfrak{c}_{a,b;q,p}(n,k)$.

5.2. **Elliptic r-restricted Stirling numbers of the first kind.** The *r-restricted (signless) Stirling numbers of the first kind* (these are usually called r-Stirling numbers of the first kind but we adopt the terminology from [38, see the sequences A143491, A143492 and A143493] to avoid possible confusion with the q-Stirling numbers), which we denote by $\mathfrak{c}^{(r)}(n,k)$, are defined, for all positive r, by the number of permutations of the set $\{1,\dots,n\}$ having k cycles, such that the numbers $1,2,\dots,r$ are in distinct cycles. For $r=1$ (or $r=0$) they reduce to the usual Stirling numbers of the first kind. They are treated with some detail in [4], where it is shown that the r-restricted Stirling numbers of the first kind have the following generating function

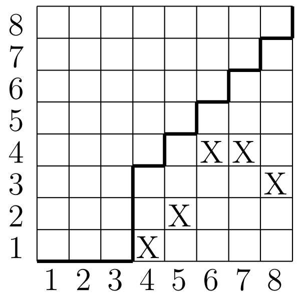
$$\sum_{k=0}^{n} \mathfrak{c}^{(r)}(n,k) z^k = \begin{cases} z^r (z+r)(z+r+1)\cdots(z+n-1), & n \geq r \geq 0, \\ 0, & \text{otherwise.} \end{cases} \tag{5.10}$$

As in subsection 3.2, let $\mathsf{St}_n^{(r)}$ denote the board $\mathsf{St}_n^{(r)} = B(c_1,\dots,c_n)$ such that $c_i = 0$ for $i = 1,\dots,r$ and $c_i = i-1$, for $i = r+1,\dots,n$. Then for $B = \mathsf{St}_n^{(r)}$ and $q \to 1$ the product formula (5.2) becomes

$$z^r (z+r)(z+r+1)\cdots(z+n-1) = \sum_{k=0}^{n} f_{n-k}(1;\mathsf{St}_n^{(r)}) z^k, \tag{5.11}$$

which is the generating function for the r-restricted Stirling numbers of the first kind. Thus we can identify $\mathfrak{c}^{(r)}(n,k)$ with $f_{n-k}(1;\mathsf{St}_n^{(r)})$. We can construct a bijection between file placements of $n-k$ rooks in $\mathsf{St}_n^{(r)}$ and permutations of n numbers with k cycles, such that the numbers $1,2,\dots,r$ are in distinct cycles. We start from the right-most rook. If this rook is placed in the cell (α,l), then place α to the left of l in the cycle notation. Then move to the second rook to the left. If this second rook is in the cell (β,l), then put β to the left of α in the same cycle ending with l, but if this rook is in (β,k) for $k \neq l$, then construct a new cycle with β and k and put β to the left of k. While iterating this procedure, if there is a rook in (l,h), then append h to the right-most place of the cycle ending with l. The numbers which never occurred in the cycle construction after reading all the rooks in the file placement are fix-points, i.e., cycles of one element. For example, given the file placement in Figure 8, the right-most rook gives (8 3) in the cycle notation, and the second right-most rook gives (7 4). The third right-most rook puts 6 to the left of 7 and gives (6 7 4). The fourth rook gives (5 2) and the last rook puts 1 to the right of the cycle ending with 4 and gives (6 7 4 1). Thus the permutation of $1,\dots,8$ corresponding to the given file placement in Figure 8 is (6 7 4 1)(5 2)(8 3).

Conversely, given a permutation in a cycle notation, we construct a file placement as follows. In each cycle, put the smallest number at the end of cycle. Then start from the left-most number, say α, find the left-most number to the right of α which is smaller than α, say β. Then this places a rook in the cell (α,β). Continue this procedure to the right. For example, given a permutation (6 7 4 1)(5 2)(8 3), we place rooks in $(6,4)$, $(7,4)$, $(4,1)$, $(5,2)$ and $(8,3)$, which recovers the file placement in Figure 8.



$\Longleftrightarrow$ (6 7 4 1) (5 2) (8 3)

FIGURE 8. A file placement with $r = 3$, $n = 8$, $n - k = 5$, and corresponding permutation in cycle notation.

We can define an elliptic analogue of the r-restricted Stirling numbers of the first kind by using the product formula for the elliptic file numbers with board $\mathsf{St}_n^{(r)}$ as the generating function :

$$([z]_{a,b;q,p})^r \prod_{i=1}^{n-r} [z+r+i-1]_{aq^{2(1-i-r)},bq^{1-i-r};q,p} = \sum_{k=0}^{n} f_{n-k}(a,b;q,p;\mathsf{St}_n^{(r)})([z]_{a,b;q,p})^k. \tag{5.12}$$

Let $\mathfrak{c}^{(r)}_{a,b;q,p}(n,k)$ denote $f_{n-k}(a,b;q,p;\mathsf{St}_n^{(r)})$. Then, by distinguishing whether there is a rook or not in the last column, we can deduce the recurrence relation of $\mathfrak{c}^{(r)}_{a,b;q,p}(n,k)$, namely, for $k \geq r-1$,

$$\mathfrak{c}^{(r)}_{a,b;q,p}(n+1,k) = [n]_{aq^{-2n},bq^{-n};q,p}\mathfrak{c}^{(r)}_{a,b;q,p}(n,k) + W_{aq^{-2n},bq^{-n};q,p}(n)\mathfrak{c}^{(r)}_{a,b;q,p}(n,k-1). \tag{5.13}$$

This recursion uniquely determines $\mathfrak{c}^{(r)}_{a,b;q,p}(n,k)$ with the conditions

$$\mathfrak{c}^{(r)}_{a,b;q,p}(n,k) = 0 \qquad \text{for } k < r-1 \text{ or } k > n,$$
$$\mathfrak{c}^{(r)}_{a,b;q,p}(r-1,r-1) = 1.$$

5.3. **Abel boards and weighted forests.** Let A_n denote the *Abel board*, the $[n-1]\times[n]$ board with column heights $(0,n,\dots,n)$. For $B = \mathsf{A}_n$, the product formula involving the file numbers (5.2), when $q \to 1$, becomes

$$z(z+n)^{n-1} = \sum_{k=0}^{n} f_{n-k}(1;\mathsf{A}_n)z^k.$$

These polynomials are a special case of the general Abel polynomials $z(z+\alpha n)^{n-1}$, which we consider separately in the discussion following equation (5.18). The coefficient $f_{n-k}(1;\mathsf{A}_n) = t_{n,k} = \binom{n-1}{k-1}n^{n-k}$ counts the number of labeled forests on n vertices composed of k rooted trees [30]. Goldman and Haglund explained this equality bijectively in [20]. More precisely, they construct a bijection between the set

$$R_{n,k} = \{(Q,u),\ u \in \{1,2,\dots,n\}\},$$

where Q is a file placement of $n-k$ rooks on A_n, and

$$F_{n,k} = \{\text{marked rooted forests of } k \text{ rooted trees on } n \text{ labeled vertices}\},$$

where a *marked rooted forest* is a forest of rooted trees with one distinguished vertex in the forest (the mark), via constructing bijections between $R_{n,k}$ and $Q_{n,k}$, and between $Q_{n,k}$ and $F_{n,k}$, where $Q_{n,k}$ is a class of "marked" partial endofunctions. Thus Goldman and Haglund actually proved $nf_{n-k}(1;\mathsf{A}_n) = nt_{n,k}$. We present a new proof which establishes $f_{n-k}(1;\mathsf{A}_n) = t_{n,k}$ directly. In the following, we describe a bijection between

$$\mathcal{R}_{n,k} = \{\text{file placements of } n-k \text{ rooks on } \mathsf{A}_n\}$$

and

$$\mathcal{F}_{n,k} = \{\text{rooted forests of } k \text{ rooted trees on } n \text{ labeled vertices}\}.$$

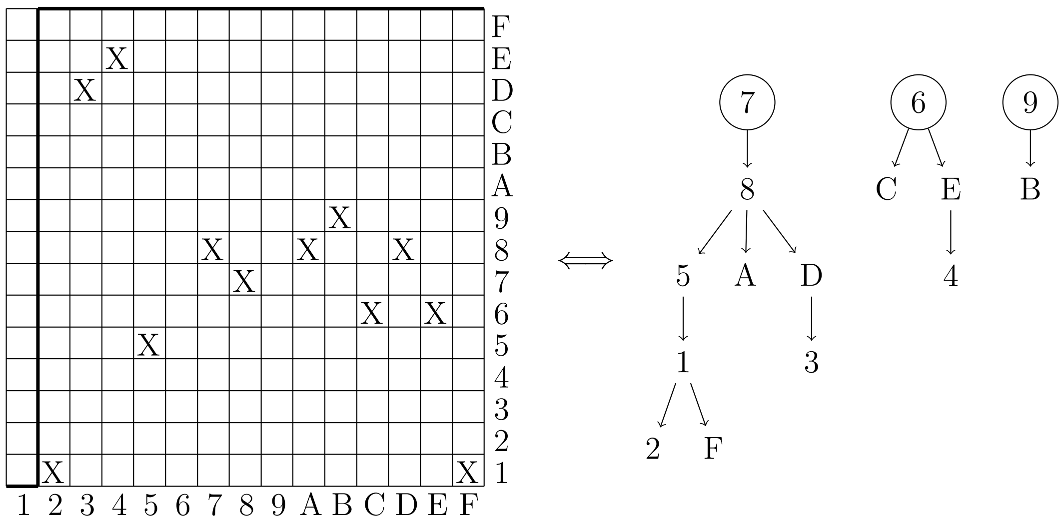


FIGURE 9. A file placement with $n = 15$, $k = 3$, and corresponding forest.

The bijection is best described by considering an example, see Figure 9, where for nicer layout, we have labeled the indices by hexadecimal digits. (Moreover, in the forest, we have circled the labels of the roots of the trees.) In this example, we have $n = 15$ and $k = 3$; we thus consider a 12-file placement of rooks on the A_{15} board. We shall successively transform this file placement into a forest of $15 - 12 = 3$ labeled trees. The rooks are in positions $(2,1)$, $(F,1)$, $(5,5)$, $(C,6)$, $(E,6)$, $(8,7)$, $(7,8)$, $(A,8)$, $(D,8)$, $(B,9)$, $(3,D)$, $(4,E)$, listed from bottom to top, left to right. Now we first identify the indices of the $k = 3$ empty columns, which are 1, 6, 9. We let 1 (which is special since the first column is always empty) be the preliminary label of the root of the first tree, while the other two numbers, 6 and 9, be the labels of the respective roots of the second and third trees. Note that these other roots shall not change whereas the first root can change. Next we look at the positions of all the rooks, first row-wise from bottom to top and then from left to right within each row. Now we interpret a rook in position (i,j) as a directed edge (forming a simple path) from vertex j to vertex i. After this we continue the path by looking for rooks in positions (l,i), in which case the path continues to go from vertex i to vertex l, etc. We transitively collect all paths in the file placement (in the prescribed order) and finally obtain an ordered collection of maximal chains (of paths which cannot be continued), and of disjoint cycles (where vertices already visited are reached again). Numbers occurring in the cycles can also appear in the chains. In our example, we have the following seven chains and two cycles.

maximal chains:
$1 \to 2$, $1 \to F$, $6 \to C$, $6 \to E \to 4$, $7 \to 8 \to A$, $7 \to 8 \to D \to 3$, $9 \to B$.

cycles: (5), $(7\,8)$.

It is important to insist that the minimal elements of the respective cycles are listed first. In the algorithm, the cycles are indeed obtained in the above order, i.e., by left-to-right increasing order of their minimal elements. Now, say we have obtained l cycles, $\gamma_1, \dots, \gamma_l$, listed by increasing order of their minimal elements. We then reverse the order of the l cycles, i.e., write out $\gamma_l, \dots, \gamma_1$ in decreasing order of their minimal elements while keeping the minimal elements of each cycle at the first position. In our example, the two cycles are thus relisted as $(7\,8)$, (5).

Now we form a new chain using all the labels from left to right appearing in the complete list of cycles, here $7 \to 8 \to 5$, and place this in the first tree before 1 (i.e., we also put an edge leading from the last vertex 5 to 1). Hence, 7 is now the new root of the first tree and we have a path leading to 1. (After a short moment of reflection, this part of the correspondence is easily noticed to be reversible, since 5 is the minimal element before 1, the minimal element before 5 is 7, etc., thus all the cycles can readily be determined.)

What remains to be done is to translate the maximal chains obtained from the file placement to form trees in the forest. This is done in the obvious way (and is clearly reversible); see Figure 9 for the result.

Since there is at most one rook in each column of the file placement, it is guaranteed that each vertex in the corresponding directed graph has at most one predecessor, i.e. the resulting graph is indeed a forest. Thus, we can conclude that each forest of n labeled vertices composed of k components corresponds to exactly one $(n-k)$-rook file placement on the board A_n and vice versa.

For the Abel board A_n, the product formula in Theorem 5.4 becomes

$$[z]_{a,b;q,p}([z+n]_{aq^{-2n},bq^{-n};q,p})^{n-1} = \sum_{k=0}^{n} f_{n-k}(a,b;q,p;\mathsf{A}_n)([z]_{a,b;q,p})^k. \tag{5.14}$$

As explained above, we can interpret the coefficient $f_{n-k}(a,b;q,p;\mathsf{A}_n)$ as the weighted sum of labeled forests on n vertices composed of k rooted trees. In the above algorithm for obtaining the forest, we weight the edge $j \to i$ by $\prod_{l=1}^{j-1} w_{a,b;q,p}(l)$ which in the file placement corresponds to the product of the weights of uncancelled cells above (i,j). If there is an empty column containing no rooks, then we weight the vertex corresponding to such a column by $\prod_{l=1}^{n} w_{a,b;q,p}(l)$. This yields a weighted forest of k rooted labeled trees on n vertices corresponding to a given $(n-k)$-file placement.

The coefficients in (5.14) have a nice closed form

$$f_{n-k}(a,b;q,p;\mathsf{A}_n) = \binom{n-1}{k-1} \left(W_{aq^{-2n},bq^{-n};q,p}(n)\right)^{k-1} \left([n]_{aq^{-2n},bq^{-n};q,p}\right)^{n-k}, \tag{5.15}$$

which is easy to prove directly combinatorially, or by using the identity

$$[z+n]_{aq^{-2n},bq^{-n}} = [n]_{aq^{-2n},bq^{-n}} + W_{aq^{-2n},bq^{-n};q,p}(n)[z]_{aq^{-2n},bq^{-n}}$$

(which is the $(y,z,a,b) \mapsto (n, z+n, aq^{-2n}, bq^{-n})$ case of Equation (3.8b)), together with the classical binomial theorem.

The above bijection easily extends to the case of r-restricted Abel boards $\mathsf{A}_n^{(r)} = B(0,\dots,0,n,\dots,n)$ of r columns of height zero and $n-r$ columns of height n. Say, we consider a file placement on $\mathsf{A}_n^{(r)}$. Then the bijection transforms a file placement of $n-k$ rooks on this board to a forest of k components where the first r numbers $1,2,\dots,r$ are in distinct trees and the $r-1$ numbers $2,\dots,r$ are roots. Now, by interchanging the labels 1 and r we immediately obtain a forest of n vertices of k labeled trees, where the first r numbers $1,2,\dots,r$ are in distinct trees and where moreover the first $r-1$ numbers $1,2,\dots,r-1$ are all roots (among the k roots of the forest). The number of such forests is $f_{n-k}(1;\mathsf{A}_n^{(r)}) = t_{n,k}^{(r)} = \binom{n-r}{k-r} n^{n-k}$. Given the analogy to the r-restricted Stirling numbers of the first and second kinds and of the r-restricted Lah numbers, it seems appropriate to refer to $t_{n,k}$ as *Abel numbers* and to $t_{n,k}^{(r)}$ as *r-restricted Abel numbers*.

For the r-restricted Abel board $\mathsf{A}_n^{(r)}$, the product formula in Theorem 5.4 becomes

$$([z]_{a,b;q,p})^r([z+n]_{aq^{-2n},bq^{-n};q,p})^{n-r} = \sum_{k=r-1}^{n} f_{n-k}(a,b;q,p;A_n^{(r)})([z]_{a,b;q,p})^k. \tag{5.16}$$

The coefficients in (5.16) have a nice closed form

$$f_{n-k}(a,b;q,p;\mathsf{A}_n^{(r)}) = \binom{n-r}{k-r} \left(W_{aq^{-2n},bq^{-n};q,p}(n)\right)^{k-r} \left([n]_{aq^{-2n},bq^{-n};q,p}\right)^{n-k}. \tag{5.17}$$

Lastly, we consider the general Abel board $\mathsf{A}_{\alpha n,n} = B(0,\alpha n,\dots,\alpha n) = [n-1]\times[\alpha n]$ for a positive integer α. The coefficients of the Abel polynomials

$$z(z+\alpha n)^{n-1} = \sum_{k=0}^{n} f_{n-k}(1;\mathsf{A}_{\alpha n,n}) z^k \tag{5.18}$$

have a very simple combinatorial interpretation. They count the number of forests of n labeled vertices composed of k rooted trees where each of the vertices can have one of α colors (distinct vertices may have the same color), where the k roots must all have the first color. This number of course is α^{n-k} times the usual monocolor case, since each of the $n-k$ vertices which are not roots can assume one of α colors. We therefore have $f_{n-k}(1;\mathsf{A}_{\alpha n,n}) = t_{\alpha,n,k} = \binom{n-1}{k-1}(\alpha n)^{n-k}$. This interpretation also easily comes out of the file placement model as follows. If there is a rook in position $(i,(c-1)n+j)$ for $1 \le i,j \le n$ and $1 \le c \le \alpha$, then form a directed path from j to i and assign color c to the vertex i.

Even more generally, we could consider the Abel board $\mathsf{A}_{\alpha n,n}$ with $\alpha = \frac{m}{n}$, i.e., $\mathsf{A}_{m,n} = [n-1]\times[m]$ for a positive integer m. Even here it is not difficult to give a combinatorial interpretation (which is consistent with the file placement model). The coefficients of the polynomials

$$z(z+m)^{n-1} = \sum_{k=0}^{n} f_{n-k}(1; \mathsf{A}_{m,n}) z^k \tag{5.19}$$

count the number of forests of n labeled vertices composed of k rooted trees where each of the vertices can have one of $\lceil \frac{m}{n} \rceil$ colors (distinct vertices may have the same color), the k roots must all have the first color, but only the successors of $1, 2, \ldots, m - \lfloor \frac{m-1}{n} \rfloor n$ are allowed to assume the highest color $\lceil \frac{m}{n} \rceil$. Here $\lceil x \rceil := \min\{y \in \mathbb{Z} : y \geq x\}$ and $\lfloor x \rfloor := \max\{y \in \mathbb{Z} : y \leq x\}$ are the ceiling and floor functions, respectively.

For instance, if $m = 4$ and $n = 3$, Equation (5.19) becomes $z(z+4)^2 = z^3 + 8z^2 + 16z$. Accordingly, using the vertices $1, 2, 3$ we form colored forests containing exactly k trees where we may use two colors, say, black and white, to color the vertices but only the successors of 1 can be white. There is exactly one forest containing three trees, each consisting of only a root. There are 8 such forests containing two trees, see Figure 10, and there are 16 such forests containing exactly one tree, see Figure 11, where for convenience primed labels indicate white vertices.

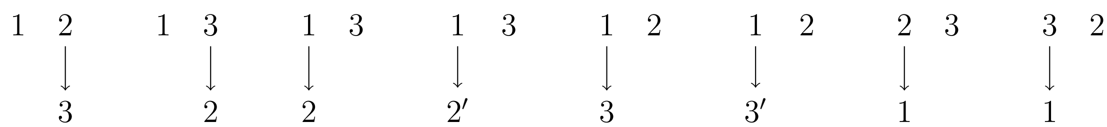


Figure 10. Colored forests of two trees corresponding to the board $\mathsf{A}_{4,3}$.

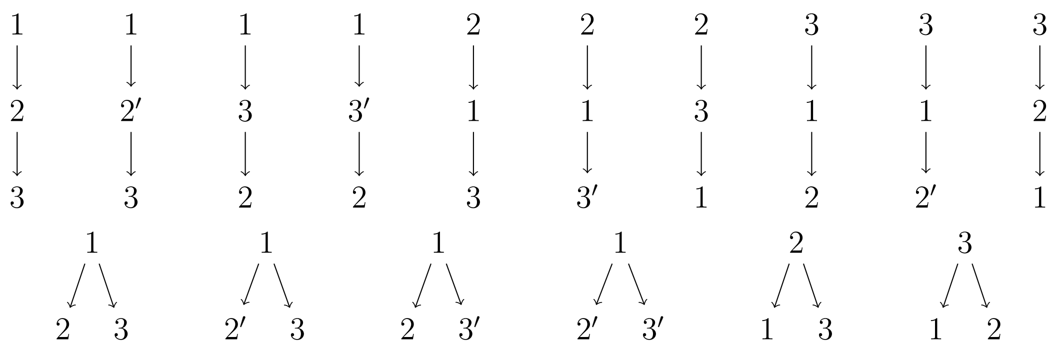


Figure 11. Colored forests of one tree corresponding to the board $\mathsf{A}_{4,3}$.

The above generalized Abel case involving file placements on the board $\mathsf{A}_{m,n}$ can even be extended to the r-restricted case where for a nonnegative integer r we consider the board $\mathsf{A}_{m,n}^{(r)} = [n-r] \times [m]$. We have the same combinatorial interpretation for the colored forests but with the additional restriction that the numbers $1, 2, \ldots, r$ are in different trees and $1, 2 \ldots, r-1$ are roots.

In the elliptic case, we have

$$([z]_{a,b;q,p})^r ([z+m]_{aq^{-2m},bq^{-m};q,p})^{n-r} = \sum_{k=0}^{n} f_{n-k}(a,b;q,p;\mathsf{A}_{m,n}^{(r)}) ([z]_{a,b;q,p})^k \tag{5.20a}$$

and for the coefficients we have the explicit formula

$$f_{n-k}(a,b;q,p;\mathsf{A}^{(r)}_{m,n}) = \binom{n-r}{k-r} \left(W_{aq^{-2m},bq^{-m};q,p}(m)\right)^{k-r} \left([m]_{aq^{-2m},bq^{-m};q,p}\right)^{n-k}. \quad (5.20b)$$

5.4. **Elliptic analogue of generalized Stirling numbers of the first kind.** Upon the introduction of the generalized $(\mathfrak{p},q)$-Stirling numbers of the second kind (see Section 4.1 for the elliptic analogue of them), Remmel and Wachs [32] also introduced and studied the generalized $(\mathfrak{p},q)$-Stirling numbers of the first kind, denoted by $\mathfrak{c}^{\hat{\imath},\hat{\jmath}}_{n,k}(\mathfrak{p},q)$. In [32], it is shown that $\mathfrak{c}^{\hat{\imath},\hat{\jmath}}_{n,k}(\mathfrak{p},q)$ satisfy $\mathfrak{c}^{\hat{\imath},\hat{\jmath}}_{0,0}(\mathfrak{p},q)=1$, $\mathfrak{c}^{\hat{\imath},\hat{\jmath}}_{n,k}(\mathfrak{p},q)=0$ if $k<0$ or $k>n$, and

$$\mathfrak{c}^{\hat{\imath},\hat{\jmath}}_{n+1,k}(\mathfrak{p},q) = \mathfrak{c}^{\hat{\imath},\hat{\jmath}}_{n,k-1}(\mathfrak{p},q) + [\hat{\imath}+n\hat{\jmath}]_{\mathfrak{p},q}\mathfrak{c}^{\hat{\imath},\hat{\jmath}}_{n,k}(\mathfrak{p},q).$$

It is also shown that

$$([z]_{\mathfrak{p},q}+[\hat{\imath}]_{\mathfrak{p},q})([z]_{\mathfrak{p},q}+[\hat{\imath}+\hat{\jmath}]_{\mathfrak{p},q})\cdots([z]_{\mathfrak{p},q}+[\hat{\imath}+(n-1)\hat{\jmath}]_{\mathfrak{p},q}) = \sum_{k=0}^{n} \mathfrak{c}^{\hat{\imath},\hat{\jmath}}_{n,k}(\mathfrak{p},q)([z]_{\mathfrak{p},q})^k. \quad (5.21)$$

$\mathfrak{c}^{\hat{\imath},\hat{\jmath}}_{n,k}(\mathfrak{p},q)$ can be computed as a weighted sum of file placements in $B_{\hat{\imath},\hat{\jmath},n}$. Recall that $B_{\hat{\imath},\hat{\jmath},n} = B(\hat{\imath},\hat{\imath}+\hat{\jmath},\dots,\hat{\imath}+(n-1)\hat{\jmath})$.

Given a file placement $Q\in\mathcal{F}_k(B)$, for B a skyline board, define

$$w_{\mathfrak{p},q,B}(Q) = q^{\alpha_B(Q)}\mathfrak{p}^{\beta_B(Q)},$$

where

$$\alpha_B(Q) = \text{ the number of cells in } B \text{ that lie above some rook } \mathbf{r} \text{ in } Q,$$
$$\beta_B(Q) = \text{ the number of cells in } B \text{ that lie below some rook } \mathbf{r} \text{ in } Q.$$

If we define

$$\tilde{f}_k(\mathfrak{p},q;B) = \sum_{Q\in\mathcal{F}_k(B)} w_{\mathfrak{p},q,B}(Q),$$

then for $0\le k\le n$ it is shown in [32] that

$$\mathfrak{c}^{\hat{\imath},\hat{\jmath}}_{n,k}(\mathfrak{p},q) = \tilde{f}_{n-k}(\mathfrak{p},q;B_{\hat{\imath},\hat{\jmath},n}).$$

We now establish an elliptic analogue of $\mathfrak{c}^{\hat{\imath},\hat{\jmath}}_{n,k}(\mathfrak{p},q)$ by modifying the weight function $w_{\mathfrak{p},q,B}(Q)$ to an elliptic function. Given a skyline board B, define

$$\tilde{f}_k(a,b;q,p;B) = \sum_{Q\in\mathcal{F}_k(B)} \widetilde{wt}_f(Q),$$

where

$$\widetilde{wt}_f(Q) = \prod_{(i,j)\in\mathcal{A}_B(Q)} w_{a,b;q,p}(i-j), \quad (5.22)$$

and $\mathcal{A}_B(Q)$ is the set of cells in B that lie above some rook $\mathbf{r}$ in Q.

Proposition 5.7. *For a skyline board $B=B(c_1,\dots,c_n)$, we have*

$$\prod_{i=1}^{n}\left([z]_{a,b;q,p}+[c_i]_{aq^{2(i-1-c_i)},bq^{i-1-c_i};q,p}\right) = \sum_{k=0}^{n} \tilde{f}_{n-k}(a,b;q,p;B)([z]_{a,b;q,p})^k. \quad (5.23)$$

Proof. We consider the extended board B_z and compute the sum

$$\sum_{P\in\mathcal{F}_n(B_z)} \overline{wt}_f(P)$$

in two different ways, where

$$\overline{wt}_f(P)=\prod_{(i,j)\in P}\overline{w}_{a,b;q,p;B_z}(i,j),$$

$$\overline{w}_{a,b;q,p;B_z}(i,j)=\begin{cases} W_{aq^{2(i-1-c_i)},bq^{i-1-c_i};q,p}(c_i-j), & \text{if } (i,j)\in B,\\ W_{a,b;q,p}(-j), & \text{if } (i,j) \text{ is below the ground.}\end{cases}$$

Recall that the line separating the board B and the extended part of B_z is called the ground, and the row coordinates below the ground are $0,-1,-2,\dots,1-z$, from top to bottom. To obtain the left-hand side of (5.23), we place n rooks column by column. In i-th column, possible placements of a rook above the ground contribute

$$1+W_{aq^{2(i-1-c_i)},bq^{i-1-c_i};q,p}(1)+\dots+W_{aq^{2(i-1-c_i)},bq^{i-1-c_i};q,p}(c_i-1)=[c_i]_{aq^{2(i-1-c_i)},bq^{i-1-c_i};q,p}$$

and possible placements below the ground give

$$1+W_{a,b;q,p}(1)+\dots+W_{a,b;q,p}(z-1)=[z]_{a,b;q,p}.$$

Sum of $[c_i]_{aq^{2(i-1-c_i)},bq^{i-1-c_i};q,p}$ and $[z]_{a,b;q,p}$ gives the i-th factor in the left-hand side of (5.23).

To get the right-hand side of (5.23), we start with a file placement $Q\in\mathcal{F}_{n-k}(B)$ and extend it to a file placement of n rooks by placing k rooks below the ground. In each empty column, possible placements of a rook below the ground give $[z]_{a,b;q,p}$, as computed above, hence placements of k rooks below the ground will give the factor $([z]_{a,b;q,p})^k$. Note that $\overline{w}_{a,b;q,p;B_z}(i,j)=W_{aq^{2(i-1-c_i)},bq^{i-1-c_i};q,p}(c_i-j)$ for $(i,j)\in B$ is defined so that $\overline{w}_{a,b;q,p;B_z}(i,j)$ equals to $\prod_{t=j+1}^{c_i} w_{a,b;q,p}(i-t)$ for (i,t) being the coordinates of the cells above the rook in (i,j). Thus,

$$\begin{aligned}\sum_{P\in\mathcal{F}_n(B_z)}\overline{wt}_f(P)&=\sum_{k=0}^{n}\left(\sum_{Q\in\mathcal{F}_{n-k}(B)}\prod_{(i,j)\in Q}\overline{w}_{a,b;q,p;B}(i,j)\right)([z]_{a,b;q,p})^k\\&=\sum_{k=0}^{n}\tilde{f}_{n-k}(a,b;q,p;B)([z]_{a,b;q,p})^k.\end{aligned}$$ □

If we apply Proposition 5.7 to the board $B_{\hat{\imath},\hat{\jmath},n}$, then we get

$$\prod_{s=1}^{n}\left([z]_{a,b;q,p}+[\hat{\imath}+(s-1)\hat{\jmath}]_{aq^{-2(\hat{\imath}+(s-1)(\hat{\jmath}-1))},bq^{-(\hat{\imath}+(s-1)(\hat{\jmath}-1))};q,p}\right)$$
$$=\sum_{k=0}^{n}\tilde{f}_{n-k}(a,b;q,p;B_{\hat{\imath},\hat{\jmath},n})([z]_{a,b;q,p})^k.$$

We can define $\mathfrak{c}^{\hat{\imath},\hat{\jmath}}_{n,k}(a,b;q,p) := \tilde{f}_{n-k}(a,b;q,p;B_{\hat{\imath},\hat{\jmath},n})$ which is an elliptic analogue of $\mathfrak{c}^{\hat{\imath},\hat{\jmath}}_{n,k}(\mathfrak{p},q)$. Then, by considering whether there is a rook or not in the last column of $B_{\hat{\imath},\hat{\jmath},n}$, we get the recurrence relation

$$\mathfrak{c}^{\hat{\imath},\hat{\jmath}}_{n+1,k}(a,b;q,p) = \mathfrak{c}^{\hat{\imath},\hat{\jmath}}_{n,k-1}(a,b;q,p) + [\hat{\imath}+n\hat{\jmath}]_{aq^{-2(\hat{\imath}+n(\hat{\jmath}-1))},bq^{-(\hat{\imath}+n(\hat{\jmath}-1))};q,p}\mathfrak{c}^{\hat{\imath},\hat{\jmath}}_{n,k}(a,b;q,p)$$

which determines $\mathfrak{c}^{\hat{\imath},\hat{\jmath}}_{n,k}(a,b;q,p)$ with the conditions

$$\mathfrak{c}^{\hat{\imath},\hat{\jmath}}_{0,0}(a,b;q,p) = 1, \text{ and } \quad \mathfrak{c}^{\hat{\imath},\hat{\jmath}}_{n,k}(a,b;q,p) = 0 \text{ if } k<0 \text{ or } k>n.$$

Remark 5.8. In [32], Remmel and Wachs showed that for $s^{\hat{\imath},\hat{\jmath}}_{n,k}(\mathfrak{p},q) = (-1)^{n-k}\mathfrak{c}^{\hat{\imath},\hat{\jmath}}_{n,k}(\mathfrak{p},q)$, the two matrices $||s^{\hat{\imath},\hat{\jmath}}_{n,k}(\mathfrak{p},q)||$ and $||S^{\hat{\imath},\hat{\jmath}}_{n,k}(\mathfrak{p},q)||$ are inverses of each other (we refer to Section 4.1 for the definition of $S^{\hat{\imath},\hat{\jmath}}_{n,k}(\mathfrak{p},q)$), namely, for all $0 \le r \le n$,

$$\sum_{k=r}^{n} S^{\hat{\imath},\hat{\jmath}}_{n,k}(\mathfrak{p},q)s^{\hat{\imath},\hat{\jmath}}_{k,r}(\mathfrak{p},q) = \chi(r=n). \tag{5.24}$$

Notice that the elliptic weights of the uncancelled cells used in (4.3) to define the rook numbers are different from the elliptic weights used in (5.22), that is, the former ones depend on the value $\hat{\jmath}$ whereas the latter ones only depend on the coordinates of the cells, as opposed to them being the same in the q-case (or $\mathfrak{p},q$-case). Hence, the cancellation occurring in the process of proving (5.24) does not occur in the same way in the elliptic case, and as a consequence, we do not have the property that the matrices $||s^{\hat{\imath},\hat{\jmath}}_{n,k}(a,b;q,p)||$ and $||S^{\hat{\imath},\hat{\jmath}}_{n,k}(a,b;q,p)||$ are inverses of each other, for $s^{\hat{\imath},\hat{\jmath}}_{n,k}(a,b;q,p) = (-1)^{n-k}\mathfrak{c}^{\hat{\imath},\hat{\jmath}}_{n,k}(a,b;q,p)$. However, if we set $\hat{\imath}=0$ and $\hat{\jmath}=1$, then the elliptic weights of the cells used in the rook and file numbers are equal, and thus we can show that the two matrices $||s^{0,1}_{n,k}(\mathfrak{p},q)||$ and $||S^{0,1}_{n,k}(\mathfrak{p},q)||$ are inverses of each other.

6. Future perspectives

6.1. **Elliptic rook numbers on augmented boards.** In [29], Miceli and Remmel introduced a generalized rook model by considering rook placements on *augmented boards* and proved the corresponding product formula as well as the q-analogue of it. This product formula can be specialized to all the known product formulas including the i-creation model of Goldman and Haglund [20]. In a separate paper [36] we provide an elliptic extension of the q-case of this model. In particular, the elliptic extension of the product formula in [36] includes our elliptic product formula of the i-creation model as a special case.

6.2. **Elliptic rook theory for matchings.** By considering rook placements on shifted Ferrers boards subject to a suitable modification of rook cancellation, Haglund and Remmel [23] developed a q-rook theory for matchings of graphs. We provide an elliptic extension of this rook theory in [37]. We actually consider a more general model there related to matchings on certain graphs which we call "$\mathbf{l}$-lazy graphs" with respect to a N-dimensional vector of positive integers $\mathbf{l}=(l_1,\dots,l_N)$. These matchings correspond to rook placements on $\mathbf{l}$-shifted boards for which we essentially employ the same rook cancellation as in the ordinary shifted case considered by Haglund and Remmel. The elliptic case is more intricate than the q-case and requires a very careful choice of weights. Our factorization theorem for elliptic rook numbers for matchings of $\mathbf{l}$-lazy graphs in [37] generalizes the factorization theorem by Haglund and Remmel already in the ordinary case and the q-case. In the simplest case, our formula can be used to deduce an elliptic extension of the numbers of perfect matchings of the complete graph K_{2n}.

6.3. **Elliptic analogue of the hit numbers.** Upon the introduction of the rook numbers, Kaplansky and Riordan [26] also defined the hit numbers. Let $\mathcal{H}_{n,k}(B)$ be the set of all placements of n rooks on $[n]\times[n]$ such that exactly k of these rooks lie on B. Then $h_{n,k}(B) := |\mathcal{H}_{n,k}(B)|$ is called the k-th *hit number* of B. Kaplansky and Riordan [26] showed

$$\sum_{k=0}^{n} r_{n-k}(B)k!(z-1)^{n-k} = \sum_{k=0}^{n} h_{n,k}(B)z^k. \tag{6.1}$$

Garsia and Remmel defined in [17] the q-hit numbers of a Ferrers board $B\subseteq[n]\times[n]$ algebraically by the equation

$$\sum_{k=0}^{n} r_{n-k}(q;B)[k]_q!z^k \prod_{i=k+1}^{n}(1-zq^i) = \sum_{k=0}^{n} h_{n,k}(q;B)z^{n-k} \tag{6.2}$$

and proved the existence of the statistic $\mathrm{stat}_{n,B}(P)$ such that

$$h_{n,k}(q;B) = \sum_{P\in\mathcal{H}_{n,k}(B)} q^{\mathrm{stat}_{n,B}(P)}.$$

They did not give a specific description for $\mathrm{stat}_{n,B}(P)$, but later, Dworkin [13] and Haglund [22] independently found combinatorial descriptions. Hence the natural quest is to establish an elliptic analogue of the relation (6.2) and define elliptic hit numbers accordingly.

Similarly, together with the file numbers, Garsia and Remmel also defined the fit numbers. Let $\mathcal{F}_{n,k}(B)$ be the set of all file placements of n rooks on $[n]\times[n]$ such that exactly k of these rooks lie on B. Then $f_{n,k}(B) := |\mathcal{F}_{n,k}(B)|$ is called the k-th *fit number* of B. Garsia and Remmel (cf. [5]) showed

$$\sum_{k=0}^{n} f_{n-k}(B)n^k(z-1)^{n-k} = \sum_{k=0}^{n} f_{n,k}(B)z^k. \tag{6.3}$$

It would be interesting to find an elliptic analogue of (6.3). However, as a matter of fact, not even a q-analogue of (6.3) has so far been established.

It should also be mentioned that Haglund and Remmel [23] defined and obtained results for hit and q-hit numbers for rook placements on the shifted board B_{2n}. It would be interesting to find extensions of their results, either to the elliptic setting or simply to results for $\mathbf{l}$-shifted boards $B_N^{\mathbf{l}}$ (where rook placements are identified as partial maximal matchings on $\mathbf{l}$-lazy graphs) as we considered in [37].

6.4. **Elliptic hypergeometric series identities.** The coefficients $c_k(q)$ in the expansion

$$P(z;q)=\sum_{k=0}^{n} c_k(q)[z]_q[z-1]_q\cdots[z-k+1]_q \tag{6.4}$$

are uniquely determined by $P(z;q)$. In particular, we have (see [24])

$$c_k(q)=\frac{1}{[k]_q!}\,\mathfrak{z}_0\,\Delta^{(k)}P(z;q) \tag{6.5a}$$

where $\mathfrak{z}_0$ denotes the evaluation at $z=0$ and

$$\Delta^{(k)}=(\epsilon-1)(\epsilon-q)\cdots(\epsilon-q^{k-1}), \tag{6.5b}$$

with ϵ the 1-shift operator, i.e., $\epsilon P(z;q)=P(z+1;q)$. In fact, the formula (6.5a) for the coefficients can be simply established by expanding (6.5b) by means of the q-binomial theorem

$$(x+y)(x+qy)\cdots(x+q^{k-1}y)=\sum_{j=0}^{k} q^{\binom{j}{2}}\begin{bmatrix}k\\j\end{bmatrix}_q y^j x^{k-j}. \tag{6.6}$$

Taking $P(z;q)=\prod_{i=1}^{n}[z+b_i-i+1]_q$, where $B=B(b_1,\dots,b_n)\subseteq[n]\times\mathbb{N}$ is a Ferrers board, the coefficients in (6.4) become $c_k(q)=r_{n-k}(q;B)$, due to Proposition 2.2. Thus one has

$$r_{n-k}(q;B)=\frac{1}{[k]_q!}\,\mathfrak{z}_0\,\Delta^{(k)}\prod_{i=1}^{n}[z+b_i-i+1]_q. \tag{6.7}$$

In [21, Equations (24) and (49)], Haglund has used an inversion argument involving the q-Chu–Vandermonde summation to arrive at an identity equivalent to (6.7). For "regular" boards (these are Ferrers boards whose shape correspond to Dyck paths), he was then able to express q-rook numbers in terms of multiples of basic hypergeometric series of Karlsson–Minton type (see [18] for the terminology). By analytic continuation such formulas can also be obtained for some other boards which are not "regular" (such as the staircase board St_n whose q-rook numbers are Stirling numbers of the second kind which also admit a basic hypergeometric series representation of Karlsson–Minton type, see (3.29)). Haglund further combined his results with identities involving q-hit numbers to obtain transformation formulae for basic hypergeometric series of Karlsson–Minton type.

Now it would be very nice if one could extend the analysis we described leading to (6.7) to the elliptic setting, given that we were able to establish an elliptic analogue of the product formula in Proposition 2.2, namely Theorem 3.8, and also given that elliptic analogues of basic hypergeometric series of Karlsson–Minton type exist (see [33] or [18, Chapter 11]). At first glance, the elliptic extension of this approach may appear straightforward. What is needed is an explicit formula for the coefficients $c_k(a,b;q,p)$ in

$$P(z;a,b;q,p)=\sum_{k=0}^{n} c_k(a,b;q,p)[z]_{a,b;q,p}[z-1]_{aq^2,bq;q,p}\cdots[z-k+1]_{aq^{2(k-1)},bq^{k-1};q,p}, \tag{6.8}$$

and then take

$$P(z;a,b;q,p)=\prod_{i=1}^{n}[z+b_i-i+1]_{aq2(i-1-b_i),bq^{i-1-b_i};q,p}.$$

Although we so far failed to find a correct elliptic extension of (6.5), a solution of this problem does not seem to be completely out of reach.

For the reader to get a feeling for some of the difficulties in the elliptic setting, compare Carlitz' compact formula for the q-Stirling numbers of the second kind in (3.28) (which, as mentioned, can also be written in terms of basic hypergeometric series of Karlsson–Minton type, see (3.29)) with the (not so uniform) expressions for the elliptic Stirling numbers of the second kind which we listed in (3.27). The latter appear not to be instances of a multiple of an elliptic hypergeometric series of Karlsson–Minton type. Nevertheless they may be instances of a series which is close to being elliptic hypergeometric[2].

6.5. **Relations to algebraic varieties.** Ding introduced the *length function* which was first used as the length of rook matrices by Solomon in his work on the Iwahori ring of $M_n(F_q)$ [39]. Given a Ferrers board $B=B(b_1,\dots,b_n)$ and a k-rook placement $P\in\mathcal{N}_k(B)$, the length function $l_B(P)$ is defined by

$$l_B(P)=\sum_{(i,j)\in P}(n-i+j-1+\gamma_{(i,j)}),$$

where $\gamma_{(i,j)}$ is the number of rooks which are in the south-east region of the rook in (i,j). Then, if we let $C_{B,k}=\sum_{i=1}^{n}b_i-\frac{k(k+1)}{2}$, we have

$$u_B(P)+l_B(P)=C_{B,k}.$$

Thus, if we define the rook length polynomial by

$$rl_k(q;B)=\sum_{P\in\mathcal{N}_k(B)}q^{l_B(P)},$$

then the relation to the q-rook number is

$$r_k(q;B)=q^{C_{B,k}}rl_k(q^{-1};B).$$

In [11], Ding studied the geometric implication of rook length polynomials by introducing partition varieties. Partition varieties are projective varieties which have cellular decomposition analogous to the cellular decomposition of the Grassmannian into Schubert cells. These partition varieties have CW-complex structure with the rook length polynomials being their Poincaré polynomials of cohomology. In [12], Ding even generalized the study of partition varieties by replacing the Borel subgroup of upper triangular matrices by more general parabolic subgroups of the general linear group. For this purpose he introduced γ-compatible partitions, γ-compatible rook placements and γ-compatible rook length polynomials, where γ is a composition.

[2]Following [18, Chapter 11], a series $\sum_{k\geq 0}c_k$ is called *elliptic hypergeometric* if and only if the quotient c_{k+1}/c_k is an elliptic (i.e., meromorphic and doubly periodic) function in k, where k is viewed as a complex variable.

It would be interesting to reveal any connection between elliptic analogues of rook numbers and algebraic varieties.

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