RIGHT-JUMPS AND PATTERN AVOIDING PERMUTATIONS

CYRIL BANDERIER AND JEAN-LUC BARIL AND CÉLINE MOREIRA DOS SANTOS

Abstract. We study the iteration of the process "a particle jumps to the right" in permutations. We prove that the set of permutations obtained in this model after a given number of iterations from the identity is a class of pattern avoiding permutations. We characterize the elements of the basis of this class and we enumerate these "forbidden minimal patterns" by giving their bivariate exponential generating function: we achieve this via a catalytic variable, the number of left-to-right maxima. We show that this generating function is a D-finite function satisfying a nice differential equation of order 2. We give some congruence properties for the coefficients of this generating function, and we show that their asymptotics involves a rather unusual algebraic exponent (the golden ratio $(1+\sqrt{5})/2$) and some unusual closed-form constants. We end by proving a limit law: a forbidden pattern of length n has typically $(\ln n)/\sqrt{5}$ left-to-right maxima, with Gaussian fluctuations.

1 Introduction

In computer science, many algorithms related to sorting a permutation were analyzed and it was shown that their behaviors are linked to nice combinatorial properties (see e.g. [18]). Their complexity can be analyzed in terms of memory needed, or number of key operations (like comparisons or pointer swaps). An important family of algorithms, like the so-called "insertion" algorithms, or "in situ" permutations, are quite efficient in terms of number of pointer swaps (but are not the fastest ones in terms of comparisons). Because of this higher cost, they were much less studied than the faster stack sorting algorithms. Like for the stack algorithms, instead of seeing them as a couple input/output, we can see them like a *process*: input/set of intermediate steps. This opens a full realm of questions on such processes, and they often lead to nice links with other parts of mathematics (like the link between trees, birth and death processes, random walks, in probability theory, or permutations and Young tableaux in algebraic combinatorics). Our article will investigate such a link.

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Cyril Banderier: LIPN UMR-CNRS 7030, Université de Paris Nord, 93430 Villetaneuse, France. http://lipn.fr/~banderier.

J.-L. Baril and C. Moreira: LE2I UMR-CNRS 6306, Université de Bourgogne, 21078 Dijon, France. http://jl.baril.u-bourgogne.fr/, http://le2i.cnrs.fr/-Celine-Moreira-Dos-Santos-.

Another motivation to analyze such processes is coming from bioinformatics. Indeed, in genomics, a crucial study is to estimate the similarity of two genomes. This consists of finding the length of a shortest path of evolutionary mutations that transform one genome into another. Usually, the main operations used in the rearrangement of a genome are of three different types: substitutions (one gene is replaced with another), insertions (a gene is added) and deletions (a gene is removed). For instance, we refer to [14, 21] for an explanation of these operations, and the notions of transposons or jumping genes.

As the problem is too hard in full generality, many simpler mathematical models of the genome were given: one of them is using permutations of $\{1, 2, \ldots, n\}$ where each gene is assigned to a number [12]. Following the idea of transposition mutations [14], our motivation is to find some combinatorial properties in terms of pattern avoiding permutations whenever one element is deleted and inserted in a position located on its right. This operation will be called a right-jump.

$$1\ 2\ 3\ 4\ 5\ 6\ 7 \rightsquigarrow 1\ 2\ 4\ 5\ 6\ 3\ 7$$

FIGURE 1. A right-jump in the permutation $\sigma = 1\ 2\ 3\ 4\ 5\ 6\ 7$. In this article, we investigate the structure of permutations obtained after several iterations of such right-jumps.

This operation is a variant of the genome duplication, which consists in copying a part of the original genome inserted into itself, followed by the loss of one copy of each of the duplicated genes. In particular, it is comparable to the whole duplication-random loss model studied in [10]. Although there are many connections between these models, it is surprising that the behavior of their combinatorial properties depends of different parameters: the right-jump model reveals some links with *left-to-right maximum* statistics [3, 5], while the whole duplication-random loss model reveals links with *descent* statistics [4, 6, 7, 8, 10, 20].

In the literature, such right-jumps in permutation are also found in the domain of sorting theory. Indeed, it corresponds (modulo a mirror symmetry) to the *insertion-sorting algorithms* [18] on permutations. Since the seminal work of Knuth on this subject, many articles related to sorting with a stack exhibit some links with pattern avoiding permutations. While for insertion sorting algorithms (also the subject of a vivid literature), only one study exhibits some links with pattern avoiding permutations: more precisely, in his thesis [19], Magnùsson proves that the set of permutations that can be sorted with one step of the insertion-sorting operator is the class of permutations avoiding the three patterns 321, 312 and 2143.

Plan of the article. In Section 2, we recall some basic facts on permutations patterns. In Section 3, we generalize the result of Magnùsson by studying the iteration of right-jumps in terms of pattern avoiding permutations: we prove that the set of permutations obtained from the identity after a given number of right-jumps is the class of permutations avoiding some patterns, which we characterize. In Section 4, we enumerate these forbidden patterns by giving their bivariate exponential generating function (involving an additional parameter: the number of left-to-right maxima), and we give the corresponding asymptotics and limit law. *En passant*, we also give some modular congruences for our main enumeration sequence. In Section 5, we conclude with several possible prolongations of this work.

2 Patterns in permutations

In this section, we give some classical definitions and properties on patterns in permutations. $\mathcal{S}_n$ will be the set of permutations of length n. The *graphical representation* of a permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ is the set of points in the plane at coordinates (i, σ_i) for $i \in [\![n]\!]$[1]. For instance, the permutation $5\ 3\ 6\ 2\ 1\ 4\ 8\ 7$ has a graphical representation illustrated in Figure 2. A *left-to-right maximum* of $\sigma \in \mathcal{S}_n$ is a value σ_i, $1 \leq i \leq n$, such that $\sigma_j \leq \sigma_i$ for $j \leq i$. A value σ_i of σ, $1 \leq i \leq n$ which is not a left-to-right maximum will be called a *non-left-to-right-maximum* of σ. For instance, if $\sigma = 5\ 3\ 6\ 2\ 1\ 4\ 8\ 7$ then the left-to-right maxima are $5, 6, 8$ and the non-left-to-right-maxima are $1, 2, 3, 4, 7$.

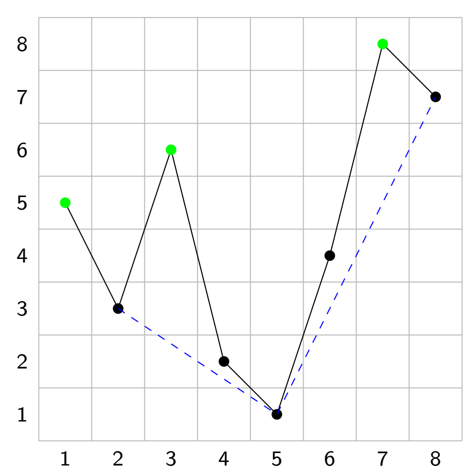

FIGURE 2. The graphical representation of $\sigma = 5\,3\,6\,2\,1\,4\,8\,7$. We show an occurrence of a pattern 213 with a blue dashed line; the left-to-right maxima are green and the non-left-to-right-maxima are black.

[1]In this article, we write $[\![n]\!]$ for $\{1, 2, \ldots, n\}$.

A permutation π of length k, is a *pattern* of a permutation $\sigma \in \mathcal{S}_n$ if there is a subsequence of σ which is order-isomorphic to π, *i.e.*, if there is a subsequence $\sigma_{i_1} \ldots \sigma_{i_k}$ of σ with $1 \leq i_1 < \cdots < i_k \leq n$ and such that $\sigma_{i_\ell} < \sigma_{i_m}$ whenever $\pi_\ell < \pi_m$. We write $\pi \prec \sigma$ to denote that π is a pattern of σ. A permutation σ that does not contain π as a pattern is said to *avoid* π. For example, $\sigma = 2413$ contains the patterns 231, 132, 213 and 312, but σ avoids the patterns 123 and 321. The set of all permutations avoiding the patterns $\pi_1, \ldots, \pi_k$ is denoted $\mathrm{Avoid}(\pi_1, \ldots, \pi_k)$. We say that $\mathrm{Avoid}(\pi_1, \ldots, \pi_k)$ is a *class* of pattern avoiding permutations of *basis* $\{\pi_1, \ldots, \pi_k\}$. For instance, we refer to the book of Kitaev [16] and Bóna [5] to deepen these notions. A set $\mathcal{C}$ of permutations is *stable* for the involvement relation $\prec$ if, for any $\sigma \in \mathcal{C}$, for any $\pi \prec \sigma$, then we also have $\pi \in \mathcal{C}$.

Now, we formulate a definition that is crucial for the present study.

Definition 1 (Permutation basis and basis permutations)**.** *If a set $\mathcal{C}$ of permutations is stable for the involvement relation $\prec$, then $\mathcal{C}$ is a class of pattern avoiding permutations: $\mathcal{C} = \mathrm{Avoid}(\mathcal{B})$. The basis $\mathcal{B}$ of forbidden patterns is then given by*

$$\mathcal{B} = \{\sigma \notin \mathcal{C}, \forall \pi \prec \sigma \text{ with } \pi \neq \sigma, \pi \in \mathcal{C}\}\,.$$

In other words, the basis $\mathcal{B}$ is the set of minimal permutations σ that do not belong to $\mathcal{C}$, where minimal is intended in the sense of the pattern-involvement relation $\prec$ on permutations, that is: if $\pi \prec \sigma$ and $\pi \neq \sigma$ then $\pi \in \mathcal{C}$. Notice that $\mathcal{B}$ might be infinite. We call basis permutations *the permutations belonging to $\mathcal{B}$.*

Equipped with these definitions, our mission consists now in giving a description of the "basis permutations" belonging to the basis $\mathcal{B}_p$ (i.e. the permutations which are the "minimal forbidden patterns") for the set $\mathcal{C}_p$ of permutations at distance at most p from the identity.

3 Iteration of right-jumps in permutations: a structural description of the forbidden patterns

In this section we study the iteration of right-jumps in terms of pattern avoiding permutations. We establish that the set $\mathcal{C}_p$ of permutations obtained from the identity after at most p right-jumps is a class of permutations avoiding some patterns that we characterize.

Lemma 1 (Characterization of the distance)**.** *A permutation obtained from the identity after p right-jumps contains at most p non-left-to-right-maxima.*

Proof. This first lemma belongs to the category of claims for which the proof could be in one word: "trivial", or in a boring half-page if we want to write a rigorous proof. Well, let us go for this boring half-page proof by induction. The result holds for $p = 1$; indeed a right-jump transformation of the identity permutation creates the permutation $1\ 2\ \ldots (i-1)\ (i+1) \ldots (j-1)\ i\ j \ldots n$ for $1 \leq i < j$, where i is the only one non-left-to-right-maximum.

Now, let us assume that each permutation π obtained from the identity after $(p-1)$ right-jumps contains at most $p-1$ non-left-to-right-maxima. Let σ be a permutation obtained from the identity after p right-jumps. Using the recurrence hypothesis, σ is obtained from a permutation π with at most $p-1$ non-left-to-right-maxima by moving an element π_i, $1 \leq i \leq n$, in a position located on its right.

We distinguish two cases: (1) π_i is a non-left-to-right-maximum, and (2) π_i is a left-to-right maximum.

Case (1): Since π_i is a non-left-to-right-maximum, there is $j < i$ such that π_j is a left-to-right maximum satisfying $\pi_j > \pi_i$. Since we move π_i on its right, π_j remains on the left of π_i in σ which implies that π_i is a non-left-to-right-maximum in σ. Using the same argument, any non-left-to-right-maximum π_k in π remains a non-left-to-right-maximum in σ. Moreover, let π_k be a left-to-right maximum in π, *i.e.*, $\pi_j < \pi_k$ for all $j < k$. Since the right-jump transformation moves on the right a non-left-to-right-maximum, all values on the left of π_k in σ are yet lower than π_k, which proves that π_k remains a left-to-right maximum in σ. Therefore, σ contains at most $p-1$ non-left-to-right-maxima (as π does).

Case (2): π_i is a left-to-right maximum, *i.e.*, $\pi_j < \pi_i$ for all $j < i$. Since π_i is moved on its right, any left-to-right maximum located on the left of π_i in π remains also a left-to-right maximum in σ. On the other hand, any left-to-right maximum located on the right of π_i in π is greater than π_i and thus, it remains a left-to-right maximum in σ. Therefore, the number of left-to-right maxima in σ is at least the number of left-to-right maxima in π minus one (we do not consider π_i). This means that the number of non-left-to-right-maxima in σ is at most p.

Considering the two previous cases, an induction completes the proof. □

We now derive our first enumeration result for the set $\mathcal{D}_p$ of permutations at distance p from the identity (i.e. the set of permutations reachable from the identity with p right-jumps, but that one cannot reach with less than p right-jumps: $\mathcal{C}_p = \cup_{k \in [\![p]\!]} \mathcal{D}_k$, and this union is disjoint).

Theorem 1 (Permutations after p right-jumps)**.** *The set $\mathcal{D}_p$ of permutations at distance p from the identity is the set of permutations with exactly p non-left-to-right-maxima. Accordingly, the number $d_{n,p}$ of permutations of length n in $\mathcal{D}_p$ is counted by the Stirling numbers $s(n, n-p)$:*

$$d_{n,p} = s(n, n-p) = \sum_{0 \leq j \leq h \leq p} (-1)^j \binom{h}{j} \binom{n-1+h}{p+h} \binom{n+p}{p-h} \frac{(j-h)^{p+h}}{h!} .$$

Proof. After considering Lemma 1, it suffices to prove that any permutation σ with at most p non-left-to-right-maxima can be obtained from the identity after p right-jumps. Let σ be a permutation with $p \geq 1$ non-left-to-right-maxima. Let us assume that the leftmost non-left-to-right-maximum is σ_i and let $j < i$ be the position of the smallest left-to-right maximum σ_j such that $\sigma_j > \sigma_i$. Then we set $\sigma' = \sigma_1 \dots \sigma_{j-1}\sigma_i\sigma_j \dots \sigma_{i-1}\sigma_{i+1} \dots \sigma_n$. Since we have $\sigma_j > \sigma_i$ and also $\sigma_i > \sigma_{j-1}$ (if σ_{j-1} exists), σ_i becomes a left-to-right maximum in σ'. Thus, σ' contains exactly $p-1$ non-left-to-right-maxima and by construction, σ can be obtained from σ' by a right-jump. This proves that permutations at distance p from the identity are exactly the permutations with $n-p$ left-to-right-maxima, which are known to be counted by $s(n, n-p)$ the signless Stirling number of the first kind (see [13] for the closed-form formula due to Schlömilch, and sequence A094638 in [24] for many occurrences of the corresponding triangular array). For instance, the values of $d_{n,p}$ for $n = 7$ and $0 \leq p < 7$ are 1, 21, 175, 735, 1624, 1764, 720. □

The following corollary says a little more on the lattice structure associated to our process "a particle jumps to the right".

Corollary 1 (Changing the starting point and sorting algorithms)**.** *Let σ and π be two permutations and $t_{\sigma^{-1}\cdot\pi}$ be the number of non-left-to-right-maxima in $\sigma^{-1} \cdot \pi$, then $t_{\sigma^{-1}\cdot\pi}$ right-jumps are necessary to obtain π from σ. In particular, if $t_{\sigma^{-1}}$ is the number of non-left-to-right-maxima in σ^{-1}, then $t_{\sigma^{-1}}$ right-jumps are necessary and sufficient to sort by insertion the permutation σ into the identity.*

Proof. Let σ be a permutation and t_σ be the number of its non-left-to-right-maxima, then t_σ right-jumps are necessary and sufficient to obtain σ from the identity. Therefore, $t_{\sigma^{-1}\cdot\pi}$ transformations are sufficient and necessary to obtain $\sigma^{-1} \cdot \pi$ from the identity. We set $t = t_{\sigma^{-1}\cdot\pi}$ and let $\chi_0 = Id, \chi_1, \dots, \chi_{t-1}, \chi_t = \sigma^{-1}\cdot\pi$ be a shortest path between the identity and $\sigma^{-1}\cdot\pi$. Now, let us prove that if a permutation β is obtained from α by one right-jump, then for any permutation γ, $\gamma \cdot \beta$ is also obtained from $\gamma \cdot \alpha$ by one right-jump. Indeed, if we have $\alpha = \alpha_1\alpha_2 \dots \alpha_n$ then β can be written $\beta = \alpha_1 \dots \alpha_{i-1}\alpha_{i+1} \dots \alpha_{j-1}\alpha_i\alpha_j \dots \alpha_n$. Composing by a permutation γ, we obtain $\gamma \cdot \alpha = \gamma(\alpha_1)\gamma(\alpha_2) \dots \gamma(\alpha_n)$ and $\gamma \cdot \beta = \gamma(\alpha_1) \dots \gamma(\alpha_{i-1})\gamma(\alpha_{i+1}) \dots \gamma(\alpha_{j-1})\gamma(\alpha_i)\gamma(\alpha_j) \dots \gamma(\alpha_n)$ which proves that $\gamma \cdot \beta$ is also obtained from $\gamma \cdot \alpha$ by one right-jump. So if we compose by σ at each step of the above shortest path, then we obtain a shortest path of $t_{\sigma^{-1}\cdot\pi}$ right-jumps from σ to π, which completes the proof. □

Since the set $\mathcal{C}_p$ of permutations obtained after p right-jumps is stable for the relation $\prec$, $\mathcal{C}_p$ is also a class $\mathrm{Avoid}(\mathcal{B}_p)$ of pattern avoiding permutations where $\mathcal{B}_p$ is the basis consisting of minimal (relatively to the pattern-involvement relation $\prec$) permutations σ that are not in $\mathcal{C}_p$ (see Definition 1). Theorem 2 gives the explicit description of these basis permutations.

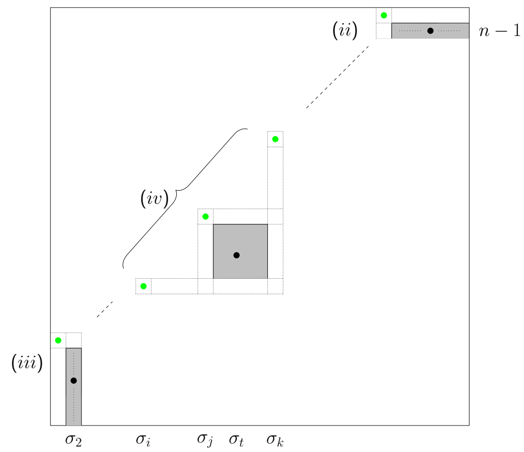

FIGURE 3. An illustration of Theorem 2 that characterizes the basis permutations of $\mathcal{B}_p$. Condition (i) states such a basis permutation has $p+1$ non-left-to-right maxima (drawn in black here, while left-to-right maxima are drawn in green color), condition (ii) states that $n-1$ is not a left-to-right maximum, condition (iii) states that σ_2 is not a left-to-right maximum, and condition (iv) states that there is a "higher" non-left-to-right maximum between 3 left-to-right maxima.

Theorem 2 (Structural description of the basis permutations)**.** *A permutation $\sigma \in \mathcal{S}_n$ belongs to the basis $\mathcal{B}_p$ of forbidden patterns, if and only if the following conditions hold:*

(i) *σ contains exactly $p+1$ non-left-to-right-maxima.*

(ii) *$n-1$ is a non-left-to-right-maximum.*

(iii) *σ_2 is a non-left-to-right-maximum.*

(iv) *For any three left-to-right maxima, σ_i, σ_j and σ_k (with $i<j<k$) such that there is no left-to-right maximum between them, there exists a non-left-to-right-maximum σ_t (with $j<t<k$) satisfying $\sigma_t > \sigma_i$.*

Proof. Let $\sigma \in \mathcal{S}_n$ be a permutation belonging to the basis $\mathcal{B}_p$, *i.e.*, $\sigma \notin \mathcal{C}_p$ and $\pi \prec \sigma$ implies $\pi \in \mathcal{C}_p$. Throughout this proof, we refer to Figure 3 for an illustration of the three conditions (ii), (iii) and (iv).

- First, the deletion of a non-left-to-right-maximum in σ decreases the number of non-left-to-right-maxima by one exactly. Therefore, the minimality of σ implies that σ necessarily contains exactly $p+1$ non-left-to-right-maxima, which proves (i).

- For a contradiction, assume that (ii) is not satisfied, *i.e.*, $n-1$ is a left-to-right maximum. Since n is always a left-to-right maximum, n is on the right of $n-1$ in σ. Thus, the permutation π obtained by deleting n from σ also contains $p+1$ non-left-to-right-maxima (a non-left-to-right-maximum on the right of n in σ remains a non-left-to-right-maximum on the right of $n-1$ in π). Therefore, π does not belong to $\mathcal{C}_p$ which gives a contradiction with the minimality of σ.

- For a contradiction, assume that (iii) is not satisfied, *i.e.*, σ_2 is a left-to-right maximum and thus, σ_1 is smaller than σ_2. Thus, the permutation π obtained by deleting σ_1 from σ also contains $p+1$ non-left-to-right-maxima. Indeed, a non-left-to-right-maximum σ_i in σ such that $\sigma_i < \sigma_1$ becomes a non-left-to-right-maximum $\sigma_i < \sigma_2 - 1$ in π. Moreover, a non-left-to-right-maximum σ_i in σ such that $\sigma_i > \sigma_1$ (there is ℓ, $2 \leq \ell < i$, with $\sigma_i < \sigma_\ell$) becomes a non-left-to-right-maximum $\sigma_i - 1$ in π with $\sigma_i - 1 < \sigma_\ell - 1$. Therefore, π does not belong to $\mathcal{C}_p$ which gives a contradiction with the minimality of σ.

- For a contradiction, assume that (iv) is not satisfied; *i.e.*, there are (i,j,k), $1 \leq i < j < k \leq n$, such that σ_i, σ_j and σ_k are three consecutive left-to-right maxima of σ (consecutive means that there is no other left-to-right maximum between σ_i and σ_j and between σ_j and σ_k), and such that there is no non-left-to-right-maximum σ_ℓ, $j < \ell < k$, verifying $\sigma_i < \sigma_\ell$. Let π be the permutation obtained from σ by deleting σ_j. It is clear that any non-left-to-right-maximum on the left of σ_j in σ remains a non-left-to-right-maximum in π. Let σ_ℓ, $\ell > k$, be a non-left-to-right-maximum on the right of σ_k in σ. If $\sigma_\ell < \sigma_j$ then $\sigma_\ell < \sigma_k - 1$ and σ_ℓ remains a non-left-to-right-maximum in π. If $\sigma_\ell > \sigma_j$ then there is σ_t, $t \geq k$, such that $\sigma_\ell \leq \sigma_t \geq \sigma_k$, and thus, there is $\sigma_t - 1$ on the left of $\sigma_k - 1$ in π with $\sigma_t - 1 > \sigma_k - 1$ which means that $\sigma_\ell - 1$ is a non-left-to-right-maximum in π. Let σ_ℓ $(j < \ell < k)$ be a non-left-to-right-maximum between σ_j and σ_k in σ. Assuming that (iv) is not satisfied, we deduce that $\sigma_\ell < \sigma_i$, and σ_ℓ remains a non-left-to-right-maximum in π. Finally, π contains also $p+1$ non-left-to-right-maxima which gives a contradiction with the minimality of σ.

Conversely, let σ be a permutation satisfying (i), (ii), (iii) and (iv) and π be a permutation obtained by deleting σ_i, $1 \leq i \leq n$, from σ. Let us prove that π belongs to $\mathcal{C}_p$, that is, π contains at most p non-left-to-right-maxima.

- If σ_i is a non-left-to-right-maximum of σ, then π has exactly p non-left-to-right-maxima and thus, $\pi \in \mathcal{C}_p$.

- Now, let us assume that σ_i is a left-to-right maximum of σ. If $\sigma_i = n$, then (ii) implies that $n-1$ is a non-left-to-right-maximum of σ and a left-to-right maximum in π; this implies that π contains p non-left-to-right-maxima and thus, $\pi \in \mathcal{C}_p$. If $\sigma_i = \sigma_1$, then (iii) implies that σ_2 is a non-left-to-right-maximum of σ and $\sigma_2 - 1$ is a left-to-right maximum in π; this implies that π contains p non-left-to-right-maxima and thus, $\pi \in \mathcal{C}_p$. If there exists (j,k), $1 \leq j < i < k \leq n$, such that σ_j and σ_k are left-to-right maxima (we choose j the greatest possible and k the lowest possible with this property). Then, (iv) implies that there is σ_ℓ, $i < \ell < k$, such that $\sigma_j < \sigma_\ell < \sigma_i$ (we choose the lowest possible $\ell > i$). Thus, σ_ℓ is a non-left-to-right-maximum in σ and becomes a left-to-right maximum in π, which implies that π contains exactly p non-left-to-right-maxima, and thus $\pi \in \mathcal{C}_p$.

Finally, the permutation π necessarily belongs to $\mathcal{C}_p$, which completes the proof. □

Corollary 2 (Length of the forbidden patterns)**.** *Permutations of $\mathcal{B}_p$ have length $\leq 2(p+1)$ and $\geq p+2$. As a consequence, $\mathcal{B}_p$ is a finite set.*

Proof. Conditions (ii), (iii) and (iv) of Theorem 2 imply that the number of left-to-right maxima in a basis permutation is at most the number of non-left-to-right-maxima. Since a basis permutation of $\mathcal{B}_p$ has $p+1$ non-left-to-right-maxima, its length is at most $2(p+1)$. □

For instance, the basis for $p = 0, 1, 2$ are respectively $\mathcal{B}_0 = \{21\}$, $\mathcal{B}_1 = \{312, 321, 2143\}$, and $\mathcal{B}_2 = \{4123, 4132, 4213, 4231, 4312, 4321, 21534, 21543, 31254, 32154, 31524, 31542, 32514, 32541, 214365\}$.

4 Enumerative results for basis permutations

In order to obtain a recursive formula for the number $b_{n,p}$ of permutations of length n in the basis $\mathcal{B}_p$, we present the following preliminary lemma.

Lemma 2 (A recursive description)**.** *Let $\sigma \in \mathcal{S}_n$ be a basis permutation having $p \geq 1$ non-left-to-right-maxima and such that $\sigma_{\ell+1} = n$, $\ell \geq 0$. Let α be the subsequence $\sigma_1\sigma_2\ldots\sigma_\ell$ and π be the permutation in $\mathcal{S}_\ell$ isomorphic to α. Then, π is a basis permutation with $p-n+\ell+1$ non-left-to-right-maxima.*

Proof. Any permutation σ can be uniquely written $\sigma = \alpha n \beta$ where α and β are two subsequences of $[\![n]\!]$. Let ℓ be the length of α and let $\pi = \pi_1\pi_2\ldots\pi_\ell$ be the permutation of $[\![\ell]\!]$ that is isomorphic to the subsequence α. Let us prove that π is minimal.

Since σ is minimal, it satisfies the three conditions (ii), (iii) and (iv) of Theorem 2. Since all elements in β are non-left-to-right-maxima in σ, π contains exactly $p-(n-1-\ell)$ non-left-to-right-maxima and thus, $n-p-1$ left-to-right maxima.

- The condition (iii) of Theorem 2 on σ does not involve the part $n\beta$. Therefore, π satisfies (iii).

- The deletion of $n\beta$ in σ preserves the condition (iv) on π. Thus, π satisfies (iv).

- Let σ_i, σ_j and $\sigma_{\ell+1} = n$, $1 \leq i < j \leq \ell$, be the last three left-to-right maxima of σ. After the deletion of $n\beta$, the two left-to-right maxima of σ, σ_i and σ_j, are respectively transported in π into π_i and $\pi_j = \ell$. Condition (iv) on σ ensures that there is σ_k, between σ_j and n such that $\sigma_k > \sigma_i$. The greatest value σ_k satisfying this property is then transported in π into $\ell-1$, which proves that $\ell-1$ is on the right of ℓ in π. Thus, π satisfies (ii).

Using Theorem 2, the permutation π is a basis permutation with $p+\ell-n+1$ non-left-to-right-maxima. □

Theorem 3 (An infinite recursion)**.** *The number $b_{n,p}$ of basis permutations of length n in $\mathcal{B}_p$ (or equivalently having exactly $p+1$ non-left-to-right-maxima) is given by the following recurrence relation (for $p < n-2$):*

$$b_{n,p} = \sum_{\ell=0}^{p-1} (\ell+1)! \cdot \binom{n-2}{\ell} \cdot b_{n-\ell-2,p-\ell-1}$$

anchored with $b_{n,p} = 0$ if $p < (n-2)/2$ or $p > n-2$, and $b_{n,n-2} = (n-1)!$ for $n > 1$.

Proof. Any permutation σ of length $n \geq 1$ contains at least one left-to-right maximum and thus, at most $n-1$ non-left-to-right-maxima which implies that $b_{n,p} = 0$ for $n \leq p+1$. Using the proof of Corollary 2, we also have $b_{n,p} = 0$ for $n > 2(p+1)$. Moreover, the basis permutations of length n with $n-1$ non-left-to-right-maxima are the permutations of the form $n\alpha$ where $\alpha \in \mathcal{S}_{n-1}$. So, we have $b_{n,n-2} = (n-1)!$ for $n > 1$.

Now, let us prove the recursive relation. Let $\sigma \in \mathcal{S}_n$ be a basis permutation with $p+1$ non-left-to-right-maxima. We consider its unique decomposition $\sigma = \alpha n \beta$ where α and β are some subsequences of $[\![n]\!]$. Let $\ell+2$, $\ell \geq 0$, be the length of $n\beta$ and let π be the permutation in $\mathcal{S}_{n-\ell-2}$ isomorphic to α. Using Lemma 2, π is minimal with $p-\ell-1$ non-left-to-right-maxima. So, we can associate to $\sigma = \alpha n \beta$ the pair (π, γ) where $\pi \in \mathcal{S}_{n-\ell-2}$ is minimal with $p-\ell-1$ non-left-to-right-maxima and $\gamma \in \mathcal{S}_{\ell+1}$ is isomorphic to β.

Conversely, let π be a basis permutation of length $n-\ell-2$ with $p-\ell-1$ non-left-to-right-maxima and $\gamma \in \mathcal{S}_{\ell+1}$. We construct a basis permutation σ of length n with $p+1$ non-left-to-right-maxima as follows. From $\gamma \in \mathcal{S}_{\ell+1}$, we construct a subsequence β of $[\![n-1]\!]$ of length $\ell+1$ such that β contains the value $n-1$ and such that β is isomorphic to γ. Since $n-1$ belongs to β, its position in β also is the position of the greatest value of γ. So, β is characterized by the choice of ℓ values among $[\![n-2]\!]$. Now, we define the unique subsequence α of $[\![n-2]\!] \backslash X$ isomorphic to π where X is the set of values used in β. This construction ensures that $\sigma = \alpha n \beta$ is a basis permutation of length n with $p+1$ non-left-to-right-maxima, and so $\sigma \in \mathcal{B}_p$. So, there are $\binom{n-2}{\ell}$ possibilities to choose β and $(\ell+1)!$ possibilities to choose γ and $b_{n-\ell-2,p-\ell-1}$ possibilities to choose a basis permutation $\pi \in \mathcal{S}_{n-\ell-2}$ with $p-\ell-1$ non-left-to-right-maxima. Varying ℓ from 0 to $p-1$, we obtain the recursive formula. □

Theorem 3 allows us to find the bivariate exponential generating function for the number of basis permutations with respect to the number of non-left-to-right-maxima.

Theorem 4 (Closed-form for the bivariate generating function)**.** *Consider the bivariate exponential generating function* $B(x,y) = \sum_{n\geq 0, p\geq 0} b_{n,p} \frac{x^n y^p}{n!}$ *where the coefficient of* $\frac{x^n y^p}{n!}$ *is the number* $b_{n,p}$ *of basis permutations of length* n *in* $\mathcal{B}_p$*. Then, we have*

$$B(x,y) = \frac{1}{2y}\frac{\sqrt{1+4/y}-1}{\sqrt{1+4/y}}\cdot(1-xy)^{\frac{1}{2}(1+\sqrt{1+4/y})} + \frac{1}{2y}\frac{\sqrt{1+4/y}+1}{\sqrt{1+4/y}}\cdot(1-xy)^{\frac{1}{2}(1-\sqrt{1+4/y})} - \frac{1}{y}\,.$$

Proof. Setting $F_p(x) := \sum_{n\geq 0} b_{n,n-p}\frac{x^n}{n!}$ and $F(x,y) := \sum_{p\geq 0} F_p(x) y^p$, we have $B(x,y) = F(xy, 1/y)$. (We work with the generating function $F_p(x)$ of the $(b_{n,n-p})$'s rather than the generating function of the $(b_{n,p})$'s because then the derivation of the proof is simpler).

Taking the second derivative of $F(x,y)$ with respect to x gives

$$\partial_x^2 F(x,y) = \partial_x^2\left(\sum_{p\geq 0} F_p(x)y^p\right) = \partial_x^2 F_0(x) + \partial_x^2 F_1(x) y + \partial_x^2 F_2(x) y^2 + \sum_{p\geq 3}\partial_x^2 F_p(x) y^p\,. \quad (1)$$

Now, the recursive relation of Theorem 3 for $b_{n+2,n-p+2}$ implies for $p \geq 3$:

$$\begin{aligned}\partial_x^2 F_p(x) &= \sum_{n\geq 0} b_{n+2,n-p+2}\frac{x^n}{n!} = \sum_{n\geq 0}\frac{x^n}{n!}\sum_{\ell=0}^{n-p+1}(\ell+1)!\cdot\binom{n}{\ell}\cdot b_{n-\ell,(n-\ell)-(p-1)}\\ &= \sum_{n\geq 0}(n+1)!\frac{x^n}{n!}\cdot\sum_{n\geq 0} b_{n,n-p+1}\frac{x^n}{n!} = \frac{1}{(1-x)^2}F_{p-1}(x).\end{aligned}$$

Plugging this recurrence into the differential equation (1) (and using $F_0(x) = F_1(x) = 0$) gives:

$$\partial_x^2 F(x,y) = \partial_x^2 F_2(x) y^2 + \sum_{p\geq 2}\frac{y}{(1-x)^2}F_p(x)y^p.$$

It thus remains to simplify F_2; well, the initial conditions of Theorem 3 ($b_{n,n-2} = (n-1)!$ and $b_{n,p} = 0$ for $n < 2$), implies that

$$F_2(x) = \sum_{n\geq 2} b_{n,n-2}\frac{x^n}{n!} = \sum_{n\geq 2}\frac{(n-1)!}{n!}x^n = -\ln(1-x) - x\,.$$

This leads to the main differential equation:

$$\partial_x^2 F(x,y) = \partial_x^2 F_2(x) y^2 + \frac{y}{(1-x)^2}F(x,y) = \frac{y}{(1-x)^2}(y + F(x,y))\,. \quad (2)$$

First, by "plug & prove", the solutions of $\partial_x^2 F(x) = yF(x)/(1-x)^2$ are a linear combination of $(1-x)^\alpha$, with $\alpha = (1+\sqrt{1+4y})/2$, or $\alpha = (1-\sqrt{1+4y})/2$. Now, $F(x,y) = -y$ is a trivial particular solution of the non homogeneous differential equation (2), so the general solution of this differential equation is of the form:

$$F(x,y) = K(y)\cdot(1-x)^{\frac{1}{2}(1+\sqrt{1+4y})} + L(y)\cdot(1-x)^{\frac{1}{2}(1-\sqrt{1+4y})} - y.$$

Since $F(0,y)=0$ and $\frac{\partial F(x,y)}{\partial x}\rfloor_{x=0}=0$, we respectively deduce the two equations

$$K(y)+L(y)-y=0 \qquad \text{and} \qquad -K(y)\cdot(1+\sqrt{1+4y})-L(y)\cdot(1-\sqrt{1+4y})=0\,,$$

thus we obtain

$$K(y)=\frac{y}{2}\frac{\sqrt{1+4y}-1}{\sqrt{1+4y}} \qquad \text{and} \qquad L(y)=\frac{y}{2}\frac{\sqrt{1+4y}+1}{\sqrt{1+4y}}\,.$$

This gives

$$F(x,y)=\frac{y}{2}\frac{\sqrt{1+4y}-1}{\sqrt{1+4y}}\cdot(1-x)^{\frac{1}{2}(1+\sqrt{1+4y})}+\frac{y}{2}\frac{\sqrt{1+4y}+1}{\sqrt{1+4y}}\cdot(1-x)^{\frac{1}{2}(1-\sqrt{1+4y})}-y\,,$$

and therefore the theorem, as $B(x,y)=F(xy,1/y)$. □

Theorem 5 (Asymptotics)**.** *The exponential generating function for the number of basis permutations with respect to their length is given by*

$$B(x)=B(x,1)=\sum_{n\geq 0} b_n\frac{x^n}{n!}=\frac{\sqrt{5}-1}{2\sqrt{5}}\cdot(1-x)^{\frac{1+\sqrt{5}}{2}}+\frac{\sqrt{5}+1}{2\sqrt{5}}\cdot(1-x)^{\frac{1-\sqrt{5}}{2}}-1.$$

It is a D-finite transcendental function satisfying the following differential equation

$$B(x)-(1-x)^2\partial_x^2 B(x)+1=0, \qquad B(0)=B'(0)=0\,. \tag{3}$$

Equivalently, its coefficients b_n satisfy the recurrence

$$b_{n+2}=2nb_{n+1}+(1+n-n^2)b_n, \qquad b_0=b_1=0, b_2=1\,,$$

and the asymptotics is given by

$$b_n\sim\frac{\phi}{\sqrt{5}\,\Gamma(\phi-1)}\frac{1}{n^{2-\phi}}(1+o(1))\,,$$

where ϕ is the golden ratio $\phi=(1+\sqrt{5})/2$, and $\Gamma(z):=\int_0^{+\infty}t^{z-1}\exp(-t)dt$ is the Euler Gamma function.

Accordingly, a permutation of length has a probability asymptotically 0 to be an element of the basis of forbidden pattern, however, this probability is not "very small" as it decays only polynomially:

$$\mathrm{Prob}(s\in\mathcal{S}_n \textit{ belongs to } \cup_p\mathcal{B}_p)=\frac{b_n}{n!}\approx 0.499/n^{.381}(1+o(1))\,.$$

Proof. Setting $y=1$ in the bivariate exponential generating function given in Theorem 4 gives $B(x)$. If a function $B(z)=\sum b_n z^n$ is D-finite (it is satisfying a linear differential equations, with polynomial coefficients in z), then its coefficients b_n are polynomially recursive (or "P-recursive"): they satisfy a linear recurrence, with polynomial coefficients in n. See e.g. [13, 25] for more on these two equivalent notions. Starting from the building blocks

$(1-x)^a$, which are D-finite, and then using the closure properties of D-finite functions (by sum and product) gives the differential equation (3) (this is e.g. implemented in the Gfun Maple package, see [23]). The recurrence is obtained by extracting the coefficient of x^n on both sides of the differential equation. The asymptotics follows from a singularity analysis (see [13]) on each term of the shape $(1-x)^a$, indeed, for any $a \in \mathbb{R}$ which is not an integer, one has:

$$[x^n](1-x)^a = \frac{1}{\Gamma(-a)n^{1+a}} \left(1 + \frac{1}{2}a(a+1)\frac{1}{n} + O(\frac{1}{n^2})\right) . \qquad \square$$

Note that $B(x,y)$ is D-finite in the variable x:

$$1 + yB(x,y) - (1-xy)^2 \partial_x^2 B(x,y) = 0 \qquad \text{with } B(0,y) = (\partial_x B)(0,y) = 0,$$

but it is not D-finite in the variable y. This follows from a saddle point analysis on $B(1,y) = \sum_n \beta_n y^n$, indeed the asymptotics of β_n then involves arbitrarily large $(\ln n)^d$, while the asymptotics of a D-finite function can only have a finite sum of such powers of log, see [13]. This argument is thus similar to a proof that $(1-y)^{1-y}$ is not D-finite.

Remark [Irrational critical exponent]: In combinatorics and in statistical physics, most of the asymptotics of integer sequences are of the shape $b_n \sim Cn^\alpha A^n$, and the exponent α which appears there is a key quantity: its value is often the signature of some universal phenomena (in physics, it is called "critical exponent"). For D-finite sequences, the theory implies that it is an algebraic number, however, this exponent is very often -3/2, or a dyadic number (for the reasons explained in [2]), or a rational number (due to a results on G-functions). Indeed, a theorem (resulting from the works of Katz, André, Chudnovky & Chudnovsky, see [9]) states that G-functions (D-finite functions with integer coefficients and non zero radius of convergence) have a rational critical exponent. Now, instead of considering the exponential generating function $B(x) = \sum b_n x^n/n!$, we may consider its inverse Borel transform, i.e. the ordinary generating function $\sum b_n x^n$. It is also a D-finite function, because D-finite functions are closed by Hadamard product, and therefore the Borel transform (and the inverse Borel transform) of a D-finite function is D-finite (i.e. if the sequence b_n is P-recursive, so are $n!b_n$ and $b_n/n!$). We have thus a new D-finite function with integer coefficients and irrational critical exponent (involving the golden ratio ϕ), but this is not contradicting the G-function theorem, because, due to the multiplication by $n!$, we now have a 0 radius of convergence. In conclusion, we have here one of the few examples in combinatorics of a problem leading to an irrational critical exponent. Other examples are given via the KPZ formula in physics, or via quantities related to quadtrees, see [13].

For such a combinatorial structure $\mathcal{B}_p$, it could be possible that the complementary set has a nicer structure, those permutations not in the basis are for sure counted by $u_n = n! - b_n$; this sequence satisfies $u_{n+3} = (n+1)(n^2-n-1)u_n - (3n^2+3n-1)u_{n+1} + 3(n+1)u_{n+2}$, which is still a quite nice recurrence but of one order more than the recurrence for b_n, so it is a heuristic confirmation than b_n is a more fundamental sequence.

The first values of b_n (the number of basis permutations of length n) are $1, 2, 7, 32$, 179, 1182, 8993, 77440 for $2 \leq n \leq 9$. We added this sequence to the On-line Encyclopedia of Integer Sequences [24]:

$p\backslash n$	2	3	4	5	6	7	8	9	10	11	$\#\mathcal{B}_p$
0	1										1
1		2	1								3
2			6	8	1						15
3				24	58	18	1				101
4					120	444	244	32	1		841
5						720	3708	3104	700	50	8232
6							5040	33984	39708	13400	78732
Σ	1	2	7	32	179	1182	8993	77440	744425	7901410	

TABLE 1. Number $b_{n,p}$ of basis permutations of length n (the "minimal forbidden patterns" of $\mathcal{B}_p$, or equivalently, with $p+1$ non-left-to-right-maxima) where $2 \leq n \leq 11$ and $0 \leq p \leq 6$ (OEIS A265163). The last column contains $\beta_p := \sum_n b_{n,p}$ (OEIS A265164); the last line contains $b_n := \sum_p b_{n,p}$ (OEIS A265165).

There is a vast literature in number theory analyzing the modular congruences of famous sequences (Pascal triangle, Fibonacci, Catalan, Motzkin, Apéry numbers [11, 26, 22, 15]). The properties of $b_n \bmod m$ are sometimes called "supercongruences" when m is the power of a prime number: many articles considered $m = 2^r$, or $m = 3^r$. We now give a result which holds for any m (not necessarily the power of a prime number).

Theorem 6 (Supercongruences for D-finite functions)**.**
Any P-recurrence $P_0(n)u_n = \sum_{i=1}^r P_i(n)u_{n-i}$, for which the polynomial $P_0(n)$ is ultimately invertible $\bmod m$ (i.e. $\gcd(P_0(n), m) = 1$, for all n large enough) is ultimately periodic[2] $\bmod m$, and there is an algorithm to get this period.

In particular, recurrences such that $P_0(n) = 1$ are periodic. Accordingly, our sequence $b_n \bmod m$ is periodic for any m.

[2] In the sequel, we will omit the word "ultimately": a periodic sequence of period p is thus a sequence for which $u_{n+p} = u_n$ for all large enough n. Some authors use the terminology "eventually periodic" instead.

Proof. Indeed, as the leading term P_0 is invertible, we can write:

$$u_n \bmod m = \sum_{i=1}^{r} \frac{P_i(n) \bmod m}{P_0(n) \bmod m} (u_{n-i} \bmod m) \,,$$

in which each term has just a finite set of possible values. What is more, for any polynomial $P(n)$ with integer coefficients, $P(n) \bmod m$ is of period p, for some $p|m$. (This follows from the fact that the sum and the product is preserving periodicity $\bmod m$, as we did not require in the definition of "period" that m is the smallest m such that the sequence is m periodic). Therefore, one can then construct a Markov chain (an automaton) listing all the possible $2r$-tuples of values $\bmod m$ for the u_{n-i} and their coefficients P_i, where the recurrence dictates the transitions in this Markov Chain. The pigeon-hole principle implies that there is a loop in this finite graph, and this gives our period.

Besides, starting with the Ansatz that $u_n = u_{n+p}$ for $n \in [n_1+1, n_1+p]$ (such a p can be found by brute-force, as n_1 has to be smaller than the number of states in the automaton, and p has to be smaller than m^{2r}), it is enough to check that this property goes on for $r+1$ steps to prove that u_n is p periodic. □

This theorem explains the periodic behavior of $b_n \bmod m$. By brute-force computation, we can get $b_n \bmod m$, for any given m. For example $b_n \bmod 15$ is periodic of period 12: for $n \geq 9$, one has $b_n \bmod 15 = (10, 5, 10, 10, 0, 10, 5, 10, 5, 5, 0, 5)^\infty$. The period can be quite large, for example $b_n \bmod 3617$ has period 26158144. We computed the period of $b_n \bmod m$ for all $m \leq 4000$; this sequence of periods is given by OEIS A265166 and seems to satisfy some nice congruences:

Conjecture 1 (Explicit periods for the D-finite sequence b_n)**.**
We write $\mathrm{Period}(b_n \bmod m) = p$ *if and only if* $\exists n_1 \in \mathbb{N}$ *such that for all* $n > n_1$, $b_{n+p} = b_n \bmod m$*, and there is no smaller* $p > 0$ *for which this holds.*
Let $b_{n+2} = 2nb_{n+1} + (1 + n - n^2)b_n, \qquad b_0 = b_1 = 0, b_2 = 1$*, then*

a) $\mathrm{Period}(b_n \bmod m) = 1$ *if and only if* m *is a product (possibly reduced to one single factor) of (non necessarily distinct) primes in* $0, 1, 4 \bmod 5$.
b) $\mathrm{Period}(b_n \bmod m) = 2$ *if and only if* $m/2$ *is a product (possibly reduced to one single factor) of (non necessarily distinct) primes in* $0, 1, 4 \bmod 5$.
c) *If the period is not 1, then it is an even number.*
d) *For any prime* p, $\mathrm{Period}(b_n \bmod p) | 2p(p-1)$.
e) *For any prime* p *not in* $0, 1, 4 \bmod 5$, *(and* $p^r \neq 4$*),*

$$\mathrm{Period}(b_n \bmod p^r) = p^r \, \mathrm{Period}(b_n \bmod p) \,.$$

f) *If* $m = p_1^{e_1} \dots p_k^{e_k}$ *(where the* p_i*'s are distinct primes), then*

$$\mathrm{Period}(b_n \bmod m) = \mathrm{lcm}(\mathrm{Period}(b_n \bmod p_1^{e_1}), \dots, \mathrm{Period}(b_n \bmod p_k^{e_k})) \,,$$

where lcm *stands for the least common multiple (it is possible to reformulate this formula using the Carmichael function* λ*, or the Euler totient function* ϕ*).*

Some of these claims are obviously interdependent. We did not try to prove the above conjectures, as they are a little bit too far from the main topics of our article, however, we do believe that they are interesting as they are typical of phenomena happening for many recurrences in general, and it would be nice to have a methodology to prove all this type of properties on the full class of D-finite sequences (following the cycles in the automata corresponding to the one mentioned in the proof of Theorem 6 should prove the multiplicative properties mentioned in our conjecture).

Nota bene: it is not always the case that P-recursive sequences are periodic mod p. E.g., it was proven than Motzkin numbers are not periodic mod m [17], and it seems that

$$(n+3)(n+2)u_n = 8(n-1)(n-2)u(n-2) + (7n^2+7n-2)u(n-1)\,, \quad u_0 = 0, u_1 = 1\,,$$

is also not periodic mod m, for any $m > 2$ (this P-recursive sequence counts a famous class of permutations, the Baxter permutations). This is coherent with our Theorem 6, as the leading term in the recurrence (the factor $(n+3)(n+2)$) is not invertible mod m, for infinitely many n.

We end by proving the following limit law:

Theorem 7 (Limit Law)**.** *In the model where all permutations of length* n *are equidistributed, a random permutation of length* n *in* $\cup\mathcal{B}_p$ *is typically a member of* $\mathcal{B}_p$*, for* $p \sim n - (\ln n)/\sqrt{5}$*, with Gaussian fluctuations. Equivalently, the average number of left-to-right-maxima in a random basis permutation is* $p \sim (\ln n)/\sqrt{5}$ *with Gaussian fluctuations.*

Proof. This follows from the closed-expression for $B(x,y)$, or from a singularity analysis of the differential equation. Indeed, the average and standard deviation follow from the computation of $\partial_y B(x,y)$ and $\partial_y^2 B(x,y)$ at $y=1$. The Gaussian limit law follows from the quasi-power theorem of Hwang applied to a variable exponent perturbation or to our non-confluent differential equation (see Theorem IX.11 and Theorem IX.18 from [13]). □

As a random permutation of $\mathcal{S}_n$ has $\ln n$ left-to-right maxima on average, the above theorem quantifies to what extent the right-jump process kills the left-to-right maxima when one starts from the identity permutation.

5 Conclusion

In this article, we analyzed the iteration of the process "a particle jumps to the right" in a permutation, and we gave the typical properties of the patterns which are not reached after p moves. This is one of the first enumerations of an infinite basis (of the forbidden patterns, for a permutation class). We expect our approach (introducing a catalytic variable and getting a D-finite function) to work in many other cases. However, we already know a nice permutation class for which the basis is non D-finite. Indeed, as an extension of this work, an interesting question is to consider a model in which both right-jumps and left-jumps are allowed: this is a very natural process, also related to sorting algorithms and bioinformatics process. In a for-coming work, we show that for this new model, the basis of forbidden minimal patterns for permutations obtained by p iterations of the process is related to Young Tableaux with 2 equal first rows (but it is no more D-finite, unlike the pure right-jump iteration process that we considered in this article).

Another natural question is: is it the case that using e.g. the Foata correspondence between records and cycles in permutations, there is an elegant process corresponding to a "particle jumps to the right", with permutations at distance p from identity being then counted in terms of cycles in the permutation?

It is interesting that the asymptotics of the process analyzed in the present article are not classical, but involve nice constants, and nice recurrences. To get some more direct "bijective" proofs (or the "proof from the Book") of our formulae is also an interesting question: as a credo, it cannot be the case that such nice formulae are only reached by solving differential equations (like we did in this article). It may be the case that a generating tree [1] approach leads to the simple recurrence we get for b_n.

Last but not least, we already mentioned a vast literature of publications in number theory analyzing the modular congruences of famous sequences (Pascal triangle, Fibonacci, Catalan, Motzkin, Apéry numbers, ..., [11, 26]). It seems to us that our approach to tackle them at the level of D-finite functions is new (see also [22, 15]), and it would be worth to analyze these properties in full generality. Is it possible to get faster proof than our mixture of Anzatz + brute force proof? We proved that $f_n \bmod m$ (where m can be any integer) is a periodic function (of period bounded by a polynomial in m), and we can make explicit this period and the values of $f_n \bmod m$ for any given m. The question is then: is it possible to give an "a priori" direct formula for $f_n \bmod m$? We do not think that such a "universal" formula exists, but it is natural to ask what its "cost" (in the sense of complexity theory): it may be possible that there is a formula (or an algorithm) less costly than a polynomial-time algorithm.

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E-mail address: cyril.banderier at lipn.univ-paris13.fr

E-mail address: barjl at u-bourgogne.fr, celine.moreira at u-bourgogne.fr